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A note on some dynamic crack problems in linear viscoelasticity

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THE PROBLEM of a semi-infinite crack propagating at constant speed in a linear viscoelastic medium under plane-strain conditions and with particular time-dependent loadings in considered. Explicit results for crack tip stresses and displacements are given for short and long times. The analysis is to some extent heuristic, but is supported by an exact analysis of the corresponding mode 3 problems (Appendix 1) and a formal factorisation (Appendix 2). Also considered is the situation of a crack propagating steadily in a viscoelastic strip. Singular perturbation methods are used in the limit when a dimensionless parameter $\varepsilon_1 = v\tau/L \ll 1$. v is the crack speed, τ a representative relaxation time of the medium and L a length associated with the problem (half strip width).

Rozpatruje się problem półnieskończonego pęknięcia rozchodzącego się ze stałą prędkością w liniowym ośrodku lepkosprężystym w warunkach płaskich naprężeń i przy szczególnego typu zależnych od czasu obciążeniach. Podaje się wyrażenia jawne na naprężenia i przesunięcia dla końcówki pęknięcia, dla czasów krótkich i długich. Analiza jest do pewnego stopnia heurystyczna lecz jest wspomagana przez ścisłą analizę odpowiednich problemów typu 3 (Uzupełnienie 1) oraz formalną faktoryzację (Uzupełnienie 2). Rozpatrzono też przypadek jednostajnej propagacji pęknięcia w lepkosprężystym pasku. Wykorzystano metody rozwinięć osobliwych w granicy, gdy parametr bezwymiarowy $\varepsilon_1 = v\tau/L \ll 1$; v jest prędkością pęknięcia, τ — reprezentatywnym czasem relaksacji ośrodka a L jest długością charakterystyczną związaną z problemem (połowa szerokości paska).

Рассматривается проблема полубесконечной трещины распространяющейся в линейной вязко-упругой среде в условиях плоских напряжений и при частного типа, зависящих от времени, нагрузках. Приведены явные выражения для напряжения и перемещения на концах трещины, для коротких и длинных отрезков времени. Анализ в некоторой мере евристический, но вспомагается точным анализом соответствующих проблем типа 3 (Дополнение 1) и формальной факторизацией (Дополнение 2). Рассмотрен тоже случай одномерного распространения трещины в вязко-упругой полосе. Использованы методы особых разложений в пределе, когда безразмерный параметр $\varepsilon_1 = v\tau/L < 1$; v является скоростью трещины, τ — характеристическим временем релаксации среды, а L является характеристической длиной, связанной с проблемой (половина ширины полосы).

Introduction

As FAR as the stress analysis of moving crack problems in viscoelastic media is concerned there has been to our knowledge only one attempt [1] which considered transient motion and included the inertia terms in the analysis. The problem considered in [1] was that of a semi-infinite crack which suddenly appeared and propagated rectilinearly with uniform velocity under mode 3 conditions. The mode 3 [anti-plane or longitudinal shear] assumption has been criticised by KNAUSS [2] on the grounds that a crack doesn't grow rectilinearly in a viscoelastic solid under the experimental conditions of longitudinal shear. Nevertheless, we expect the main features of the results of [1] to be indicative of what would happen in the much more complicated plane-strain situation. In this note we give an approximate solution of some mode 1 (plane-strain) problems, the solutions assumed valid for small and large times, and compare these solutions with exact solutions given in Appendix 1 for the mode 3 situation. Also, as a means of comparison we reconsider and slightly generalise the steady-state problem of a crack growing in a viscoelastic strip [3].

1. Analysis

Before considering the dynamic crack problem we state briefly some well-known results of the linear theory of viscoelasticity (see for example CHRISTENSEN [4]). The stress-strain relations for isotropic viscoelasticity can be written

$$s_{ij} = \int_{-\infty}^{t} G_1(t-\tau)\frac{de_{ij}(\tau)}{d\tau}\,d\tau \tag{1.1}$$

and

$$\sigma_{kk} = \int_{-\infty}^{t} G_2(t-\tau)\frac{d\varepsilon_{kk}(\tau)}{d\tau}\,d\tau, \tag{1.2}$$

where

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\,\delta_{ij}\sigma_{kk}, \quad s_{ii} = 0, \tag{1.3}$$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\,\delta_{ij}\varepsilon_{kk}, \quad e_{ii} = 0. \tag{1.4}$$

The usual summation convention is employed above and the infinitesimal strain ε_{ij} is defined by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}). \tag{1.5}$$

An alternative form to Eqs. (1.1) and (1.2) using differential operators would be

$$H_1\left(\frac{d}{dt}\right)s_{ij} = P_1\left(\frac{d}{dt}\right)e_{ij}, \tag{1.6}$$

$$H_2\left(\frac{d}{dt}\right)\sigma_{kk} = P_2\left(\frac{d}{dt}\right)\varepsilon_{kk}, \tag{1.7}$$

where H_1, H_2, P_1 and P_2 are functions of the operator $\frac{d}{dt}$.

To the above constitutive equations must be added the equation of motion

$$\frac{\partial\sigma_{ij}}{\partial x'_j} = \varrho\frac{\partial^2 u_i}{\partial t^2}, \tag{1.8}$$

where ϱ is the density and the x'_j are stationary Cartesian coordinates.

Problems in which the boundary (i.e. a semi-infinite crack) moves with velocity v are to be considered, so put

$$x_1 = x'_1 - vt, \quad x_2 = x'_2, \quad x_3 = x'_3. \tag{1.9}$$

We now apply the Laplace transform over time and the Fourier transform over x, defined by

$$\bar{f}(x_1, p) = \int_0^\infty e^{-pt} f(x_1, t)\,dt,$$
(1.10)
$$\bar{\bar{f}}(s, p) = \int_{-\infty}^{\infty} e^{isx_1} \bar{f}(x_1, p)\,dx_1.$$

Then Eq. (1.8) becomes

(1.11) $$\frac{d\bar{\bar{\sigma}}_{i2}}{dx_2} - is\bar{\bar{\sigma}}_{i1} = \varrho(p+ivs)^2 \bar{\bar{u}}_i, \quad i = 1, 2, 3,$$

while Eqs. (1.6) and (1.7) become

(1.12) $$H_1(p+ivs)\bar{\bar{s}}_{ij} = P_1(p+ivs)\bar{\bar{e}}_{ij}, \quad H_2(p+ivs)\bar{\bar{\sigma}}_{kk} = P_2(p+ivs)\bar{\bar{\varepsilon}}_{kk}$$

and comparison with Eqs. (1.1) and (1.2) gives

(1.13) $$p\bar{G}_1(P) \equiv P_1(p)/H_1(p); \quad p\bar{G}_2(P) \equiv P_2(p)/H_2(p).$$

Equations (1.11) and (1.12) are identical with the transformed elastic equations so the formulation of the elastic problem of BAKER [5], for example, can be used here with the replacements

(1.14)
$$3\lambda + 2\mu \equiv (p+ivs)\bar{G}_2(p+ivs),$$
$$2\mu \equiv (p+ivs)\bar{G}_1(p+ivs),$$

where λ and μ are the Lamé constants. Also the elastic wave speeds $c_1^2 = (\lambda+2\mu)/\varrho$ and $c_2^2 = \mu/\varrho$ are replaced by the corresponding functions of the transformed variables from Eqs. (1.14).

