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Homogenization of fissured Reissner-like plates(*)
Part III. Some particular cases and illustrative example

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THE LAST part of the paper deals with three important particular cases of homogenization of fissured Reissner-like plates. An illustrative example of homogenization of a plate weakened by fissures distributed parallely to one of the in-plane coordinate axes is also given

W ostatniej części pracy rozpatrzono trzy ważne przypadki szczególne homogenizacji spękanych płyt Reissnera. Podano również konkretny przykład homogenizacji płyty osłabionej szczelinami równoległymi do jednej z osi płaszczyzny płyty.

В последней части работы рассмотрены три важные частные случаи гомогенизации пластин Рейсснера с трещинами. Приведен конкретный пример гомогенизации пластины с трещинами, параллельными одной из ее осей.

1. Introduction

In the first part of the paper [3] the method of two-scale asymptotic expansions was used to derive the general formulae describing various homogenized models of Reissner-like plates. From a rigorously mathematical viewpoint such an approach is formal. A rigorous study of the convergence as $\varepsilon \to 0$ has been presented in Part II, [4].

The last part of the paper is concerned with two problems. First, in Sect. 2, we discuss three particular cases of homogenization of fissured plates. These cases are determined by the bending cracking mode, shear cracking mode and tension cracking mode. For the first case we show that the curvature tensor influences the membrane forces and vice-versa, the moment tensor is influenced by the in-plane strain tensor.

Section 3 presents an example of homogenization of the Reissner-like plate weakened by fissures distributed parallely to one of the in-plane coordinate axes. Flexural fissures are studied. Though the fissures considered are not microfissures and the basic cell not connected, yet the derived homogenization formulae can effectively be applied.

Roman numerals refer to the relevant sections, equations, references and figures of the first and second part of the paper. The same notations as previously will be used here.

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2. Some particular cases

Basic cell problems for all admissible fissuring modes have been formulated in [3] as a set of variational inequalities ($\mathscr{P}^{\alpha}_{\text{loc}}$), $\alpha = 1, 2$, cf. formulae ((I.3.17)–(I.3.28)). Below we shall investigate particular cases of these problems connected with the three cracking modes which have been considered in Sect. I.2 and illustrated by Figs. I.3, I.4 and I.5.

2.1. Bending cracking mode (Fig. I.3)

Let us first examine the local problem $\mathscr{P}^1_{\text{loc}}$. It now reads

find $\mathbf{v}^1 \in [H^1_{\text{per}}(Y)]^2$, $\boldsymbol{\varphi}^1 \in K^{\text{bc}}$ such that

$$(2.1)\qquad \int\limits_{YF} \big(G_{\alpha\beta\lambda\mu}\varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1) + E_{\alpha\beta\lambda\mu}\gamma^y_{\lambda\mu}(\mathbf{v}^1)\big)\varrho^y_{\alpha\beta}(\boldsymbol{\psi}-\boldsymbol{\varphi}^1)\,dy \geqslant L_2(\boldsymbol{\psi}-\boldsymbol{\varphi}^1)\;\forall\boldsymbol{\psi}\in K^{\text{bc}}_{YF},$$

$$(2.2)\qquad \int\limits_{YF} \big(A_{\alpha\beta\lambda\mu}\gamma^y_{\lambda\mu}(\mathbf{v}^1) + E_{\alpha\beta\lambda\mu}\varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1)\big)\gamma^y_{\alpha\beta}(\mathbf{z})\,dy = L_1(\mathbf{z}), \qquad \forall\,\mathbf{z}\in[H^1_{\text{per}}(Y)]^2.$$

Since the local displacement field $\mathbf{v}^1$ is now an element of the function space, the variational inequality (I.3.17) takes the form of the variational equation (2.2).

