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On the rate-independent limit of visco-plastic constitutive equations

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RATE-INDEPENDENT plasticity is developed here as a limiting case of the finite strain theory of elastic-visco-plastic materials. The analysis of large bending and extension of an isotropic incompressible elastic-plastic block is given as an example of the use of the derived rate-independent constitutive equations.

Opracowano niezależną od prędkości teorię plastyczności jako przypadek graniczny teorii odkształceń skończonych materiałów sprężysto-lepko-plastycznych. Jako przypadek zastosowania wyprowadzonych tu niezależnych od prędkości równań konstytutywnych przeprowadzono analizę silnego zginania i rozciągania izotropowego nieściśliwego sprężysto-plastycznego bloku.

Разработана, независящая от скорости, теория пластичности как предельный случай теории конечных дефомации упруго-вязкопластических материалов. Как случай применения, выведенных здесь независящих от скорости определяющих уравнений, проведен анализ сильного изгиба и растяжения изотропного несжимаемого упруго-пластического блока.

1. Introduction

IN THE RECENT paper by LEHMANN [1] the finite strain theory of elastic-visco-plastic materials is obtained by extension of a theory of elasto-plasticity. Lehmann's approach is based on the assumption (often used in microscopic theories of work-hardening) that stress can be decomposed into the rate-independent (inviscid) stress and the exceeding (viscous) stress. Of course, the rate-independent theory is then obtained simply by assuming that the exceeding stress is zero.

An opposite point of view is used here. The elastic-visco-plastic theory is regarded as basic and the rate-independent plasticity is developed as a convenient approximative limit case of the rate-dependent theory. Our approach is justified by the fact that response of actual inelastic materials is always rate-sensitive. Only in a narrow range of loading rates can some materials be regarded as rate-independent. This idealization, particularly in the form of the classical theory of plasticity, has been successful in analyses of many engineering problems.

Constitutive equations of rate-independent plasticity were obtained as a limit of the visco-plastic constitutive law by PERZYNA and WOJNO [2] and KRATOCHVÍL and DILLON [3]. In both papers intuitive arguments were used so as to reach the limit. In this note we attempt to specify more explicitly the conditions under which such limit procedure can be realized.

To obtain a rate-independent limit of elastic-visco-plastic constitutive equations two types of assumptions have to be introduced. First, a sufficient smoothness of the functions in the elastic-visco-plastic constitutive equations has to be guaranteed. Second, the existence of the limit has to be assumed. As to the second, rather strong and unpleasant assumption we can remark the following. The papers [4, 5] treat visco-plastic constitutive equations which are a special case of the constitutive equations considered here. We get them from our equations (2.2), (2.3) accepting the infinitesimal strain approach, the convexity of the yield surface, and some other mild assumptions of a technical nature. In [4, 5] it is shown that the solution of a boundary value problem for these special constitutive equations converges in a certain sense for zero viscosity to a solution of the same boundary value problem of the classical flow theory of plasticity. Therefore, in this special case the existence of the rate-independent limit in the global sense is guaranteed. We cannot follow the same procedure here, as for the finite strain the existence and uniqueness of the solutions are not proved and, moreover, the existence of the limit in local sense is needed. The existence of such a limit for the general class of inelastic materials considered in this paper seems to be a deeply rooted problem we have not yet been able to overcome successfully.

In Sect. 2 we briefly describe elastic-visco-plastic constitutive equations for finite strain deformations. The internal variable approach is employed. The main assumptions and the derived constitutive equations are summarized in Lemma in Sect. 3. The derived constitutive equations are of the form usually assumed in the finite strain theories of invicid plasticity (e.g. [6]). The special case of isotropic incompressible elastic-plastic constitutive equations is then considered and used in an example of a large bending.

2. Elastic-visco-plastic materials

Mechanical and thermal treatment of an elastic-visco-plastic material induces deformation and changes in its physical properties. At finite strain the deformation of the material, described by the *deformation gradient* $\mathbf{F}$, may be resolved into an *elastic part* $\mathbf{E}$ and an *inelastic part* $\mathbf{P}$ according to the relation $\mathbf{F} = \mathbf{EP}$ (e.g. [8, 9]). The material time derivative of this relation yields the decomposition rule ([8, 9])

$$\mathbf{L} = \mathbf{L}_E + \mathbf{L}_P, \tag{2.1}$$

where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the velocity gradient and $\mathbf{L}_E = \dot{\mathbf{E}}\mathbf{E}^{-1}$, $\mathbf{L}_P = \mathbf{E}\dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{E}^{-1}$ are rates of elastic and inelastic deformations respectively (the superposed dot means material time derivative). The change of physical properties can be described in terms of *structural parameters* α_i, $i = 1, \ldots, n$; α_i may be interpreted, e.g. as scalar quantities which appear in the theories of work-hardening [10–12] (when convenient, we use the notation $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$; an extension of the present consideration to the case of the structural parameters of higher tensorial rank is readily possible).

The class of elastic-visco-plastic materials described in detail in [8, 13] is characterized by six constitutive functions which give the values of the density of internal energy u, the Cauchy stress tensor $\mathbf{T}$, the density of entropy η, the heat flux vector $\mathbf{h}$, the rate of the inelastic deformation $\mathbf{L}_P$, and the rate of the structural parameter $\boldsymbol{\alpha}$, if values of the elastic part of the deformation gradient (¹) $\mathbf{E}$, the temperature θ, the temperature gradient $\mathbf{g}$, and the structural parameter $\boldsymbol{\alpha}$ are specified. The set of variables $\mathbf{E}, \theta, \mathbf{g}, \boldsymbol{\alpha}$ will be denoted by $\boldsymbol{\omega}$, i.e. $\boldsymbol{\omega} \equiv (\mathbf{E}, \theta, \mathbf{g}, \boldsymbol{\alpha})$.