We consider situations where a semi-infinite crack propagates in an infinite viscoelastic medium under mode 1 conditions (in Appendix 1 exact solutions to some mode 3 situations are given). Typical boundary conditions in the moving coordinate system are:

(1.15)
$$\sigma_{12} = 0, \qquad -\infty < x_1 < \infty, \quad x_2 = 0, \quad t > 0,$$
$$\sigma_{22} = -\sigma(x_1, t) \qquad x_1 < 0, \quad x_2 = 0, \quad t > 0,$$
$$u_2 = 0, \qquad x_1 > 0, \quad x_2 = 0, \quad t > 0.$$

The stress $\sigma(x_1, t)$ on the crack faces is assumed to be known. Taking the Fourier and Laplace transforms of these boundary conditions gives

(1.16) $$\bar{\bar{\sigma}}_{22} = -g(s, p) + H_+(s),$$

where

$$H_+(s) = \int_0^\infty e^{isx_1} \bar{h}(x_1, p)\,dx_1$$

and $\bar{h}(x_1, p) = \int_0^\infty e^{-pt} h(x_1, t)\,dt$ with $h(x_1, t)$ the unknown σ_{22} stress ahead of the crack tip. Also

$$g(s, p) = \int_{-\infty}^{0} e^{isx_1} dx_1 \int_0^\infty \sigma(x_1, t) e^{-pt}\,dt.$$

If we define

$$\bar{u}_2 = 0, \quad x_1 > 0, \quad \text{on} \quad x_2 = 0, \tag{1.17}$$
$$= j(x_1), \quad x_1 < 0, \quad \text{on} \quad x_2 = 0,$$

then

$$\bar{\bar{u}}_2 \equiv J_-(s) = \int_{-\infty}^{0} e^{isx_1} j(x_1)\,dx_1 \quad \text{on} \quad x_2 = 0. \tag{1.18}$$

We assume that the behaviour of the stresses and displacements as $x_1 \to \pm\infty$ are such that $H_+(s)$ and $J_-(s)$ are functions regular in overlapping half-planes of the complex s plane. Then, solving the equations of motion (1.11) in terms of potentials and following the algebra of [5] leads to the Wiener-Hopf equation

$$-g(s,p) + H_+(s) = K(s,p)J_-(s), \tag{1.19}$$

where

$$K(s,p) = \left\{ \frac{(s^2+\gamma_2^2)^2}{4\gamma_1\gamma_2} - s^2 \right\} \frac{4c_2^2(s)\gamma_2\mu(s)}{v^2\left(s - \frac{ip}{v}\right)^2} \tag{1.20}$$

with

$$\gamma_j^2 = s^2 + (p+ivs)^2/c_j^2, \quad j = 1, 2,$$

and

$$3\varrho c_1^2 = (p+ivs)[\overline{G}_2(p+ivs) + 2\overline{G}_1(p+ivs)],$$
$$2\varrho c_2^2 = (p+ivs)\overline{G}_1(p+ivs), \quad 2\mu = (p+ivs)\overline{G}_1(p+ivs).$$

The fact that c_1^2 and c_2^2 are now complicated functions of s makes the factorisation, $K = K_+K_-$, of K into the product of functions regular and non-zero in respective half-planes difficult in general. Nevertheless, if we assume that this essential step in the solution of the functional equation (1.19) has been made, then Eq. (1.19) can be rearranged as

$$L(s) = K_-(s,p)J_-(s) + C_-(s) = \frac{H_+(s)}{K_+(s,p)} - C_+(s), \tag{1.21}$$

where

$$\frac{g(s,p)}{K_+(s,p)} = C_+(s) + C_-(s). \tag{1.22}$$

In each of the above expressions the plus subscript denotes regularity in some upper region of the complex s plane and the minus subscript regularity in some lower region. The two regions are assumed to have a common strip of regularity. The sum split (1.22) can be effected by Cauchy's theorem when g/K_+ is regular in some strip of the complex s plane (see e.g. NOBLE [6] for more details).

Our intention here is to give an approximate solution of the dynamic crack problem in a viscoelastic medium (i.e. an approximate solution of Eq. (1.19)) for mode 1 conditions and guide the solution by comparison with the exact mode 3 solution given in Appendix 1. To do this we note that BAKER [5] has given the factorisation $K = K_+K_-$ in the elastic case, we amend this a little and write it as

$$(1.23)\qquad \begin{aligned} K_+(s,p) &= [(1-v/\bar{c}_2)s+ip/\bar{c}_2]^{1/2}F_+(s)\,(1-v/\bar{c}_2)^{-1/2},\\ K_-(s,p) &= \frac{-4\bar{c}_2^2\bar{\mu}}{v^2R_1}[(1+v/\bar{c}_2)s-ip/\bar{c}_2]^{1/2}F_-(s)\,(1-v/\bar{c}_2)^{1/2}\end{aligned}$$

with

(1.24)

$$F_\lambda^\pm(s) = \frac{s\pm\dfrac{ip}{c_R\mp v}}{s\pm\dfrac{ip}{\bar{c}_2\mp v}}\exp\left[\frac{1}{\pi}\int\limits_{(\bar{c}_1\mp v)^{-1}}^{(\bar{c}_2\mp v)^{-1}}\tan^{-1}\left\{\frac{\left[w^2-\dfrac{(1\pm vw)^2}{2\bar{c}_2^2}\right]^2}{w^2\left[w^2-\dfrac{(1\pm vw)^2}{\bar{c}_1^2}\right]^{1/2}\left[\dfrac{(1\pm vw)^2}{\bar{c}_2^2}-w^2\right]^{1/2}}\right\}\frac{dw}{w\mp\dfrac{is}{p}}\right]$$

and

$$R_1 = \frac{(1-v^2/\bar{c}_1^2)^{1/2}(1-v^2/\bar{c}_2^2)^{1/2}}{(1-v^2/\bar{c}_1^2)^{1/2}(1-v^2/\bar{c}_2^2)^{1/2}-(1-v^2/2\bar{c}_2^2)^2},$$

where in Eqs. (1.23) and (1.24) $\bar{c}_1, \bar{c}_2$ and $\bar{\mu}$ are elastic constants and so independent of s or p. Our expression (1.23) has the same properties as the factorisation of Baker who considered the elastic case, we have merely multiplied by the factor $(1-v/\bar{c}_2)^{-1/2}$ for $K_+(s)$ and divided by it for $K_-(s)$.

In our subsequent analysis of the viscoelastic problem we will need the lim $|s| \to \infty$ of $K_+(s,p)$ and $K_-(s,p)$ in their respective half planes of regularity. We note that if the material behaves like a solid for short times, then c_1^2, c_2^2 and μ defined in Eq. (1.20) each tend to constants as $|s| \to \infty$ (to see this use the definitions (2.12), (2.13)) and we denote these constants by c_{10}^2, c_{20}^2 and μ_0. They are those wave speeds and moduli associated with the small time elastic moduli of the body. From Eqs. (1.23) and (1.24) with $v > 0$ we deduce the result

$$(1.25)\qquad \begin{aligned} &\lim_{|s|\to\infty} K_+(s,p) \to s_+^{1/2},\\ &\lim_{|s|\to\infty} K_-(s,p) \to \frac{-4c_{20}^2\mu_0(1-v^2/c_{20}^2)^{1/2}s_-^{1/2}}{v^2R_{10}},\end{aligned}$$

where R_{10} is the expression R_1 with $\bar{c}_1$ and $\bar{c}_2$ replaced by c_{10} and c_{20} and

$$(1.26)\qquad \begin{aligned} 3\varrho c_{10}^2 &= \lim_{\zeta\to\infty}[\zeta\bar{G}_2(\zeta)+2\zeta\bar{G}_1(\zeta)],\\ 2\varrho c_{20}^2 &= \lim_{\zeta\to\infty}\zeta\bar{G}_1(\zeta), \quad 2\mu_0 = \lim_{\zeta\to\infty}\zeta\bar{G}_1(\zeta).\end{aligned}$$

The behaviour (1.25) can be compared with the results (A.5) (Appendix 1) of the corresponding mode 3 problem which has the factor $(1-v^2/c^2)^{1/2}$ [c being the short time shear wave speed c_{20}] in place of the velocity factors above which include the Rayleigh factor in the denominator of R_{10}. To proceed further with the analysis we consider particular loadings chosen so that the sum-split (1.22) is easily made. It is expected that these loadings will illustrate the influence of the viscoelastic properties of the body on the propagating crack.

