Arch. Mech., 32, 1, pp. 111-123, Warszawa 1980

On the effective transport coefficients
Part II. The effective viscosity of suspensions

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THE EFFECTIVE viscosity of suspensions depends on the viscosities of suspended particles and of the ambient fluid and also on the structure of suspension. In the first part of the paper the conditions necessary for determining the scalar effective viscosity are examined. In the second part an approximate method (being a modification of cell model approach) for calculating the effective viscosity is presented.

Lepkość efektywna zawiesiny zależy od lepkości fazy spójnej, lepkości fazy rozproszonej oraz geometrii zawiesiny. W pierwszej części tej pracy przedyskutowano warunki konieczne dla zdefiniowania skalarnego współczynnika lepkości efektywnej. W drugiej części pracy przedstawiono przybliżoną metodę obliczenia tego współczynnika, będącą zmodyfikowaną wersją modelu komórkowego.

Эффективная вязкость суспензии зависит от вязкостив нешней фазы, от вязкости дисперсной фазы, а также от геометрии суспензни. В первой части этой работы обсуждены необходимые условия для определения скалярного коэффициента эффективной вязкости. Во второй части работы представлен приближенный метод расчета этого коэффициента, будучий модифицированным вариантом ячеечной модели.

1. Introduction

THE MAIN distinction between composite materials which were discussed in the first part of this paper and suspensions, what will be presently discussed here, is that in the former case the distribution of inclusions is determined by the process of manufacture whereas in the latter the distribution of particles depends on the bulk flow. Therefore one cannot assume the form of the distribution function but it should result from the solution of the considered hydrodynamical problem. The first steps in this direction were done in series of papers by BATCHELOR and his co-workers [1, 2]. The relation between a distribution function and a type of flow is proposed and it enables, at least in principle, to consider flows and distribution functions lacking isotropy. The isotropy is the necessary condition for introducing the notion of a scalar effective viscosity.

The results obtained so far are confined to the cases of non-interacting suspended particles, and to the first-order effects of hydrodynamic interactions. In the latter case the interations between pairs of particles are considered, and thus results concern only dilute suspensions. It was shown that for some types of flows i.e. a simple straining motion, flows with strong Brownian motions, the distribution of spheres is an isotropic one, and for these cases the effective viscosity up to the term ϕ^2(ϕ — the volume density of suspended spherical particles) has been calculated. The above approach gives a deep insight into the flow of suspensions, but it is not well suited to deal with dense suspensions.

In this paper we assume a priori that the flow of suspension considered can be treated as a flow of a simple Newtonian fluid and thus that there exists for it an effective viscosity. Keeping in mind the fact that this assumption is based on the statistical properties of flowing suspensions, we named it statistical similarity. Provided this assumption is obeyed, the so-called momentum or Maxwell approach, and the energy or Einstein approach (see Sect. 2) lead to the same results.

It should be noted that in contrast with the papers mentioned above, where the suspension occupied the whole space, we consider the finite region with an appropriate homogeneous boundary condition, thus the complications related to non-convergent integrals are avoided.

Due to Brownian motion and, first of all, to hydrodynamic interactions between suspended particles, any flow of suspension which is steady in the macroscopic scale is unsteady in the scale of individual particles. This unsteadiness contributes to the dissipation of energy, and thus to the value of the effective viscosity coefficient.

In the second part of this paper we introduce, similarily as we have done it in [3], a new version of cell model. The model proposed in this paper is in many respects similar to other cell models and shares their efficacies and their deficiencies. It allows, however, to calculate the effective viscosity coefficient not only for monodispersed, but also for polydispersed suspensions. Apart from the effective viscosity, it gives also a dispersion of this magnitude, which can be related to experimental scattering.

2. Statistical similarity

As in [3], by the statistical similarity we understand that for a particulate suspension there exists a constitutive relation of Newtonian form. In the case of suspensions it is much more restrictive than in the case of solid particulate materials with fixed structures. The similarity of flows of suspensions is valid only in the bulk of the flow, far from the walls and only when the flow itself does not impose any ordering of particles and, especially, does not change their concentration.

The notion of similarity of flow fields (here only Stokes flows will be considered) includes variations of parameters not only spatial but temporal as well. Thus the proper averaging procedure would be the time averaging of quantities of interest at a given point. The relevant question, how to apply an ensemble average to derive the bulk equations of motion for fluid suspensions, has been considered by several authors, most recently by HINCH [4].