The localization of (2.1) and (2.2) yields

$$(2.3)\qquad \frac{\partial}{\partial y_\beta}\left(A_{\alpha\beta\lambda\mu}\frac{\partial v^1_\lambda}{\partial y_\mu} + E_{\alpha\beta\lambda\mu}\frac{\partial \varphi^1_\lambda}{\partial y_\mu}\right) = 0 \quad \text{in} \quad YF,$$

$$(2.4)\qquad \frac{\partial}{\partial y_\beta}\left(E_{\alpha\beta\lambda\mu}\frac{\partial v^1_\lambda}{\partial y_\mu} + G_{\alpha\beta\lambda\mu}\frac{\partial \varphi^1_\lambda}{\partial y_\mu}\right) = 0 \quad \text{in} \quad YF.$$

More information on localization the reader may find in Part II. Multiplying Eq. (2.3) by e and knowing that $E_{\alpha\beta\lambda\mu} = eA_{\alpha\beta\lambda\mu}$, $G_{\alpha\beta\lambda\mu} = e^2A_{\alpha\beta\lambda\mu} + D_{\alpha\beta\lambda\mu}$ we readily arrive at

$$(2.5)\qquad \frac{\partial}{\partial y_\beta}\left(A_{\alpha\beta\lambda\mu}\frac{\partial v^1_\lambda}{\partial y_\mu}\right) = 0 \quad \text{in} \quad YF,$$

$$(2.6)\qquad \frac{\partial}{\partial y_\beta}\left(A_{\alpha\beta\lambda\mu}\frac{\partial \varphi^1_\lambda}{\partial y_\mu}\right) = 0 \quad \text{in} \quad YF.$$

Equation (2.5) implies

$$(2.7)\qquad a_Y(\mathbf{v}^1, \mathbf{v}^1) = 0 \Rightarrow \mathbf{v}^1 = \mathbf{v}^1(x).$$

Thus the local problem (2.1) and (2.2) reduces to

find $\boldsymbol{\varphi}^1 \in K^{\text{bc}}_{YF}$ such that

$$(2.8)\qquad a_G(\boldsymbol{\varphi}^1, \boldsymbol{\psi}-\boldsymbol{\varphi}^1) \geqslant L_2(\boldsymbol{\psi}-\boldsymbol{\varphi}^1)\;\forall\,\boldsymbol{\psi}\in K^{\text{bc}}_{YF} \qquad (\mathscr{P}^{\text{ben}}_{\text{loc}}).$$

Similarly, it can easily be shown that in the case considered the local problem $\mathscr{P}^2_{\text{loc}}$ is trivial, that is $w^1 = w^1(x)$. The elastic potential W is

$$(2.9)\qquad W(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega}) = \frac{1}{2|Y|}\int\limits_{YF} \{A_{\alpha\beta\lambda\mu}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\lambda\mu} + 2E_{\alpha\beta\lambda\mu}\,\varepsilon_{\alpha\beta}[\varkappa_{\lambda\mu} + \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}))]$$

(2.9) [cont.]
$$+G_{\alpha\beta\lambda\mu}[\varkappa_{\alpha\beta}+\varrho^{y}_{\alpha\beta}(\boldsymbol{\varphi}^1(\boldsymbol{\epsilon},\boldsymbol{\varkappa}))][\boldsymbol{\varkappa}_{\lambda\mu}+\varrho^{y}_{\lambda\mu}(\boldsymbol{\varphi}^1(\boldsymbol{\epsilon},\boldsymbol{\varkappa}))]+H_{\alpha\beta}\omega_\alpha\omega_\beta\}dy.$$