The briefly described material model represents a rate-sensitive material. This is caused by the fact that the considered constitutive equations for $\mathbf{L}_P$ and $\dot{\boldsymbol{\alpha}}$, $\mathbf{L}_P = \bar{\mathbf{L}}_P(\boldsymbol{\omega})$ and $\dot{\boldsymbol{\alpha}} = a(\boldsymbol{\omega})$ are not homogeneous in the first time derivatives, i.e. a change in time scale (or similarly a change in deformation and heating rates) influences the response of the material. Therefore, in order to obtain a theory of rate-independent plastic behaviour the constitutive equations for rates $\mathbf{L}_P$ and $\dot{\boldsymbol{\alpha}}$ have to be modified. As a starting point of the modification we use a special class of elastic-visco-plastic materials with well-defined elastic range. We assume:

a. There exists a continuous scalar function $f = \bar{f}(\boldsymbol{\omega})$; the set of $\boldsymbol{\omega}$ such that $\bar{f}(\boldsymbol{\omega}) = 0$ will be referred to as the *generalized yield surface*.

b. For all $\boldsymbol{\omega}$ such that $\bar{f}(\boldsymbol{\omega}) \leqslant 0$ the constitutive function $\bar{\mathbf{L}}_P$ satisfies $\bar{\mathbf{L}}_P(\boldsymbol{\omega}) = 0$; for $\boldsymbol{\omega}$ close to the generalized yield surface, such that $\bar{f}(\boldsymbol{\omega}) > 0$, $\mathbf{L}_P$ is governed by the equation $\mathbf{L}_P = \lambda \bar{f}(\boldsymbol{\omega})\bar{\mathbf{G}}(\boldsymbol{\omega})$, i.e. we can write

$$\mathbf{L}_P = \lambda \bar{f}(\boldsymbol{\omega}) h(\lambda \bar{f}(\boldsymbol{\omega}))\mathbf{G}(\boldsymbol{\omega}), \tag{2.2}$$

where λ is a positive scalar parameter (a constant for a given material, $h(\cdot)$ represents the Heaviside function ($h(x) = 0$ for $x < 0$, $h(x) = 1$ for $x \geqslant 0$), and $\mathbf{G}(\boldsymbol{\omega})$ is a second-order tensor function of $\boldsymbol{\omega}$.

c. The constitutive equations for the rates $\dot{\alpha}_i$ have the form

$$\dot{\alpha}_i = \operatorname{tr}(\mathbf{A}_i(\boldsymbol{\omega})\mathbf{L}_P), \quad i = 1, \ldots, n, \tag{2.3}$$

where $\mathbf{A}_i(\boldsymbol{\omega})$ are second-order tensor functions of $\boldsymbol{\omega}$.

In the assumptions (a)–(c) the consequences of the principle of material indifference has not been respected. Using the standard procedure [8] we can get the frame indifferent form of Eqs. (2.2) and (2.3)

$$\mathbf{L}_P = \lambda \hat{f}(\overline{\boldsymbol{\omega}}) h(\lambda \hat{f}(\overline{\boldsymbol{\omega}}))\mathbf{E}\hat{\mathbf{G}}(\overline{\boldsymbol{\omega}})\mathbf{E}^T, \tag{2.4}$$

$$\dot{\alpha}_i = \lambda \hat{f}(\overline{\boldsymbol{\omega}}) h(\lambda \hat{f}(\overline{\boldsymbol{\omega}}))\operatorname{tr}[\hat{\mathbf{G}}(\overline{\boldsymbol{\omega}})\hat{\mathbf{A}}_i(\overline{\boldsymbol{\omega}})\mathbf{C}_E], \tag{2.5}$$

where $\mathbf{C}_E = \mathbf{E}^T\mathbf{E}$ is the right Cauchy-Green elastic tensor,

$$\overline{\boldsymbol{\omega}} \equiv (\mathbf{C}_E, \theta, \bar{\mathbf{g}}, \boldsymbol{\alpha}), \quad \bar{\mathbf{g}} = \mathbf{E}^T\mathbf{g}, \quad \text{and} \quad \hat{f}(\overline{\omega}) = \bar{f}(\omega'),$$

$$\hat{\mathbf{G}}(\overline{\boldsymbol{\omega}}) = \mathbf{G}(\boldsymbol{\omega}'), \quad \hat{\mathbf{A}}_i(\overline{\boldsymbol{\omega}}) = \mathbf{A}_i(\boldsymbol{\omega}'); \quad \boldsymbol{\omega}' \equiv (\mathbf{C}_E^{1/2}, \theta, \mathbf{C}_E^{-1/2}\bar{\mathbf{g}}, \boldsymbol{\alpha}).$$

The assumptions (a)–(c) guarantee the fact that the material model behaves during thermo-mechanical processes which remain in the elastic range, i.e. $\hat{f}(\boldsymbol{\omega}) \leqslant 0$, as a thermo-elastic material. The reason is that for $\boldsymbol{\omega}$ such that $\hat{f}(\overline{\boldsymbol{\omega}}) \leqslant 0$, there are no inelastic changes, both $\mathbf{L}_P$ and $\dot{\boldsymbol{\alpha}}$ are zero. The constitutive equation (2.3) or (2.5) provides that the structural changes occur only during inelastic deformation. Recovery, aging and quenching effects are excluded by these assumptions. By the constitutive assumption (2.2) or (2.4) we require that for $\overline{\boldsymbol{\omega}}$ close to the generalized yield surface with $\hat{f}(\overline{\boldsymbol{\omega}})$ positive, the rate of

(¹) The assumption that $\mathbf{F}$ and $\mathbf{P}$ appear in the constitutive equations only through $\mathbf{E} = \mathbf{F}\mathbf{P}^{-1}$ is treated in [8] and [13] as a characteristic feature of elastic-visco-plastic materials.

inelastic deformation is proportional to f; $\mathbf{G}$ is an orientation factor. The parameter λ can be interpreted as an inverse scalar measure of the viscosity of the material. Materials with higher λ react more rapidly to the excess of f above zero, i.e. the materials with higher λ are less viscous. Looking for a description of rate-independent (i.e. viscous free) behaviour in the next section we determine the form of the constitutive equations (2.4) and (2.5) in the limiting case $\lambda \to \infty$.