1.1.

$$(1.27)\quad \text{(ii)}\qquad \sigma(x_1, t) = T \quad \text{for} \quad t > 0 \quad \text{so that} \quad g(s,p) = \frac{T}{isp}.$$

The pole at $s = 0$ lies in the plus region since we need $\operatorname{Im} s < 0$ for the half transform $g(s,p)$ to exist.

The factorisation (1.22) can now be made by inspection giving

$$(1.28)\qquad \begin{aligned} C_+(s) &= \frac{T}{isp}\left[\frac{1}{K_+(s,p)} - \frac{1}{K_+(0,p)}\right],\\ C_-(s) &= \frac{T}{isp\,K_+(0,p)}. \end{aligned}$$

To complete the solution of Eq. (1.21) a generalised form of Liouville's theorem is applied to that equation so as to show that $L \equiv 0$ with the results

$$(1.29)\qquad J_-(s) = \frac{-C_-(s)}{K_-(s,p)}, \quad H_+(s) = C_+(s)K_+(s,p);$$

see [7] for a brief account of how the argument goes for crack problems and [6] for the method in general.

Our main interest is the behaviour of stress and displacement at the crack tip and this can be obtained via Tauberian theorems by taking the limit as $|s| \to \infty$ in the respective half-planes of regularity of the expressions in Eq. (1.29). The results are

$$(1.30)\qquad \lim_{|s|\to\infty} H_+(s) \to \frac{-Ts_+^{-1/2}}{ipK_+(0,p)}$$

and

$$\lim_{|s|\to\infty} J_-(s) \to \frac{T(1-v^2/c_{20}^2)^{-1/2}v^2R_{10}s_-^{-3/2}}{4\mu_0 c_{20}^2 ipK_+(0,p)}.$$

From the expressions (1.30) and appropriate Tauberian theorems (cf. [7] Eqs. (4.10) and (4.11)) the time transforms of the stress and displacement at the crack tip can be found as

$$(1.31)\qquad \overline{h}(x_1, p) = \frac{T}{pK_+(0,p)}\,\frac{e^{i\pi/4}x_1^{-1/2}}{\pi^{1/2}}$$

and

$$(1.32)\qquad \overline{u}_2(x_1, p) = \frac{T(1-v^2/c_{20}^2)^{-1/2}v^2R_{10}(-x_1)^{1/2}e^{\pi i/4}}{2\pi^{1/2}c_{20}^2\mu_0 pK_+(0,p)}.$$

To proceed further with the solution, the full factorisation of $K(s,p)$ (defined in Eqs. (1.20)) is needed since in order to invert the transforms (1.31) and (1.32), we require $K_+(0,p)$ for all p. To do this for all p would be, we think, a complicated task. However, by analogy with the results in Appendix 1, we assert that when the long and short time behaviour of the medium is such that the moduli and wave speeds are finite and non-zero (instantaneous and long-time elastic behaviour), then the factorisations $\lim\limits_{p\to 0} K_\pm(s,p)$

and $\lim_{p\to\infty} K_{\pm}(s,p)$ follow from the factorisation (1.23), (1.24) with $\bar{c}_1, \bar{c}_2$ and $\bar{\mu}$ replaced by the corresponding long time values when $p \to 0$ and short time values when $p \to \infty$. A formal justification of this result is outlined in Appendix 2. It then follows that

$$\text{(1.33)}\qquad \begin{aligned} &\lim_{p\to\infty} K_+(0,p) \to \frac{(ip)^{1/2}}{c_{20}^{1/2}} F_{0+}(0)\,(1-v/c_{20})^{-1/2},\\ &\lim_{p\to 0} K_+(0,p) \to \frac{(ip)^{1/2}}{c_{21}^{1/2}} F_{1+}(0)\,(1-v/c_{21})^{-1/2}, \end{aligned}$$

where

$$F_+(0) = \frac{\bar{c}_2 - v}{\bar{c}_R - v} \exp\left[\frac{1}{\pi}\int_{(\bar{c}_1-v)^{-1}}^{(\bar{c}_2-v)^{-1}} \tan^{-1}\left\{\frac{\left[w^2 - \dfrac{(1+vw)^2}{2\bar{c}_2^2}\right]^2}{w^2\left[w^2 - \dfrac{(1+vw)^2}{\bar{c}_1^2}\right]^{1/2}\left[\dfrac{(1+vw)^2}{\bar{c}_2^2} - w^2\right]^{1/2}}\right\}\frac{dw}{w}\right],$$

$\bar{c}_R$ is the Rayleigh velocity associated with the zero of the denominator of R_L (defined in Eq. (1.24)). In Eq. (1.33) the subscripts zero and one on F_+ and on c_2 mean that the expressions $\bar{c}_2, \bar{c}_1$ and $\bar{c}_R$ should be replaced by $c_{20}, c_{10}, c_{R0}; c_{21}, c_{11}, c_{R1}$ etc. where subscript zero denotes short-time wave-speeds (i.e. $p = \infty$), (see the definition (1.26)) and subscript one denotes long-time wavespeeds ($p = 0$), i.e.

$$\text{(1.34)}\qquad \begin{aligned} &3\varrho c_{11}^2 = \lim_{\zeta\to 0}[\zeta\bar{G}_2(\zeta) + 2\zeta\bar{G}_1(\zeta)],\\ &2\varrho c_{21}^2 = \lim_{\zeta\to 0}\zeta\bar{G}_1(\zeta); \quad 2\mu_1 = \lim_{\zeta\to 0}\zeta\bar{G}_1(\zeta). \end{aligned}$$

Using the relations (1.33) in Eqs. (1.31) and (1.32) and inverting gives on $x_2 = 0$

$$\text{(1.35)}\qquad \begin{aligned} &\sigma_{22} \sim T x_1^{-1/2} t^{1/2} \frac{2c_{20}^{1/2}(1-v/c_{20})^{1/2}}{\pi F_{0+}(0)},\\ &u_2(x_1,t) \sim \frac{T(-x_1)^{1/2} v^2 R_{10}}{c_{20}^{3/2}\mu_0 F_{0+}(0)}\frac{t^{1/2}}{\pi}(1+v/c_{20})^{-1/2} \end{aligned}$$

for small t, and

$$\text{(1.36)}\qquad \begin{aligned} &\sigma_{22} \sim \frac{2T}{\pi}\frac{x_1^{-1/2} t^{1/2} c_{21}^{1/2}(1-v/c_{21})^{1/2}}{F_{1+}(0)},\\ &u_2(x_1,t) \sim \frac{T(-x_1)^{1/2} v^2 t^{1/2} c_{21}^{1/2}(1-v/c_{21})^{1/2} R_{10}}{\pi c_{20}^2 \mu_0 (1-v^2/c_{20}^2)^{1/2} F_{1+}(0)} \end{aligned}$$

for large t. It is of interest to evaluate the flow of energy into the crack tip, this we do by calculating the work done at the crack tip from the above limiting stress and displacement distributions. The results are:

for small time

$$\text{(1.37)}\qquad G = \frac{tT^2}{\pi\mu_0}\frac{v^2 R_{10}}{[F_{0+}(0)]^2 c_{20}}\frac{(1-v/c_{20})^{1/2}}{(1+v/c_{20})^{1/2}};$$

for large time

$$\text{(1.38)}\qquad G = \frac{tT^2}{\pi\mu_0}\frac{c_{21}}{c_{20}^2}\frac{v^2 R_{10}}{[F_{1+}(0)]^2}\frac{(1-v/c_{21})^{1/2}}{(1-v^2/c_{20}^2)^{1/2}}.$$

Both these expressions are linear in t as we might expect, Eq. (1.37) simply being the elastic solution for all time with the short time elastic constants. The presence of $[F_{0+}(0)]^2$ in the denominator gives rise to the Rayleigh factor $v-c_{R0}$ in the numerator so that $G \to 0$ as $v \to c_{R0}$. In Eq. (1.38) on the other hand we find that G has a velocity factor involving $\dfrac{(v-c_{R1})^2}{(v-c_{R0})}$ (the term $(v-c_{R0})$ being a factor of the denominator of R_{10}). Since $c_{R1} < c_{R0}$ by virtue of the fact that the long time moduli are less than the short time ones, G in Eq. (1.38) will tend to zero as $v \to c_{R1}$. For intermediate times we have a transition between these two extreme behaviours. To get an approximate curve for the stress intensity factor versus time for all time, it might be possible to replace t in the long time result by $(t+t_0)$ where t_0 is some threshold time estimated from the exact mode 3 results given in Appendix 1 (cf. [1]).