The homogenized constitutive equations are

$$\mathfrak{N}_{\alpha\beta} = A_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}+\frac{1}{|Y|}\int\limits_{YF} E_{\alpha\beta\lambda\mu}[\varkappa_{\lambda\mu}+\varrho^{y}_{\lambda\mu}(\boldsymbol{\varphi}^1(\boldsymbol{\epsilon},\boldsymbol{\varkappa}))]dy, \tag{2.10}$$

$$\mathfrak{M}_{\alpha\beta} = E_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}+\frac{1}{|Y|}\int\limits_{YF} G_{\alpha\beta\lambda\mu}[\varkappa_{\lambda\mu}+\varrho^{y}_{\lambda\mu}(\boldsymbol{\varphi}^1(\boldsymbol{\epsilon},\boldsymbol{\varkappa}))]dy, \tag{2.11}$$

$$\mathfrak{Q}_\alpha = H_{\alpha\beta}\omega_\beta \quad \text{(unchanged)}. \tag{2.12}$$

The homogenized constitutive equations (2.10) and (2.11) are coupled. To corroborate this statement let us return to the mid-plane displacement $\mathbf{u}^0 = \mathbf{v}^0+e\boldsymbol{\varphi}^0$. Taking account of (I.2.31) we obtain the identities

$$A_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}+E_{\alpha\beta\lambda\mu}\varkappa_{\lambda\mu} = A_{\alpha\beta\lambda\mu}\tilde{\gamma}_{\lambda\mu}(\mathbf{u}^0), \tag{2.13}$$

$$E_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}+G_{\alpha\beta\lambda\mu}\varkappa_{\lambda\mu} = D_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^0)+E_{\alpha\beta\lambda\mu}\tilde{\gamma}_{\lambda\mu}(\mathbf{u}^0). \tag{2.14}$$

The bending moment ($\tilde{\mathfrak{M}}_{\alpha\beta}$), referred to the mid-plane of the homogenized plate, can be written as follows (see Eq. (I.2.42)):

$$\tilde{\mathfrak{M}}_{\alpha\beta} = \mathfrak{M}_{\alpha\beta}-e\mathfrak{N}_{\alpha\beta}. \tag{2.15}$$

Then

$$\mathfrak{N}_{\alpha\beta} = A_{\alpha\beta\lambda\mu}\left[\tilde{\gamma}_{\lambda\mu}(\mathbf{u}^0)+\frac{e}{|Y|}\int\limits_{YF}\varrho^{y}_{\lambda\mu}(\boldsymbol{\varphi}^1(\boldsymbol{\epsilon},\boldsymbol{\varkappa}))dy\right], \tag{2.16}$$

$$\tilde{\mathfrak{M}}_{\alpha\beta} = D_{\alpha\beta\lambda\mu}\left[\varrho_{\lambda\mu}(\boldsymbol{\varphi}^0)+\frac{1}{|Y|}\int\limits_{YF}\varrho^{y}_{\lambda\mu}(\boldsymbol{\varphi}^1(\boldsymbol{\epsilon},\boldsymbol{\varkappa}))dy\right]. \tag{2.17}$$

Coupling is absent if $e = 0$. Otherwise the curvature tensor influences the membrane forces ($\mathfrak{N}_{\alpha\beta}$) and vice versa, the moment tensor is influenced by the in-plane Ω strain tensor $\boldsymbol{\epsilon}$.

We pass to the strong formulation of the local problem $\mathscr{P}^{\text{ben}}_{\text{loc}}$. The localization procedure is similar to that used in our paper [2] and in the second part of the present contribution. Therefore we shall only adduce the final results provided that all functions are sufficiently regular. The local problem $\mathscr{P}^{\text{ben}}_{\text{loc}}$ yields the following relations and conditions:

(i) the equilibrium equation

$$G_{\alpha\beta\lambda\mu}\frac{\partial^2\varphi^1_\lambda}{\partial y_\beta\partial y_\mu} = 0 \quad \text{in} \quad YF; \tag{2.18}$$

(ii) $\boldsymbol{\varphi}^1$ is Y-periodic;

(iii) $m_{\alpha\beta}n_\alpha$ are opposite at the opposite sides of the basic cell Y, where

$$m_{\alpha\beta} = G_{\alpha\beta\lambda\mu}\varrho^{y}_{\lambda\mu}(\boldsymbol{\varphi}^1)+m^0_{\alpha\beta},\ m^0_{\alpha\beta} = E_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}+G_{\alpha\beta\lambda\mu}\varkappa_{\lambda\mu}, \tag{2.19}$$