3. Rate-independent limit

We obtain the rate-independent form of the constitutive equations (2.4) and (2.5) under the assumptions that the considered thermo-mechanical processes and the functions $\hat{f}$, $\hat{\mathbf{G}}$, $\hat{\mathbf{A}}_i$ are sufficiently smooth and the limit of Eq. (2.5) for $\lambda \to \infty$ exists. Note that inelastic deformation $\mathbf{P}$ does not appear in the right hand sides of Eqs. (2.4) and (2.5) (see the footnote ([1])). Therefore, the equation (3.1) for the rate of inelastic deformation $\mathbf{L}_P$ plays no active role in the following consideration. It is used only to derive its limiting form which is needed in Eq. (2.1).

Consider a sequence of elastic-visco-plastic materials characterized by Eqs. (2.4) and (2.5) with different values of λ. We suppose that for all $\overline{\boldsymbol{\omega}}$ the functions $\hat{\mathbf{G}}(\overline{\boldsymbol{\omega}})$ and $\hat{\mathbf{A}}_i(\overline{\boldsymbol{\omega}})$ are continuous and $\hat{f}(\overline{\boldsymbol{\omega}})$ has continuous partial derivatives; $\hat{f}(\cdot)$, $\hat{\mathbf{G}}(\cdot)$ and $\hat{\mathbf{A}}_i(\cdot)$ are assumed to be independent of λ. Suppose further that the right hand side of the constitutive equation (2.5) is such that the existence of the solution $\boldsymbol{\alpha}(t)$ in the interval $\langle t_1, t_2\rangle$ is guaranteed (e.g. the right hand side of Eq. (2.5) is continuous and bounded in $Q = \langle t_1, t_2\rangle x(-\infty, \infty)$ and Lipschitz-like continuous in Q with respect to $\boldsymbol{\alpha}$).

Suppose that we are given in the time interval $\langle t_1, t_2\rangle$ for different values of λ a common thermo-mechanical process $\mathbf{C}_E(t)$, $\theta(t)$ $\overline{\mathbf{g}}(t)$ with continuous time derivatives and at $t = t_1$ a common initial value of the structural parameter $\boldsymbol{\alpha}_0$, i.e. $\boldsymbol{\alpha}(t_1) = \boldsymbol{\alpha}_0$. Then under the introduced conditions in the interval $\langle t_1, t_2\rangle$ there exists for all λ a solution $\boldsymbol{\alpha}(t)$ of Eq. (2.5) with the continuous time derivative $\dot{\boldsymbol{\alpha}}(t)$. As in general, the solution $\boldsymbol{\alpha}(t)$ depends on the value of λ in the sequence of the materials, we write $\boldsymbol{\alpha}^\lambda(t)$. Further we denote

$$\boldsymbol{\omega}^\lambda \equiv (\mathbf{C}_E, \theta, \overline{\mathbf{g}}, \boldsymbol{\alpha}^\lambda), \quad f^\lambda(t) \equiv \hat{f}(\boldsymbol{\omega}^\lambda(t)),$$
$$\mathbf{G}^\lambda(t) \equiv \hat{\mathbf{G}}(\boldsymbol{\omega}^\lambda(t)), \quad \mathbf{A}_i^\lambda(t) \equiv \hat{\mathbf{A}}_i(\boldsymbol{\omega}^\lambda(t)),$$
$$(\partial f^\lambda/\partial \mathbf{C}_E)(t) \equiv (\partial \hat{f}/\partial \mathbf{C}_E)(\boldsymbol{\omega}^\lambda(t)),$$

etc. Then we write Eqs. (2.4) and (2.5) in the form

$$\mathbf{L}_P^\lambda = \lambda f^\lambda h(\lambda f^\lambda)\mathbf{E}\mathbf{G}^\lambda\mathbf{E}^T, \tag{3.1}$$

$$\dot{\alpha}_i^\lambda = \lambda f^\lambda h(\lambda f^\lambda)\,\mathrm{tr}[\mathbf{G}^\lambda\mathbf{A}_i^\lambda\mathbf{C}_E]. \tag{3.2}$$

Time differentiation of f^λ yields the identity

$$\dot{f}^\lambda \equiv \mathrm{tr}\left(\frac{\partial f^\lambda}{\partial \mathbf{C}_E}\dot{\mathbf{C}}_E\right) + \frac{\partial f^\lambda}{\partial \theta}\dot{\theta} + \frac{\partial f^\lambda}{\partial \overline{\mathbf{g}}}\cdot\dot{\overline{\mathbf{g}}} + \sum_{i=1}^{n}\frac{\partial f^\lambda}{\partial \alpha_i}\dot{\alpha}_i^\lambda. \tag{3.3}$$