1.2.

(ii) $\sigma(x_1, t) = \delta(x_1+vt)H(t) \equiv \delta(x_1')H(t)$, where H is the Heaviside step function, δ the delta function. In this case $g(s, p) = 1/(is+P/v)$ for $\operatorname{Im} s < P/v$.

The factorisation (1.22) gives

$$C_+(s) = \frac{1}{(is+P/v)}\left[\frac{1}{K_+(s,p)} - \frac{1}{K_+\left(\dfrac{ip}{v}, p\right)}\right],$$

$$C_-(s) = \frac{1}{(is+P/v)} \cdot \frac{1}{K_+\left(\dfrac{ip}{v}, p\right)}.$$

The result (1.29) still holds and in place of Eq. (1.30) we have

$$\lim_{|s|\to\infty} H_+(s) \to \frac{s_+^{-1/2}}{iK_+\left(\dfrac{ip}{v}, p\right)},$$

$$\lim_{|s|\to\infty} J_-(s) \to \frac{(1-v^2/c_{20}^2)^{-1/2} v^2 R_{10} s_-^{-3/2}}{4c_{20}^2\mu_0 iK_+\left(\dfrac{ip}{v}, p\right)}.$$

As in Example 1 the time transform of the stress and displacement of the crack tip can be determined and written

$$\bar{h}(x_1, p) = \frac{x_1^{-1/2} i^{1/2}}{\pi^{1/2} K_+(ip/v, p)}.$$

and

$$\bar{u}_2(x_1, p) = \frac{(1-v^2/c_{20}^2)^{-1/2} v^2 R_{10} i^{1/2} (-x_1)^{1/2}}{\pi^{1/2} 2c_{20}^2 \mu_0 K_+(ip/v, p)}.$$

For the solution for all time we again need the full factorisation of $K(s, p)$. To determine the solution for short and long times we calculate $K_+(ip/v, p)$ by analogy with the exact solution of Appendix 1 by first evaluating $K_+(s, p)$ from the elastic factorisation (1.23) in the limits $p \to 0$ and $p \to \infty$, then replacing s by ip/v and again taking the appropriate limit.

This procedure or an application of the results of Appendix 2 gives

$$\lim_{p\to\infty} K_+\left(\frac{ip}{v}, p\right) \to \frac{(ip)^{1/2}}{v^{1/2}} G_0(1-v/c_{20})^{-1/2},$$

and

$$\lim_{p\to 0} K_+\left(\frac{ip}{v}, p\right) \to \frac{(ip)^{1/2}}{v^{1/2}} G_1(1-v/c_{21})^{-1/2}.$$

The subscript zero refers to short times and subscript one to long times as described following Eq. (1.33). Also

$$G_j = \lim_{p\to\infty} F_{j+}\left(\frac{ip}{v}\right) = \frac{c_{Rj}(c_{2j}-v)}{c_{2j}(c_{Rj}-v)} \exp\left[\frac{1}{\pi}\int_{(c_{1j}-v)^{-1}}^{(c_{2j}-v)^{-1}} \frac{\tan^{-1} f(w)\,dw}{(w+1/v)}\right], \tag{1.39}$$

where j takes the value 0 or 1 and

$$f(w) = \frac{\left[w^2 - \dfrac{(1+vw)^2}{2c_{2j}^2}\right]^2}{w^2\left[w^2 - \dfrac{(1+vw)^2}{c_{1j}^2}\right]^{1/2}\left[\dfrac{(1+vw)^2}{c_{2j}^2} - w^2\right]^{1/2}}. \tag{1.40}$$

Inverting the transforms one gets for the stress and displacement at the crack tip

$$\begin{aligned} \sigma_{22} &\sim \frac{1}{\pi}(x_1)^{-1/2} v^{1/2} t^{-1/2} \frac{(1-v/c_{20})^{1/2}}{G_0}, \\ u_2 &\sim \frac{v^2 R_{10}}{2\pi\mu_0 c_{20}^2}(1+v/c_{20})^{-1/2}(-x_1)^{1/2} t^{-1/2} \frac{v^{1/2}}{G_0} \end{aligned} \tag{1.41}$$

on $x_2 = 0$ for a small time.

For large time one gets

$$\sigma_{22} \sim \frac{1}{\pi}(x_1)^{-1/2} v^{1/2} t^{-1/2} \frac{(1-v/c_{21})^{1/2}}{G_1} \tag{1.42}$$

and

$$u_2 \sim \frac{v^2 R_{10}}{2\pi\mu_0 c_{20}^2} \frac{(1-v/c_{21})^{1/2}}{(1-v^2/c_{20}^2)^{1/2}} (-x_1)^{1/2} t^{-1/2} \frac{v^{1/2}}{G_1}.$$

From these results one obtains the energy flow into the crack tip as

$$G = \frac{1}{4\pi} \frac{v^3 R_{10}}{\mu_0 c_{20}^2} \frac{t^{-1}}{G_0^2} \frac{(1-v/c_{20})^{1/2}}{(1+v/c_{20})^{1/2}} \tag{1.43}$$

for small t, and

$$G = \frac{1}{4\pi} \frac{v^3 R_{10}}{\mu_0 c_{20}^2} \frac{t^{-1}}{G_1^2} \frac{(1-v/c_{21})}{(1-v^2/c_{20}^2)^{1/2}} \tag{1.44}$$

for t large.

1.3.

(iii) $$\sigma(x_1, T) = \underset{(\lambda>0)}{e^{\lambda x_1}} \quad \text{for} \quad t > 0 \quad \text{then} \quad g(s, p) = \frac{T}{p(is+\lambda)},$$

where the pole at $s = i\lambda$ $(\lambda > 0)$ lies in the plus region. The steps of the solution are almost identical with those of examples (i) and (ii); merely replace $pK_+(0, p)$ in example (i) by $pK_+(i\lambda, p)$. Assuming as before that the factorisation for long and short times is essentially the factorisation (1.23) with the wave speeds replaced by their corresponding long and short time values, or alternatively using the results of Appendix 2, gives

$$\lim_{p\to\infty} K_+(i\lambda, p) \to \left(\frac{ip}{c_{20}}\right)^{1/2} F_{0+}(0)(1-v/c_{20})^{-1/2}$$

and

$$\lim_{p\to 0} K_+(i\lambda, p) \to [i\lambda]^{1/2}.$$

The short time behaviour is thus just like that of example (i) as we might expect. Of more interest here is the long time behaviour which from Eqs. (1.31) and (1.32) can be seen to be

$$\sigma_{22} = \frac{Tx_1^{-1/2}}{\lambda^{1/2}\pi^{1/2}},$$

(1.45)

$$u_2 = \frac{T(1-v^2/c_{20}^2)^{-1/2}v^2R_{10}(-x_1)^{1/2}}{2\lambda^{12}\pi^{1/2}c_{20}^2\mu_0}$$

at the crack tip on $x_2 = 0$. Further in this connection it is worth noting that the factorisation $K(s, 0) = K_+(s, 0)K_-(s, 0)$ can be made directly from Eq. (1.20) with $p = 0$. Writing

(1.46)

$$K_+(s, 0) = s_+^{1/2},$$

$$K_-(s, 0) = \frac{-4\tilde{c}_2^2\tilde{\mu}s_-^{1/2}(1-v^2/\tilde{c}_2^2)^{1/2}}{v^2\tilde{R}_1},$$

where $\tilde{R}_1$ is the same as R_1 of Eq. (1.24) with $\bar{c}_1, \bar{c}_2, \bar{\mu}$ replaced by $\tilde{c}_1, \tilde{c}_2, \tilde{\mu}$ and which are the same as the wave speeds and moduli defined following Eq. (1.20) but with p replaced by zero. In particular for use in the next section we remind the reader that

(1.47) $$R_{1i} = \frac{(1-v^2/c_{1i}^2)^{1/2}(1-v^2/c_{2i}^2)^{1/2}}{(1-v^2/c_{1i}^2)^{1/2}(1-v^2/c_{2i}^2)^{1/2}-(1-v^2/2c_{2i}^2)^2},$$

where $i = 0$ or 1 and c_{10}, μ_0 etc. are the effective "short time" wave speeds and moduli and c_{11}, μ_1 etc. the "long time" wave speeds and moduli.

