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Wave propagation in thermo-viscous materials with hidden variables

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SHOCK waves and acceleration waves in heat-conducting viscous materials are considered. The material properties are expressed through response functions dependent on the temperature, the deformation gradient and the hidden variables and through an evolution function dependent also on the temperature gradient and the velocity gradient (Sect. 2, 3). The investigation of the propagation condition shows that the theory allows for the existence of shock waves (Sect. 4), acceleration waves and higher order waves (Sect. 5). Finally (Sect. 6), the paper presents a model of heat-conducting viscous fluid accounting for wave propagation and meanwhile providing Fourier's law and Navier-Stokes' law as asymptotic limits.

Rozpatrzono fale uderzeniowe i fale przyśpieszenia w materiałach lepkich przewodzących ciepło. Własności materiałowe przedstawione są przez funkcje reakcji zależne od temperatury, gradientu deformacji i zmiennych wewnętrznych, a poprzez funkcję ewolucji również od gradientów temperatury i prędkości (punkty 2, 3). Rozpatrywany warunek propagacji pokazuje, że teoria dopuszcza istnienie fal uderzeniowych (punkt 4), fal przyspieszenia i fal wyższego rzędu (punkt 5). Wreszcie (punkt 6) podano model przewodzącego ciepło płynu lepkiego. Model ten uwzględnia propagację fal, a jednocześnie zawiera prawo Fouriera i prawo Naviera-Stokesa jako asymptotyczne przypadki graniczne.

Рассмотрены ударные волны и волны ускорения в вязких теплопроводящих материалах. Материальные свойства представлены функциями отклика, зависящими от температуры, градиента деформаций и неявных переменных, а через функцию эволюции также от градиентов температуры и скорости (пункты 2, 3). Рассмотрение условия распространения показывает, что теория допускает существование ударных волн (пункт 4), волн ускорения и волн высшего порядка (пункт 5). Наконец (пункт 6) дается модель теплопроводящей, вязкой жидкости, причем эта модель учитывает распространение волн и одновременно содержит закон Фурье и закон Навье-Стокса как асимптотические предельные случаи.

1. Introduction

IT IS WELL known that Fourier's law of heat conduction and Navier-Stokes' law of viscosity rule out the possibility of wave propagation. This paradox has been given a great deal of solutions in the literature concerning temperature and acceleration waves in heat-conducting materials. Of course, the properties of any solution are closely related to the statement adopted for the second law of thermodynamics. In connection with theories involving the second law in the form of the Clausius-Duhem inequality, I mention, for example, the papers by GURTIN and PIPKIN [1] and by myself [2] about materials with fading memory. Accounts for temperature and acceleration waves are delivered in the papers by KOSIŃSKI and PERZYNA [3] and by KOSIŃSKI [4] using the model of materials with hidden — or internal — variables [5, 6]. On the other hand, apart from the paper by COLEMAN, GREENBERG and GURTIN [7], little attention has been paid to wave propagation in viscous materials.

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The aim of this paper is to exhibit a thermodynamic theory of thermo-viscous materials allowing for the existence of shocks and waves of any order. To this purpose the continuum at hand is regarded as a material with hidden variables (Sects. 2, 3). The present procedure is not standard as the response function and the evolution function are defined on different domains; namely the evolution function depends on the temperature gradient and the velocity gradient while the response function does not. In spite of being unusual, such a difference is not at all new since it is utilised in refs. [3, 4] and in the paper by SULICIU [8] in connection with the temperature gradient.

The main features of the present theory may be summarised as follows. First, the introduction of hidden variables does not change significantly the propagation condition of shocks (Sect. 4) whereas it gives rise to new terms affecting the growth of the shock amplitude as it is shown by a detailed investigation of shock propagation in thermo-viscous fluids [9]. Roughly speaking, the material turns out to be elastic as to the propagation condition and thermo-viscous as to the growth of the amplitude. Second, it is shown that acceleration waves may exist; moreover, qualitatively new terms, due to viscosity and heat conduction, appear in the propagation condition (Sect. 5). Furthermore, it is proved that the propagation condition for higher order waves is equal to that for acceleration waves; this is the counterpart of analogous results found by ERICKSEN [10], TRUESDELL [11], COLEMAN and GURTIN [12], and COLEMAN, GREENBERG and GURTIN [7] in connection with hyperelastic materials, elastic materials, simple materials with fading memory, and Maxwellian materials, respectively.

Of course, any well-grounded constitutive theory of thermo-viscous materials, in addition to being compatible with wave propagation, must deliver Fourier's law and Navier-Stokes' law when stationary phenomena are considered. Section 6, which deals with thermo-viscous fluids, shows that, in a sense, such is the case with respect to the present theory while the general properties of shocks and waves described above hold again.

It is a quite unusual result provided by the theory that transverse acceleration waves in thermo-viscous fluids are possible just as in ordinary elastic materials. This fact seems to be unavoidable in the sense that the existence of acceleration waves in thermo-viscous fluids leads naturally to the existence of transverse acceleration waves. So the theory exhibits a peculiar property for testing experimentally the validity of the model.

2. Systems with hidden variables

Throughout $\mathbf{R}$, $\mathbf{R}^+$, $\mathbf{R}^{++}$ stand for the real numbers, the positive real numbers and the strictly positive real numbers, respectively. A superposed dot denotes material time differentiation. A dot between two vectors or tensors means the inner product. The symbols $\mathbf{Y}, \mathbf{Z}, \mathbf{A}, \mathbf{\Sigma}$ denote finite-dimensional real normed vector spaces while $\mathbf{L}(\mathbf{Y}, \mathbf{A})$ designates the normed vector space of all linear maps from $\mathbf{Y}$ into $\mathbf{A}$; $\mathscr{V}$ stands for the ordinary three-dimensional vector space.

A system with hidden variables $\{\mathbf{y}_0, \mathbf{z}_0, \boldsymbol{\alpha}_0, \mathbf{U}, \mathbf{V}, \boldsymbol{\sigma}, \mathbf{h}\}$ on $\mathbf{Y}\times\mathbf{Z}\times\mathbf{A}$ consists of a ground value $(\mathbf{y}_0, \mathbf{z}_0, \boldsymbol{\alpha}_0)$ of the independent variables $(\mathbf{y}, \mathbf{z}, \boldsymbol{\alpha}) \in \mathbf{Y}\times\mathbf{Z}\times\mathbf{A}$, $\boldsymbol{\alpha} \in \mathbf{A}$ being

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the vector value of the hidden variables, together with an open connected neighbourhood $\mathbf{U}\times\mathbf{V}$ of $(\mathbf{y}_0, \mathbf{z}_0)$ and the maps

$$\boldsymbol{\sigma} \in C^2(\mathbf{U}\times\mathbf{A}, \boldsymbol{\Sigma}), \quad \mathbf{h} \in C^2(\mathbf{U}\times\mathbf{V}\times\mathbf{A}, \mathbf{A}) \tag{2.1}$$

while $\dim\mathbf{A} \leqslant \dim\mathbf{Y}+\dim\mathbf{Z}$(1). The growth of the hidden variables is determined by the triple $(\mathbf{y}, \mathbf{z}, \boldsymbol{\alpha})$ via the evolution function $\mathbf{h}$ whereas the response of the system depends only on the pair $(\mathbf{y}, \boldsymbol{\alpha})$ via the response function $\boldsymbol{\sigma}$.

The first step is now to precise the zero rate condition for the hidden variables; to this end it is convenient to introduce a map $\mathbf{E}: \mathbf{Y}\times\mathbf{Z} \to \mathbf{A}$ subject to the following restriction.

I. *Corresponding to each pair* $(\mathbf{y}, \mathbf{z}) \in \mathbf{U}\times\mathbf{V}$ *there is just one value of the hidden variables* $\mathbf{E}(\mathbf{y}, \mathbf{z}) \in \mathbf{A}$ *such that*

$$\mathbf{h}\big(\mathbf{y}, \mathbf{z}, \mathbf{E}(\mathbf{y}, \mathbf{z})\big) = \mathbf{0}$$

while

$$\mathbf{E}(\mathbf{y}_0, \mathbf{z}_0) = \boldsymbol{\alpha}_0 .$$

The set of hidden variables

$$\mathbf{B} = \{\mathbf{E}(\mathbf{y}, \mathbf{z}) \colon (\mathbf{y}, \mathbf{z}) \in \mathbf{U}\times\mathbf{V}\}, \quad \mathbf{E} \in C^2(\mathbf{U}\times\mathbf{V}, \mathbf{B}),$$

is open in $\mathbf{A}$, *and there is a subset* $\mathbf{W} \subset \mathbf{U}\times\mathbf{V}$ *such that* $(\mathbf{y}_0, \mathbf{z}_0) \in \mathbf{W}$ *and the restriction* $\hat{\mathbf{E}} = \mathbf{E}|_{\mathbf{W}}$ *of* $\mathbf{E}$ *to* $\mathbf{W}$ *is a bijection from* $\mathbf{W}$ *onto* $\mathbf{B}$ *whose inverse* $\hat{\mathbf{E}}^{-1} \in C^2(\mathbf{B}, \mathbf{W})$.

The response function $\boldsymbol{\sigma}^* \in C^2(\mathbf{U}\times\mathbf{B}, \boldsymbol{\Sigma})$ is defined by

$$\boldsymbol{\sigma}^*(\mathbf{y}, \boldsymbol{\alpha}) = \boldsymbol{\sigma}\big(\mathbf{y}, \mathbf{E}(\mathbf{y}, \mathbf{z})\big), \quad (\mathbf{y}, \mathbf{z}) \in \mathbf{U}\times\mathbf{V}. \tag{2.2}$$

The subsequent developments are greatly simplified if the response function $\boldsymbol{\sigma}$ and the evolution function $\mathbf{h}$ satisfy the following requirements.