(iv) the internal Signorini-type conditions

$$\text{(2.20)} \qquad [\![\varphi_N^1]\!] \geqslant 0, \quad \overset{1}{m}_N = \overset{2}{m}_N = m_N \leqslant 0, \quad m_N[\![\varphi_N^1]\!] = 0, \quad \text{on} \quad F,$$

where $\overset{\sigma}{m}_N = m_{\alpha\beta}|_\sigma N_\alpha N_\beta$.

The proof of the above assertions is based on the identity

$$\text{(2.21)} \qquad a_G(\boldsymbol{\varphi}^1, \boldsymbol{\psi}) - L_2(\boldsymbol{\psi}) = \int\limits_{YF} G_{\alpha\beta\lambda\mu} \frac{\partial^2 \varphi_\lambda^1}{\partial y_\alpha \partial y_\mu} \psi_\beta dy + \sum_{A=1}^{4} \int\limits_{\Gamma_A} m_{\alpha\beta} n_\beta \psi_\alpha ds$$

$$+ \int\limits_F [(\overset{1}{m}_N \overset{1}{\psi}_N - \overset{2}{m}_N \overset{2}{\psi}_N) + (\overset{1}{m}_T \overset{1}{\psi}_T - \overset{2}{m}_T \overset{2}{\psi}_T)] ds,$$

where

$$\overset{\sigma}{m}_T = m_{\alpha\beta}|_\sigma N_\alpha T_\beta, \quad \overset{\sigma}{\psi}_N = \psi_\alpha|_\sigma N_\alpha, \quad \overset{\sigma}{\psi}_\Gamma = \psi_\alpha|_\sigma T_\alpha.$$

2.2. Shear cracking mode (Fig. I.4)

A simple analysis of the local problem $\mathscr{P}_{\text{loc}}^1$ shows that now $\boldsymbol{\varphi}^1 = \boldsymbol{\varphi}^1(x)$ and $\mathbf{v}^1 = \mathbf{v}^1(x)$. The local transverse displacement w^1 is a solution of the local problem $\mathscr{P}_{\text{loc}}^2$ which now reads

find $w^1 \in K_{YF}^{\flat}$ such that

$$\text{(2.22)} \qquad a_H(w^1, u - w^1) \geqslant L_3(u - w^1)\ \forall\, u \in K_{YF}^{\flat} \qquad (\mathscr{P}_{\text{loc}}^{\text{shear}})$$

The homogenized constitutive equations are

$$\text{(2.23)} \qquad \mathfrak{N}_{\alpha\beta} = A_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu} + E_{\alpha\beta\lambda\mu}\varkappa_{\lambda\mu},$$

$$\text{(2.24)} \qquad \mathfrak{M}_{\alpha\beta} = E_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu} + G_{\alpha\beta\lambda\mu}\varkappa_{\lambda\mu},$$

$$\text{(2.25)} \qquad \mathfrak{Q}_\alpha = \frac{1}{|Y|} \int\limits_{YF} H_{\alpha\beta}\left(\omega_\beta + \frac{\partial w^1(\boldsymbol{\omega})}{\partial y_\beta}\right) dy.$$

Similarly as before, displacement $\mathbf{u}^0 = \mathbf{v}^0 + e \cdot \boldsymbol{\varphi}^0$ can be used instead of $\mathbf{v}^0$. Then Eqs. (2.23) and (2.24) transform, respectively, into

$$\text{(2.26)} \qquad \mathfrak{N}_{\alpha\beta} = A_{\alpha\beta\lambda\mu}\tilde{\gamma}_{\lambda\mu}(\mathbf{u}^0),$$

$$\text{(2.27)} \qquad \widetilde{\mathfrak{M}}_{\alpha\beta} = D_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^0).$$

Obviously, Eq. (2.25) remains unchanged. Thus we see that the problem considered is unaffected by e and we can put $e = 0$. Stretching and bending effects are uncoupled.