If we define c_R to be that root of the denominator of $\tilde{R}_1$ which is least for all s (i.e. the root when $s = 0$, if the long time elastic constants and wave speeds are least), then for $v < c_R$, $K_-(s, 0)$ will be analytic in $\operatorname{Im} s < 0$; $s_-^{1/2}$ has a branch cut from $i0$ to $i\infty$ and $s_+^{1/2}$ a cut from $-i0$ to $-i\infty$. A special case of the factorisation (1.46) has been used in [3] where an asymptotic analysis of a variety of steady state problems has been made. A slight generalisation of one of these problems is considered in the next section. Note also that from the definition $\overline{G}(ivs) = \int_0^\infty e^{-ivst}G(t)\,dt$, the transforms $\overline{G}(ivs)$ are analytic functions in $\operatorname{Im} s < 0$ and hence c_1^2, c_2^2 etc. are from Eq. (1.20), so we don't anticipate any difficulty with the above argument provided $v < c_R$ and c_R is in turn less than any of the instantaneous wave speeds.

1.4. A crack propagating steadily in a viscoelastic strip

Here we consider the problem where fixed displacements are applied to the sides of the strip $x_2 = \pm 1$ and a crack propagates on the x_1' axis with uniform velocity v. As in the previous examples we use coordinates moving with the crack tip and define $x_1 = x_1' - vt$. On account of the steady-state assumption the stress and displacement field depend only on x_1 and x_2. The boundary conditions of the problem can thus be written as

$$\begin{aligned} &\text{on} \quad x_2 = \pm 1, \quad u_2 = \pm u_{20}, \quad u_1 = 0 \quad \text{for all} \quad x_1, \\ &\text{on} \quad x_2 = 0, \quad \sigma_{22} = 0 = \sigma_{12} \quad \text{for} \quad x_1 < 0 \quad \text{(stress-free crack)}, \\ &\text{and on} \quad x_2 = 0, \quad u_2 = 0 \quad \text{for} \quad x_1 > 0 \quad \text{(from symmetry)}, \end{aligned} \tag{1.48}$$

u_{20} is a constant and the viscoelastic properties of the medium depend on a small parameter ε through the relaxation functions G_1 and G_2 of the form defined in Eq. (1.48) below.

In [3] an asymptotic method was outlined and applied to certain steady moving boundary problems. Results were given explicitly for the standard linear solid. The key ingredient in the analysis was a dimensionless parameter $\varepsilon_1 = v\tau/L$ ($\varepsilon_1 \ll 1$) where v was the crack speed, τ the relaxation time of the medium and L a length associated with the problem (half the strip width say). Here we will present the asymptotic method in a slightly different way and apply it to media where the moduli have small relaxation times. Thus a typical relaxation function $G(t)$ might be written as

$$G(t) = G_0 + \sum_{j=3}^{N} G_j \exp(-t/\varepsilon t_j), \tag{1.49}$$

where G_j and t_j are constants and ε a small parameter. Note that the sum $j = 3$ to N ($N > 3$) is chosen so as not to clash with the previous use of the relaxation functions G_1 and G_2. In [3] an argument was given suggesting that for steady state situations (such as that of

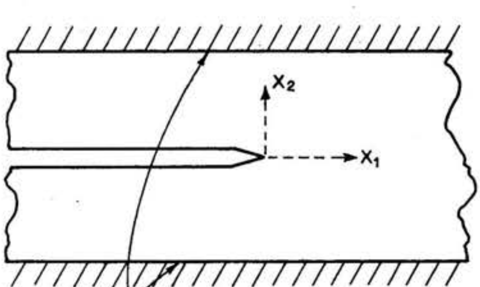

FIG. 1.

a crack propagating steadily in a displacement loaded strip, Fig. 1) the stress analysis problem can be viewed as a singular perturbation problem.

If we put $\varepsilon = 0$ in the viscoelastic relaxation functions as a first approximation to the problem, $G(t) \equiv G_0$, we get an elastic strip problem whose solution is well known. However, if a formal perturbation expansion is attempted it is soon seen that the expansion is a singular one. This suggests that the elastic solution formed by putting $\varepsilon = 0$ is valid at distances $\delta \gg \varepsilon$ from the crack tip. The influence of this "outer" solution is transmitted to the "inner" solution through matching conditions near the crack tip where $\varepsilon \ll \delta < 1$, and since both inner and outer approximations are valid in these regions they must be asymptotically equivalent there. Thus the inner limit of the "outer solution" must match with the outer limit of the "inner" solution. (cf. VAN DYKE [8] for details of the method of matched asymptotic expansions). One key feature of the present problem is that the zero-order outer solution is just the solution for the elastic strip and the inner limit of this solution (i.e. the solution near the crack tip) has the form

$$\begin{aligned} \sigma_{22} &\sim A_1 x_1^{-1/2}, \\ u_2 &\sim A_2(-x_1)^{1/2}. \end{aligned} \tag{1.50}$$

The constants A_1 and A_2 are known from the solutions [9, 7]. We give here the slightly more general result for a strip which has elastic moduli varying in the direction perpendicular to the crack direction; we quote from [7] correcting an obvious misprint there.

$$A_1 = (2\pi)^{-1/2} \cdot 4U_{20}\mu_1^{1/2} \frac{c_{21}}{v} \frac{(1-v^2/c_{21}^2)^{1/4}}{R_{11}^{1/2}} \left\{ \int_{-1}^{1} \frac{dx_2}{(\lambda+2\mu)_1} \right\}^{-1/2} \tag{1.51}$$

and

$$A_2 = \frac{A_1}{2} \frac{R_{11}}{(1-v^2/c_{21}^2)^{1/2}} \frac{v^2}{c_{21}^2}.$$

Note that the subscripts one on the c_2, μ etc. refer to the "long" time wave speeds and moduli defined in Eq. (1.34). R_{11} is the corresponding value of R_1, Eq. (1.24).