II. *There is a positive constant* χ *such that*

$$|\boldsymbol{\sigma}(\mathbf{y}, \boldsymbol{\alpha}+\boldsymbol{\beta}) - \boldsymbol{\sigma}(\mathbf{y}, \boldsymbol{\alpha})| \leqslant \chi|\boldsymbol{\beta}|, \quad \mathbf{y} \in \mathbf{U}, \quad \boldsymbol{\alpha}, \boldsymbol{\alpha}+\boldsymbol{\beta} \in \mathbf{A}. \tag{2.3}$$

III. *There is a map* $\boldsymbol{\Lambda} \in L(\mathbf{A}, \mathbf{A})$ *and a positive constant* δ *such that*

$$|\mathbf{h}(\mathbf{y}, \mathbf{z}, \boldsymbol{\alpha}+\boldsymbol{\beta}) - \mathbf{h}(\mathbf{y}, \mathbf{z}, \boldsymbol{\alpha}) - \boldsymbol{\Lambda}\boldsymbol{\beta}| \leqslant \delta|\boldsymbol{\beta}|, \quad (\mathbf{y}, \mathbf{z}) \in \mathbf{U}\times\mathbf{V}, \quad \boldsymbol{\alpha}, \boldsymbol{\alpha}+\boldsymbol{\beta} \in \mathbf{A}, \tag{2.4}$$

while $\boldsymbol{\Lambda}+\delta\mathbf{I}_{\mathbf{A}}$ *is negative definite.*

So, in view of II, III, we have the uniform Lipschitz conditions

$$\boldsymbol{\sigma}(\mathbf{y}, \cdot) \in \mathrm{Lip}\,\chi, \quad \mathbf{h}(\mathbf{y}(, \mathbf{z}, \cdot) \in \mathrm{Lip}(|\boldsymbol{\Lambda}|+\delta), \quad \mathbf{y} \in \mathbf{U}, \quad \mathbf{z} \in \mathbf{V}.$$

A *path* is a bounded and piecewise continuously differentiable map $\boldsymbol{\pi}$ from $\mathbf{R}$ into $\mathbf{U}\times\mathbf{V}$. If $\boldsymbol{\pi}$ is a path and $t \in \mathbf{R}$, then $\boldsymbol{\pi}(t) \in \mathbf{U}\times\mathbf{V}$ is termed the value of $\boldsymbol{\pi}$ at time t. A history is a function defined on $\mathbf{R}$ with values in $\mathbf{U}\times\mathbf{V}$. Given a path $\boldsymbol{\pi}$ and a time $t \in \mathbf{R}$, the history of $\boldsymbol{\pi}$ up to time t, $\boldsymbol{\pi}\,(\cdot)$, is defined by $\boldsymbol{\pi}\,(\zeta) = \boldsymbol{\pi}(t-\zeta)$, $\zeta \in \mathbf{R}^+$. A path $\boldsymbol{\pi}$ is closed if there exist two times t_1, t_2 $(t_1 \leqslant t_2)$ such that

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(t_1), \quad t \leqslant t_1,$$
$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(t_2), \quad t \geqslant t_2,$$

(1) In ref. [13] such a requirement is shown to be strictly related to the existence of a unique entropy function.

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and, moreover, $\boldsymbol{\pi}(t_1) = \boldsymbol{\pi}(t_2)$. A *process* is a pair $(\boldsymbol{\pi}, \boldsymbol{\alpha})$, where $\boldsymbol{\pi}$ is a path and $\boldsymbol{\alpha} \in \mathbf{A}$; it is said to be closed if $\boldsymbol{\pi}$ is closed.

A path $\boldsymbol{\pi} = (\mathbf{y}, \mathbf{z})$ determines the growth of the hidden variables through the evolution equation

$$\dot{\boldsymbol{\alpha}}(t) = \mathbf{h}\big(\boldsymbol{\pi}(t), \boldsymbol{\alpha}(t)\big), \quad \boldsymbol{\alpha}(t_0) = \boldsymbol{\alpha}^0. \tag{2.5}$$

For any given continuously differentiable path $\boldsymbol{\pi}$ the solution of Eq. (2.5) exists and is unique. Moreover, the hidden variables $\boldsymbol{\alpha}(t)$ are independent of the present value of the path $\boldsymbol{\pi}(t)$ just as it happens in standard theories. Such a topic is considered by LUBLINER [14], and by KOSIŃSKI and WOJNO [15] within a comparison between hidden variables and fading memory approaches.

The solution $\boldsymbol{\alpha}$ of Eq. (2.5) is endowed with the property of asymptotic stability. For, consider the hidden variables $\boldsymbol{\alpha}, \boldsymbol{\alpha}+\boldsymbol{\beta} \in \mathbf{A}$ corresponding to the paths $\boldsymbol{\pi}, \boldsymbol{\pi}+\boldsymbol{\nu}$, that is to say

$$\begin{aligned} \dot{\boldsymbol{\alpha}} &= \mathbf{h}(\boldsymbol{\pi}, \boldsymbol{\alpha}), \\ \dot{\boldsymbol{\alpha}}+\dot{\boldsymbol{\beta}} &= \mathbf{h}(\boldsymbol{\pi}+\boldsymbol{\nu}, \boldsymbol{\alpha}+\boldsymbol{\beta}). \end{aligned} \tag{2.6}$$

Subtraction allows us to write the evolution equation for the difference $\boldsymbol{\beta}$ as

$$\dot{\boldsymbol{\beta}} = \{\mathbf{h}(\boldsymbol{\pi}, \boldsymbol{\alpha}+\boldsymbol{\beta}) - \mathbf{h}(\boldsymbol{\pi}, \boldsymbol{\alpha}) + \boldsymbol{\Lambda}\boldsymbol{\beta}\} + \boldsymbol{\gamma} - \boldsymbol{\Lambda}\boldsymbol{\beta},$$

where $\boldsymbol{\gamma} = \mathbf{h}(\boldsymbol{\pi}+\boldsymbol{\nu}, \boldsymbol{\alpha}+\boldsymbol{\beta}) - \mathbf{h}(\boldsymbol{\pi}, \boldsymbol{\alpha}+\boldsymbol{\beta})$. Letting $-m < 0$ denote the largest real part of the eigenvalues of $\boldsymbol{\Lambda}$, account of III and application of Gronwall's inequality yield the estimate

$$|\boldsymbol{\beta}(t)| \leqslant |\boldsymbol{\beta}(t_0)| \exp(-(t-t_0)m) + \delta \int_{t_0}^{t} \exp(-(t-s)m)|\boldsymbol{\beta}(s)|ds + \int_{t_0}^{t} \exp\big(-(t-s)m\big)|\boldsymbol{\gamma}(s)|ds. \tag{2.7}$$

Hence a routine procedure establishes that

$$|\boldsymbol{\beta}(t)| \leqslant |\boldsymbol{\beta}(t_0)| \exp(-(m-\delta)\,(t-t_0)) + \frac{1}{m-\delta} \max_{t_0 \leqslant s \leqslant t} |\boldsymbol{\gamma}(s)| \{1 - \exp\big(-(m-\delta)(t-t_0)\big)\}. \tag{2.8}$$

Notice that $m-\delta > 0$ because of the negative definiteness of $\boldsymbol{\Lambda} + \delta \mathbf{I}_{\mathbf{A}}$.

The inequality (2.8) gives an estimate of the difference $\boldsymbol{\beta}$ at time t in terms of its initial value $\boldsymbol{\beta}(t_0)$ and of the difference path $\boldsymbol{\nu}$ via the quantity $\boldsymbol{\gamma}$. In the instance of equal paths, that is $\boldsymbol{\nu} \equiv \mathbf{0}$ and then $\boldsymbol{\gamma} \equiv \mathbf{0}$, it follows that

$$|\boldsymbol{\beta}(t)| \leqslant |\boldsymbol{\beta}(t_0)| \exp\big(-(m-\delta)\,(t-t_0)\big),$$

whereby the difference between the hidden variables, arising from different initial values, decreases in time at least as $\exp\big(-(m-\delta)\,(t-t_0)\big)$. Accordingly, letting $\boldsymbol{\pi}(t_0) = \hat{\mathbf{E}}^{-1}(\boldsymbol{\alpha}')$, $\boldsymbol{\alpha}' \in \mathbf{B}$, and assuming that the constant path $\boldsymbol{\pi} = \boldsymbol{\pi}(t_0)$ may occur, the solution of the evolution equation

$$\dot{\boldsymbol{\alpha}} = \mathbf{h}(\boldsymbol{\pi}(t_0), \boldsymbol{\alpha}), \quad \boldsymbol{\alpha}(t_0) \neq \boldsymbol{\alpha}',$$

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must satisfy the condition of asymptotic stability

$$\lim_{t\to\infty} \boldsymbol{\alpha}(t) = \boldsymbol{\alpha}'.$$

This result lends operative meaning to the assignment of the initial condition for the hidden variables; we can get the initial value $\boldsymbol{\alpha}'$ at time t simply by holding the path $\boldsymbol{\pi}$ equal to $\hat{\mathbf{E}}^{-1}(\boldsymbol{\alpha}')$ up to time t. Incidentally, it is this fact which suggests the definition of closed process.