2.3. Tension cracking mode (Fig. I.5)

Now $[\![\varphi_n^\varepsilon]\!] = 0$ and hence $[\![v_n^\varepsilon]\!] = [\![u_n^\varepsilon]\!]$ on F^ε. As previously, the problem is unaffected by e and we can assume $e = 0$. The only unilateral kinematical condition is imposed by

$[\![v_n^\varepsilon]\!] \geqslant 0$. Local problems $\mathscr{P}_{\mathrm{loc}}^{\alpha}$ imply that $w^1 = w^1(x)$ and $\boldsymbol{\varphi}^1 = \boldsymbol{\varphi}^1(x)$. The local displacement $\mathbf{u}^1 = \mathbf{v}^1$ is a solution of the following variational inequality

$$\left.\begin{array}{l}\text{find } \mathbf{u}^1 \in K_{YF}^{\mathrm{pc}} \text{ such that} \\ a_A(\mathbf{u}^1, \mathbf{z}-\mathbf{u}^1) \geqslant L_1(\mathbf{z}-\mathbf{u}^1)\ \forall\, \mathbf{z} \in K_{YF}^{\mathrm{pc}}\end{array}\right| (\mathscr{P}_{\mathrm{loc}}^{\mathrm{ten}}), \tag{2.28}$$

where

$$L_1(\mathbf{z}) = -\int_{YF} A_{\alpha\beta\lambda\mu}\,\varepsilon_{\lambda\mu}\,\gamma_{\alpha\beta}^{y}(\mathbf{z})\,dy. \tag{2.29}$$

The homogenized constitutive equations are given by

$$\mathfrak{N}_{\alpha\beta} = \frac{1}{|Y|}\int_{YF} A_{\alpha\beta\lambda\mu}\Big(\varepsilon_{\lambda\mu} + \gamma_{\lambda\mu}^{y}\big(\mathbf{u}^1(\boldsymbol{\epsilon})\big)\Big)\,dy, \tag{2.30}$$

$$\mathfrak{M}_{\alpha\beta} = D_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}, \qquad \mathfrak{Q}_\alpha = H_{\alpha\beta}\omega_\beta. \tag{2.31}$$

Similarly as in the preceding case, the stretching and bending effects are uncoupled. The homogenized plate problem is conventional, that is linear within the framework of moderately thick plates. On the other hand, the homogenized membrane problem is nonlinear.

Finally we observe that other cases may be investigated in a similar way.

3. Example. Homogenization of a plate weakened by fissures of the bending type and periodically distributed along parallel lines

Let us consider an elastically isotropic homogeneous plate weakened by fissures of the bending type, distributed along parallel lines $x_2 = \mathfrak{n}\varepsilon b$, $\mathfrak{n} = \pm 1, \pm 2, \ldots$, see Fig. 1.

In this case the fissures intersect the boundary $\Gamma = \partial\Omega$ of the plate; thus the boundary of the $Y_{\varepsilon,i}$ cells with dimensions $a \times \varepsilon b$ is intersected. Therefore the assumption concerning connectedness is violated, see Sect. I.3. Nevertheless it is meaningful to homogenize such a plate by using the results of the first part of the paper. We observe that the plate considered is not kinematically variable since it is clamped along Γ.

Figure 1 b represents the geometry of the basic cell. We observe that the distance l_1 determining the position of fissure F may be arbitrary provided that $0 < l_1 < b$. A similar homogenization problem has been solved in our paper [2]. However, only the simplest Kirchhoff plate has been examined there. Therefore discussion of the present case will be limited to general formulations and results, detailed calculations being omitted.