Let the volume G be divided as previously into G_1 occupied by the particles with viscosity $\tilde{\mu}$ and G_2 occupied by the ambient fluid with viscosity μ. We assume that both inside the particles and in the ambient fluid the Stokes equation holds and that on the boundary Γ_1 of the particles the velocity and the tangential component of normal stress are continuous:

$$[v_i]_{\Gamma_1} = 0, \tag{2.1}$$

$$[\sigma_{ij}n_j - n_i n_k n_l \sigma_{kl}]_{\Gamma_1} = 0, \tag{2.2}$$

and further that the particles are not deformable, that is

$$v_i n_i = 0 \quad \text{on} \quad \Gamma_1. \tag{2.3}$$

We may define

$$\overline{v_{i,j}} = \frac{1}{V} \oint_{\Gamma} v_i n_j dS. \tag{2.4}$$

For any function φ continuous in G we have

$$\int_G (\varphi\sigma_{ij})_{,j} dV = \int_{G_1} \varphi_{,j}\sigma_{ij} dV + \int_{G_2} \varphi_{,j}\sigma_{ij} dV + \oint_{\Gamma_1} [\sigma_{ij} n_j \varphi] dS = \oint_{\Gamma} \varphi\sigma_{ij} n_j dS. \tag{2.5}$$

The above formula differs from that for composite materials (the occurrence of the integral $\oint_{\Gamma_1} [\sigma_{ij} n_j \varphi] ds$), due to the allowed here discontinuity of normal components of stress.

The expression (2.5) can be used for obtaining the rate of dissipation of kinetic energy:

$$D = \int_G (v_i \sigma_{ij})_{,j} dV = \oint_{\Gamma} v_i \sigma_{ij} n_j dS. \tag{2.6}$$

The average stress is defined by

$$\overline{\sigma_{ij}} = \frac{1}{V} \oint_{\Gamma} x_i \sigma_{jk} n_k dS. \tag{2.7}$$

(It should be noted that this formula differs from the volume average of σ_{ij} over the region G).

Now, the assumption of statistical similarity reads

$$\overline{\sigma}_{ij} = 2\mu^* \overline{\alpha}_{ij} - \delta_{ij} \overline{p}, \tag{2.8}$$

where

$$\alpha_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad \overline{\alpha}_{ij} = \frac{1}{V} \int_G \alpha_{ij} dV,$$

$$\overline{p} = \frac{1}{V} \int_G p dV.$$

Here μ^* is assumed to be a scalar magnitude independent of $\overline{\alpha}_{ij}$. This assumption is equivalent to the so-called momentum approach.

An alternative way of introducing the effective viscosity through the energy dissipation is (the so-called energy approach)

$$D = 2\mu^* \overline{\alpha}_{ij} \overline{\alpha}_{ij} V. \tag{2.9}$$

The notion of the effective viscosity is physically meaningful if the definitions of μ^* given by Eqs. (2.8) and (2.9) coincide. Here again, as in the case of heat conductivity, the positive answer is obtained only for two types of homogeneous boundary conditions.

For

$$\alpha_{ij}|_{\Gamma} = \alpha_{ij}^0 = \text{const} \tag{2.10}$$

or

(2.11) $$\sigma^0_{ij}|_\Gamma = 2\mu\alpha^0_{ij} - p^0\delta_{ij},$$

we get from Eq. (2.7):

(2.12) $$\bar{\sigma}_{ij} = \frac{1}{V}\sigma^0_{jk}\oint_\Gamma x_i n_k dS = \sigma^0_{ij},$$

(i.e. in this case the averaged stress is equal to the imposed one). Equation (2.6) gives

(2.13) $$D = \sigma^0_{ij}\oint_\Gamma v_i n_j dS = \sigma^0_{ij}\bar{v}_{i,j} = \bar{\sigma}_{ij}\bar{\alpha}_{ij},$$

and hence from Eqs. (2.8) the relation (2.9) follows.

Homogeneous boundary relations for velocity read

(2.14) $$v_i|_\Gamma = c_{ij}x_j, \quad c_{ij} = c_{ji}, \quad c_{ii} = 0.$$

From Eq. (2.4) it follows that in this case the averaged velocity gradient remains unchanged:

(2.15) $$\bar{v}_{i,j} = c_{ij}.$$

Further, according to Eq. (2.7),

(2.16) $$D = c_{ik}\oint_\Gamma x_k\sigma_{ij}n_j dS = \bar{v}_{i,k}\bar{\sigma}_{ik},$$

and again the effective viscosity which follows from Eqs. (2.8) and (2.9) coincide.

No restrictions were imposed so far on the region G. Physically, however, G should be understood as large enough compared with the characteristic length of the flow. In this sense, locally, we can always assume homogeneous boundary conditions on Γ.

