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# BRIEF NOTES

## Partial opening of a crack in an unbounded elastic medium

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UNBOUNDED elastic medium containing a crack is subject to the action of uniformly distributed pressure $q$ and two concentrated forces $P$ acting in the opposite directions. The problem of partial opening of the crack is analyzed as a function of the ratio $P/q$.

CONSIDER the plane state of strain in an unbounded elastic medium containing a horizontal crack $|x| < a$, $y = 0$ of length $2a$ (i.e. the crack has a form of a strip of breadth $2a$ extending to $\pm\infty$ in the direction of $z$); the crack remains closed under the action of a uniformly distributed compressive load $q$ applied at $y = \pm\infty$, Fig. 1. Let the medium

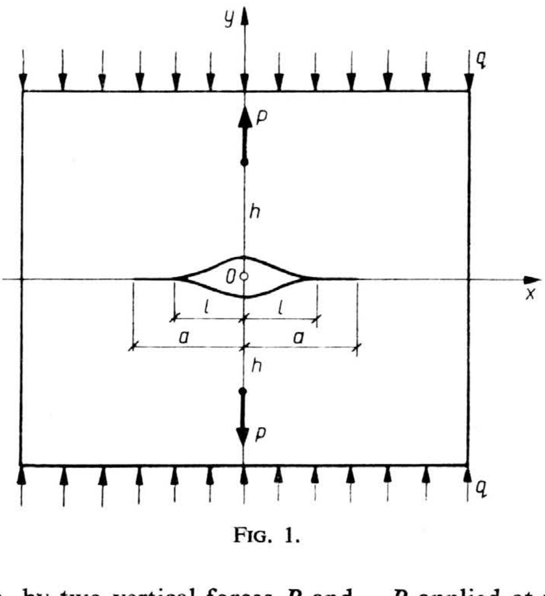


FIG. 1.

be loaded, in addition, by two vertical forces $P$ and $-P$ applied at the distances $h$ from the origin of the coordinate system $x$, $y$, $z$ and producing tensile stresses $\sigma_{yy}$ in the region between the points $(0, h)$ and $(0, -h)$ (Fig. 1).

Let us consider the following three questions:

1. What is the value $P_0$ of force $P$ at which the crack begins to open at point 0?

2. For $P > P_0$ the crack will open along the distance $2l$, between the points $(l, 0)$ and $(-l, 0)$. What is the dependence $l = l(P)$?

3. At which value $P_a$ of force $P$ the entire crack $2a$ will be opened?

To answer these questions, let us first solve the simple auxiliary problem which consists in determining the stress intensity factors (SIF) $K_I$ at the crack tips $x = \pm a$, $y = 0$ produced by simultaneous action of compressive loads $q$ acting at infinity and forces $P$, $-P$ applied at points $(0, h)$ and $(0, -h)$. The value of $K_I$ is found from the well-known formula (cf., e.g., Eq. (95) in [1])

$$K_I = \frac{1}{\sqrt{\pi a}} \int_{-a}^{a} p_2(x) \sqrt{\frac{a+x}{a-x}}\, dx, \tag{1}$$

where $p_2(x)$ is the distribution of stresses $\sigma_{yy}(x, 0)$ produced by external loads in a solid body (without the crack).

In the present case the function $p_2(x)$ is easily found to be (cf. [2])

$$p_2(x) = P\left[\frac{1-2\nu}{2\pi(1-\nu)} \frac{h}{x^2+h^2} + \frac{1}{\pi(1-\nu)} \frac{h^3}{(x^2+h^2)^2}\right] - q. \tag{2}$$