To obtain the zero-order inner solution, we define the inner coordinates (X_1, X_2) by

$$x_1 = \varepsilon X_1, \quad x_2 = \varepsilon X_2 \tag{1.52}$$

and write

$$\sigma_{22} = \varepsilon^{-1/2} T_{22}, \quad u_2 = \varepsilon^{1/2} U_2. \tag{1.53}$$

Recall that if the crack is stress free, then applying a Fourier transform over x_1 as in Eq. (1.19) leads to the functional equation

$$H_+(s) = K(s, 0) J_-(s), \tag{1.54}$$

where H_+ and J_- are defined in Eqs. (1.16) and (1.18). If in these expressions we refer to the inner coordinate X_1 and replace s by s_1/ε, we set

$$H_+(s) \equiv \varepsilon^{1/2} \int_0^\infty e^{is_1X_1} T_{22}(X_1, 0) dX_1 = \varepsilon^{1/2} \overline{T}_+(s_1) \tag{1.55}$$

similarly

$$J_-(s) \equiv \varepsilon^{3/2} \int_0^{-\infty} e^{is_1X_1} U_2(X_1, 0) dX_1 = \varepsilon^{3/2} \bar{U}_- .$$

Also for the relaxation functions we get

$$\bar{G}(ivs) = \bar{G}(ivs_1/\varepsilon) = \varepsilon \int_0^{\infty} e^{-ivs_1t_1} G(\varepsilon t_1) dt_1 \tag{1.56}$$

and from the definition (1.49), $G(\varepsilon t_1)$ is independent of ε so $ivs\,\bar{G}(ivs)$ is a function only of s_1 when written in terms of s_1 and does *not* depend on ε. Hence the moduli and wave speeds defined from Eq. (1.14) depend only on s_1 for solids modelled by the definition (1.49). Using these results in Eq. (1.54) gives the functional equation

$$\bar{T}_+ = K_1(s_1) \bar{U}_-, \tag{1.57}$$

where

$$K_1(s_1) \equiv \frac{K(s_1/\varepsilon)}{\varepsilon}.$$

Now by analogy with the factorisation (1.46) we can deduce tha

$$K_1(s_1) = K_{1+}(s_1) K_{1-}(s_1),$$

with

$$\begin{aligned} K_{1+}(s_1) &= s_{1+}^{1/2}, \\ K_{1-}(s_1) &= \frac{-4[\tilde{c}_2(s_1/\varepsilon)]^2 \tilde{\mu}(s_1/\varepsilon) s_{1-}^{1/2} \left(1 - v^2/\tilde{c}_2^2\left(\frac{s_1}{\varepsilon}\right)\right)^{1/2}}{v^2 \tilde{R}_1(s_1/\varepsilon)} . \end{aligned} \tag{1.58}$$

We repeat that there is no explicit dependence on ε in the expressions $\tilde{R}_1(s_1/\varepsilon)$, $\tilde{c}_2(s_1/\varepsilon)$ etc. because of the form of the relaxation functions $G(t)$, Eq. (1.49) and the argument following Eq. (1.56).

It remains now to solve the functional equation (1.57) subject to the matching requirements that the far field should match with Eq. (1.50) written in inner coordinates. This leads to the requirement that

$$U_2 \sim A_2(-X)^{1/2} \quad \text{as} \quad X \to -\infty, \tag{1.59}$$

and

$$T_{22} \sim A_1 X^{-1/2} \quad \text{as} \quad X \to +\infty .$$

These matching conditions will be satisfied if the transforms have the behaviour

$$\left.\begin{aligned} \bar{U}_- &\sim \frac{-\pi^{1/2}}{2} s_{1-}^{-3/2} A_2 e^{\pi i/4} \\ \bar{T}_+ &\sim A_1 \pi^{1/2} e^{\pi i/4} s_{1+}^{-1/2} \end{aligned}\right\} \quad \text{as } s_1 \to 0. \tag{1.60}$$

Using Eq. (1.58) in Eq. (1.57) gives

$$N(s_1) \equiv \frac{\bar{T}_+}{K_+} = K_- \bar{U}_- . \tag{1.61}$$

The function $N(s_1)$ defined by both sides of Eq. (1.61) is analytic in the whole s_1 plane except possibly at $s_1 = 0$, and for large s_1 each side of Eq. (1.61) is bounded on account of the usual condition that the stress should be no more singular than $r^{-1/2}$ at the crack tip. Matching the stress boundary condition on $\overline{T}_+$ from Eq. (1.60) and using Liouville's theorem specifies $N(s_1)$ as

$$N(s_1) = \frac{A_1 \pi^{1/2} e^{\pi i/4}}{s}. \tag{1.62}$$

Then, from Eq. (1.61) the transforms $\overline{T}_+$ and $\overline{U}_-$ are determined, and using Tauberian theorems the stress and displacement at the crack tip can be determined. The resulting expressions are on $X_2 = 0$, $|x_1| \ll 1$

$$T_{22} \sim A_1 X_1^{-1/2}$$

(1.63) and

$$U_2 \sim \frac{A_1(-X_1)^{1/2} v^2 R_{10}}{2c_{20}^2 \mu_0 (1-v^2/c_{20}^2)^{1/2}}.$$

Referring these expressions to the (x_1, x_2) coordinate system and evaluating the energy flow to the crack tip via a local work argument at the crack tip gives the result

$$G = \frac{\pi A_1^2 v^2 R_{10}}{4\mu_0 c_{20}^2 (1-v^2/c_{20}^2)^{1/2}}. \tag{1.64}$$

If we now substitute in for A_1 given from Eq. (1.51) and simplify, we get the result

$$G = 2U_{20}^2 \frac{\mu_1}{\mu_0} \frac{c_{21}^2}{c_{20}^2} \frac{R_{10}}{R_{11}} \frac{(1-v^2/c_{21}^2)^{1/2}}{(1-v^2/c_{20}^2)^{1/2}} \left\{ \int_{-1}^{1} \frac{dx_2}{(\lambda+2\mu)_1} \right\}^{-1}. \tag{1.65}$$

This expression can be simplified a little particularly if we write

$$G = \frac{2U_{20}^2}{\int_{-1}^{1} \frac{dx_2}{(\lambda+2\mu)_1}}$$

for the elastic strip; then the expression (1.65) becomes

$$\frac{G}{G_E} = \frac{\mu_1}{\mu_0} \cdot \frac{c_{21}^2}{c_{20}^2} \cdot \frac{(1-v^2/c_{10}^2)^{1/2}}{(1-v^2/c_{11}^2)^{1/2}} \cdot \frac{\{(1-v^2/c_{11}^2)^{1/2}(1-v^2/c_{21}^2)^{1/2} - (1-v^2/2c_{21}^2)^2\}}{\{(1-v^2/c_{10}^2)^{1/2}(1-v^2/c_{20}^2)^{1/2} - (1-v^2/2c_{20}^2)^2\}}. \tag{1.66}$$

Again we remind the reader that the subscript zero refers to short times (see Eq. (1.26)) and the subscript one to long times (see Eq. (1.34)). This result should be the same apart from some misprints with the result derived in [3] for the standard linear solid. Note the presence of the Rayleigh factor top and bottom of the expression (1.66), hence G tends to zero as v tends to the long time Rayleigh velocity. Note further that Eq. (1.66) arises from only the first terms in the asymptotic expansions; we expect, however, that the Rayleigh factors will be present in higher order terms also. We have derived the result (1.66) using Eq. (1.51) as if in the long time limit the material were elastically inhomogeneous. The argument should still work for this case provided the crack propagated in a homogeneous viscoelastic layer (thickness $|x_2| \leqslant h < 1$) such that the limit $\lim_{\varepsilon \to 0} h/\varepsilon \to \infty$, for the inner coordinates (X_1, X_2) so the inner problem would still be as described above.

2. Concluding remarks

The main results of this paper relate to the short and long time behaviour of constant velocity dynamic crack propagation in a linear viscoelastic solid. For the short time behaviour the results of Eqs. (1.35) and (1.41) for the various loadings are intuitively clear. For early enough times we expect the material to behave like an elastic solid with the corresponding "short time" moduli. The limit (1.25) eventually leads to the result that the crack tip field will always possess a velocity factor which is like the usual Rayleigh factor encountered in the elastic problem but involving the "short time" moduli and wave speeds. A result of this kind should also apply to the case of non-uniformly moving cracks. To see this use the differential operator form of the constitutive equations (1.6) and (1.7), and substitute into Eq. (1.8). An eigenfunction expansion in coordinates based at the moving crack tip will then result in equations governing the coefficient of the $r^{\frac{1}{2}}$ (leading term) in the crack tip displacement which depend on the highest derivatives in the differential equation. The highest derivative terms in Eqs. (1.6) and (1.7) involve the "short time" moduli and the resulting equations are then just as in the elastic situation.

More intriguing are the results (1.36) and (1.42), for the long time behaviour, which show the presence of a factor $(C_{R1}-v)$ in the numerator of both stress and displacement at the crack tip in addition to the presence of the factor R_{10} (discussed above). We remind the reader that C_{R1} denotes the Rayleigh velocity calculated from the "long time" wave speeds and R_{10} is in terms of the "short time" wave speeds. (The denominator of R_{10} has a zero at C_{R0}, the "short time" Rayleigh velocity).