3. The effective viscosity of mono- and polydispersed systems

The model described in Sect. 4 Part I is used here to calculate the effective viscosity of suspensions. We assume that each suspended spherical particle of radius a is enveloped by a reference sphere (of radius R) concentric with it.

As previously we introduce the characteristic length of the suspension ($l^3 = 3\phi_m/4\pi n$), and the non-dimensional variables $s = a/l$, $Z = R/l$.

Further, instead of the Cartesian coordinates x_i, and the radius r, $r^2 = x_i x_i$, we use $z_i = x_i/l$, $z = r/l$.

Now, our assumptions similar to these used in [3] are:

(i) the velocity field in the particle-free region, i.e. outside the reference spheres and inside the reference spheres in the shells $1 \leqslant z \leqslant Z$ is the same as the imposed one;

(ii) the velocity field inside reference spheres of radii $Z \leqslant 1$ is calculated as the solution of the system of Stokes equations with the boundary conditions imposed on $Z = 1$.

The second part of (i) means that for "large" radii of reference spheres, greater than the characteristic length of the suspension, the velocity field remains unchanged. This allows us to introduce (ii), and to assume the boundary conditions for all reference spheres at $Z = 1$.

As the quantity of interest will be the average rate of energy dissipation and as the only component of motion contributing to the dissipation (in the problem considered) is the straining motion, we take the boundary condition on the l-sphere in the form

$$\begin{gathered} v_i^0 = c_{ij}x_j \quad \text{or} \quad w_i^0 = c_{ij}z_j, \\ c_{ij} = c_{ji}, \quad c_{ii} = 0, \quad w_i = v_i/l. \end{gathered} \tag{3.1}$$

Here we quote the solution in the form given by KELLER, RUBENFELD and MOLYNEAUX [5]:

$$\begin{aligned} w_i(\mathbf{z}) &= \frac{1}{M(s,\lambda)}\{K(s,z,\lambda)c_{ij}n_j + L(s,z,\lambda)c_{lj}n_l n_j n_i\}, \\ p(\mathbf{z}) &= p_0 + \frac{\tilde{\mu}}{M(s,\lambda)}P(s,z,\lambda)c_{ij}n_i n_j, \quad z < s, \\ p(\mathbf{z}) &= p_0 + \frac{\mu}{M(s,\lambda)}P(s,z,\lambda)c_{ij}n_i n_j, \quad s \leqslant z \leqslant 1, \end{aligned} \tag{3.2}$$

λ denotes the viscosity ratio, $\lambda = \tilde{\mu}/\mu$.

The coefficients $K(s, z, \lambda)$, $L(s, z, \lambda)$, $P(s, z, \lambda)$ and $M(s, \lambda)$ introduced above with the aid of the auxiliary functions

$$\begin{aligned} \phi_0 &= 5s^7 - 7s^5 + 2, \\ \phi_1 &= -[5(2-5\lambda)s^7 + 21s^5\lambda + 4(1+\lambda)], \\ \phi_2 &= s^5[5\lambda s^2 - (2+5\lambda)], \\ \phi_3 &= (1-\lambda)s^5 + \lambda, \\ \phi_4 &= 5(1-\lambda)s^7 + 2 + 5\lambda, \end{aligned} \tag{3.3}$$

were expressed by

$$K = \left(-\frac{5z^3}{s^2} + 3z\right)\phi_0, \quad L = 2\frac{z^3}{s^2}\phi_0, \quad P = -21\frac{z^2}{s^2}\phi_0, \quad \text{for} \quad 0 \leqslant z \leqslant s, \tag{3.4}$$

and for $s \leqslant z \leqslant 1$

$$\begin{aligned} K &= \phi_1 z - 5\frac{z^3}{s^2}\phi_2 + 4\frac{s^5}{z^4}\phi_3, \\ L &= 2\phi_2\frac{z^3}{s^2} + 2\frac{s^3}{z^2}\phi_4 - 10\frac{s^5}{z^4}\phi_3, \\ P &= -\left(21\frac{z^2}{s^2}\phi_2 - 4\frac{s^3}{z^3}\phi_4\right), \end{aligned} \tag{3.5}$$

and for $0 \leqslant z \leqslant 1$

$$M = 4(1-\lambda)s^{10} - 5(2-5\lambda)s^7 - 42\lambda s^5 + 5(2+5\lambda)s^3 - 4(1+\lambda). \tag{3.6}$$

An important property of this solution is that under the assumed conditions the net force and torque acting on spherical particles vanish.

