# A non-Riemannian geometrical theory of imperfections in a Cosserat continuum

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STARTING with a basic recognition of Cosserat continua, the author introduces a space of non-metric connection which describes imperfections in a Cosserat continuum. When the consideration is restricted to micropolar materials, the space is specialized to that of metric connection and then there appear no imperfections other than dislocations and disclinations. The identities for the geometric objects of the space are transformed into the basic expressions for dislocations and disclinations in micropolar materials. The total Burgers vector is re-examined in terms of geometry and it is shown that a dislocation source is converted into a distribution of disclinations.

Wychodząc z podstawowego określenia kontinuów Cosseratów autor wprowadza przestrzeń niemetryczną, która opisuje defekty w kontinuum Cosseratów. Gdy rozważania ograniczają się do mikropolarnych materiałów, przestrzeń specyfikuje się jako przestrzeń metryczną i wtedy nie występują defekty inne niż dyslokacje i dysklinacje. Tożsamości charakteryzujące obiekty geometryczne przestrzeni przekształca się na podstawowe wyrażenia dla dyslokacji i dysklinacji w materiałach mikropolarnych. Całkowity wektor Burgersa jest badany z geometrycznego punktu widzenia i pokazuje się, że źródło dyslokacji zamienia się w rozkład dysklinacji.

Исходя из основного определения континуумов Коссера, автор вводит неметрическое пространство, которое описывает дефекты в континуум Коссера. Когда рассуждения ограничиваются микрополярными материалами, тогда пространство специфицируется как метрическое пространство и тогда не выступают дефекты другие, чем дислокации и дисклинации. Тождества, характеризующие геометрические объекты пространства, преобразуются в основные выражения для дислокаций и дисклинаций в микрополярных материалах. Полный вектор Бюргерса исследовался с геометрической точки зрения и оказывается, что источник дислокаций изменяется в распределение дисклинаций.

## 1. Introduction

THE non-Riemannian plasticity theory has been established by KONDO and his collaborators [1]. In his extensive works, Kondo introduced a space of linear connection whose torsion and R.-C. curvature tensors are compared with a distribution of imperfections in crystals. Its fundamental metric tensor and coefficient of linear connection are given with the aid of tearing. Tearing is a step in which we cut materials into small pieces and bring them into the natural, or free, state.

The aim of the present paper is to develop an application of the theory to the problem of imperfections in a Cosserat continuum. E. and F. COSSERAT [2] themselves stated their basic recognition of deformable bodies in he following words: "L'ensemble continue de trois dimensions de trièdres sera ce que nous appellerons un milieu déformable". In Cosserat materials, trièdres — trihedrons — play the role of points in the ordinary materials. The present theory starts with this basic recognition of Cosserat materials.

SAKATA [3] developed a theory of zero- and one-diemnsional tearings and obtained a space of non-metric connection which describes a field of imperfection. ANTHONY [4] discussed the lattice connection and elastic metric and stated that there are some cases where the former is non-metric with respect to the latter (see also [5]).

In the present paper we shall develop an elementary exploration for the derivation of a space of non-metric connection which is required for the description of imperfections in a Cosserat continuum. It is also the aim of the present paper to derive the equations which we have used in the analyses for the stress and couple-stress fields due to dislocations and disclinations in a micropolar continuum [6, 7].

When micropolar materials are assumed, no imperfections, other than dislocations and disclinations, appear. The total Burgers vector [8] is re-examined with the aid of the geometry and it will be shown that a dislocation source [9] is converted into a distribution of disclinations.

## 2. Tearing of Cosserat materials

Consider a three-dimensional Cosserat continuum. A Cosserat continuum is an ensemble of trihedrons. A thrihedron is composed of a point and three vectors drawn therefrom. When materials are deformed, those trihedrons not only undergo rigid translations but also change the directions and lengths of their vectors.

Assume an orthogonal Cartesian coordinate system with respect to which the position of a point is stated by $x^\varkappa$. Throughout this paper, Greek indices take 1, 2 or 3 and Einstein's summation convention is used for indices appearing twice in one expression. Let $C$ be a point in the body and $e_\varkappa$ be three vectors drawn therefrom. We denote a trihedron composed of $C$ and $e_\varkappa$ by $[c; e_\varkappa]$. In geometrical terminologies, $e_\varkappa$ construct a reference frame. We denote it by $\{e_\varkappa\}$.

We shall start with a special case where trihedrons are attached to the material-lines so that they deform and rotate together with the deformations of the material. This is the case of the materials of constrained rotation.