3. Thermo-viscous materials with hidden variables

One way of describing the evolution of a body $\mathscr{B}$ is to suppose it consists of particles labelled by the positions they occupy in a reference configuration $\mathscr{R}$; $\mathbf{x}(\mathbf{X}, t)$ denotes the position vector of the particle $\mathbf{X}$ at time t while $\boldsymbol{\pi}(\mathbf{X}, \cdot)$ is the path of the particle $\mathbf{X}$. Accordingly $\boldsymbol{\pi}$ and $\boldsymbol{\alpha}$ map $\mathscr{R}\times\mathbf{R}$ into $\mathbf{U}\times\mathbf{V}$ and $\mathbf{A}$, respectively. To save writing, however, the dependence of $\boldsymbol{\pi}$ and $\boldsymbol{\alpha}$ on $\mathbf{X}$ is often understood and not written.

A particle of a thermo-viscous body is characterised by identifying $\mathbf{y}\in\mathbf{U}$ with the pair $(\theta, \mathbf{F})$ and $\mathbf{z}\in\mathbf{V}$ with the pair $(\mathbf{G}, \dot{\mathbf{F}})$; here $\theta\in\mathbf{R}^{++}$ stands for the temperature, $\mathbf{G}$ the material temperature gradient, and $\mathbf{F}$ the deformation gradient. Meanwhile, the response $\boldsymbol{\sigma}$ is identified with the set of quantities

$$\boldsymbol{\sigma} = (e, \mathbf{S}, \mathbf{Q}, \eta),$$

where e is the internal energy density, $\mathbf{S}$ is the Piola-Kirchhoff stress tensor, $\mathbf{Q}$ is the material heat flux vector and η is the entropy density(²). These quantities enter the balance equations in the following way. Denote by $\mathscr{P}$ an arbitrary domain in $\mathscr{R}$ and by $\mathbf{N}$ the unit outward normal to the bounding surface $\partial\mathscr{P}$ of $\mathscr{P}$. Without any loss in generality the mass density in the reference configuration ϱ_0 is assumed to be uniform. Then, letting $\mathbf{v}(\mathbf{X}, t)$, $\mathbf{f}(\mathbf{X}, t)$, and $r(\mathbf{X}, t)$ stand for the velocity, the body force density, and the energy source density, respectively, in absence of discontinuity surfaces the balance of momentum and energy is expressed by

$$(3.1)\qquad \begin{aligned} &\frac{d}{dt}\int_{\mathscr{P}} \mathbf{v}\,dV = \int_{\partial\mathscr{P}} \mathbf{S}\mathbf{N}\,dA + \int_{\mathscr{P}} \mathbf{f}\,dV,\\ &\frac{d}{dt}\int_{\mathscr{P}} \left(e+\frac{1}{2}v^2\right)dV = \int_{\partial\mathscr{P}} (\mathbf{v}\mathbf{S}-\mathbf{Q})\cdot\mathbf{N}\,dA + \int_{\mathscr{P}} (\mathbf{f}\cdot\mathbf{v}+r)\,dV, \end{aligned}$$

where the mass density ϱ_0 has been dropped out. Under suitable smoothness assumptions the differential forms of the balance laws (3.1) are

$$(3.2)\qquad \begin{aligned} &\dot{\mathbf{v}} = \nabla\cdot\mathbf{S}+\mathbf{f},\\ &\dot{e} = \mathbf{S}\cdot\dot{\mathbf{F}}-\nabla\cdot\mathbf{Q}+r, \end{aligned}$$

∇ being the material gradient operator.

(²) Precisely, $\mathbf{S} = \mathbf{T}(\mathbf{F}^{-1})^T/\varrho$, ϱ being the mass density and $\mathbf{T}$ the Cauchy stress tensor, while $\mathbf{Q} = \mathbf{F}^{-1}\mathbf{q}/\varrho$, $\mathbf{q}$ being the spatial heat flux vector.

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The response function $\boldsymbol{\sigma}$ must be compatible with the second law of thermodynamics. Unfortunately, this assertion has not a unique mathematical counterpart since the current literature exhibits several statements of the second law. Among such statenents the Clausius-Duhem inequality appears to be the most restrictive one. Accordingly, this paper aims to provide a theory of wave propagation compatible with the Clausius-Duhem inequality whereby

$$\frac{d}{dt}\int_{\mathscr{P}} \eta dV \geqslant -\int_{\partial\mathscr{P}} \frac{1}{\theta}\mathbf{Q}\cdot\mathbf{N}dA+\int_{\mathscr{P}}\frac{r}{\theta}dV \tag{3.3}$$

is assumed to hold for any domain $\mathscr{P}\subset\mathscr{R}$ and for any $\mathbf{C}^1$ path on $\mathscr{R}\times\mathbf{R}$. The differential counterpart is

$$\dot{\eta}\geqslant -\nabla\cdot\left(\frac{\mathbf{Q}}{\theta}\right)+\frac{r}{\theta}. \tag{3.4}$$

On introducing the free energy $\psi = e-\theta\eta$ and substituting Eq. $(3.2)_2$ it folows that

$$-(\dot{\psi}+\eta\dot{\theta})+\mathbf{S}\cdot\dot{\mathbf{F}}-\frac{1}{\theta}\mathbf{Q}\cdot\mathbf{G}\geqslant 0 \tag{3.5}$$

must hold at any particle $\mathbf{X}\in\mathscr{R}$ for any $\mathbf{C}^1$ path on $\mathbf{R}$. Suppose now that the path $\boldsymbol{\pi} = (\theta, \mathbf{F}, \mathbf{G}, \dot{\mathbf{F}})$ and the time t are given. Letting $\varphi\in\mathbf{R}, \mathscr{F}\in\mathbf{L}(\mathscr{V},\mathscr{V})$, and $\mathscr{G}\in\mathscr{V}$ set $\mathbf{v} = (\varphi, \mathscr{F}, \mathscr{G}, \dot{\mathscr{F}})$. It is always possible to find $\mathbf{C}^2$ histories $\varphi^t(\cdot), \mathscr{F}^t(\cdot)$ and $\mathscr{G}^t(\cdot)$ in such a way that they vanish identically up to time $t-\varepsilon$, $\varepsilon > 0$, and $|\varphi+|\mathscr{F}|$ is small enough at any time while $\dot{\varphi}(t)$, and $\dot{\mathscr{F}}(t)$ are arbitrary. Then the choice of ι small enough ε makes $\int_{t-\varepsilon}^{t}|\boldsymbol{\gamma}(s)|ds$ as small as we please. Correspondingly, Eq. (2.8) tels us that the change of the hidden variables $\boldsymbol{\beta}$ is bounded at any time and hence the estimate (2.7) allows us to say that at time t it is as small as we wish. So, in connection vith the history $(\boldsymbol{\pi}+\mathbf{v})^t(\cdot)$ the Clausius-Duhem inequality (3.5) can be written in the forr

$$-(\psi_\theta+\eta+n)(\dot{\theta}+\dot{\varphi})-(\psi_{\mathbf{F}}-\mathbf{S}+\mathfrak{S})\cdot(\dot{\mathbf{F}}\cdot\dot{\mathscr{F}})-\left(\psi_{\boldsymbol{\alpha}}\cdot\mathbf{h}+\frac{1}{\theta}\mathbf{Q}\cdot\mathbf{G}+\omega\right)\geqslant 0. \tag{3.5'}$$

Since n, $|\mathfrak{S}|$, and ω may be made as small as we please, the arbitrariness of $\dot{\varphi}(t)$ and $\dot{\mathscr{F}}(t)$ allows us to conclude that the inequality (3.5)′ holds only if

$$\eta = -\psi_\theta, \quad \mathbf{S} = \psi_{\mathbf{F}}, \tag{3.6}$$

$$\psi_{\boldsymbol{\alpha}}\cdot\mathbf{h}+\frac{1}{\theta}\mathbf{Q}\cdot\mathbf{G}\leqslant 0. \tag{3.7}$$

Obviously, the conditions (3.6) and (3.7) are also sufficient for the Clausius-Duhem inequality (3.5) to hold.

The second law has been examined by DAY [13] through the assunption that the Clausius integral is non-positive for any closed process starting from an equilibrium state. An analogous procedure cannot be applied here because $\mathbf{z}$ contains the rate type quantity $\dot{\mathbf{F}}$ and then the properties 1.4, 1.5 of [13] are no longer true.

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4. Shock waves

Denote by $\mathscr{S}(t)$ a surface which divides $\mathscr{R}$ into the regions $\mathscr{R}^+(t)$, $\mathscr{R}^-(t)$ and forms a common boundary between them. Define the unit normal $\mathbf{N}$ to $\mathscr{S}$ to be directed from $\mathscr{R}^-$ to $\mathscr{R}^+$. Letting $\xi(\cdot,\ \cdot)$ be any function defined on $(\mathscr{R}\times\mathbf{R})\setminus\mathscr{S}$, the field $\xi(\cdot,t)\setminus$ is assumed to be continuous within $\mathscr{R}^+$ and $\mathscr{R}^-$; the symbols $\xi^+(t)$ and $\xi^-(t)$ stand for the definite limits of $\xi(\mathbf{X},t)$ as $\mathbf{X}$ approaches a point on $\mathscr{S}$ along paths lying entirely in $\mathscr{R}^+$ and $\mathscr{R}^-$, respectively. The surface $\mathscr{S}(t)$ is said to be singular with respect to the field $\xi(\cdot,t)$ at time t if

$$[\xi](t) := \xi^-(t)-\xi^+(t) \neq 0.$$

The singular surface $\mathscr{S}(t)$ is a wave front if its speed of propagation U_N is different from zero.