According to (i), for $z \in [1, Z]$,

$$w_i(z) = c_{ij} z_j. \tag{3.7}$$

Let us consider now the volume-averaged rate-of-strain tensor $\overline{\alpha}_{ij}$ (compare Sect. 2). For the reference spheres of radius $Z \leqslant 1$, we have

$$\overline{\alpha}_{ij}(s, Z) = \frac{1}{V(Z)} \int\limits_K \alpha_{ij} dV = \beta(s, Z) c_{ij}, \tag{3.8}$$

where $V(Z) = \frac{4}{3}\pi Z^3$, K the sphere of radius Z, and

$$\beta(s, Z) = \frac{1}{M}\left[\left(-\frac{6}{5}\phi_0 - \phi_1 + \frac{21}{5}\phi_2\right)\left(\frac{s}{Z}\right)^3 - \frac{21}{5}\phi_2\left(\frac{Z}{s}\right)^2 + \phi_1\right], \quad s \leqslant Z \leqslant 1.$$

In the reference sphere of a radius equal to the characteristic length l $(Z = 1)$:

$$\overline{\alpha}_{ij}(s, 1) = c_{ij}. \tag{3.9}$$

A similar condition holds for reference spheres of a radius $Z > 1$; the volume-averaged rate-of-strain tensor $\overline{\alpha}_{ij}$ is by Eq. (3.9), and (i) equal to the imposed c_{ij}.

From the obtained velocity field in the reference sphere with an arbitrary Z, the expression for the mean value of the tensor $\overline{\alpha}_{ij}$ over a set of all reference spheres is calculated:

$$\langle \overline{\alpha}_{ij} V \rangle / \langle V \rangle, \tag{3.10}$$

where $\langle\ \rangle$ denotes averaging with the distribution function $f(s, Z)$ to the nearest neighbour particle (see Part I). Further, we introduce the kinematic coefficient B which is aimed at adjusting the mean rate-of-strain tensor to the imposed one.

The boundary conditions (3.1) are changed to

$$w_i = B^{-1} c_{ij} z_j, \tag{3.11}$$

where the kinematic coefficient B is by definition equal to

$$B(s, \lambda) = \frac{\langle \overline{\alpha}_{ij} V \rangle}{c_{ij} \langle V \rangle} = \frac{1}{M(As^3+1)} \Bigg\{ \phi_1 + \left(-\frac{6}{5}\phi_0 + \frac{21}{5}\phi_2\right) As^3 \tag{3.12}$$
$$+ \left(-\phi_1 + M - \frac{21}{5} A\phi_2 s^{-2}\right) e^{A(s^3-1)} - \frac{21}{5}\phi_2 A^{-2/3} s^{-2} e^{As^3}\left[\Gamma\left(\frac{8}{3}, As^3\right) - \Gamma\left(\frac{8}{3}, A\right)\right]\Bigg\}.$$

Here $A = -8\phi_m \ln(1-s^3)/s^3$, and $\Gamma(x, y)$ denotes the incomplete gamma function, $\Gamma(x, y) = \int\limits_y^\infty e^{-t} t^{x-1} dt$.

Using the boundary conditions (3.11), the demanded property

$$\langle \overline{\alpha}_{ij} V \rangle = c_{ij} \langle V \rangle \tag{3.13}$$

follows.

The effective viscosity will be calculated by the energy method.

For that purpose we calculate the energy dissipation $\overline{D(s, Z)}$ averaged over a set of all reference spheres:

$$\langle \overline{D}V \rangle = \frac{4}{3}\pi l^3 \int \overline{D(s, Z)} Z^3 f(s, Z) dZ. \tag{3.14}$$

Thus, for the number density n, we have in the unit volume

$$n\langle \overline{D}V \rangle = \left(\frac{4}{3}\right)\pi n l^3 \langle \overline{D(s, Z)} Z^3 \rangle = \phi_m \langle \overline{D(s, Z)} Z^3 \rangle. \tag{3.15}$$