When a material piece, including trihedron $[C; e_\lambda]$, is torn apart from the body, it changes its shape and, in the same time $e_\varkappa$ change their directions and lengths. Let $\bar{e}_\varkappa$ be the vectors so changed from $e_\varkappa$. Since, $e_\varkappa$ and $\bar{e}_\varkappa$ are vectors in three-dimensional space, we should have

$$(2.1) \qquad e_\varkappa = B_\varkappa^\lambda \bar{e}_\lambda \quad \text{and} \quad \bar{e}_\varkappa = \bar{B}_\varkappa^\lambda e_\lambda,$$

where $B_\varkappa^\lambda$ is the tensor of transformation, $\bar{B}_\varkappa^\lambda$ its inverse transformation so that $B_\varkappa^\lambda \bar{B}_\lambda^\mu = \delta_\varkappa^\mu$ and $B_\lambda^\varkappa \bar{B}_\mu^\lambda = \delta_\mu^\varkappa$, $\delta_\mu^\varkappa$ being the Kronecker delta.

PROPOSITON (I) requires that: $\{e_\varkappa\}$ is so chosen that $\{\bar{e}_\varkappa\}$ becomes orthonormal so that

$$(2.2) \qquad \bar{e}_\varkappa \cdot \bar{e}_\lambda = \delta_{\varkappa\lambda}.$$

By the substitution of Eqs. $(2.1)_2$ and $(2.2)$,

$$(2.3) \qquad g_{\varkappa\lambda} \overset{\text{def}}{=} e_\varkappa \cdot e_\lambda = B_\varkappa^\nu B_\lambda^\mu \delta_{\nu\mu}.$$
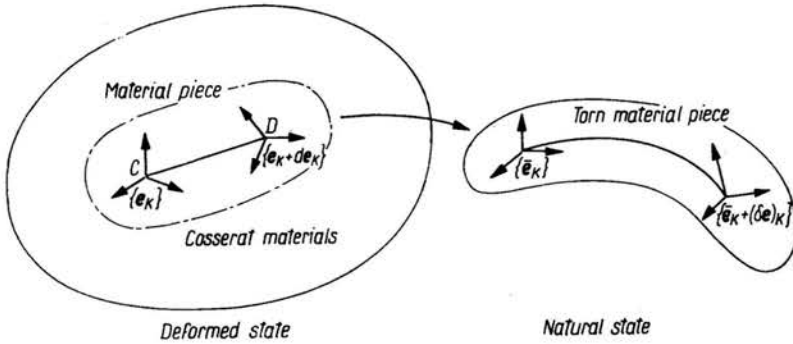
This will be used as the fundamental metric tensor.

FIG. 1. Tearing of Cosserat materials. A matrial piece including two trihedrons ($[C;\ \mathbf{e}_\varkappa]$ and $[D; \mathbf{e}_\varkappa+d\mathbf{e}_\varkappa]$) is torn apart from the body.

We assume the trihedron $[D; \mathbf{e}_\varkappa+d\mathbf{e}_\varkappa]$, which is placed in the neighbourhood of $[C; \mathbf{e}_\varkappa]$. We cut out a material piece, including those two trihedrons, from the body and bring it into the natural state. This is a naturalization of Cosserat materials with respect to a vector line-element connecting $C$ to $D$.

Let $\{\mathbf{e}_\varkappa\}$ and $\{\mathbf{e}_\varkappa+d\mathbf{e}_\varkappa\}$ be brought into $\{\bar{\mathbf{e}}_\varkappa\}$ and $\{\bar{\mathbf{e}}_\varkappa+(\delta e)_\varkappa\}$, respectively, by the naturalization of the materials with respect to $\overrightarrow{CD}$. In the special case mentioned before, $\{\mathbf{e}_\varkappa\}$ and $\{\bar{\mathbf{e}}_\varkappa\}$ are connected by Eq. (2.1), and $\{\mathbf{e}_\varkappa+d\mathbf{e}_\varkappa\}$ and $\{\bar{\mathbf{e}}_\varkappa+(\delta e_\varkappa)\}$ should also be connected by a similar relation. Note that $\{\bar{\mathbf{e}}_\varkappa+(\delta e)_\varkappa\}$ is not necessarily orthonormal, although $\{\bar{\mathbf{e}}_\varkappa\}$ is orthonormal by Proposition (I).