For later use it is worth writing down some general relations connected with waves. First, Maxwell's theorem asserts that if $[\xi]=0$, then

$$[\nabla\xi] = ([\nabla\xi]\cdot\mathbf{N})\mathbf{N}. \tag{4.1}$$

Second, the growth of the jump $[\xi]$ is determined by

$$\frac{\delta[\xi]}{\delta t} = [\dot{\xi}]+U_N\mathbf{N}\cdot[\nabla\xi], \tag{4.2}$$

where $\frac{\delta}{\delta t}$ stands for the displacement derivative. In particular, if $[\xi]=0$, the compatibility relation

$$[\dot{\xi}]+U_N\mathbf{N}\cdot[\nabla\xi] = 0 \tag{4.2$'$}$$

must hold.

Now suppose that the wave front $\mathscr{S}(t)$ is singular with respect to $\mathbf{F}$ and θ (shock wave). In such a case the evolution equation is not defined at $\mathscr{S}(t)$; in other words, if $\mathbf{X}\in\mathscr{S}(t)$ and $t_0<t$, we may write

$$\dot{\boldsymbol{\alpha}}(\zeta) = \mathbf{h}\big(\boldsymbol{\pi}(\zeta),\boldsymbol{\alpha}(\zeta)\big), \quad \boldsymbol{\alpha}(t_0) = \boldsymbol{\alpha}^0,$$

provided $\zeta\in[t_0,t)$. However, the property whereby $\boldsymbol{\alpha}(\zeta)$ does not depend on $\boldsymbol{\pi}(\zeta)$ suggests to define $\boldsymbol{\alpha}(t)$ by continuity. This observation warrants the following:

DEFINITION. *A wave $\mathscr{S}(t)$ is said to be a shock wave if*:

S1. *the functions* $\mathbf{x}(\cdot,\ \cdot)$, $\boldsymbol{\alpha}(\cdot,\ \cdot)$ *are continuous on* $\mathscr{R}\times\mathbf{R}$,

S2. *the functions* $\dot{\mathbf{x}}(\cdot,\ \cdot)$, $\mathbf{F}(\cdot,\ \cdot)$, $\theta(\cdot,\ \cdot)$, $\dot{\boldsymbol{\alpha}}(\cdot,\ \cdot)$, $\nabla\boldsymbol{\alpha}(\cdot,\ \cdot)$ *and the derivatives of higher order suffer jump discontinuities across* $\mathscr{S}(t)$ *but are continuous functions on* $(\mathscr{R}\times\times\mathbf{R})\setminus\mathscr{S}(t)$.

The aim of this section is to investigate whether, and how, the present theory accounts for the existence of shocks. To this end note first that in view of the balance equations (3.1) and of the boundedness of $\mathbf{f}$ and r, Kotchine's theorem yields the usual jump relations

$$U_N[\mathbf{v}] = -[\mathbf{S}]\mathbf{N}, \tag{4.3}$$

$$U_N\left[e+\frac{1}{2}v^2\right] = -[\mathbf{v}\mathbf{S}-\mathbf{Q}]\cdot\mathbf{N}. \tag{4.4}$$

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Moreover, the Clausius-Duhem inequality (3.3) leads to

$$U_N[\eta] \geqslant \mathbf{N}\cdot\left[\frac{\mathbf{Q}}{\theta}\right]. \tag{4.5}$$

Denoting by a bar the mean value of a field variable on the two sides of the front, that is to say

$$\bar{\xi} = \frac{1}{2}(\xi^+ + \xi^-),$$

by virtue of Eq. (4.3) it follows that

$$U_N[v^2] = -\bar{\mathbf{v}}\cdot([\mathbf{S}]\mathbf{N}). \tag{4.6}$$

On the other hand, Maxwell's theorem enables us to introduce the shock amplitude $\mathbf{s}$ related to $\mathfrak{F} = [\mathbf{F}]$ by

$$\mathfrak{F} = \mathbf{s}\otimes\mathbf{N},$$

where $\otimes$ denotes the tensor product. So, Eq. (4.4) and the compatibility condition

$$[\mathbf{v}] = -U_N\mathfrak{F}\mathbf{N} = -U_N\mathbf{s}$$

imply that the Hugoniot relation

$$U_N([e] - \bar{\mathbf{S}}\cdot\mathfrak{F}) - \mathbf{N}\cdot[\mathbf{Q}] = 0 \tag{4.7}$$

must hold at the shock.

With a view to determining the shock propagation condition it is convenient to write the jump $[\mathbf{S}]$ in the form

$$[\mathbf{S}] = \mathbf{S}_\theta\vartheta + \mathbf{S}_\mathfrak{F}\mathfrak{F},$$

where ϑ denotes the jump $[\theta]$. For any component S^{ij} of $\mathbf{S}$ the derivatives S^{ij}_θ, $S^{ij}_\mathfrak{F}$ are evaluated at a suitable point $(\theta^+ + l\vartheta, \mathbf{F}^+ + l\mathfrak{F}, \boldsymbol{\alpha})$, where $l \in [0, 1]$ depends on the component under consideration. Analogously we write

$$[\mathbf{Q}] = \mathbf{Q}_\theta\vartheta + \mathbf{Q}_\mathfrak{F}\mathfrak{F}, \quad [e] = e_\theta\vartheta + e_\mathfrak{F}\mathfrak{F}.$$

Then, if $\mathbf{I} \in \mathbf{L}(\mathscr{V}, \mathscr{V})$ is the identity tensor, Eqs. (4.3), and (4.4) yield

$$\begin{cases} (-U_N^2\mathbf{I} + \mathbf{N}\mathbf{S}_\mathfrak{F}\mathbf{N})\mathbf{s} + \mathbf{S}_\theta\mathbf{N}\vartheta = \mathbf{Q}, \\ (U_N(e_\mathfrak{F}\mathbf{N} - \bar{\mathbf{S}}\mathbf{N}) - \mathbf{N}\mathbf{Q}_\mathfrak{F}\mathbf{N})\cdot\mathbf{s} + (U_N e_\theta - \mathbf{N}\cdot\mathbf{Q}_\theta)\vartheta = 0. \end{cases} \tag{4.8}$$

In principle, the possible values of the shock speed U_N and the relation between $\mathbf{s}$ and ϑ are now a direct consequence of the system (4.8). Unfortunately, the derivatives $\mathbf{S}_\theta$, $\mathbf{S}_\mathfrak{F}$, $\mathbf{Q}_\theta$, $\mathbf{Q}_\mathfrak{F}$, e_θ, $e_\mathfrak{F}$ depend on $\mathbf{s}$ and ϑ thus making the system (4.8) nonlinear. For the sake of simplicity attention is confined to the linear approximation — infinitesimal shocks — whereby the derivatives are evaluated at $(\theta^+, \mathbf{F}^+, \boldsymbol{\alpha})$ and $\bar{\mathbf{S}}$ is replaced by $\mathbf{S}^+$.

Infinitesimal shocks

A non-trivial solution of the system (4.8) exists only if

$$\det\left(\frac{\mathbf{N}\mathbf{S}_\mathfrak{F}\mathbf{N} - U_N^2\mathbf{I}}{(e_\mathfrak{F}\mathbf{N} - \bar{\mathbf{S}}\mathbf{N})U_N - \mathbf{N}\mathbf{Q}_\mathfrak{F}\mathbf{N}} \,\middle|\, \frac{\mathbf{S}_\theta\mathbf{N}}{e_\theta U_N - \mathbf{N}\cdot\mathbf{Q}_\theta}\right) = 0. \tag{4.9}$$

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A significant particular case of the propagation condition (4.9) is provided by non-heat conductors. Indeed, on substituting $\mathbf{Q} \equiv 0$ and assuming $U_N \neq 0$ the condition (4.9) becomes

$$\det(\mathbf{B} - U_N^2 \mathbf{I}) = 0, \tag{4.10}$$

where $\mathbf{B} = \mathbf{N}\mathbf{S}_{\mathfrak{F}}\mathbf{N} + (\mathbf{S}_\vartheta \mathbf{N}) \otimes \mathbf{b}$ and $\mathbf{b} = (\mathbf{S}\mathbf{N} - e_{\mathfrak{F}}\mathbf{N})/e_\vartheta$. Meanwhile $\mathbf{s}$ is an eigenvector of $\mathbf{B} - U_N^2\mathbf{I}$ and $\vartheta = \mathbf{b} \cdot \mathbf{s}$. Therefore, the existence of shock waves is ensured by the symmetry and the positive definiteness of $\mathbf{B} \in \mathbf{L}(\mathscr{V}, \mathscr{V})$. On appealing to Eq. $(3.6)_3$ it follows that

$$\mathbf{N}\mathbf{S}_{\mathfrak{F}}\mathbf{N} = \mathbf{N}\psi_{\mathbf{FF}}\mathbf{N}$$

and hence $\mathbf{N}\mathbf{S}_{\mathfrak{F}}\mathbf{N}$ is clearly symmetric. Also, a straightforward calculation gives

$$(\mathbf{S}_\vartheta \mathbf{N}) \otimes \mathbf{b} = \theta \psi_{\theta \mathbf{F}} \mathbf{N} \otimes \psi_{\theta \mathbf{F}} \mathbf{N}.$$

The symmetry of $\mathbf{B}$ is now proved. As to the positive definiteness of $\mathbf{B}$ observe that

$$\mathbf{s} \cdot \mathbf{B}\mathbf{s} = \mathbf{s} \cdot (\mathbf{N}\psi_{\mathbf{FF}}\mathbf{N})\mathbf{s} + \theta\left((\psi_{\theta\mathbf{F}}\mathbf{N}) \cdot \mathbf{s}\right)^2.$$

The last term is evidently non-negative. Accordingly, the positive definiteness of the acoustic tensor $\mathbf{N}\psi_{\mathbf{FF}}\mathbf{N}$ implies the positive definiteness of $\mathbf{B}$.