The effective viscosity we are looking for can be expressed in the form

$$\frac{\mu^*}{\mu} = \int_s^\infty \left\{1 - \phi_m Z^3 + \phi_m \frac{\overline{D(s, Z)}}{2\mu c_{ij} c_{ij}} Z^3\right\} f(s, Z) dZ = \int_s^\infty H f(s, Z) dZ, \tag{3.16}$$

or

$$\frac{\mu^*}{\mu} = 1 - \phi_m \left(s^3 + \frac{1}{A}\right) + \frac{\phi_m}{B^2} \left\{ e^{A(s^3-1)} \left[b_1 s^{10} + b_2 s^8 + b_3 s^6 \right.\right.$$
$$\left. + b_5 s - b_6 + b_7 s^{-2} + b_8 s^{-4} + \frac{B^2}{A} \right] + e^{As^3} [b_1 s^{10} A^{7/3} G_{-4/3} + b_2 s^8 A^{5/3} G_{-2/3} + b_3 s^6 A G_0$$
$$\left. + b_5 s A^{-2/3} G_{5/3} + b_7 s^{-2} A^{-5/3} G_{8/3} + b_8 s^{-4} A^{-7/3} G_{10/3}] + b_4 s^3 + b_6 \left(s^3 + \frac{1}{A}\right)\right\},$$

where

$$G_\alpha = \Gamma(\alpha, As^3) - \Gamma(\alpha, A),$$

$$b_1 = -96\phi_3^2 M^{-2}, \qquad b_2 = \frac{192}{5}\phi_3\phi_4 M^{-2},$$

$$b_3 = -\frac{24}{5}\phi_4^2 M^{-2}, \qquad b_4 = -[(2\phi_1\phi_4 + 18\phi_2\phi_3)/5] M^{-2},$$

$$b_5 = \frac{12}{5}\phi_2\phi_4 M^{-2}, \qquad b_6 = \phi_1^2 M^{-2},$$

$$b_7 = -\frac{42}{5}\phi_1\phi_2 M^{-2}, \qquad b_8 = \frac{111}{5}\phi_2^2 M^{-2}.$$

The use of the explicitly given distribution function enables us to compute also the higher moments of the magnitude of interest. The most important is the value of the dispersion σ, which characterizes the fluctuations of the bulk viscosity in the suspension. This value is given by the expression

$$\frac{\sigma^2}{\mu^2} = \frac{(\mu^2)^* - (\mu^*)^2}{\mu^2}, \tag{3.17}$$

where the second moment is

$$\frac{(\mu^2)^*}{\mu^2} = \int_s^\infty H^2 f(s, Z) dZ.$$

As described in Part I, the approach can be generalized to calculate the effective viscosity of polydispersed suspensions. In the particular case of the suspension built up of spherical particles of radius a_1, the viscosity ratio λ_1, with the number density n_1 and of spherical particles of radius a_2, the viscosity ratio λ_2, with the number density n_2, we arrived at the following formula:

$$\frac{\mu^*}{\mu} = \sum_{i,j} \int_{\nu_{4i}}^{\nu_{5i}} d_i d_j \Bigg\{ 1 - \phi_m Z_{ij}^3 + \frac{\phi_m}{B^2} \Bigg[b_{1i} \left(\frac{s}{Z_{ij}}\right)^{10} + b_{2i} \left(\frac{s}{Z_{ij}}\right)^{8} + b_{3i} \left(\frac{s}{Z_{ij}}\right)^{6} + b_{4i} \left(\frac{s}{Z_{ij}}\right)^{3} + b_{5i} \left(\frac{s}{Z_{ij}}\right) + b_{6i} + b_{7i} \left(\frac{Z_{ij}}{s}\right)^{2} + b_{8i} \left(\frac{Z_{ij}}{s}\right)^{4} \Bigg] Z_{ij}^3 \Bigg\} f_{ij}(\phi; Z) dZ \tag{3.18}$$

$$+ \sum_{i,j} \int_{\nu_{5i}}^{\infty} d_i d_j \Bigg\{ 1 - \phi_m Z_{ij}^3 + \frac{\phi_m}{B^2} \Bigg[b_{1i} \left(\frac{sL}{l_i}\right)^{10} + b_{2i} \left(\frac{sL}{l_i}\right)^{8} + b_{3i} \left(\frac{sL}{l_i}\right)^{6} + b_{4i} \left(\frac{sL}{l_i}\right)^{3} + b_{5i} \left(\frac{sL}{l_i}\right) + b_{6i} + b_{7i} \left(\frac{l_i}{sL}\right)^{2} + b_{8i} \left(\frac{l_i}{sL}\right)^{4} \Bigg] + \phi_m \left[Z_{ij}^3 - \left(\frac{l_i}{L}\right)^3 \right] \Bigg\} f_{ij}(\phi; Z) dZ.$$

Above, the following notation is used (for details — see Part I):

$$\nu_{4i} = \frac{a_i}{L}, \quad \nu_{5i} = \frac{l_i}{L}, \quad s^3 = \frac{\phi}{\phi_m}, \quad d_i = \frac{n_i}{n_1 + n_2} \text{ for } i = 1, 2, \; L^3 = d_1 l_1^3 + d_2 l_2^3, \tag{3.19}$$

$b_{1i}, b_{2i}, \ldots, b_{8i}$ are defined by Eqs. (3.16) with λ_i entering the formulae (3.3), and (3.6) for $\phi_0, \phi_1, \ldots, \phi_4$, and M in place of λ.