Let $\bar{B}_\varkappa^\lambda+d\bar{B}_\varkappa^\lambda$ be the value of $\bar{B}_\varkappa^\lambda$ at $D$. Then, we have

$$(2.4) \qquad \bar{\mathbf{e}}_\varkappa+(\delta\mathbf{e})_\varkappa = (\bar{B}_\varkappa^\lambda+d\bar{B}_\varkappa^\lambda)\cdot(\mathbf{e}_\lambda+d\mathbf{e}_\lambda).$$

If follows from Eqs. $(2.1)_2$ and (2.4) that

$$(2.5) \qquad \mathscr{D}\mathbf{e}_\lambda \overset{\text{def}}{=} B_\lambda^\varkappa(\delta\mathbf{e})_\varkappa = d\mathbf{e}_\lambda+B_\lambda^\varkappa d\bar{B}_\varkappa^\mu\,\mathbf{e}_\mu.$$

In the case where $B_\varkappa^\lambda$ is a continuously differentiable function of the position, the last equation is transformed into

$$(2.6) \qquad \mathscr{D}\mathbf{e}_\lambda = d\mathbf{e}_\lambda+dx^\nu B_\lambda^\varkappa(\partial_\nu\bar{B}_\varkappa^\mu)\mathbf{e}_\mu,$$

where the Cartesian coordinates of $C$ and $D$ are $x^\varkappa$ and $x^\varkappa+dx^\varkappa$, respectively.

We shall enter into the general case where trihedrons are relaxed from the materials lines so that $e_{n's}$ can change their directions and lengths independently of the deformations of the latter. In this general case, $\{\mathbf{e}_K+d\mathbf{e}_K\}$ is brought into $\{\bar{\mathbf{e}}_K+(\delta\bar{\mathbf{e}}_K)\}$, where $(\delta\bar{e})_K-(\delta e)_K = \mathbf{0}_K$.

If the distance between $C$ and $D$ is sufficiently small, this additional term is proportional to $dx^\varkappa$ and, moreover, it is also proportional to $\mathbf{e}_\varkappa$ themselves. Therefore, this additional term can be put as $-dx^\nu B_\lambda^\mu Q_{\nu\mu}^{:\varkappa}\mathbf{e}_\varkappa$ and Eq. (2.6) is replaced by $D\mathbf{e}_\lambda \overset{\text{def}}{=} B_\lambda^K(\delta\bar{e})_L$ so that

$$(2.7) \qquad \mathscr{D}\mathbf{e}_\lambda = d\mathbf{e}_\lambda-dx^\nu\Gamma_{\nu\lambda}^{\varkappa}\mathbf{e}_\varkappa,$$

where

$$(2.8) \qquad \Gamma_{\nu\mu}^\varkappa = -B_\mu^\tau(\partial_\nu\bar{B}_\tau^\varkappa)+Q_{\nu\mu}^{:\varkappa}.$$

$Q_{\nu\mu}^{\cdot\cdot\varkappa}$ is a tensor which is connected to the basic physical recognition of the materials. For example, if micropolar materials are assumed, $dx^\nu Q_{\nu\mu}^{\cdot\cdot\varkappa}$ becomes a tensor of rotation, and if materials of the higher grade are considered, it includes pure strains.

PROPOSITION (II) requires that Eq. (2.7) be interpreted as the absolute differential of $\{e_\varkappa\}$.

$\Gamma_{\nu\mu}^{\cdot\varkappa}$ is employed as the coefficient of linear connection. Note that the coefficient of linear connection is given in conjunction with *free* trihedrons, as compared to the metric tensor which is introduced in connection with constrained trihedrons.

$g_{\varkappa\lambda}$ and $\Gamma_{\nu\mu}^{\cdot\varkappa}$ enable us to assume a space of linear connection. Details will be studied in the next section.

## 3. Space of non-metric connection

We assume a space of linear connection ($L_3$) whose metric tensor is $g_{\varkappa\lambda}$ and the coefficient of linear connection is $\Gamma_{\nu\mu}^{\cdot\lambda}$. A geometrical version of the naturalization of the material line $\overline{CD}$ is that a point undergoes a Cartan displacement along $\overline{CD}$, and the absolute differential of the reference frame by $\mathcal{D}e_\lambda$. Details of the geometry of the space of linear connection can be found, e.g. in [10, 11].

By the substitution of Eqs. (2.4) and (2.9),

(3.1) $$\nabla_\nu g_{\mu\lambda} = \partial_\nu g_{\mu\lambda} - \Gamma_{\nu\lambda}^{\cdot\rho} g_{\rho\lambda} - \Gamma_{\nu\mu}^{\cdot\rho} g_{\rho\mu} = -2Q_{\nu(\mu\lambda)},$$

where $\nabla$ means the absolute derivative in relation to $\Gamma_{\nu\mu}^{\cdot\lambda}$,

(3.2) $$Q_{\nu\mu\lambda} = Q_{\nu\mu}^{\cdot\cdot\rho} g_{\rho\lambda},$$

and ( ) means to calculate the symmetric part with respect to the indices enclosed. Equation (3.1) means that, as far as $Q_{\nu(\mu\lambda)} \neq 0$, $L_3$ emerges as a space of non-metric connection.