From the definition $\mathbf{C}_E = \mathbf{E}^T\mathbf{E}$ we have $\dot{\mathbf{C}}_E = \mathbf{E}^T\mathbf{L}_E^T\mathbf{E}+\mathbf{E}^T\mathbf{L}_E\mathbf{E}$, hence denoting $\mathbf{L}^\lambda = \mathbf{L}_E+\mathbf{L}_P^\lambda$ (see Eq. (2.1)) and using Eqs. (3.2) and (3.3) we get

$$\dot{f}^\lambda = H^\lambda - K^\lambda \lambda f^\lambda h(\lambda f^\lambda), \tag{3.4}$$

where

$$H^\lambda \equiv 2\,\mathrm{tr}\left[\frac{\partial f^\lambda}{\partial \mathbf{C}_E}\mathbf{E}^T\mathbf{L}^\lambda\mathbf{E}\right]+\frac{\partial f^\lambda}{\partial\theta}\dot{\theta}+\frac{\partial f^\lambda}{\partial\bar{\mathbf{g}}}\cdot\dot{\bar{\mathbf{g}}},$$

$$K^\lambda \equiv 2\,\mathrm{tr}\left[\frac{\partial f^\lambda}{\partial \mathbf{C}_E}\mathbf{C}_E\mathbf{G}^\lambda\mathbf{C}_E\right]-\sum_{i=1}^{n}\frac{\partial f^\lambda}{\partial\alpha_i}\mathrm{tr}[\mathbf{G}^\lambda\mathbf{A}_i^\lambda\mathbf{C}_E].$$

In Eq. (3.4) we assume that there exist a common $\delta > 0$ and $\varepsilon > 0$, such that

$$|K^\lambda| \geqslant \delta, \quad |\mathrm{tr}[\mathbf{G}^\lambda\mathbf{A}_i^\lambda\mathbf{C}_E]| \geqslant \varepsilon \tag{3.5}$$

for at least one $i \in \{i, \ldots, n\}$. In the following the symbol lim always means the limit for $\lambda \to \infty$ and $\bar{h}$ is a function on R^1 such that $\bar{h}(0) = 1$ and $\bar{h}(x) = 0$ for $x \neq 0$.

LEMMA. If in the time interval $\langle t_1, t_2\rangle$ there exists a uniform $\lim\dot{\boldsymbol{\alpha}}^\lambda(t)$, then for all $t\in\langle t_1, t_2\rangle$ we have

$$\mathbf{L}_P = \frac{H}{K}h(f)\bar{h}(\dot{f})\mathbf{E}\mathbf{G}\mathbf{E}^T, \tag{3.6}$$

$$\dot{\alpha}_i = \frac{H}{K}h(f)\bar{h}(\dot{f})\,\mathrm{tr}[\mathbf{G}\mathbf{A}_i\mathbf{C}_E], \tag{3.7}$$

where $\mathbf{L}_P \equiv \lim \mathbf{L}_P^\lambda$, $\boldsymbol{\alpha} \equiv \lim\boldsymbol{\alpha}^\lambda$, $f \equiv \hat{f}(\boldsymbol{\omega})$, $\mathbf{G} \equiv \hat{\mathbf{G}}(\boldsymbol{\omega})$, $\mathbf{A}_i \equiv \hat{\mathbf{A}}_i(\boldsymbol{\omega})$, $\boldsymbol{\omega} \equiv (\mathbf{C}_E, \theta, \bar{\mathbf{g}}, \boldsymbol{\alpha})$, $\mathbf{L} = \mathbf{L}_E + \mathbf{L}_P$ and

$$H \equiv 2\,\mathrm{tr}\left[\frac{\partial f}{\partial \mathbf{C}_E}\mathbf{E}^T\mathbf{L}\mathbf{E}\right]+\frac{\partial f}{\partial\theta}\dot{\theta}+\frac{\partial f}{\partial\bar{\mathbf{g}}}\cdot\dot{\bar{\mathbf{g}}},$$

$$K \equiv 2\,\mathrm{tr}\left[\frac{\partial f}{\partial \mathbf{C}_E}\mathbf{C}_E\mathbf{G}\mathbf{C}_E\right]-\sum_{i=1}^{n}\frac{\partial f}{\partial\alpha_i}\mathrm{tr}[\mathbf{G}\mathbf{A}_i\mathbf{C}_E].$$

P r o o f. As we have $\alpha^\lambda(t_1) = \boldsymbol{\alpha}_0$ for all λ, the existence of the solutions $\boldsymbol{\alpha}^\lambda$ of Eq. (3.2) with the continuous time derivative and the existence of uniform $\lim\dot{\boldsymbol{\alpha}}^\lambda$ imply the following consequences valid in $\langle t_1, t_2\rangle$. There exists uniform $\lim\boldsymbol{\alpha}^\lambda = \boldsymbol{\alpha}$, $\dot{\boldsymbol{\alpha}}$ continuous, such that $\dot{\boldsymbol{\alpha}} = \lim\dot{\boldsymbol{\alpha}}^\lambda$. It means $\lim\boldsymbol{\omega}^\lambda = \boldsymbol{\omega}$, $\lim f^\lambda = f$, $\lim\dot{f}^\lambda = \dot{f}$ and $\dot{f}$ is continuous. Using Eq. $(3.5)_2$ we see further that there exists continuous $|\lim\mathrm{tr}[\mathbf{G}^\lambda\mathbf{A}_i^\lambda\mathbf{C}_E]| = |\mathrm{tr}[\mathbf{G}\mathbf{A}_i\mathbf{C}_E]| \geqslant \varepsilon$. By Eq. (3.2) this implies the existence of the continuous $\lim\lambda f^\lambda h(\lambda f^\lambda)$. We denote

$$\varphi \equiv \lim\lambda f^\lambda h(\lambda f^\lambda). \tag{3.8}$$

For $t\in\langle t_1, t_2\rangle$ we have $0 \leqslant \varphi(t) < \infty$. We see that

$$\mathbf{L}_P \equiv \lim\mathbf{L}_P^\lambda = \varphi\mathbf{E}\mathbf{G}\mathbf{E}^T, \tag{3.9}$$

$$\dot{\alpha}_i = \lim\dot{\alpha}_i^\lambda = \varphi\,\mathrm{tr}[\mathbf{G}\mathbf{A}_i\mathbf{C}_E], \tag{3.10}$$

$$\dot{f} = H - \varphi K. \tag{3.11}$$

Finally, the form of the limiting function φ will be derived.