The claim that the acoustic tensor is positive definite is to be distinguished from the corresponding claim in the theory of elasticity. Indeed, the derivatives $\psi_{\mathbf{FF}}$, as well as all coefficients in Eq. (4.9), depend also on the hidden variables and then on the history of θ, $\mathbf{F}$, $\mathbf{G}$, and $\dot{\mathbf{F}}$.

5. Acceleration waves

The independence of $\boldsymbol{\alpha}(t)$ of $\boldsymbol{\pi}(t)$ suggests the following to be introduced:

DEFINITION. *A wave $\mathscr{S}(t)$ is said to be an acceleration wave if*:

A1. *the functions* $\dot{\mathbf{x}}(\cdot,\cdot), \mathbf{F}(\cdot,\cdot), \theta(\cdot,\cdot), \boldsymbol{\alpha}(\cdot,\cdot)$ *are continuous on* $\mathscr{R} \times \mathbf{R}$,

A2. *the functions* $\ddot{\mathbf{x}}(\cdot,\cdot), \dot{\mathbf{F}}(\cdot,\cdot), \dot{\theta}(\cdot,\cdot), \mathbf{G}(\cdot,\cdot), \dot{\boldsymbol{\alpha}}(\cdot,\cdot), \nabla\boldsymbol{\alpha}(\cdot,\cdot)$ *and the derivatives of higher order suffer jump discontinuities across* $\mathscr{S}(t)$ *but are continuous functions on* $(\mathscr{R} \times \mathbf{R})/\mathscr{S}(t)$.

Since $[\dot{\mathbf{x}}] = \mathbf{0}$ at $\mathscr{S}(t)$, the jump relations (4.3), and (4.4) reduce to

$$[\mathbf{S}]\mathbf{N} = 0, \tag{5.1}$$

$$U_N[e] = \mathbf{N} \cdot [\mathbf{Q}]. \tag{5.2}$$

U_N being now the speed of propagation of the acceleration wave. The continuity of θ, $\mathbf{F}$, and $\boldsymbol{\alpha}$ makes $\mathbf{S}$, e, and $\mathbf{Q}$ continuous across the wave front and then both Poisson's condition (5.1) and Eq. (5.2) hold identically. As to the entropy balance, the inequality (4.5) also holds identically because of the continuity of θ, $\mathbf{Q}$, and η.

Look now at the balance equations (3.2). On assuming that $\mathbf{f}$ and r are continuous across the wave front, the standard procedure yields

$$\begin{aligned} \mathbf{a} &= -\frac{1}{U_N}[\dot{\mathbf{S}}]\mathbf{N}, \\ [\dot{e}] &= -\frac{1}{U_N}(\mathbf{S}\mathbf{N}) \cdot \mathbf{a} + \frac{1}{U_N}[\dot{\mathbf{Q}}] \cdot \mathbf{N}, \end{aligned} \tag{5.3}$$

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where $\mathbf{a} := [\ddot{\mathbf{x}}]$. To go further we need to introduce explicit expressions for $[\dot{e}]$, $[\dot{\mathbf{Q}}]$, and $[\dot{\mathbf{S}}]$ in terms of θ, $\mathbf{F}$, $\boldsymbol{\alpha}$ and $[\mathbf{G}]$, $[\dot{\theta}]$, $[\dot{\mathbf{F}}]$. For example we have

$$[\dot{e}] = e_\theta[\dot{\theta}] + e_{\mathbf{F}} \cdot [\dot{\mathbf{F}}] + e_\alpha \cdot [\mathbf{h}].$$

The continuity of $\theta, \mathbf{F}, \boldsymbol{\alpha}$ makes $e_\theta, e_{\mathbf{F}}, e_\alpha$ continuous across the wave front and allows the jump $[\mathbf{h}]$ to be written as

$$[\mathbf{h}] = \mathbf{h}_{[\mathbf{G}]}[\mathbf{G}] + \mathbf{h}_{[\dot{\mathbf{F}}]}[\dot{\mathbf{F}}] \tag{5.4}$$

where, for any component h^ν of $\mathbf{h}$, the derivatives $h^\nu_{[\mathbf{G}]}$, $h^\nu_{[\dot{\mathbf{F}}]}$ are evaluated at a suitable point $(\theta, \mathbf{F}, \mathbf{G}^+ + k[\mathbf{G}], \dot{\mathbf{F}}^+ + k[\dot{\mathbf{F}}], \boldsymbol{\alpha})$, $k \in [0, 1]$. For the sake of simplicity assume now that the function $\mathbf{h}$ depends linearly on $\mathbf{G}$ and $\dot{\mathbf{F}}$; otherwise the same conclusions could be attained in the case of infinitesimal waves. Moreover, to save writing, replace the symbols $\mathbf{h}_{[\mathbf{G}]}, \mathbf{h}_{[\dot{\mathbf{F}}]}$ by $\mathbf{h}_{\mathbf{G}}$ and $\mathbf{h}_{\dot{\mathbf{F}}}$, respectively. Denoting by Θ the jump $[\dot{\theta}]$ and making use of the compatibility conditions yield

$$[\dot{e}] = e_\theta\Theta - \frac{1}{U_N} e_{\mathbf{F}} \cdot (\mathbf{a}\otimes\mathbf{N}) - \frac{1}{U_N} e_\alpha \cdot \{\mathbf{h}_{\mathbf{G}}\mathbf{N}\Theta + \mathbf{h}_{\dot{\mathbf{F}}}\mathbf{a}\otimes\mathbf{N}\}.$$

Analogous relations hold for $[\dot{\mathbf{Q}}]$ and $[\dot{\mathbf{S}}]$. On substituting into Eq. (5.3) and rearranging the terms it follows that

$$\begin{aligned} (\mathfrak{Q} - U_N^2\mathbf{I})\mathbf{a} + \mathfrak{Z}\Theta &= \mathbf{0}, \\ \mathfrak{X} \cdot \mathbf{a} + q\Theta &= 0, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} \mathfrak{Q} &= \mathbf{N}(\mathbf{S}_{\mathbf{F}} + \mathbf{S}_\alpha \mathbf{h}_{\dot{\mathbf{F}}})\mathbf{N}, \\ \mathfrak{Z} &= -U_N\mathbf{S}_\theta\mathbf{N} + \mathbf{N}\mathbf{S}_\alpha\mathbf{h}_{\mathbf{G}}\mathbf{N}, \\ \mathfrak{X} &= U_N(\mathbf{S}\mathbf{N} - e_{\mathbf{F}}\mathbf{N} - e_\alpha \mathbf{h}_{\mathbf{F}}\mathbf{N}) + \mathbf{N}\mathbf{Q}_{\mathbf{F}}\mathbf{N} + \mathbf{N}\mathbf{Q}_\alpha \mathbf{h}_{\dot{\mathbf{F}}}\mathbf{N}, \\ q &= e_\theta U_N^2 - (e_\alpha \cdot \mathbf{h}_{\mathbf{G}}\mathbf{N} + \mathbf{N} \cdot \mathbf{Q}_\theta) U_N + \mathbf{N} \cdot \mathbf{Q}_\alpha \mathbf{h}_{\mathbf{G}}\mathbf{N}. \end{aligned}$$

Consequently the speed of propagation U_N must be the solution of the determinantal equation

$$\det\left(\begin{array}{c|c} \mathfrak{Q} - U_N^2\mathbf{I} & \mathfrak{Z} \\ \hline \mathfrak{X} & q \end{array}\right) = 0. \tag{5.6}$$

To sum up the results obtained so far we can write the following:

THEOREM. *The speed of propagation U_N of an infinitesimal acceleration wave traveling in the direction* $\mathbf{N}$ *through thermo-viscous materials with hidden variables must satisfy* Eq. (5. 6). *The relation between the amplitudes* $\mathbf{a}$ *and* Θ *is given by* Eq. (5.5).

Setting aside a detailed analysis of Eq. (5. 6) observe how the problem at hand simplifies in some special cases. First, if the stress $\mathbf{S}$ is independent of the temperature θ and of the hidden variables $\boldsymbol{\alpha}$, and hence $\mathfrak{Z} = \mathbf{0}$, Eqs. (5.5) exhibit a thermal wave solution $\mathbf{a} = 0$, $\Theta \neq 0$. Letting $\mathbf{j}$ and $\mathbf{K}$ stand for the quantities $\frac{1}{2}(e_\alpha \mathbf{h}_{\mathbf{G}} + \mathbf{Q}_\theta)$ and $\mathbf{Q}_\alpha \mathbf{h}_{\mathbf{G}}$, respectively, the existence of thermal waves implies that U_N satisfies the equation $q = 0$, that is

$$e_\theta U_N^2 - 2\mathbf{j} \cdot \mathbf{N}U_N + \mathbf{N} \cdot \mathbf{K}\mathbf{N} = 0. \tag{5.7}$$

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The derivative e_θ represents the heat capacity of the body per unit mass and it is natural to assume that it is a strictly positive quantity. Then a sufficient condition for thermal waves to exist is the negative definiteness of the tensor **K**.

Apart from the different meaning of the terms involved, a result like Eq. (5.7) has already been obtained by GURTIN and PIPKIN [1] and by myself [2] with recourse to memory functionals in order to account for the response of the material. Moreover, as to materials with hidden variables, the same result has been obtained by KOSIŃSKI and PERZYNA [3].