$L_3$ has two important geometric objects: one is the torsion tensor which is given by

(3.3) $$S_{\nu\mu}^{\cdot\cdot\lambda} \overset{\text{def}}{=} \Gamma_{[\nu\mu]}^{\cdot\lambda}$$

and the other, the Riemann-Christoffel curvature tensor given by

(3.4) $$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa} \overset{\text{def}}{=} 2(\partial_{[\nu}\Gamma_{\mu]\lambda}^{\cdot\varkappa} + \Gamma_{[\nu|\rho|}^{\cdot\varkappa}\Gamma_{\mu]\lambda}^{\cdot\rho}),$$

where [ ] means to calculate the antisymmetric part with respect to the indices enclosed and, if indices have to be singled out, the sign | | is used. We assume

(3.5) $$R_{\nu\mu\lambda\varkappa} = R_{\nu\mu\lambda}^{\cdot\cdot\cdot\rho} g_{\rho\varkappa} \quad \text{and} \quad S_{\nu\mu\lambda} = S_{\nu\mu}^{\cdot\cdot\rho} g_{\rho\lambda}.$$

The physical counterparts of those objects are well known (see [1]): the torsion tensor $S_{\nu\mu}^{\cdot\cdot\lambda}$ is compared with a distribution of dislocations, and the antisymmetric part of the R.-C. curvature tensor $R_{[\nu\mu][\lambda\varkappa]}$ with disclinations. We have no definite physical counterpart of $R_{[\nu\mu](\lambda\varkappa)}$. This may appear in materials of the higher grade, as will be stated later.

We put

(3.6) $$\alpha^{\lambda\varkappa} = \varepsilon^{\lambda\mu\nu} S_{\nu\mu}^{\cdot\cdot\varkappa} \quad \text{and} \quad S_{\nu\mu}^{\cdot\cdot\varkappa} = \frac{1}{2}\varepsilon_{\nu\mu\lambda}\alpha^{\lambda\varkappa}$$

and

(3.7) $$\theta^{\lambda\varkappa} = \frac{1}{4}\,\varepsilon^{\lambda\nu\mu}\varepsilon^{\tau\varkappa\rho}R_{\nu\mu\tau\rho} \quad \text{and} \quad R_{(\nu\mu)[\lambda\varkappa]} = \varepsilon_{\nu\mu\tau}\varepsilon_{\lambda\rho\varkappa}\theta^{\tau\rho},$$

where $\varepsilon^{\lambda\nu\mu}$ is Eddington's permutation symbol. $\alpha^{\lambda\varkappa}$ is the dislocation density tensor and $\theta^{\lambda\varkappa}$ the disclination density tensor.

Form Eq. (2.8),

(3.8) $$S_{\lambda\varkappa}^{\cdot\cdot\mu} = \bar{B}_\rho^\mu\,\partial_{[\lambda}B_{\varkappa]}^\rho + Q_{[\lambda\varkappa]}^{\cdot\cdot\mu}.$$

From the third identity and Eq. (3.1) it is well known that

(3.9) $$R_{\nu\mu(\lambda\varkappa)} = 2\nabla_{[\nu}Q_{\mu](\lambda\varkappa)} + 2S_{\nu\mu}^{\cdot\cdot\rho}Q_{\rho(\lambda\varkappa)}.$$

Moreover, since the R.-C. curvature tensor calculated from the first term of the right hand side of Eq. (2.8) vanishes, we get

(3.10) $$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa} = 2\nabla_{[\nu}Q_{\mu]\lambda}^{\cdot\cdot\varkappa} - 2Q_{[\nu|\rho|}^{\cdot\varkappa}Q_{\mu]\lambda}^{\cdot\cdot\rho} + 2S_{\nu\mu}^{\cdot\cdot\rho}Q_{\rho\lambda}^{\cdot\cdot\varkappa},$$

from which, when $Q_{\nu(\lambda\varkappa)} = 0$,

(3.11) $$R_{\nu\mu[\lambda\varkappa]} = 2\nabla_{[\nu}Q_{\mu][\lambda\varkappa]} + 2S_{\nu\mu}^{\cdot\cdot\rho}Q_{\rho[\lambda\varkappa]}.$$

Details of the derivation of those equations can be found, for example, by [10]. Equations (3.8) to (3.11) will be used in the later calculation.