(i) If $\varphi > 0$ on $M \subset \langle t_1, t_2\rangle$, we find from Eq. (3.8) that $h = 1$ for $\boldsymbol{\lambda}$ greater than some λ_0 for every $t \in M$ (λ_0 depends on $t \in M$). $\varphi < \infty$ implies $f = 0$ on M, so that $\dot{f} = 0$ on M (the continuity of φ implies that M is relatively open in $\langle t_1, t_2\rangle$). From Eqs. $(3.5)_1$ and (3.11) we get $\varphi = H/K$, hence, as $f = 0$, $\dot{f} = 0$, we can write formally $\varphi = (H/K)$ $h(f)\bar{h}(\dot{f})$.

(ii) The case $f > 0$ for some $t \in \langle t_1, t_2\rangle$ is a contradiction with $\varphi < \infty$.

(iii) If $\varphi = 0$, $f = 0$, $\dot{f} = 0$ on $N \subset \langle t_1, t_2\rangle$, from Eq. (3.11) follows $H = 0$, hence, again $\varphi = (H/K)h(f)\bar{h}(\dot{f})$.

(iv) If $\varphi = 0$, $f = 0$, $\dot{f} < 0$ for some $t_0 \in (t_1, t_2\rangle$, we get $f > 0$ for some $t_{00} < t_0$, hence, it is a contradiction with $\varphi < \infty$. If this situation takes place in $t = t_1$, we can again write $\varphi = (H/K)h(f)\bar{h}(\dot{f})$.

(v) The case $\varphi = 0, f = 0, \dot{f} > 0$ is similarly possible to (iv) only for $t = t_2$, and again $\varphi = (H/K)h(f)\bar{h}(\dot{f})$.

(vi) For $\varphi = 0, f < 0$ on $P \subset \langle t_1, t_2\rangle$ it is $\varphi = (H/K)h(f)\bar{h}(\dot{f})$. Note that $t_1 \cup M \cup N \cup \cup P \cup t_2 = \langle t_1, t_2\rangle$. The use of the relations (i) to (vi) in Eqs. (3.1) and (3.2) completes the proof.

REMARK. The lemma deals with the continuous processes in $\langle t_1, t_2\rangle$. However, the results can be extended to discontinuities in $\mathbf{C}_E$, θ, $\bar{\mathbf{g}}$. Consider a process in $\langle t_1, t_2\rangle$ with a discontinuity in $t_3 \in (t_1, t_2)$ such that the left-hand limits (i.e. $t \to t_{3-}$) and the right-hand limits (i.e. $t \to t_{3+}$) of $\dot{\mathbf{C}}_E$, $\dot{\theta}$, $\dot{\bar{\mathbf{g}}}$ exist and $\mathbf{C}_E$, θ, $\bar{\mathbf{g}}$ remain continuous in $\langle t_1, t_2\rangle$. First we use the lemma for $\langle t_1, t_3\rangle$ defining $\dot{\mathbf{C}}_E(t_3)$, $\dot{\theta}(t_3)$, $\dot{\bar{\mathbf{g}}}(t_3)$ as the corresponding left-hand limiting values. Then we use the final limiting value $\boldsymbol{\alpha}(t)$ as the initial value for the process in $\langle t_3, t_2\rangle$, defining now $\dot{\mathbf{C}}_E(t_3)$, $\dot{\theta}(t_3)$, $\dot{\bar{\mathbf{g}}}(t_3)$ as the corresponding right hand limits. It can be easily seen that in this case we obtain a discontinuity in $\dot{\boldsymbol{\alpha}}$ in $t = t_3$, but $\boldsymbol{\alpha}$ remains continuous. This follows from the form of Eqs. (3.6) and (3.7). In the visco-plastic case despite a discontinuity in $\dot{\mathbf{C}}_E$, $\dot{\theta}$, $\dot{\bar{\mathbf{g}}}$ both $\boldsymbol{\alpha}$ and $\dot{\boldsymbol{\alpha}}$ would remain continuous in t_3 (in the visco-plastic case $\dot{\boldsymbol{\alpha}}$ is a function of $\mathbf{C}_E$, θ, $\bar{\mathbf{g}}$, but not of $\dot{\mathbf{C}}_E$, $\dot{\theta}$, $\dot{\bar{\mathbf{g}}}$).