Second, consider the non-heat conductors $\mathbf{Q} \equiv 0$, and suppose that the free energy function is expressed as

$$\psi = \psi'(\theta) + \psi''(\mathbf{F}, \boldsymbol{\alpha}),$$

while $\mathbf{h}$ is independent of $\mathbf{F}$, that is $\mathbf{h}_\mathbf{F} = 0$. In such a case the vector $\mathfrak{X}$ vanishes and then Eq. (5.5) bears evidence of the solution $\mathbf{a} \neq 0, \Theta = 0$ provided that

$$\det\{\mathbf{N}(\mathbf{S}_\mathbf{F} + \mathbf{S}_\alpha \mathbf{h}_{\dot{\mathbf{F}}})\mathbf{N} - U_N^2 \mathbf{I}\} = 0. \tag{5.8}$$

This in turn means that purely mechanical acceleration waves — $\mathbf{a} \neq 0, \Theta = 0$ — may exist provided the tensor $\mathbf{N}(\mathbf{S}_\mathbf{F} + \mathbf{S}_\alpha \mathbf{h}_{\dot{\mathbf{F}}})\mathbf{N}$ has at least one real positive eigenvalue. Since $\mathbf{N}\mathbf{S}_\mathbf{F}\mathbf{N}$ is the acoustic tensor, the present theory exhibits the additional contribution $\mathbf{N}\mathbf{S}_\alpha \mathbf{h}_{\dot{\mathbf{F}}}\mathbf{N}$ accounting for viscosity through hidden variables. In principle, this term allows the existence of acceleration waves even in purely viscous materials, that is when $\mathbf{S}_\mathbf{F} = 0$. However, in general, the tensor $\mathbf{N}\mathbf{S}_\alpha \mathbf{h}_{\dot{\mathbf{F}}}\mathbf{N}$ may be not even symmetric. Such is not the case of the model of thermo-viscous fluid described in Sect. 6 where this tensor accounts for the existence of longitudinal and transverse acceleration waves.

Waves of higher order

Let $n \geqslant 2$ be an integer.

DEFINITION. *A singular surface* $\mathscr{S}(t)$ *of the time dependent fields* $\mathbf{x}(\cdot, t)$, $\theta(\cdot, t)$, *and* $\boldsymbol{\alpha}(\cdot, t)$ *is called a wave of order* $n+1$ *if the following conditions hold*:

n1. *the functions* $\dot{\mathbf{x}}(\cdot,\cdot), \mathbf{F}(\cdot,\cdot), \theta(\cdot,\cdot), \boldsymbol{\alpha}(\cdot,\cdot)$ *and their first* $n-1$ *derivatives are continuous on* $\mathscr{R} \times \mathbf{R}$,

n2. *the* n^{th} *derivatives of* $\dot{\mathbf{x}}, \mathbf{F}, \theta, \boldsymbol{\alpha}$ *and the derivatives of higher order suffer jump discontinuities across* $\mathscr{S}(t)$ *but are continuous functions on* $(\mathscr{R} \times \mathbf{R}) \setminus \mathscr{S}(t)$.

The analysis of a wave of order $n+1$ needs the requirements

$$\boldsymbol{\sigma} \in \mathbf{C}^{n+1}(\mathbf{U} \times \mathbf{A}, \boldsymbol{\Sigma}), \quad \mathbf{h} \in \mathbf{C}^n(\mathbf{U} \times \mathbf{V} \times \mathbf{A}, \mathbf{A}).$$

Also, for the sake of simplicity assume that $\mathbf{f}$, r and their first $n-1$ derivatives are continuous functions on $\mathscr{R} \times \mathbf{R}$.

Consider the balance equations (3.2) and differentiate them with respect to $\mathbf{X}$ $n-1$ times; since $[\nabla^{n-1}\mathbf{f}] = 0$ and $[\nabla^{n-1}r] = 0$, it follows that

$$[\nabla^{n-1}\dot{\mathbf{v}}] = [\nabla^{n-1}(\nabla \cdot \mathbf{S})], \tag{5.9}$$

$$[\nabla^{n-1}\dot{e}] = [\nabla^{n-1}(\mathbf{S} \cdot \dot{\mathbf{F}})] - [\nabla^{n-1}(\nabla \cdot \mathbf{Q})]. \tag{5.10}$$

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Property n1 implies that $[\nabla^{n-1}\mathbf{S}]=0$ and hence

$$[\nabla^{n-1}(\mathbf{S}\cdot\dot{\mathbf{F}})]=[\nabla^{n-1}\dot{\mathbf{F}}]\mathbf{S}.$$

Accordingly, Eq. (5.10) becomes

$$(5.10)' \qquad [\nabla^{n-1}\dot{e}]=[\nabla^{n-1}\dot{\mathbf{F}}]\mathbf{S}-[\nabla^{n-1}\nabla\cdot\mathbf{Q}].$$

Observe that, given a function ξ defined on $\mathscr{R}\times\mathbf{R}$ and subject to $[\nabla^{n-1}\xi]=0$, the iterative application of Eq. (4.2)′ yields

$$(5.11) \qquad [\nabla^{m}\xi]=(-U_N)^{-m}[\overset{(m)}{\xi}]\underbrace{\mathbf{N}\otimes\ldots\otimes\mathbf{N}}_{m},$$

where $\overset{(m)}{\xi}$ is the m^{th} time derivative of ξ. On the other hand the jump $[\overset{(n)}{\mathbf{S}}]$ can be written as

$$(5.12) \qquad [\overset{(n)}{\mathbf{S}}]=\mathbf{S}_\theta[\overset{(n)}{\theta}]+\mathbf{S}_\mathbf{F}[\overset{(n)}{\mathbf{F}}]+\mathbf{S}_\alpha\big(\mathbf{h}_\mathbf{G}[\overset{(n-1)}{\mathbf{G}}]+\mathbf{h}_\mathbf{F}[\overset{(n)}{\mathbf{F}}]\big),$$

while analogous relations hold for the functions $\overset{(n)}{e}$, $\overset{(n)}{\mathbf{Q}}$(3). Then, in view of Eqs. (5.11) and (5.12) we can write Eq. (5.9) as

$$[\overset{(n)}{\mathbf{v}}]=-\frac{1}{U_N}\{\mathbf{S}_\theta[\overset{(n)}{\theta}]+\mathbf{S}_\mathbf{F}[\overset{(n)}{\mathbf{F}}]+\mathbf{S}_\alpha\big(\mathbf{h}_\mathbf{G}[\overset{(n-1)}{\mathbf{G}}]+\mathbf{h}_\mathbf{F}[\overset{(n)}{\mathbf{F}}]\big)\}.$$

Upon substitution of the compatibility relations

$$[\overset{(n)}{\mathbf{F}}]=-\frac{1}{U_N}[\overset{(n)}{\mathbf{v}}]\otimes\mathbf{N}, \qquad [\overset{(n-1)}{\mathbf{G}}]=-\frac{1}{U_N}[\overset{(n)}{\theta}]\mathbf{N},$$

it is easily seen that

$$(5.13) \qquad (\mathfrak{Q}-U_N^2\mathbf{I})[\overset{(n)}{\mathbf{v}}]+\mathfrak{Z}[\overset{(n)}{\theta}]=\mathbf{0}.$$

A similar procedure applies to the jump relation (5.10). In fact, use of Eq. (5.11) provides

$$[\overset{(n)}{e}]=\mathbf{S}\cdot[\overset{(n)}{\mathbf{F}}]+\frac{1}{U_N}\mathbf{N}\cdot[\overset{(n)}{\mathbf{Q}}],$$

whence

$$(5.14) \qquad \mathfrak{X}\cdot[\overset{(n)}{\mathbf{v}}]+q[\overset{(n)}{\theta}]=0.$$

Looking at the system of equations (5.13) and (5.14) in the unknowns $[\overset{(n)}{\mathbf{v}}]$, $[\overset{(n)}{\theta}]$ it follows that a non-trivial solution is possible only if the determinantal equation (5.6) does hold. So we have proved that waves of higher order and acceleration waves have the same propagation condition thus establishing the counterpart of analogous theorems by ERICKSEN [10], TRUESDELL [11], COLEMAN and GURTIN [12], COLEMAN, GREENBERG and GURTIN [7]. In other words, all waves of the order $n\geqslant 1$ have the same speed of propagation. However, a remark is in order. In fact, the propagation condition (5.6) holds exactly for

(3) Since $[\overset{n-1}{\theta}]=0$, $[\overset{n-1}{\mathbf{F}}]=0$ the dependence of $\mathbf{h}$ on θ, $\mathbf{F}$ does not give rise to corresponding terms in the jump relations.

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waves of higher order and also for acceleration waves provided the amplitudes $\mathbf{a}, \Theta$ are small enough to replace the actual values of the derivatives $\mathbf{h}_{\mathbf{G}}, \mathbf{h}_{\dot{\mathbf{F}}}$ by their values at $(\theta, \mathbf{F}, \mathbf{G}^+, \dot{\mathbf{F}}^+, \boldsymbol{\alpha})$. In conclusion we can write the following:

THEOREM. *The speed of propagation U_N and the relation between the amplitudes associated with waves of higher order propagating through thermo-viscous materials with hidden variables are the same as in the corresponding case of infinitesimal acceleration waves.*

6. An example: thermo-viscous fluids

This section delivers a model of fluid with hidden variables which is compatible with the existence of shocks and acceleration waves. Meanwhile Fourier's law of heat conduction and Navier-Stokes' law of viscosity are obtained as asymptotic limits in stationary conditions. Although such a model is to be viewed as an example of the theory performed in the previous sections, the spatial description appears to be more convenient and then some results will be restated briefly in their dual form.