We stress that these results, although they involve fairly general viscoelastic moduli, have been derived in a heuristic way and moreover only at short and long times. A complete analysis based on equations (B.3) of Appendix 2 would be desirable although it would then probably only be possible to treat particular constitutive equations. We hope to do this in the future. In support of the above mentioned results, however, are the exact results of Appendix 1 which are valid for all time.

The results in Sect. (1.4) generalise and correct some misprints in [2]. As a final remark note that the energy release rates calculated in the paper have been based on local work calculations at the crack tip; the effect of the medium has been involved only in determining what these crack tip fields will be.

Appendix 1

We consider here mode 3 (longitudinal shear) analogues of the mode 1 situations of the main text to guide the approximations used there. The mode 3 situation was first considered in [1] where an exact analysis was given for certain model viscoelastic solids. We expect properties of the exact analysis given here to agree qualitatively with those of the approximate method given in the text.

Because mode 3 conditions are assumed, we have in place of Eq. (1.15) the conditions

$$\begin{aligned} \sigma_{23} &= -\sigma(x_1, t) \quad x_1 < 0, \quad x_2 = 0, \quad t > 0, \\ u_3 &= 0, \quad x_1 > 0, \quad t > 0. \end{aligned} \tag{A.1}$$

Following the procedure outlined in the main text leads to the functional equation

$$-g(s,p)+H_+(s) = K(s,p)J_-(s), \tag{A.2}$$

where H_+ and J_- are now respectively the transforms of the unknown σ_{23} stress ahead of the crack and the unknown opening u_3 of the crack. Following [1] we consider the constitutive model

$$\left(\frac{d}{dt}+\beta\right)^2 \sigma_{j3} = 2\mu_1 \left(\frac{d}{dt}+\alpha\right)^2 e_{j3}, \quad j = 1, 2, \tag{A.3}$$

where μ_1, α and β are constants. This model was originally suggested by ACHENBACH and CHAO [10] as an alternative to the standard linear solid; it is used here as it simplifies the analysis a little.

For this solid $K(s)$ can be factored as $K = K_+K_-$ where with $\varrho c^2 = \mu_1$,

$$K_+(s) = [s+iX_1]^{1/2}$$

(A.4) and

$$K_-(s) = \frac{-\mu_1(1-v^2/c^2)^{1/2}[s-i(\alpha+p)/v][s-iX_2]^{1/2}[s-iX_3]^{1/2}[s-iX_4]^{1/2}}{[s-i(p+\beta)/v]^2},$$

where

$$X_{\substack{1\\2}} = \frac{1}{2v(1-v/c)}\left[\left\{[(v/c)(2p+\beta)-(p+\alpha)]^2+\frac{4v}{c}(1-v/c)p(p+\beta)\right\}^{1/2} \pm\{v(2p+\beta)/c-(p+\alpha)\}\right],$$

$$X_{\substack{3\\4}} = \frac{1}{2v(1+v/c)}\left[\left\{\frac{v}{c}(2p+\beta)+p+\alpha\right\}\pm\left\{[(v/c)(2p+\beta)+(p+\alpha)]^2 -\frac{4v}{c}(1+v/c)p(p+\beta)\right\}^{1/2}\right]$$

Thus $\mathrm{Re}X_j > 0$, $j = 1, 2, 3, 4$ and the radicals in Eq. (A.4) have branch cuts from $s = -iX_1$ to $-i\infty$ in the lower half plane and from iX_2, iX_3 and iX_4 to $+i\infty$ in the upper half plane. Further,

$$\lim_{|s|\to\infty} K_+(s) \to s_+^{1/2}, \qquad \lim_{|s|\to\infty} K_-(s) \to -\mu_1(1-v^2/c^2)^{1/2}s_-^{1/2}. \tag{A.5}$$

We now consider particular examples which correspond to the loadings treated in the main text.

$$\text{ex(i)}\ \sigma(x_1,t) = T \quad \text{for} \quad t>0 \quad \text{so} \quad g(s,p) = \frac{T}{isp}. \tag{A.6}$$

Corresponding to Eq. (1.30) we now get

$$\lim_{|s|\to\infty} H_+(s) \to \frac{-Ts_+^{-1/2}}{ipK_+(0)}, \qquad \lim_{|s|\to\infty} J_-(s) \to \frac{T(1-v^2/c^2)^{-1/2}s_-^{-3/2}}{ipK_+(0)\mu_1} \tag{A.7}$$

with

$$K_+(0) = [iX_1]_+^{1/2}.$$

From the exact factorisation above we have $\lim_{p\to\infty} X_1 \to \frac{p}{(c-v)}$ and

$\lim_{p\to 0} X_1 \to \frac{p}{\left(\frac{c\alpha}{\beta}-v\right)}$ provided $v < c\alpha/\beta$, and we note that $\mu_1 \alpha^2/\beta^2$ is the long time shear modulus of the model (A.3) and hence $c\alpha/\beta$ is the long-time wave-speed.

Inverting Eq. (A.7) for the stress at the crack tip gives

$$\sigma_{23} = \frac{T(\pi x_1)^{-1/2}}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{pX_1^{1/2}}\, dp \tag{A.8}$$

and for the displacement

$$u_3 = 2\left(\frac{-x_1}{\pi}\right)^{1/2} \frac{(1-v^2/c^2)^{-1/2}}{\mu_1} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{pX_1^{1/2}}\, dp. \tag{A.9}$$

From the behaviour of X_1 as p tends to infinity or zero it is straightforward to deduce from Eqs. (A.8) and (A.9) that

$$\begin{aligned} \sigma_{23} &= \frac{2T}{\pi}(x_1)^{-1/2}(c-v)^{1/2}t^{1/2}, \\ u_3 &= \frac{4T}{\pi}\frac{(-x_1)^{1/2}}{\mu_1}\frac{c}{(c+v)^{1/2}}\, t^{1/2} \end{aligned} \tag{A.10}$$

at the crack tip for small t, and

$$\begin{aligned} \sigma_{23} &= \frac{2T}{\pi}(x_1)^{-1/2}\left(\frac{c\alpha}{\beta}-v\right)^{1/2} t^{1/2}, \\ u_3 &= \frac{4T}{\pi}\frac{(-x_1)^{1/2}}{\mu_1}\frac{\left(\frac{c\alpha}{\beta}-v\right)^{1/2}}{(1-v^2/c^2)^{1/2}}\, t^{1/2} \end{aligned} \tag{A.11}$$

for large time. Results for intermediate times are tabulated in [1] giving the stress intensity factor as a function of time.

From Eqs. (A.10) and (A.11) and a local work argument one gets for the energy release rate

$$G = \frac{4}{\pi}T^2\frac{c}{\mu_1}\frac{(c-v)^{1/2}}{(c+v)^{1/2}}\, t \tag{A.12}$$

for t small, and

$$G = \frac{4T^2}{\pi}\frac{1}{\mu_1}\frac{\left(\frac{c\alpha}{\beta}-v\right)t}{(1-v^2/c^2)^{1/2}} \tag{A.13}$$

for t large.

ex(ii) $\sigma(x_1, t) = T\delta(x_1+vt)H(t)$ so $g(s,p) = \dfrac{T}{i(s-ip/v)}$.