4. Results

We shall examine first the effective viscosity of dilute suspensions. The asymptotic expansion of μ^*/μ for $s \to 0$ is

$$\frac{\mu^*}{\mu} = 1 + \frac{1}{2} \frac{2+5\lambda}{1+\lambda} \phi \Bigg\{ 1 + e^{-8\phi_m} \left[\frac{5}{8\phi_m} + \frac{21}{5} - \frac{32}{25}\phi_m - \frac{1344}{25} \phi_m^2 \xi_{8/3} \right] - e^{-16\phi_m} \left(1 + \frac{168}{25} \phi_m \right) \Bigg\} + \frac{4}{5} \left(\frac{2+5\lambda}{1+\lambda} \right)^2 \phi^2 \ln \phi + \frac{21\lambda}{2(1+\lambda)} \left(\frac{\phi}{\phi_m} \right)^{5/3} \times \left\{ e^{-8\phi_m} \left[\phi_m^2 \xi_{8/3} - \frac{1}{8} \right] + e^{-16\phi_m} \left[\phi_m + \frac{1}{8} \right] \right\} + \ldots, \tag{4.1}$$

where $\xi_{8/3}$ — the numerical coefficient is given by the following series:

$$\xi_{8/3} = \sum_{n=0}^{\infty} \frac{(-1)^n (8\phi_m)^n}{n! \left(\frac{8}{3} + n \right)}$$

and its numerical value is close to 0.375.

Taking $\phi_m = 0.74$, one gets

$$\frac{\mu^*}{\mu} \cong 1 + \frac{1}{2} \frac{2+5\lambda}{1+\lambda} \phi [1 - 0.02] + \frac{4}{5} \left(\frac{2+5\lambda}{1+\lambda} \right)^2 \phi^2 \ln \phi + 0.003 \frac{\lambda}{1+\lambda} \phi^{5/3} + \ldots.$$

Practically, this result coincides with the classical Einstein-Taylor formula because the ratio of the second term in the bracket to the first one is of the order 10^{-2}. The important feature of the formula is the presence of terms proportional to $\phi^2 \ln\phi$, and $\phi^{5/3}$; it is attributed to the presence of multiple interactions in the entire two-phase medium. The appearance of these terms shows also that even for not very dense suspensions the statistics of the suspended particles has a significant influence on the effective viscosity(1).

Further discussion of the results is based on numerical results obtained using the formulae (3.16) and (3.18).

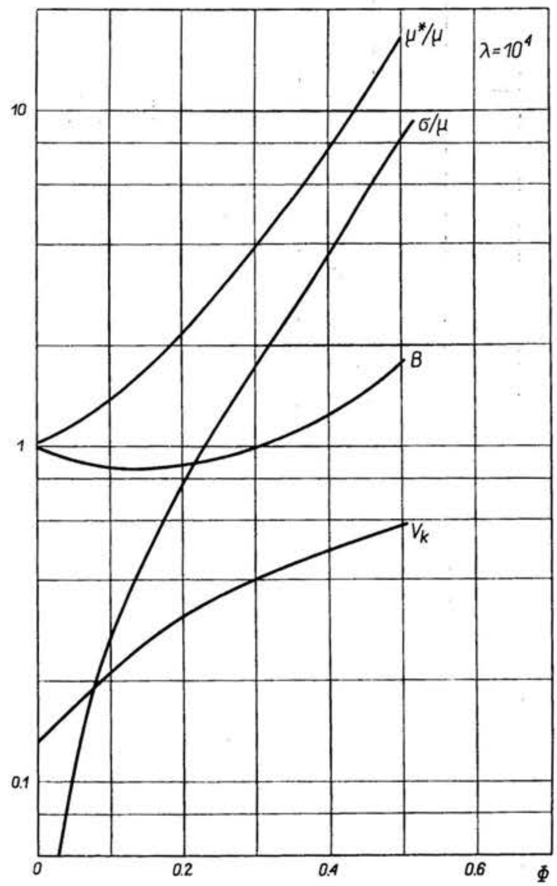

FIG. 1. The effective viscosity μ^*, the dispersion σ, the kinematic coefficient B, and the average volume of reference spheres V_k as the function of volume concentration ϕ, for rigid particles.