A material is said to be a fluid if its symmetry group is the unimodular group [5]. Accordingly, the response function $\boldsymbol{\sigma}$ and the evolution function $\mathbf{h}$ may depend on $\mathbf{F}$ only through the determinant det $\mathbf{F}$, that is through the actual density $\varrho = \varrho_0/\det\mathbf{F}$. Then, describing heat conduction and viscosity via the spatial temperature gradient $\mathbf{g}$ and the rate of strain $\mathbf{D}$, for a temperature rate independent fluid we can write

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\theta, \varrho, \boldsymbol{\alpha}),$$
$$\dot{\boldsymbol{\alpha}} = \mathbf{h}(\theta, \varrho, \mathbf{g}, \mathbf{D}, \boldsymbol{\alpha}),$$

where $\boldsymbol{\sigma}$ stands for the set of quantities $(e, \mathbf{T}, \mathbf{q}, \eta)$. To account in a simple way for heat conduction and viscosity it is convenient to suppose that the hidden variables $\boldsymbol{\alpha}$ consist of a vector $\boldsymbol{\alpha}_1$ and a symmetric tensor $\boldsymbol{\alpha}_2$, i.e. $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$. Assuming the independence of $\mathbf{h}$ of θ and ϱ, consider the pair of linear evolution equations

$$\dot{\boldsymbol{\alpha}}_1 = \frac{1}{\tau_1}(\mathbf{g}-\boldsymbol{\alpha}_1), \qquad \boldsymbol{\alpha}_1(t_0) = \boldsymbol{\alpha}_1^0,$$
(6.1)
$$\dot{\boldsymbol{\alpha}}_2 = \frac{1}{\tau_2}(\mathbf{D}-\boldsymbol{\alpha}_2), \qquad \boldsymbol{\alpha}_2(t_0) = \boldsymbol{\alpha}_2^0,$$

where $\boldsymbol{\alpha}_1^0, \boldsymbol{\alpha}_2^0 \in C^1(\mathscr{R})$(4). Property III requires that the relaxation times τ_1, τ_2 belong to $\mathbf{R}^{++}$. The map $\mathbf{E}$ is expressed by $\boldsymbol{\alpha}_1 = \mathbf{g}, \boldsymbol{\alpha}_2 = \mathbf{D}$. The obvious solutions of Eqs. (6.1) are

(6.2)
$$\boldsymbol{\alpha}_1(t) = \bar{\mathbf{g}}(t; \tau_1)+\boldsymbol{\alpha}_1^0\exp(-(t-t_0)/\tau_1), \qquad t-t_0 \in \mathbf{R}^+,$$
$$\boldsymbol{\alpha}_2(t) = \bar{\mathbf{D}}(t; \tau_2)+\boldsymbol{\alpha}_2^0\exp(-(t-t_0)/\tau_2), \qquad t-t_0 \in \mathbf{R}^+,$$

the symbol $\bar{\xi}(t; \tau)$ being defined as

$$\bar{\xi}(t; \tau) = \frac{1}{\tau}\int_{t_0}^{t} \exp(-(t-\zeta)/\tau)\xi(\zeta)d\zeta.$$

(4) According to Eqs. (6.1) we have $\dim\mathbf{A} = \dim\mathbf{Z}$.

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The second law of thermodynamics is supposed to be expressed by the Clausius-Duhem inequality

$$(6.3) \qquad -\varrho(\dot{\psi}+\eta\dot{\theta})+\mathbf{T}\cdot\mathbf{D}-\frac{1}{\theta}\mathbf{q}\cdot\mathbf{g}\geqslant 0.$$

So the response functions are compatible with thermodynamics provided that

$$-\varrho(\psi_\theta+\eta)\dot{\theta}+\left(\varrho^2\psi_\varrho\mathbf{I}+\mathbf{T}-\frac{\varrho}{\tau_2}\psi_{\alpha_2}\right)\cdot\mathbf{D}-\left(\frac{1}{\theta}\mathbf{q}+\frac{\varrho}{\tau_1}\psi_{\alpha_1}\right)\cdot\mathbf{g}+\varrho\left(\frac{1}{\tau_1}\psi_{\alpha_1}\cdot\boldsymbol{\alpha}_1+\frac{1}{\tau_2}\psi_{\alpha_2}\cdot\boldsymbol{\alpha}_2\right)\geqslant 0$$

holds for every time t and for every path $\boldsymbol{\pi}=(\theta,\varrho,\mathbf{g},\mathbf{D})$. By using the kind of argument used already in proving Eqs. (3.6) and, (3.7), we conclude that this inequality holds if and only if

$$6.4) \qquad \eta=-\psi_\theta, \quad \mathbf{T}=-\varrho^2\psi_\varrho\mathbf{I}+\frac{\varrho}{\tau_2}\psi_{\alpha_2}, \quad \mathbf{q}=-\frac{\varrho\theta}{\tau_1}\psi_{\alpha_1},$$

$$(6.5) \qquad \frac{1}{\tau_1}\psi_{\alpha_1}\cdot\boldsymbol{\alpha}_1+\frac{1}{\tau_2}\psi_{\alpha_2}\cdot\boldsymbol{\alpha}_2\geqslant 0.$$

Then a function $\psi(\theta,\varrho,\boldsymbol{\alpha})$ satisfying the inequality (6.5) makes the response functions (6.4) identically compatible with the second law of thermodynamics.

To specialise the example under consideration, look now at a free energy function ψ dependent on θ, ϱ and on the quadratic invariants $\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_2$, and $(\mathbf{I}\cdot\boldsymbol{\alpha}_2)^2$ in the form

$$(6.6) \qquad \psi(\theta,\varrho,\boldsymbol{\alpha})=\Psi(\theta,\varrho)+\frac{1}{\varrho}\left\{\frac{\varkappa\tau_1}{2\theta}\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1+\mu\tau_2\boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_2+\frac{1}{2}\lambda\tau_2(\mathbf{I}\cdot\boldsymbol{\alpha}_2)^2\right\},$$

where $\varkappa$, μ, λ are non-vanishing constants. It is a simple matter to show that the function (6.6) satisfies the inequality (6.5) if and only if

$$(6.7) \qquad \mu>0, \quad 3\lambda+2\mu\geqslant 0; \quad \varkappa>0.$$

Substitution of Eq. (6.6) into Eq. (6.4) delivers

$$(6.8) \qquad \eta=-\Psi_\theta+\frac{\varkappa\tau_1}{2\varrho\theta^2}\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1,$$

$$(6.9) \qquad \mathbf{T}=-p\mathbf{I}+2\mu\boldsymbol{\alpha}_2+\lambda(\mathbf{I}\cdot\boldsymbol{\alpha}_2)\mathbf{I}, \quad \mathbf{q}=-\varkappa\boldsymbol{\alpha}_1,$$

being $p=\varrho^2\psi_\varrho$. If $\mathbf{g}(t)=\mathbf{g}'$, $\mathbf{D}(t)=\mathbf{D}'$, $t-t_0\in\mathbf{R}^+$, it follows that

$$\lim_{t\to\infty}\big(\boldsymbol{\alpha}_1(t),\boldsymbol{\alpha}_2(t)\big)=(\mathbf{g}',\mathbf{D}').$$

This allows us to say that, except for the dependence of p on $\boldsymbol{\alpha}$ and hence on $\mathbf{g}$ and $\mathbf{D}$, when $\mathbf{g}$ and $\mathbf{D}$ are constant in time Eqs. (6.9) asymptotically become the Navier-Stokes and Fourier constitutive equations. So, the results (6.7) may be regarded as the Stokes-Duhem and Fourier inequalities.

Look at the behaviour of the thermo-viscous fluid described by Eqs. (6.6)–(6.9) as to the propagation condition for shock waves and acceleration waves.

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Shock waves

Application of Kotchine's theorem to the balance equations of mass, momentum, and energy enables us to write the jump relations

$$[\varrho U] = 0,$$
$$[\varrho U\mathbf{v}] + [\mathbf{T}]\mathbf{n} = \mathbf{0},$$
$$[\varrho Ue] + [\mathbf{v}] \cdot \overline{\mathbf{T}}\mathbf{n} - [\mathbf{q}] \cdot \mathbf{n} = 0,$$

where U is the local speed of propagation and $\mathbf{n}$ is the normal to the shock front in the actual configuration. Since $[\boldsymbol{\alpha}_1] = \mathbf{0}$, and $[\boldsymbol{\alpha}_2] = \mathbf{0}$, in view of Eq. (6.9) it follows that

$$[\mathbf{T}] = -[p]\mathbf{I}, \quad [\mathbf{q}] = \mathbf{0}.$$

Again the shock turns out to be longitudinal and, in the case of infinitesimal shocks, we find the propagation condition

$$e_\theta(p_\varrho - U^2) - p_\theta\left(e_\varrho - \frac{\mathbf{n} \cdot \mathbf{T}\mathbf{n}}{\varrho^2}\right) = 0$$

where the quantities

$$p = \varrho^2 \Psi_\varrho - \left\{\frac{\varkappa\tau}{2\theta}\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_1 + \mu\tau_2 \boldsymbol{\alpha}_2 \cdot \boldsymbol{\alpha}_2 + \frac{1}{2}\lambda\tau_2(\mathbf{I} \cdot \boldsymbol{\alpha}_2)^2\right\},$$

$$p_\theta = \varrho^2 \Psi_{\theta\varrho} + \frac{\varkappa\tau_1}{2\theta}\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_1, \quad p_\varrho = (\varrho^2\Psi_\varrho)_\varrho,$$

$$e_\theta = -\theta\Psi_{\theta\theta} - \frac{\varkappa\tau}{\varrho\theta^2}\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_1, \quad e_\varrho = (\Psi - \theta\Psi_\theta)_\varrho - \frac{1}{\varrho^2}\left\{\frac{\varkappa\tau_1}{\theta}\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_1 + \mu\tau_2\boldsymbol{\alpha}_2 \cdot \boldsymbol{\alpha}_2 + \frac{1}{2}\lambda\tau_2(\mathbf{I} \cdot \boldsymbol{\alpha}_2)^2\right\}$$

are to be evaluated in the region ahead of the shock.