For this problem following the steps outlined in the text one gets the results (A.7) with $pK_+(0)$ replaced by $K_+(ip/v)$ and

(A.14) $$K_+(ip/v) = [i(p/v+X_1)]^{1/2}.$$

Furthermore using the expression for X_1 one can deduce that

$$p/v+X_1 \to \frac{pc}{v(c-v)} \quad \text{as} \quad p\to\infty$$

and

$$\to \frac{pc\alpha/\beta}{v(c\alpha/\beta-v)} \quad \text{as} \quad p\to 0.$$

So in place of Eqs. (A.10) and (A.11) we now get

(A.15) $$\sigma_{23} = \frac{T}{\pi}(x_1)^{-1/2}\left(\frac{v}{c}\right)^{1/2}(c-v)^{1/2}t^{-1/2},$$
$$u_3 = \frac{T^2}{\pi}\frac{(-x_1)^{1/2}}{\mu_1}\frac{(vc)^{1/2}}{(c+v)^{1/2}}t^{-1/2}$$

at the crack tip for small t, and

(A.16) $$\sigma_{23} = \frac{T}{\pi}(x_1)^{-1/2}\left(\frac{v}{c\alpha/\beta}\right)^{1/2}(c\alpha/\beta-v)^{1/2}t^{-1/2},$$
$$u_3 = \frac{T^2}{\pi}\frac{(-x_1)^{1/2}}{\mu_1}\left(\frac{v}{c\alpha/\beta}\right)^{1/2}\frac{(c\alpha/\beta-v)^{1/2}}{(1-v^2/c^2)^{1/2}}t^{-1/2}$$

for large time. From Eqs. (A.15) and (A.16) one gets for small time

(A.17) $$G = \frac{T^2}{\pi}\frac{v}{\mu_1}\frac{(c-v)^{1/2}}{(c+v)^{1/2}}t^{-1}$$

and for large time

(A.18) $$G = \frac{T^2}{\pi}\frac{v}{\mu_1(c\alpha/\beta)}\frac{(c\alpha/\beta-v)}{(1-v^2/c^2)^{1/2}}t^{-1}.$$

ex(iii) $\sigma(x_1, t) = Te^{\lambda x_1}$ for $t>0$, so $g(s,p) = \dfrac{T}{p(is+\lambda)}$.

The steps of the solution are the same as before; now in Eq. (A.7) we replace $pK_+(0)$ by $pK_+(i\lambda)$, hence in Eqs. (A.8) and (A.9) we have $\lambda+X_1$ in place of X_1. It is straighforward to deduce that

$$\lim_{p\to\infty}[\lambda+X_1] \to \lambda+\frac{p}{(c-v)} \quad \text{and} \quad \lim_{p\to 0}[\lambda+X_1] \to \lambda+\frac{p}{(c\alpha/\beta-v)}.$$

From these limits it is clear that the results (A.10) and (A.12) hold for short times. The long time result follows from Eqs. (A.8) and (A.9) as

(A.19)
$$\sigma_{23} = T(\pi x_1)^{-1/2}\lambda^{-1/2},$$
$$u_{23} = 2T\left(\frac{-x_1}{\pi}\right)^{1/2} \frac{(1-v^2/c^2)^{-1/2}}{\mu_1} \lambda^{-1/2}$$

with

(A.20)
$$G = T^2 \frac{(1-v^2/c^2)^{-1/2}}{\mu_1} \lambda^{-1}.$$

We assume here as before that the crack is running at a speed less than the long-time wave speed $c\alpha/\beta$ of the medium, i.e. $v < c\alpha/\beta$. If instead we consider the situation when $v \geqslant c\alpha/\beta$, then λ in the above long time results is replaced by $\lambda + \dfrac{[v\beta/c-\alpha]}{v(1-v/c)}$.

Appendix 2

In this appendix we give a formal derivation of the factorisation of the expression $K(s, p)$ and make some deductions about the behaviour at long and short times.

From Eq. (1.20) one gets

(B.1)
$$\lim_{|s|\to\infty} K(s, p) = |s|\left\{\frac{-4c_{20}^2\mu_0(1-v^2/c_{20}^2)^{1/2}}{v^2 R_{10}}\right\}.$$

So we define

(B.2)
$$N(s, p) = \frac{-v^2 R_{10}}{4c_{20}\mu_0} \frac{K(s, p)}{\gamma_{20}}$$

with

$$\gamma_{20} = (1-v^2/c_{20}^2)^{1/2}\left[s - \frac{ip}{(v+c_{20})}\right]^{1/2}\left[s + \frac{ip}{(c_{20}-v)}\right]^{1/2}$$

and the square roots having cuts from $s = \dfrac{ip}{(c_{20}+v)}$ to $i\infty$ and $s = -ip/(c_{20}-v)$ to $-i\infty$, respectively. (We remind the reader that the subscript zero refers to short time wave speeds, i.e. $p \to \infty$, see the definition (1.26)). From the above follows the result $\lim\limits_{|s|\to\infty} N(s, p) \to 1$ where the limit is taken in the strip of regularity of $N(s, p)$. However, for all p we do not know precisely what this strip of regularity is without closer investigation of the detailed viscoelastic moduli so we denote this strip by $-d_1 < \operatorname{Im} s < d_2$, where $-d_1 \leqslant 0$ is above all the singularities (including branch points) in the lower half s-plane. Similarly, d_2 is to be below all singularities in the upper half s plane. The factorisation of N into the product of plus and minus functions then follows in a standard way as

(B.3)
$$N_+(s, p) = \exp\left\{\int_{-\infty-id_3}^{\infty-id_3} \frac{\log N(\zeta, p)d\zeta}{\zeta - s}\right\},$$
$$N_-(s, p) = \exp\left\{-\int_{-\infty+id_4}^{\infty+id_4} \frac{\log N(\zeta, p)d\zeta}{\zeta - s}\right\},$$

with $-d_1 < -d_3 < \operatorname{Im} s < d \ < d_2$.

Then

$$K_+(s,p) = \left[s + \frac{ip}{(c_{20}-v)}\right]^{1/2} N_+(s,p) \tag{B.4}$$

and

$$K_-(s,p) = -\left[s - \frac{ip}{(c_{20}+v)}\right]^{1/2} (1-v^2/c_{20}^2)^{1/2} \frac{4c_{20}^2\mu_0}{v^2 R_{10}}.$$

Clearly these expressions are consistent with the limits given in Eq. (1.25). Although this completes a formal factorisation any further analytical progress requires detailed knowledge of the viscoelastic moduli. We restrict ourselves to the behaviour for long $(p \to 0)$ and short times $(p \to \infty)$. If in the relations (B.3) we make the substitution $\zeta = p\zeta_1$, then

$$N_+(s,p) = \exp\left\{ \int_{-\infty - id_{3/p}}^{\infty - id_{3/p}} \frac{\log N(p\zeta_1, p)}{\zeta_1 - s/p} d\zeta_1 \right\}. \tag{B.5}$$

We now consider the singularities of the function $N(p\zeta_1, p)$. As $p \to \infty$ we expect that these singularities coincide with the singularities obtained for an elastic solid with the "short time" wave speed and moduli $c_{10}^2, c_{20}^2, \mu_0$. This assumption leads to the result

$$\lim_{p \to \infty} N_+(s,p) \to F_{0+}(s), \tag{B.6}$$

where F_+ is defined in Eq. (1.24) and the subscript zero is used to denote that the velocities $\bar{c}_1, \bar{c}_2, \bar{c}_R$ in Eq. (1.24) are to be replaced by c_{10}, c_{20}, c_{R0}.

On the other hand, as $p \to 0$ $(t \to \infty)$ we expect that the singularities of $K(p\zeta_1, p)$ will tend to those obtained for an elastic solid with the "long time" wave speeds and moduli $c_{11}^2, c_{21}^2, \mu_1$ (cf. Eq. (1.34)).

With this assumption the following result is obtained from Eqs. (B.5) and (B.2),

$$\lim_{p \to 0} N_+(s,p) \to \frac{\left[s + \frac{ip}{(c_{21}-v)}\right]^{1/2}}{\left[s + \frac{ip}{(c_{20}-v)}\right]^{1/2}} F_{1+}(s). \tag{B.7}$$

Note that F_{1+} is defined as F_{0+} except that the velocities are now c_{11}, c_{21} and c_{R1}. Also the above expression is a function only of s/p and $\lim_{|s| \to \infty} N_+(s,p) \to 1$ is still retained.

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