(1) In our earlier paper [8] the kinematic coefficient was introduced locally, i.e. we demanded the local adjustement of the field and we used a distribution function introduced ad hoc. Due to the different asymptotic behaviour of this function, for $s \to 0$, the term of order $\phi^2 \ln\phi$ was absent. As a result, the following expression for the dilute limit was obtained:

$$\frac{\mu^*}{\mu} = 1 + \frac{5\lambda+2}{2(1+\lambda)}\phi + o(\phi).$$

The qualitative behaviour of the investigated functions was the same as that obtained in this paper.

In Fig. 1 we illustrate the features of the functions μ^*/μ, B, V_k, and σ/μ for the viscosity ratio 10^4, as the function of ϕ (V_k denotes the averaged volume of reference spheres).

In the next figure (Fig. 2) it can be seen that the effective viscosity is a rapidly increasing function of the concentration, when $\lambda = 10^4$ and $\lambda = 1$, but it is even not monotonic

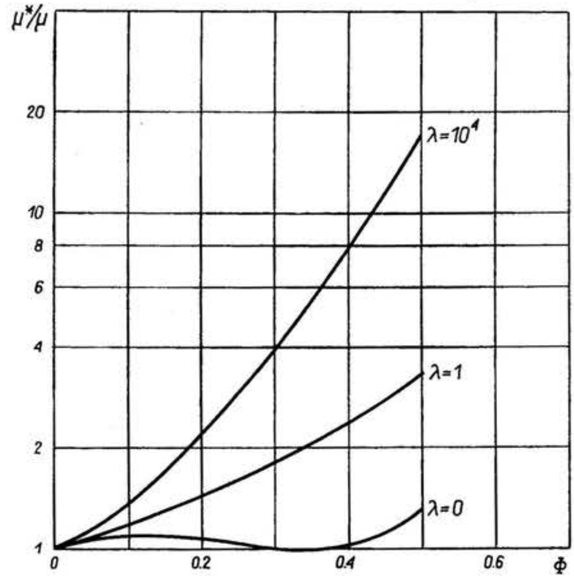

FIG. 2. The effective viscosity μ^* as the function of volume concentration ϕ, for different viscosity ratios ($\lambda = 10^4$, $\lambda = 1$, $\lambda = 0$).

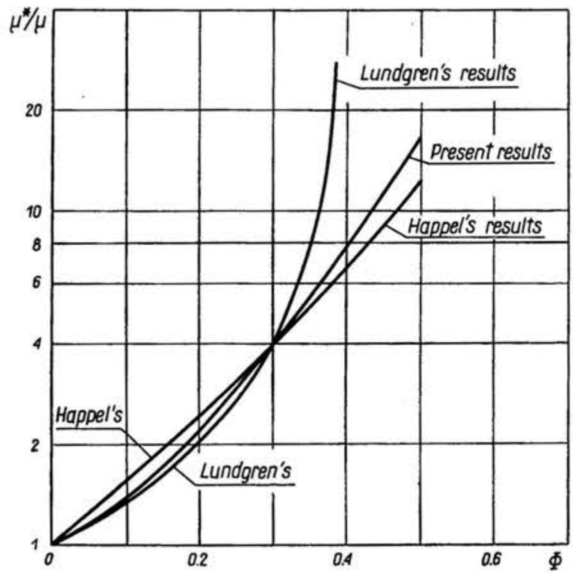

FIG. 3. The comparison of the effective viscosity coefficient for rigid suspended particles, with the theoretical results of HAPPEL [6], and LUNDGREN [7].

for $\lambda = 0$. It should be noted that the results for $\lambda = 0$, i.e. for the suspension of gas-bubbles, are doubtful because in such a suspension the deformation of particles plays probably an important role.

We have compared in Fig. 3 our results with the results of HAPPEL [6] and LUNDGREN [7], obtained by different methods. Lundgren's results seem to be too high for higher concentrations, whereas Happel's approach gives too high results for the concentration range up to 0.2.