Now the strong formulation of the local problem or the problem posed on the basic cell Y consists in finding a function $\boldsymbol{\varphi}^1 = (\varphi_\alpha^1) \in [C^2(YF)]^2$ satisfying, cf. Eqs. (2.18)–(2.20).

(i) Equilibrium equations

$$\nabla^2\varphi_\alpha^1 + \frac{1+\nu}{1-\nu}\varphi_{\beta,\beta\alpha}^1 = 0 \quad \text{in} \quad YF,$$

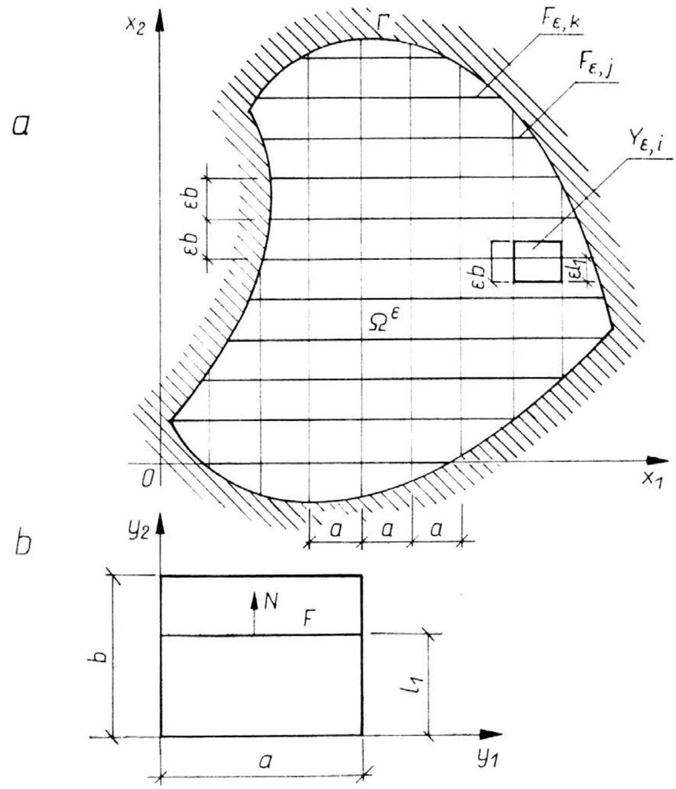

FIG. 1. Plate weakened by fissures periodically distributed along the lines $x_2 = n\varepsilon b$, $n = \pm 1, \pm 2, \ldots$

where ν is Poisson's ratio. The above form of equilibrium equations is a consequence of isotropy of the material of the plate considered, since

$$G_{\alpha\beta\lambda\mu} = G\left(\nu\delta_{\alpha\beta}\delta_{\lambda\mu} + \frac{1-\nu}{2}(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\lambda})\right),$$

where

$$G = \frac{\bar{e}\bar{R}}{L^2(1-\nu^2)}, \quad \bar{R} = \bar{e} + \frac{h^2}{12\bar{e}}.$$

(ii) Periodicity conditions

$$\begin{aligned} \varphi^1_\alpha(y_1, 0) &= \varphi^1_\alpha(y_1, b), & \varphi^1_\alpha(0, y_2) &= \varphi^1_\alpha(a, y_2), \\ m_{11}(0, y_2) &= m_{11}(a, y_2), & m_{22}(y_1, 0) &= m_{22}(y_1, b), \\ m_{12}(y_1, 0) &= m_{12}(y_1, b), & m_{21}(0, y_2) &= m_{21}(a, y_2), \end{aligned}$$

where

$$0 \leqslant y_1 \leqslant a, \quad 0 \leqslant y_2 \leqslant b.$$

(iii) Continuity conditions imposed by the bending mode considered

$$\begin{aligned} \varphi^1_1(y_1, l_1-0) &= \varphi^1_1(y_1, l_1+0), \\ m_{21}(y_1, l_1-0) &= m_{21}(y_1, l_1+0), \\ m_N := m_{22}(y_1, l_1-0) &= m_{22}(y_1, l_1+0) \end{aligned}$$