Using similar arguments as in the previous paper [8] we can get a special form of Eqs. (3.6) and (3.7) valid for isotropic elastic-plastic materials (for simplicity we assume $\bar{\mathbf{g}} \equiv 0$)

$$\mathbf{L}_P = \frac{H'}{K'} h(\bar{f})\bar{h}(\dot{\bar{f}})\overline{\mathbf{G}}\mathbf{B}_E, \tag{3.12}$$

$$\dot{\alpha}_i = \frac{H'}{K'} h(\bar{f})\bar{h}(\dot{\bar{f}})\operatorname{tr}[\overline{\mathbf{G}}\overline{\mathbf{A}}_i\mathbf{B}_E], \tag{3.13}$$

where $\mathbf{B}_E = \mathbf{E}\mathbf{E}^T$ is the left Cauchy-Green elastic tensor, $\bar{f}(\mathbf{B}_E, \theta, \boldsymbol{\alpha}) = \hat{f}(\mathbf{B}_E, \theta, \mathbf{0}, \boldsymbol{\alpha})$, $\overline{\mathbf{G}}(\mathbf{B}_E, \theta, \boldsymbol{\alpha}) = \hat{\mathbf{G}}(\mathbf{B}_E, \theta, \mathbf{0}, \boldsymbol{\alpha})$, $\overline{\mathbf{A}}_i(\mathbf{B}_E, \theta, \boldsymbol{\alpha}) = \hat{\mathbf{A}}_i(\mathbf{B}_E, \theta, \mathbf{0}, \boldsymbol{\alpha})$ and

$$H' = 2\operatorname{tr}\left[\frac{\partial \bar{f}}{\partial \mathbf{B}_E}\mathbf{B}_E\mathbf{L}\right] + \frac{\partial \bar{f}}{\partial \theta}\cdot\dot{\theta},$$

$$K' = 2\operatorname{tr}\left[\frac{\partial \bar{f}}{\partial \mathbf{B}_E}\overline{\mathbf{G}}\mathbf{B}_E^2\right] - \sum_{i=1}^{n}\frac{\partial \bar{f}}{\partial \alpha_i}\operatorname{tr}[\overline{\mathbf{G}}\overline{\mathbf{A}}_i\mathbf{B}_E].$$

Moreover, $\bar{f}$, $\bar{\mathbf{G}}$, $\bar{\mathbf{A}}_i$ and the partial derivatives of $\bar{f}$ are isotropic functions of $\mathbf{B}_E$, therefore they can be expressed as polynomials in $\mathbf{B}_E$, e.g. $\bar{f}(\mathbf{B}_E\theta, \boldsymbol{\alpha}) = f_0(\beta)$, $\partial\bar{f}/\partial\mathbf{B}_E = \varphi_0(\beta)1 + \varphi_1(\beta)\mathbf{B}_E + \varphi_2(\beta)\mathbf{B}_E^2$ where $\beta \equiv (\mathrm{I}, \mathrm{II}, \mathrm{III}, \theta, \boldsymbol{\alpha})$ is the set of the principal invariants I, II, and III of $\mathbf{B}_E$, the temperature θ, and the structural parameters $\alpha_1, \ldots, \alpha_n$, and $\varphi_0, \varphi_1, \varphi_2$ are scalar-valued functions.

The constitutive equations (3.6), (3.7) and (3.12), (3.13) are homogeneous in the first time derivatives, therefore the response of the material is independent of a change of time scale, i.e. the rate-independent theory of plastic materials is obtained. The derived constitutive equations (3.6), (3.7) or (3.12), (3.13) are of the form of the constitutive laws assumed in the theories of finite strain plasticity (e.g. [6, 7, 14]).

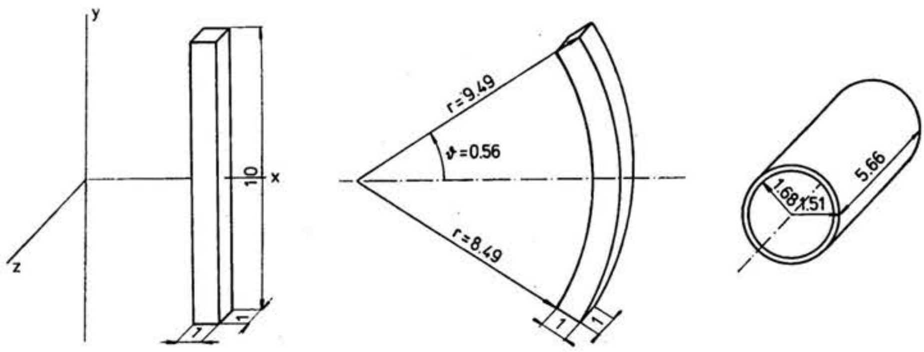
4. Large bending and extension of elastic-plastic block

The method of exact solutions of special inhomogeneous boundary value problems, known in finite elasticity for incompressible materials, can be modified for the present model. The method of solution, described in detail for the case of incompressible elastic-visco-plastic materials in [15], is of an inverse type. The deformation is specified fully at the outset and the problem is to find tractions which are necessary to maintain the deformation. This problem can be reduced to the problem of solving a system of ordinary differential equations.

As an illustration we consider large bending and extension of an isotropic, incompressible elastic-plastic block without including thermal effects. The constitutive equations are taken to be

$$\mathbf{T} = -p\mathbf{1} + \mathbf{S}, \quad \mathbf{S} = S_1\mathbf{B}_E + S_2\mathbf{B}_E^2, \tag{4.1}$$

together with Eqs. (3.12) and (3.13), where now the thermal quantities are excluded and the condition of plastic incompressibility $\operatorname{tr}\mathbf{L}_P = 0$ is obeyed. Moreover, only one scalar structural parameter α is considered. In Eq. (4.1) elastic incompressibility $\det\mathbf{B}_E = 1$ is respected, i.e. the Cauchy stress $\mathbf{T}$ is determined by $\mathbf{B}_E$ only to within an arbitrary hydro-



reference block t=1/3 t=3π/5

FIG. 1.

static pressure p, and $\mathbf{S}$ means the extra stress tensor. The scalar coefficients $\mathbf{S}_1, \mathbf{S}_2$ similarly as the scalar coefficient in the polynomial expressions for $\bar{f}$, $\overline{\mathbf{G}}$, etc., in Eqs. (3.12) and (3.13) are now scalar functions only of I, II and α.