## 4. Imperfections in Cosserat continua

Deformations of Cosserat materials are stated by means of distortions and microdistortions. The distortion is related to $B_\varkappa^\lambda$ (or $\bar{B}_\varkappa^\lambda$), for the microdistortion $Q_{\nu\mu}^{\cdot\cdot\lambda}$. The present section will introduce those terms and lead the expressions for dislocations and disclinations in terms of those distortions from the geometry of the space.

We put

(4.1) $$B_\varkappa^\lambda = \delta_\varkappa^\lambda + \gamma_\varkappa^\lambda,$$

where $\gamma_\varkappa^\lambda$ is the distortion. This implies the relative translation between two adjacent trihedrons. The relative rotation and strain between them are given by $dx^\nu\Gamma_{\nu\mu}^{\cdot\cdot\lambda}$, as has been mentioned.

It follows from Eq. (2.8) that

(4.2) $$\Gamma_{\mu[\varkappa\mu]} = -g_{\rho[\mu}B_{\varkappa]}^\tau\partial_\lambda\bar{B}_\tau^\rho + Q_{\lambda[\varkappa\mu]},$$

and

(4.3) $$\Gamma_{\lambda(\varkappa\mu)} = \partial_\lambda g_{\varkappa\mu} + Q_{\lambda(\varkappa\mu)},$$

where

(4.4) $$\Gamma_{\lambda\varkappa\mu} = \Gamma_\varkappa^{\lambda\rho}g_{\rho\mu}.$$

Equation (4.2) implies the relative rotation between two adjacent trihedrons, and Eq. (4.3) the pure strain between them. The first term of the right hand side of Eq. (4.2) is the contribution of the distortion to the relative rotation, while the second term is originated from

the micromechanics of Cosserat materials. Similar circumstances are given for the terms of Eq. (4.3). Therefore, we may call $Q_{\lambda[\nu\mu]}$ the microrotation, and $Q_{\lambda(\nu\mu)}$ the microstrain.

In the following calculations, the terms of orders higher than or equal to the second with respect to $S_{\nu\mu}^{\cdot\cdot\lambda}$, $Q_{\nu\mu}^{\cdot\cdot}$ and $\gamma_{\varkappa}^{\lambda}$ are neglected. As far as micropolar materials are concerned, the microstrain plays no essential part of the theory. Therefore, we may assume $Q_{\lambda(\nu\mu)} = 0$. In this case $L_3$ entails a space of metric connection and $R_{\nu\mu(\lambda\varkappa)}$ vanishes. This means that there are no imperfections other than dislocations and disclinations.

We put

$$(4.5) \qquad \varkappa_{\lambda}^{\cdot\mu} = \frac{1}{2} \varepsilon^{\nu\mu\varkappa} Q_{\lambda\nu\varkappa} \quad \text{and} \quad Q_{\lambda[\nu\varkappa]} = \varepsilon_{\nu\rho\varkappa} \varkappa_{\lambda}^{\cdot\rho}.$$

By substituting Eqs. (3.6)$_1$, (4.1) and (4.5) in Eq. (3.8),

$$(4.6) \qquad \alpha_{\cdot\mu}^{\rho} = \varepsilon^{\rho\lambda\varkappa} \partial_{\lambda} \gamma_{\varkappa\mu} + \varepsilon^{\rho\lambda\varkappa} \varepsilon_{\varkappa\mu\tau} \varkappa_{\lambda}^{\cdot\tau},$$

where $\gamma_{\lambda\mu} = \gamma_{\lambda}^{\rho} \delta_{\rho\mu}$. On the other hand, by substituting Eq. (3.7)$_1$ and (4.5) in Eq. (3.11),

$$(4.7) \qquad \theta^{\tau\rho} = \varepsilon^{\tau\nu\mu} \partial_{\nu} \varkappa_{\mu}^{\cdot\rho}.$$

The last two equations have been used in the analysis of the stress and couple-stress fields due to dislocations and disclinations in micropolar continua [6, 7].

When $Q_{\nu(\mu\varkappa)} \neq 0$, there appear imperfections of the higher grade. In a special case where

$$(4.8) \qquad Q_{\nu\mu\lambda} = Q_{\nu} g_{\mu\lambda},$$

we get from Eq. (3.9)

$$(4.9) \qquad R_{\nu\mu\lambda\varkappa} = \partial_{[\nu} Q_{\mu]} \delta_{\lambda\varkappa}.$$

A physical counterpart of this term is as follows: If we bring materials belonging to a closed circuit into the natural state, starting from one point on it, and going on from one point to the next. After the return to the initial point, we get a broken line in the natural state. In the case where Eq. (4.9) takes a finite value, there appears a dilatational discrepancy between materials at both ends of the broken line. This is the simplest case of imperfections of the higher grade.