(iv) Internal Signorini-type conditions on F, viz. for $y_2 = l_1$

$$[\![\varphi_2^1]\!] \geqslant 0, \quad m_N \leqslant 0, \quad m_N[\![\varphi_2^1]\!] = 0.$$

Due to isotropy, the local bending moments are

$$m_{\alpha\beta} = G[(1-\nu)\varrho^y_{\alpha\beta}(\boldsymbol{\varphi}^1)+\nu\delta_{\alpha\beta}\varrho^y_{\lambda\lambda}(\boldsymbol{\varphi}^1)]+m^0_{\alpha\beta},$$

where

$$m^0_{\alpha\beta} = G\left[(1-\nu)\left(\varkappa_{\alpha\beta}+\frac{\varepsilon_{\alpha\beta}}{R}\right)+\nu\delta_{\alpha\beta}\left(\varkappa_{\lambda\lambda}+\frac{\varepsilon_{\lambda\lambda}}{R}\right)\right],$$

and $R = \overline{R}/H$.

It can be shown that the periodic function $\boldsymbol{\varphi}^1$ satisfying (i)–(iv) has the form

$$\boldsymbol{\varphi}^1 = \begin{cases} 0, & \text{if} \quad m^0_{22} \leqslant 0, \\ y_2\underline{\mathfrak{m}}_1+\underline{\mathfrak{m}}_2, & \text{if} \quad y_2 \in [0, l_1) \quad \text{and} \quad m^0_{22} > 0, \\ y_2\underline{\mathfrak{m}}_1, & \text{if} \quad y_2 \in (l_1, b] \quad \text{and} \quad m^0_{22} > 0, \end{cases}$$

where

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \underline{\mathfrak{m}}_1 = \begin{bmatrix} 0 \\ -m^0_{22}/G \end{bmatrix}, \quad \underline{\mathfrak{m}}_2 = \begin{bmatrix} 0 \\ -m^0_{22}b/G \end{bmatrix}$$

and

$$\frac{m^0_{22}}{G} = (\varkappa_{22}+\nu\varkappa_{11})+\frac{1}{R}(\varepsilon_{22}+\nu\varepsilon_{11}).$$

Hence we obtain

$$\varrho^y_{\alpha\beta}(\boldsymbol{\varphi}^1) = \begin{cases} 0, & \text{if} \quad m^0_{22} \leqslant 0, \\ -\delta_{\alpha 2}\delta_{\beta 2}m^0_{22}/G, & \text{if} \quad m^0_{22} > 0. \end{cases}$$

The homogenized constitutive equations (2.10) defining the membrane forces take now the form

$$\mathfrak{N}_{11} = \begin{cases} B[\varepsilon_{11}+\nu\varepsilon_{22}+e(\varkappa_{11}+\nu\varkappa_{22})], & \text{if} \quad (\varkappa_{22}+\nu\varkappa_{11})+(\varepsilon_{22}+\nu\varepsilon_{11})/R \leqslant 0, \\ B\left[\left(1-\nu^2\dfrac{e}{R}\right)\varepsilon_{11}+\nu\left(1-\dfrac{e}{R}\right)\varepsilon_{22}\right]+eB(1-\nu^2)\varkappa_{11}, & \text{otherwise}, \end{cases}$$

$$\mathfrak{N}_{12} = \mathfrak{N}_{21} = B(1-\nu)\varepsilon_{12}+eB(1-\nu)\varkappa_{12},$$

$$\mathfrak{N}_{22} = \begin{cases} B[\varepsilon_{22}+\nu\varepsilon_{11}+e(\varkappa_{22}+\nu\varkappa_{11})], & \text{if} \quad (\varkappa_{22}+\nu\varkappa_{11})+(\varepsilon_{22}+\nu\varepsilon_{11})/R \leqslant 0 \\ B\left(1-\dfrac{e}{R}\right)(\varepsilon_{22}+\nu\varepsilon_{11}), & \text{otherwise}, \end{cases}$$