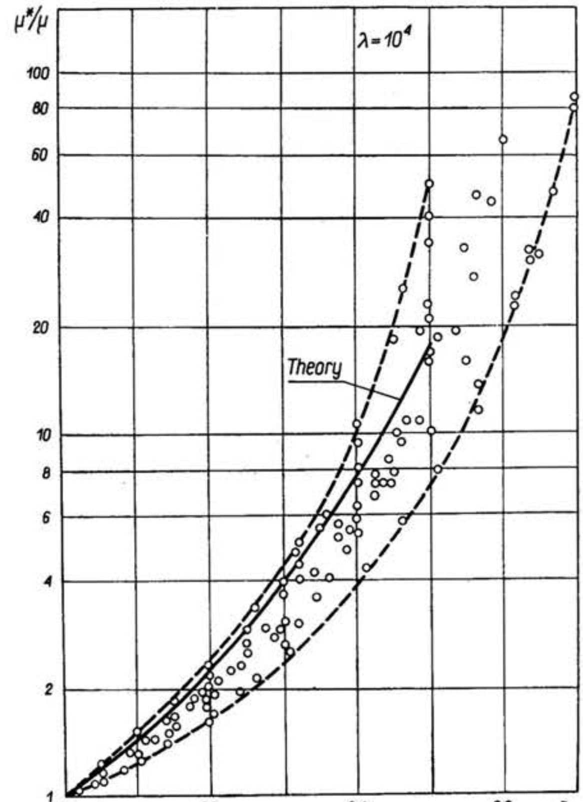

FIG. 4. The comparison of the effective viscosity coefficient μ^* with available experimental data, collected by THOMAS [9].

The next figure (Fig. 4) shows the results and the available experimental data collected by THOMAS [9].

Let us discuss briefly the results for polydispersed suspensions.

Figure 5 presents the effective viscosity of suspensions which consist mainly of suspended particles of the viscosity ratio $\lambda = 10^4$, and only 20% of suspended particles have the viscosity ratio $\lambda = 1.0$, or $\lambda = 0.0$. The presence of suspended particles with a lower viscosity ratio reduces the effective viscosity of suspension. This result is closely related to the increase of the rate of energy dissipation in a reference sphere with the rise of the viscosity ratio λ.

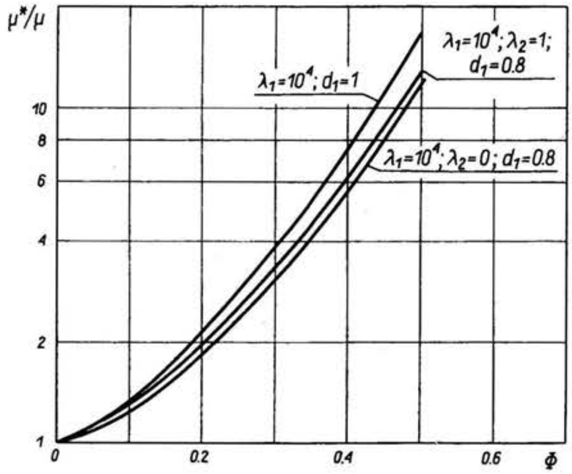

FIG. 5. The effective viscosity coefficient μ^* for suspensions built up of particles of different materials.

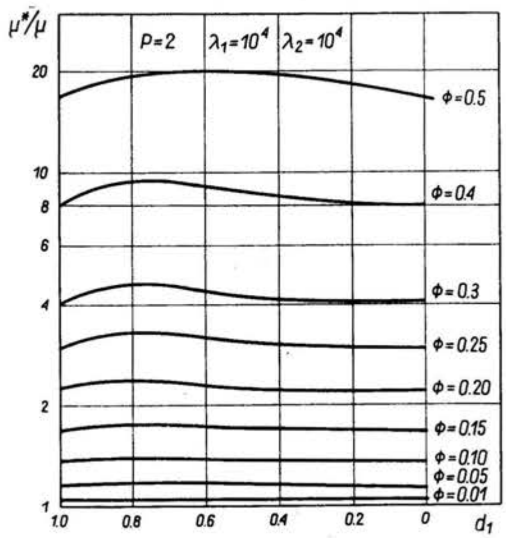

FIG. 6. The effective viscosity coefficient μ^* for suspensions built up of particles of two different radii.

The next figure (Fig. 6) shows the μ^* of a suspension, built up of particles of two different radii $\left(P = (a_2 a_1)/2\right)$. For polydispersed suspension, the only available data were reported by CHONG, CHRISTIANSEN, BAER [10], and the effective viscosity has a minimum in contrast to our results. However, it should be noted, that the radii ratio of spheres used in these experiments was about 20, and $d_1 \sim 0.2, 0.5, 0.75$. One can expect that for such suspensions the value of ϕ_m increases. In the frames of the proposed model we are unable to relate ϕ_m to the concentration ratio of fractions of binary mixture. It was suggested already in [10] that a different value of ϕ_m should be taken for polydispersed suspensions.

In [10] the changes of ϕ_m (or, in the original notation ϕ_∞) for the binary mixtures are very great, much greater that one can expect on the basis of purely geometrical considerations.

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INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received December 14, 1978.