Acceleration waves

The balance equations provide the jump relations

$$[\dot{\varrho}] + \varrho[\operatorname{div}\mathbf{v}] = 0,$$
$$\varrho[\dot{\mathbf{v}}] - [\operatorname{div}\mathbf{T}] = \mathbf{0},$$
$$\varrho[\dot{e}] - \mathbf{T} \cdot [\mathbf{D}] + [\operatorname{div}\mathbf{q}] = 0,$$

while $[\mathbf{v}] = 0$ and $[\theta] = 0$. For the problem in study a convenient form of the compatibility condition is

$$\frac{\delta[\xi]}{\delta t} = [\dot{\xi}] + U\mathbf{n} \cdot [\operatorname{grad}\ \xi].$$

Consequently $[\mathbf{T}] = 0$ and $[\mathbf{q}] = 0$ imply that

$$[\operatorname{div}\mathbf{T}] = -\frac{1}{U}[\dot{\mathbf{T}}]n, \quad [\operatorname{div}\mathbf{q}] = -\frac{1}{U}[\dot{\mathbf{q}}] \cdot \mathbf{n}.$$

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Accordingly, since $[\dot{\boldsymbol{\alpha}}_1] = [\mathbf{g}]/\tau_1$, $[\dot{\boldsymbol{\alpha}}_2] = [\mathbf{D}]/\tau_2$, account of the response functions (6.6)–(6.9) allows us to write

(6.10)

$$\left(\varrho U^2 - \frac{\mu}{\tau_2}\right)\mathbf{a} - \left\{\left(\varrho p_\varrho + \frac{\mu+\lambda}{\tau_2} + \lambda(\mathbf{I}\cdot\boldsymbol{\alpha}_2) + 2\mu\mathcal{N}\right)\mathbf{a}\cdot\mathbf{n} + 2\mu\mathcal{S}\mathbf{a}\cdot\mathbf{t}\right\}\mathbf{n} - \left(Up_\theta + \frac{\varkappa}{\theta}\boldsymbol{\alpha}_1\cdot\mathbf{n}\right)\Theta\mathbf{n} = 0$$

$$U(\varrho^2 e_\varrho - p)\mathbf{a}\cdot\mathbf{n} + \left(\varrho e_\theta U^2 - \frac{2\varkappa}{\theta}\boldsymbol{\alpha}_1\cdot\mathbf{n}U - \frac{\varkappa}{\tau_1}\right)\Theta = 0,$$

where $\mathcal{N} = \mathbf{n}\cdot\boldsymbol{\alpha}_2\mathbf{n}$ and $\mathcal{S} = |\mathbf{n}\wedge\boldsymbol{\alpha}_2\mathbf{n}|$ are the components of $\boldsymbol{\alpha}_2\mathbf{n}$, that is

$$\boldsymbol{\alpha}_2\mathbf{n} = \mathcal{N}\mathbf{n} + \mathcal{S}\mathbf{t}, \quad \mathbf{t}\cdot\mathbf{n} = 0.$$

The determinantal equation associated with Eqs. (6.10) follows straightaway.

Consider now some particular solutions of Eqs. (6.10). If $\mathcal{S} = 0$, Eqs. (6.10) bear evidence of the existence of purely mechanical transverse waves — $\mathbf{a}\cdot\mathbf{n} = 0$, $\theta = 0$. The corresponding local speed of propagation U is

$$U = \left(\frac{\mu}{\varrho\tau_2}\right)^{\frac{1}{2}}.$$

On the other hand longitudinal waves — $\mathbf{a} = (\mathbf{a}\cdot\mathbf{n})\mathbf{n}$, $\Theta \neq 0$ — are possible. In such a case the propagation condition is

$$c_4 U^4 + c_3 U^3 + c_2 U^2 + c_1 U + c_0 = 0, \tag{6.11}$$

where

$$c_4 = \varrho^2 e_\theta, \quad c_3 = -\frac{2\varkappa\varrho}{\theta}\boldsymbol{\alpha}_1\cdot\mathbf{n},$$

$$c_2 = -\left\{\frac{\varkappa\varrho}{\tau_1} + \varrho e_\theta\left(\varrho p_\varrho + \frac{2\mu+\lambda}{\tau_2} + 2\mu\mathcal{N} + \lambda(\mathbf{I}\cdot\boldsymbol{\alpha}_2)\right) + p_\theta(p - \varrho^2 e_\varrho)\right\},$$

$$c_1 = \frac{\varkappa}{\theta}\left\{\varrho^2 e_\varrho - p + 2\left(\varrho p_\varrho + \frac{2\mu+\lambda}{\tau_2} + 2\mu\mathcal{N} + \lambda(\mathbf{I}\cdot\boldsymbol{\alpha}_2)\right)\right\}\boldsymbol{\alpha}_1\cdot\mathbf{n},$$

$$c_0 = \frac{\varkappa}{\tau_1}\left\{\varrho p_\varrho + \frac{2\mu+\lambda}{\tau_2} + 2\mu\mathcal{N} + \lambda(\mathbf{I}\cdot\boldsymbol{\alpha}_2)\right\}.$$

If the temperature gradient is zero until the arrival of the wave front, we have $\boldsymbol{\alpha}_1 = 0$ and then $c_3 = 0$, $c_1 = 0$. The corresponding acceleration waves are symmetric [8].

In respect of the response functions, the presence of hidden variables involves memory effects. In particular, the example described above accounts for memory effects associated with heat conduction and viscosity. A possible connection with Fourier's and Navier-Stokes' theories may be accomplished by examining the behaviour of the speed U when the memory becomes extremely short, that is to say when $\tau_1, \tau_2 \to 0$. For the sake of simplicity let $\tau := \tau_1$ and suppose $\tau_2 = O(\tau)$. Observe first that

$$\boldsymbol{\alpha}_1 = \mathbf{g} + O(\tau), \quad \boldsymbol{\alpha}_2 = \mathbf{D} + O(\tau)$$

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as $\tau \to 0$. Hence, as $\tau \to 0$, $c_4, c_3 = O(1)$ while $c_2, c_1 = O(\tau^{-1})$ and $c_0 = O(\tau^{-2})$. Meanwhile, in view of Eq. (6.11), the speed of propagation must satisfy the condition $U = O\left(\tau^{-\frac{1}{2}}\right)$. The sought connection is then established. Indeed, Fourier's and Navier-Stokes' theories, ruling out the possibility of wave propagation, may be obtained from the present example in the limiting case $\tau \to 0$.

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References

1. M. E. GURTIN and A. C. PIPKIN, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal., **31**, 113–126, 1968.
2. A. MORRO, *Temperature waves in rigid materials with memory*, Meccanica, **12**, 72–77, 1977.
3. W. KOSIŃSKI and P. PERZYNA, *Analysis of acceleration waves in materials with internal parameters*, Arch. Mech., **24**, 629–643, 1972.
4. W. KOSIŃSKI, *Thermal waves in inelastic bodies*, Arch. Mech., **27**, 733–748, 1975.
5. B. D. COLEMAN and M. E. GURTIN, *Thermodynamics with internal state variables*, J. Chem. Phys., **47**, 597–613, 1967.
6. P. J. CHEN and M. E. GURTIN, *Growth and decay of one-dimensional shock waves in fluids with internal state variables*, Phys. Fluids, **14**, 1091–1094, 1971.
7. B. D. COLEMAN, J. M. GREENBERG and M. E. GURTIN, *Waves in materials with memory. V. On the amplitudes of acceleration waves in mild discontinuities*, Arch. Rational Mech. Anal., **22**, 333–354, 1966.
8. I. SULICIU, *Symmetric waves in materials with internal state variables*, Arch. Mech., **27**, 841–856, 1975.
9. A. MORRO, *Shock waves in thermo-viscous fluids with hidden variables*, Arch. Mech. [to appear].
10. J. L. ERICKSEN, *On the propagation of waves in isotropic incompressible perfectly elastic materials*, J. Rational Mech. Anal., **2**, 329–337, 1953.
11. C. TRUESDELL, *General and exact theory of waves in finite elastic strain*, Arch. Rational Mech. Anal., **8**, 263–296, 1961.
12. B. D. COLEMAN and M. E. GURTIN, *Waves in materials with memory. IV. Thermodynamics and the velocity of general acceleration waves*, Arch. Rational Mech. Anal., **19**, 317–338, 1965.
13. W. A. DAY, *Entropy and hidden variables incontinuum thermodynamics*, Arch. Rational Mech. Anal. **62**, 367–389, 1976.
14. J. LUBLINER, *On fading memory in materials of evolutionary type*, Acta Mechanica, **8**, 75–81, 1969.
15. W. KOSIŃSKI and W. WOJNO, *Remarks on internal variable and history descriptions of materials*, Arch. Mech., **25**, 709–713, 1973.

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