In this case we get from Eqs. (3.8) and (3.9)

$$(4.10) \qquad \alpha^{\tau\mu} = \varepsilon^{\lambda\varkappa\tau} \partial_{\lambda} \lambda_{\varkappa}^{\mu} - q^{\tau\mu} \quad \text{and} \quad r^{\tau} = \partial_{\varkappa} q^{\tau\varkappa},$$

where

$$(4.11) \qquad q^{\lambda\varkappa} = \varepsilon^{\lambda\varkappa\tau} Q_{\tau} \quad \text{and} \quad Q_{\tau} = \frac{1}{2} \varepsilon_{\tau\lambda\varkappa} q^{\lambda\varkappa},$$

and

$$(4.12) \qquad r^{\tau} \delta_{\lambda\varkappa} = \frac{1}{2} \varepsilon^{\tau\nu\mu} R_{\nu\mu\lambda\varkappa} \quad \text{and} \quad R_{\nu\mu\lambda\varkappa} = \varepsilon_{\nu\mu\tau} \delta_{\lambda\varkappa} r^{\tau}.$$

Equations (4.10) are the fundamental equations for this case.

## 5. Total dislocation

Assume a material point whose radius vector is $x^\varkappa$. If this point undergoes a Cartan displacement along a circle, the point will, after its return, have a radius vector $x^\varkappa -$

$-\frac{1}{2} R_{\nu\mu\lambda}^{\cdots\varkappa} x^\lambda df^{\nu\mu} - S_{\nu\mu}^{\cdots\varkappa} df^{\nu\mu}$, $df^{\nu\mu}$ being the bivector of a surface enclosed by the circle.

There appears a change in the radius vector. To put it back on its initial position,

(5.1)
$$B^\varkappa = \frac{1}{2} R_{\nu\mu\lambda}^{\cdots\varkappa} x^\lambda df^{\nu\mu} + S_{\nu\mu}^{\cdots\varkappa} df^{\nu\mu}$$

should be added to it.

Let a point in the real materials be mapped onto the materials in the natural state by the following way. Locate the origin of the coordinate system on the natural state and draw the radius vector of a point from this origin. The mapping of the point is located at its terminal point. After the Cartan displacement of the point has been undertaken, its radius vector is changed by $B$, and accordingly its mapping should be moved by the same vectorial amount. This means that there exists an uncertainty in the location of the mapping.

The above positional uncertainty is different from that stated by $S_{\nu\mu}^{\cdots\lambda}$. The latter is explored by the following way: Assume a circle in the real materials and map a point belonging to it onto the natural state. The circle is mapped onto the natural state from one point to the next. After the return, the mapping assumes a position which is different from the initial one. This also means an uncertainty in the location of the mapping.

Note that in the case mentioned before the origin of the coordinate system is a priori located on the natural state, and the positional uncertainty is explored with respect to it. On the contrary, in the latter case, the mapping of a material point is a priori located on the natural state, and the positional uncertainty is considered with respect to itself.

As is well known, $S_{\nu\mu}^{\cdots\lambda}$ is related to the ordinary Burgers vector. In case where $R_{\nu\mu\lambda}^{\cdots\varkappa} = 0$, $B^\varkappa$ is identical with the above vector. $B^\varkappa$ is called the total Burgers vector.

We put

(5.2)
$$G_{\nu\mu}^{\cdots\varkappa} = \frac{1}{2} R_{\nu\mu\lambda}^{\cdots\varkappa} x^\lambda + S_{\nu\mu}^{\cdots\varkappa},$$

from which we get total dislocation density tensor through

(5.3)
$$\xi^{\tau\varkappa} = \varepsilon^{\tau\nu\mu} G_{\nu\mu}^{\cdots\varkappa} \quad \text{and} \quad G_{\nu\mu}^{\cdots\varkappa} = \frac{1}{2} \varepsilon_{\nu\mu\tau} \xi^{\tau\varkappa}.$$

In what follows we shall restrict the consideration to micropolar materials.

By the substitution of Eqs. (3.7) and (5.3) in Eq. (5.2),

(5.4)
$$\xi^{\tau\varkappa} = \alpha_{\cdot}^{\tau\varkappa} + \varepsilon_{\gamma\varkappa\lambda} \theta^{\tau\gamma} x^\lambda.$$

This has been given by DEWIT [8].