The deformation that carries quasi-statically the block bounded by the planes $X = X_1$, $X = X_2$, $Y = \pm Y_0$, $Z = \pm Z_0$ into the annular wedge bounded by the cylinders $r = r_1 = (2AX_1)^{1/2}$, $r = r_2 = (2AX_2)^{1/2}$ and the planes $\vartheta = \pm\vartheta_0 = \pm BY_0$, $z = \pm z_0 = \pm Z_0/AB$ is described by (we use the rectangular Cartesian coordinates $(X^K) = (X, Y, Z)$ in the fixed reference configuration, and cylindrical coordinates $(x^K) = (r, v, z)$ in the current configuration, see Fig. 1)

$$r = (2A(t)X)^{1/2}, \quad \vartheta = B(t)Y, \quad z = Z/A(t)B(t), \tag{4.2}$$

where $A(t)$ and $B(t)$, $A(t)B(t) \neq 0$, are supposed to be continuously differentiable scalar functions of time. The tensors $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ and $\mathbf{B}_F = \mathbf{F}\mathbf{F}^T$ expressed in the physical components are then

$$L^{\langle kn\rangle} = \begin{bmatrix} \frac{\dot{A}}{2A} & -\frac{\vartheta\dot{B}}{B} & 0 \\ 0 & \frac{\dot{B}}{B}+\frac{\dot{A}}{2A} & 0 \\ 0 & 0 & -\frac{\dot{A}}{A}-\frac{\dot{B}}{B} \end{bmatrix}, \quad B_F^{\langle kn\rangle} = \begin{bmatrix} \frac{A^2}{r^2} & 0 & 0 \\ 0 & B^2r^2 & 0 \\ 0 & 0 & (AB)^{-2} \end{bmatrix}. \tag{4.3}$$

To calculate the corresponding stress field $\mathbf{T}$ from Eq. (4.1) we have to find $\mathbf{B}_E$ and α from Eq. (3.13) and the equation

$$\dot{\mathbf{B}}_E = \mathbf{L}\mathbf{B}_E + \mathbf{B}_E\mathbf{L}^T - 2\mathbf{L}_P\mathbf{B}_E, \tag{4.4}$$

where $\mathbf{L}_P$ is determined by Eq. (3.12) and $\mathbf{L}$ is given by Eq. $(4.3)_1$. Equation (4.4) follows from the definition $\mathbf{B}_E = \mathbf{E}\mathbf{E}^T$ and the decomposition rule (2.1). As $\mathbf{L}_P$ can be expressed as a polynomial in $\mathbf{B}_E$, we have $\mathbf{L}_P\mathbf{B}_E = \mathbf{B}_E\mathbf{L}_P$ and $\mathbf{L}_P = \mathbf{L}_P^T$. Due to the symmetry of $\mathbf{B}_F$ in Eq. $(4.3)_2$, only the diagonal components of $\mathbf{B}_E$ will be non-zero (for the proof see [15]), hence Eq. (4.4) is reduced to

$$\dot{B}_E^{\langle 11\rangle} = \frac{\dot{A}}{A}B_E^{\langle 11\rangle} - (L_P B_E)^{\langle 11\rangle}, \tag{4.5}$$

$$\dot{B}_E^{\langle 22\rangle} = \left(\frac{2\dot{B}}{B}+\frac{\dot{A}}{A}\right) B_E^{\langle 22\rangle} - (L_P B_E)^{\langle 22\rangle}. \tag{4.6}$$

From the incompressibility condition $\det \mathbf{B}_E = 1$ we get the component $B_E^{\langle 33\rangle}$. For known $\mathbf{B}_E$ and α the stress $\mathbf{T}$ is obtained from Eq. (4.1), where p is determined from the equilibrium equation of the quasi-static motion

$$p(r, t) = S^{\langle 11\rangle}(r, t) + \int_{r_1}^{r} \frac{S^{\langle 11\rangle}(\varrho, t) - S^{\langle 22\rangle}(\varrho, t)}{\varrho} d\varrho. \tag{4.7}$$

The normal force $N(t)$ on the faces perpendicular to the z-direction and the resultant moment $M(t)$ (the moment $M(t)$ is exerted by the normal stresses acting upon faces $\vartheta =$

$=$ const and is taken with respect to a point on the axis $r = 0$, a unit height is considered) that have to be applied to maintain the deformation are

$$(4.8) \qquad N(t) = 2\vartheta_0 \int_{r_1}^{r_2} T^{\langle 33\rangle}(\varrho, t)\varrho\, d\varrho, \quad M(t) = \int_{r_1}^{r_2} T^{\langle 22\rangle}(\varrho, t)\varrho\, d\varrho.$$

To give a numerical example the system (4.5), (4.6) and (3.13) was solved and the tractions evaluated from Eqs. (4.1), (4.7) and (4.8) for a simple model of the elasto-plastic material. For this elasto-plastic model we choose in Eq. (4.1) $\mathbf{S} = \mathbf{B}_E$ and in Eqs. (3.12), (3.13) $\bar{f} = \mathrm{II} - 10\alpha$, $\mathbf{G} = \mathbf{1} - (\mathrm{I}/3)\mathbf{B}_E^{-1}$, $\bar{\mathbf{A}} = \mathbf{1} + \mathbf{B}_E$, the initial value of $\boldsymbol{\alpha}$ is $\boldsymbol{\alpha}_0 = 1$. The calculation was performed for the motion (4.2), where $A(t) = t^{-2}$, $B(t) = t/3$ in the time interval $\langle 0.2, 2\rangle$ and $X_1 = 4$, $X_2 = 5$, $Y_0 = 5$, $Z_0 = 0.5$. The deformed body is "closest" to