Substitution of expression (2) into (1) and introduction of a new variable

$$t = \sqrt{\frac{1+x/a}{1-x/a}}, \quad dx = \frac{4at\,dt}{(1+t^2)^2}$$

leads to the formula

$$K_I = -q\sqrt{\pi a} + \frac{P}{\pi\sqrt{\pi a}\,(1-\nu)} [(1-2\nu)\eta S_1 + 2\eta^3 S_2] \tag{3}$$

with the notations $\eta = h/a$, and

$$S_1 = \frac{1}{1+\eta^2} \int_{-\infty}^{\infty} \frac{t^2 dt}{t^4 + 2\dfrac{\eta^2-1}{\eta^2+1} t^2 + 1},$$

$$S_2 = \frac{1}{(1+\eta^2)^2} \int_{-\infty}^{\infty} \frac{t^2(t^2+1)^2 dt}{\left(t^4 + 2\dfrac{\eta^2-1}{\eta^2+1} t^2 + 1\right)^2}. \tag{4}$$

Integrals (4) may easily be evaluated, e.g. by means of the theorem of residues, in view of the fact that the integrands possess (simple or multiple) singular points

$$t = \frac{\pm(1 \pm i\eta)}{\sqrt{1+\eta^2}}.$$

On calculating the necessary residues, the following result is obtained (see also [2] in a slightly different notation):

$$K_I = -q\sqrt{a\pi} + \frac{P}{2\sqrt{\pi a}\,(1-\nu)} \frac{1}{\sqrt{1+\eta^2}} \left[(3-2\nu) - \frac{1}{1+\eta^2}\right]. \tag{5}$$

This result has no physical meaning for negative values of $K_I$ since then it would correspond to overlapping edges of the crack. If $K_I > 0$, the crack will open at its entire length. It follows from Eq. (5) that then $P > P_a$, where

(6)
$$P_a = \frac{2\pi aq(1-\nu)\sqrt{1+\eta^2}}{(3-2\nu)-1/(1+\eta^2)}.$$

At small distances $x-a$ from the crack tip $x = a$ the known Irwin's fromulae may be used (cf. e.g. [1]),

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{yx}\end{Bmatrix} = \frac{K_I}{\sqrt{2\pi(x-a)}}\cos\theta\begin{Bmatrix}1-\sin\theta/2\sin3\theta/2\\ 1+\sin\theta/2\sin3\theta/2\\ \sin\theta/2\cos3\theta/2\end{Bmatrix}.$$

Vanishing of the SIF $K_I$ creates the possibility of establishing the conditions under which only a part of the crack will remain open. Denote the length of the opened portion of the crack by $2l$ (the problem is symmetric with respect to the $y$-axis), and the ratio $h/l = \lambda$. Then, Eq. (5) may be used to write the condition $K_I = 0$ in the form

(7)
$$2\pi\frac{qh}{P} = F(\lambda),$$
$$F(\lambda) = \frac{1}{1-\nu}\frac{\lambda}{\sqrt{1+\lambda^2}}\left[(3-2\nu)-\frac{1}{1+\lambda^2}\right].$$

$F$ is a monotone increasing function of variable $\lambda = h/l$; its variation from $F(0) = 0$ to $F(\infty) = (3-2\nu)/(1-\nu)$ is shown in Fig. 2.

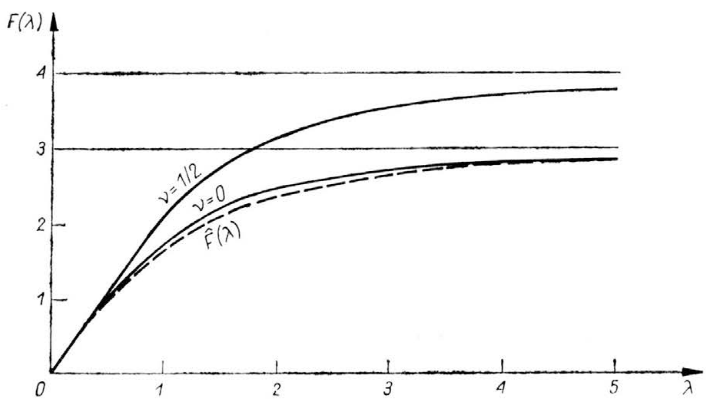


FIG. 2.