It follows from Eqs. (5.1), (5.2) and (5.4) that

(5.5)
$$B_\varkappa = \int dS_\tau \xi_{\cdot\varkappa}^\tau,$$

where $S$ is a surface enclosed by a circle with respect to which the total Burgers vector is calculated. If Eq. (5.5) is calculated with respect to a closed surface, we should have

$$(5.6) \qquad \oint dS_\tau(\alpha^\tau_{\cdot\varkappa} + \varepsilon_{\gamma\varkappa\lambda}\theta^{\tau\gamma}x^\lambda) = \int \{(\partial_\tau\alpha^\tau_{\cdot\varkappa} + \varepsilon_{\varkappa\tau\gamma}\theta^{\tau\gamma}) + \partial_\tau\theta^{\tau\gamma}\}dV,$$

where the volume integral is calculated with respect to the region enclosed by the closed surface under consideration. Equation (5.6) vanishes because of

$$(5.7) \qquad \partial_\tau\alpha^\tau_{\cdot\varkappa} + \varepsilon_{\varkappa\tau\gamma}\theta^{\tau\gamma} = 0 \quad \text{and} \quad \partial_\tau\theta^{\tau\gamma} = 0.$$

Accordingly, we arrive at the conclusion that the field of total dislocation density tensor has no sources, nor sinks, in the interior of the field. This is similar to the case of ordinary dislocations where there are no disclinations. Since Eq. (5.6) vanishes,

$$(5.8) \qquad \partial_\tau\xi^\tau_{\cdot\varkappa} = 0.$$

Before closing this section, we shall compare the total dislocation with the resultant, or total, couple due to the stress and couple-stress fields. The total couple, with respect to the origin of the coordinate system, acting on a region of the materials from the surrowndings is given by

$$(5.9) \qquad M_\varkappa = \oint dS_\tau M^\tau_{\cdot\varkappa},$$

where

$$(5.10) \qquad M^\tau_{\cdot\varkappa} = \mu^\tau_{\cdot\varkappa} + \varepsilon_{\gamma\varkappa\lambda}\sigma^{\tau\gamma}x^\lambda,$$

$\mu^\tau_{\cdot\varkappa}$ is the couple-stress tensor and $\sigma^{\lambda\varkappa}$ the stress tensor and the integration is calculated with respect to the surface enclosing the region. $M^\tau_{\cdot\varkappa}$ is the total couple-stress tensor.

If no body couples are acting, we have from Eq. (5.9)

$$(5.11) \qquad \partial_\tau M^\tau_{\cdot\varkappa} = 0,$$

which is followed by the well-known equation

$$(5.12) \qquad \partial_\tau\mu^\tau_{\cdot\varkappa} + \varepsilon_{\varkappa\tau\gamma}\sigma^{\tau\gamma} = 0 \quad \text{and} \quad \partial_\tau\sigma^{\tau\gamma} = 0.$$

Equations (5.9), (5.11) and (5.12) are compared with Eqs. (5.5), (5.8) and (5.7), respectively.

## 6. Dislocation source

We shall consider the relation between disclinations and dislocation sources.

Let $L_1$ be a closed circle and $P$ a point belonging to it. Draw a vector $v^\varkappa$ from $P$. $Q$ is its terminal point (Fig. 2). If $P$ travels on $L_1$, $Q$ draws a closed circle — $L_2$ of Fig. 2. Let $S_1$ and $S_2$ be surfaces enclosed by $L_1$ and $L_2$, respectively.

If the materials belonging to $L_1$ are brought into the natural state, the line is not closed. Its closure failure being $S^{\cdot\cdot\varkappa}_{\nu\mu}df^{\nu\mu}$, is identified as the Burgers vector if $L_1$ is taken as the Burgers circuit. In the same time, $L_2$ is also brought into a line which is not closed. Its closure failure is given by $\frac{1}{2} R^{\cdot\cdot\varkappa}_{\nu\mu\lambda}v^\lambda df^{\nu\mu} + S^{\cdot\cdot\varkappa}_{\nu\mu}df^{\nu\mu}$. This implies the Burgers vector if $L_2$ is taken as the Burgers circuit.
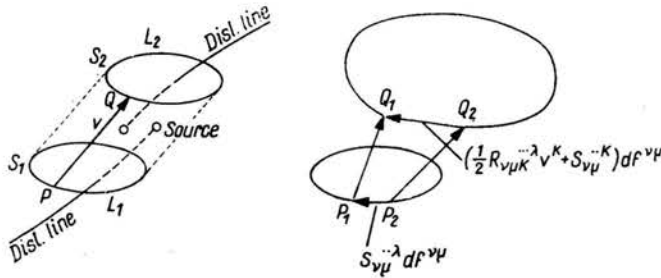
FIG. 2. Dislocation sources and the difference between the Burgers vectors.