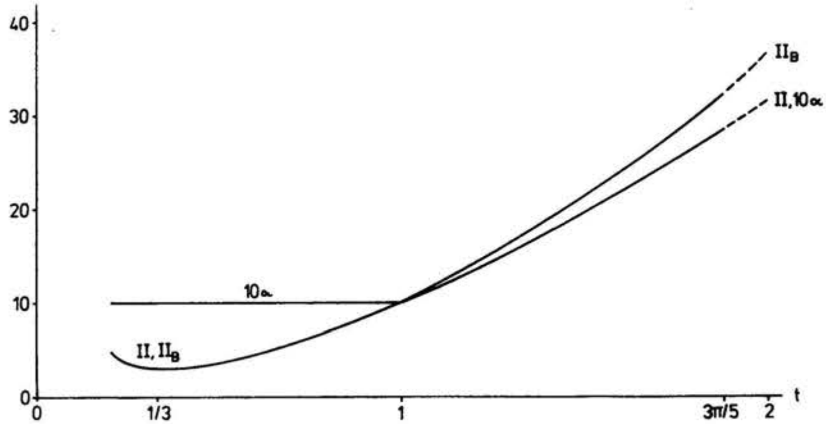


FIG. 2.

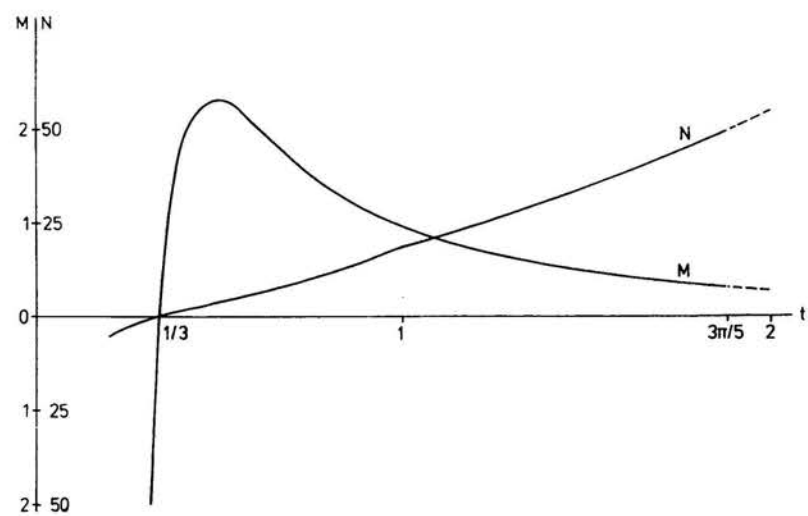


FIG. 3.

the shape of the reference rectangular block at $t = 1/3$. At $t = 3\pi/5$ the annular wedge is deformed to the complete annulus (see Fig. 1).

Figure 2 indicates that yielding begins to occur at $t = 1$. The corresponding values of the second invariant $\mathrm{II}(t)$ of $\mathbf{B}_E$, the second invariant $\mathrm{II}_B(t)$ of $\mathbf{B}_F\alpha(t)$, the force $N(t)$, and the moment $M(t)$ are shown in Figs. 2 and 3.

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References

1. TH. LEHMANN, *On the theory of large, non-isothermic, elastic-plastic and elastic-visco-plastic deformations*, Arch. Mech., **29**, 393, 1977.
2. P. PERZYNA, W. WOJNO, *Thermodynamics of a rate sensitive plastic material*, Arch. Mech, **20**, 499, 1968.
3. J. KRATOCHVÍL, O. W. DILLON, *Thermodynamics of elastic-plastic materials as a theory with internal state variables*, J. Appl. Phys. **40**, 3207, 1969.
4. J. NEČAS, L. TRÁVNÍČEK, *Evolutionary variational inequalities and applications in the plasticity*, Aplikace Matematiky [in press].
5. L. TRÁVNÍČEK, *The existence and uniqueness of boundary value problem for elasto-plastic solids*, Thesis, Charles University, Prague 1977 [in Czech].
6. TH. LEHMANN, *Einige Betrachtungen zur Thermodynamik grosser elasto-plastischer Formänderungen*, Acta Mechanica, **20**, 187, 1974.
7. E. H. LEE, *Elastic-plastic deformation at finite strains*, J. Appl. Mech., **36**, 1, 1969.
8. J. KRATOCHVÍL, *On a finite strain theory of elastic-inelastic materials*, Acta Mechanica, **16**, 127, 1973.
9. J. ZARKA, *Sur la viscoplasticité des métaux*, Mémorial de l'Artillerie francaise, **44**, 223, 1970.
10. J. R. RICE, *Inelastic constitutive relations for solids: an internal variable theory and its application to metal plasticity*, J. Mech. Phys. Solids, **19**, 433, 1971.
11. J. KRATOCHVÍL, O. W. DILLON, *Thermodynamics of crystalline elastic-visco-plastic materials*, J. Appl. Phys., **41**, 1470, 1970.
12. J. KRATOCHVÍL, R. J. DE ANGELIS, *Torsion of a titanium elasto-visco-plastic shaft*, J. Appl. Phys., **42**, 1091, 1971.
13. J. KRATOCHVÍL, *Thermodynamics of elastic-inelastic materials at finite strain*, in: Dynamika Osrodków Niesprężystych, P. PERZYNA (Editor), Ossolineum, Warsaw 1974.
14. J. MANDEL, *Director vectors and constitutive equations for plastic and visco-plastic media*, in: Problems of Plasticity, A. SAWCZUK (Editor), Noordhoff, Leyden 1974.
15. J. KRATOCHVÍL, *Dynamically possible finite deformations of isotropic, incompressible, elastic-inelastic solids with temperature independent response*, ZAMP, **23**, 949, 1972.

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