Assume the value of $q$ to be given and denote the expression $2\pi qh/P$ by $Q$. It is seen from Eq. (7) and Fig. 2 that for $Q > (3-2\nu)/(1-\nu)$ the entire crack remains closed since either the force $P$ is too small or the distance $h$ is too large. At $Q_0 = 2\pi qh/P_0 = (3-2\nu)/(1-\nu)$ Eq. (7) yields the result $\lambda = h/l \to \infty$, what means that an infinitesimal opening of the crack appears at $x = y = 0$. By increasing the value of $P$ (or by reducing the distance $h$), the crack opening length increases and reaches a certain value $l = l(Q)$. This function may be written in the form

(8)
$$l = \frac{h}{F^{-1}(Q)},$$

where $F^{-1}$ denotes the function inverse to $F$ given by Eq. (7).

To demonstrate the general character of that dependence, assume $\nu = 0$ in Eq. (7), i.e.

$$F(\lambda) = \frac{\lambda}{\sqrt{1+\lambda^2}}\left(3-\frac{1}{1+\lambda^2}\right).$$

Since $F$ is a monotone increasing function of $\lambda$, $F(0) = 0$, $dF/d\lambda|_{\lambda=0} = 2$, and $\lim_{\lambda\to\infty} F(\lambda) = 3$, it may be approximated by a simpler function

$$\hat{F}(\lambda) = \frac{2\lambda}{\sqrt{1+\frac{4}{9}\lambda^2}}$$

(dashed line in Fig. 2). Then Eq. (8) assumes the approximate form

$$l \approx \frac{P}{\pi q}\sqrt{1-Q^2/9}, \quad Q = 2\pi qh/P.$$

Graph of this approximate relation is shown in Fig. 3 where $\lambda^{-1} = l/h$ is shown as a function of the ratio $P/qh$.

As long as $l$ found from this formula is smaller than $a$, the SIF at tips $x = \pm l$ vanish so that the solution may be considered as corresponding to a partly closed cracks of

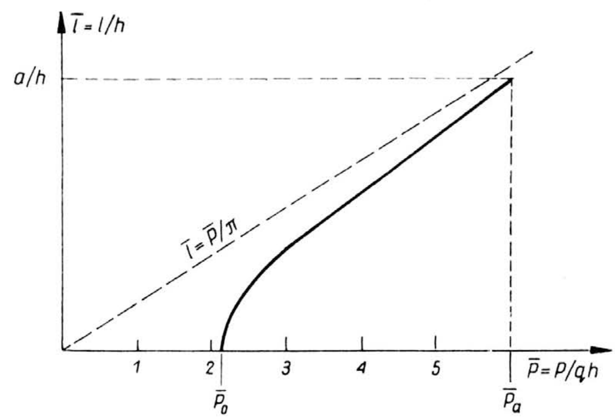


FIG. 3.

length 2a, provided the shearing stresses $\sigma_{xy}$ vanish along the crack edges and normal stresses $\sigma_{yy}$ at $l < |x| < a$ are compressive. Let us verify whether these conditions are satisfied.

Following the formula derived in [2] (pp. 462 and 470), stresses produced in the medium (containing a crack $2l$) by concentrated forces $P$, $-P$ and distributed loads $q$, $-q$ (Fig. 1) are

$$(9) \qquad \sigma_{yy}(\xi, 0) = \frac{P\xi(\xi^2-1)^{-1/2}}{2\pi l(1-\nu)} f(\xi) - \frac{q\xi}{\sqrt{\xi^2-1}},$$

where $\xi = x/l$, and

$$f(\xi) = (3-2\nu)\frac{\sqrt{1+\lambda^2}}{\xi^2+\lambda^2} - \frac{\xi^2+2\lambda^2\xi^2-\lambda^2}{(\xi^2+\lambda^2)^2}\frac{1}{\sqrt{1+\lambda^2}}.$$