When $R_{\nu\mu\lambda}^{\cdots\kappa} \neq 0$, those two Burgers vectors are not identical with each other. This means that some of the dislocation lines which pierce $S_1$ or $S_2$ are terminated at the points in the cylindrical region whose bases are $S_1$ and $S_2$. There exist sources of dislocations in the cylindrical region (see [9]). The strength of those sources is given by the difference between the two Burgers vectors; one calculated with respect to $L_1$ and the other with respect to $L_2$. From the discussion mentioned before, this should be $\dfrac{1}{2} R_{\nu\mu\lambda}^{\cdots\kappa} v^\lambda df^{\nu\mu}$.
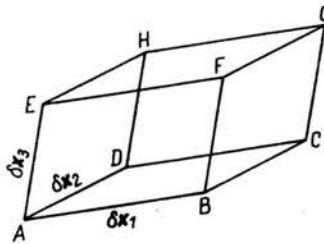


FIG. 3. Parallelepiped region.

We consider a parallelepiped region (Fig. 3). Let $\delta x_1^\kappa$, $\delta x_2^\kappa$ and $\delta x_3^\kappa$ be the vectors belonging to its sides which start from one vertex. The strength of the sources with respect to the dislocation lines piercing $ABCD$ and $EFGH$ is given from the above expression. Since $df^{\nu\mu} = 2\delta x_1^{[\nu} \delta x_2^{\mu]}$ and $v^\kappa = \delta x_3^\kappa$, it entails that $R_{\nu\mu\lambda}^{\cdots\kappa} \delta x_3^\lambda \delta x_1^{[\nu} \delta x_2^{\mu]}$. A similar expression is given with respect to $ADHE$ and $BCGF$, as well as $ABEF$ and $DCGH$. The former is $R_{\nu\mu\lambda}^{\cdots\kappa} \delta x_1^\lambda \delta x_2^{[\nu} \delta x_3^{\mu]}$ and the latter $R_{\nu\mu\lambda}^{\cdots\kappa} \delta x_2^\lambda \delta x_3^{[\nu} \delta x_1^{\mu]}$.

Therefore, the strength of the sources distributed in the parallelepiped region is given by the sum of those three terms such as follows:

$$(6.1) \qquad R_{\nu\mu\lambda}^{\cdots\kappa} \delta x_3^\lambda \delta x_1^{[\nu} \delta x_2^{\mu]} + R_{\nu\mu\lambda}^{\cdots\kappa} \delta x_1^\lambda \delta x_2^{[\nu} \delta x_3^{\mu]} + R_{\nu\mu\lambda}^{\cdots\kappa} \delta x_2^\lambda \delta x_3^{[\nu} \delta x_1^{\mu]} = \frac{1}{2} \varepsilon^{\nu\mu\lambda} R_{\nu\mu\lambda}^{\cdots\kappa} \delta V,$$

where $\delta V$ is the volume of the parallelepiped region. Thus, the density of the sources is given by

$$(6.2) \qquad R^\kappa = \frac{1}{2} \varepsilon^{\nu\mu\lambda} R_{\nu\mu\lambda}^{\cdots\kappa}.$$

In case where there are no imperfections other than dislocations and disclinations, by substituting Eq. $(3.7)_2$ the last equation is transformed into

$$(6.3) \qquad R_\kappa = \varepsilon_{\lambda\kappa\mu} \theta^{\lambda\mu}.$$

This means that a source of dislocations is converted into a distribution of disclinations. This has been discussed by DEWIT [12] from the crystallographical point of view.


## 7. Concluding summary

We have introduced a space of non-metric connection which describes a field of imperfections in a Cosserat continuum. The explorations partly overlap the pioneering works by BILBY *et al.*, SAKATA, and ANTHONY. The gist is that the metric tensor and the coefficients of connection are introduced from the different facets of the deformations of the materials. BILBY *et al.* and ANTHONY assumed them from the crystallographical point of view, and SAKATA from diakoptics. In the present paper the treatment was made from an elementary recognition of Cosserat continua.

When micropolar materials are concerned, the space becomes of metric connection. From the identities for the geometric objects of the space we have obtained some basic expressions for dislocations and disclinations in a Cosserat continuum.

In the last two sections we have discussed the total Burgers vector and dislocation sources. It has been shown that they bear certain geometrical implications.

A space of non-metric connection is required to describe imperfections of the higher grade, where microstrains are taken into consideration. Detailed explorations in this regard are beyond the scope of the present paper. Those will be made elsewhere.


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