Substituting here for $q$ the value obtained from Eq. (7),

$$q = \frac{P}{2\pi l(1-\nu)}\frac{1}{\sqrt{1+\lambda^2}}\left[(3-2\nu)-\frac{1}{1+\lambda^2}\right], \tag{10}$$

the following result is derived

$$\sigma_{yy}(\xi, 0) = -\frac{P\xi\sqrt{\xi^2-1}\,G(\xi)}{(\xi^2+\lambda^2)^2 2\pi l(1-\nu)(1+\lambda^2)^{2/3}}, \tag{11}$$

where

$$G(\xi) = (2-2\nu)\xi^2+(3-2\nu)\xi^2\lambda^2+(4-2\nu)\lambda^2+(5-2\nu)\lambda^4.$$

It is seen that the stress singularities appearing at $\xi = 1$ have really vanished.

Similarly, displacement $v^+(\xi, 0)$ of the upper edge of the crack for $|\xi| < 1$ produced by forces $P$ ([2], p. 470) is

$$v_P^+(\xi, 0) = \frac{P}{2\pi\mu}\left[(1-\nu)\log\frac{1+\sqrt{\dfrac{1-\xi^2}{1+\lambda^2}}}{1-\sqrt{\dfrac{1-\xi^2}{1+\lambda^2}}} + \frac{\lambda^2}{\sqrt{1+\lambda^2}}\frac{\sqrt{1-\xi^2}}{\xi^2+\lambda^2}\right] \tag{12}$$

and the displacement due to the uniform compression to be superposed on (12) is

$$v_q^+(\xi, 0) = -\frac{q(1-\nu)}{\mu}l\sqrt{1-\xi^2}. \tag{13}$$

Substituting for $q$ the expression (10), assuming that $\sqrt{1-\xi^2} \to 0$ and expanding the function

$$\log\frac{1+\sqrt{(1-\xi^2)/(1+\lambda^2)}}{1-\sqrt{(1-\xi^2)/(1+\lambda^2)}} = 2\sqrt{\frac{1-\xi^2}{1+\lambda^2}} + 0[(1-\xi^2)^{3/2}],$$

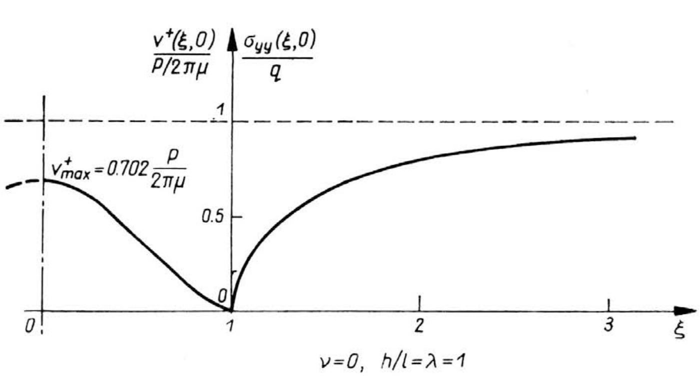


FIG. 4.

it is found that the terms involving $\sqrt{1-\xi^2}$ which appeared in Eqs. (12) and (13) cancel each other and the crack closes smoothly at the tips. This is true as long as $l < a$, i.e. the crack remains partly opened.

The diagrams shown in Fig. 4 demonstrate the crack opening (which is positive, i.e. the crack surfaces do not overlap), and the distribution of stresses along the closed segment of the crack (the stresses are compressive).

A procedure similar to that outlined above may also be applied to the analysis of more complicated cases of media containing cracks and subject to complex loads; in several cases this highly nonlinear contact problem may be solved in this simple manner.

## References

1. J. R. Rice, *Mathematical analysis in the mechanics of fracture*, in: Fracture, H. Liebowitz (ed.), Academic Press, New York—London 1968.
2. M. Matczyński, *The static problem of a crack in an elastic medium*, Arch. Mech., **22**, 4, 1970.

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