where

$$B = \frac{H^2}{L^2(1-\nu^2)} = \frac{G}{eR}.$$

The homogenized bending moments (2.11) now are

$$\mathfrak{M}_{11} = \begin{cases} G[\varkappa_{11}+\nu\varkappa_{22}+(\varepsilon_{11}+\nu\varepsilon_{22})/R], & \text{if} \quad (\varkappa_{22}+\nu\varkappa_{11})+(\varepsilon_{22}+\nu\varepsilon_{11})/R \leqslant 0 \\ & \text{(the fissure is closed)}, \\ G(1-\nu^2)(\varkappa_{11}+\varepsilon_{11}/R), & \text{otherwise (the fissure is open)}, \end{cases}$$

$$\mathfrak{M}_{12} = \mathfrak{M}_{21} = G(1-\nu)(\varkappa_{12}+\varepsilon_{12}/R),$$

$$\mathfrak{M}_{22} = \begin{cases} G[\varkappa_{22}+\nu\varkappa_{11}+(\varepsilon_{22}+\nu\varepsilon_{11})/R], & \text{if} \quad (\varkappa_{22}+\nu\varkappa_{11})+(\varepsilon_{22}+\nu\varepsilon_{11})/R \leqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

As we know, the constitutive equation for shear forces remains unchanged.

The elastic potential of the homogenized plate

$$W(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega}) = \frac{1}{2}\,[G_{\alpha\beta\lambda\mu}\big(\varkappa_{\lambda\mu}+\varrho^{y}_{\lambda\mu}(\boldsymbol{\varphi}^1)\big)\big(\varkappa_{\alpha\beta}+\varrho^{y}_{\alpha\beta}(\boldsymbol{\varphi}^1)\big) + 2E_{\alpha\beta\lambda\mu}\big(\varkappa_{\lambda\mu}+\varrho^{y}_{\lambda\mu}(\boldsymbol{\varphi}^1)\big)\,\varepsilon_{\alpha\beta}+A_{\alpha\beta\lambda\mu}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\lambda\mu}+H_{\alpha\beta}\omega_\alpha\omega_\beta],$$

written out explicitly reads

$$W(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega}) = \begin{cases} W_1(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega}), & \text{if} \quad (\varkappa_{22}+\nu\varkappa_{11})+(\varepsilon_{22}+\nu\varepsilon_{11})/R \leqslant 0, \\ W_2(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega}), & \text{otherwise} \end{cases} \tag{3.1}$$

where

$$W_1 = \frac{1}{2}\,\{G[\varkappa_{11}^2+2\nu\varkappa_{11}\varkappa_{22}+\varkappa_{22}^2+(1-\nu)(\varkappa_{12}^2+\varkappa_{21}^2)] + 2\,\frac{G}{R}\,[\varkappa_{11}\,\varepsilon_{11}+\nu(\varkappa_{11}\,\varepsilon_{22}+\varkappa_{22}\,\varepsilon_{11})+\varepsilon_{22}\varkappa_{22} + (1-\nu)(\varkappa_{12}\,\varepsilon_{12}+\varkappa_{21}\,\varepsilon_{21})]+B[\varepsilon_{11}^2+2\nu\varepsilon_{11}\,\varepsilon_{22}+\varepsilon_{22}^2 + (1-\nu)(\varepsilon_{12}^2+\varepsilon_{21}^2)+S(\omega_1^2+\omega_2^2)\},$$

$$W_2 = \frac{1}{2}\,\{G[(1-\nu^2)\varkappa_{11}^2+(1-\nu)(\varkappa_{12}^2+\varkappa_{21}^2)] + 2\,\frac{G}{R}\,[(1-\nu^2)\varkappa_{11}\,\varepsilon_{11}+(1-\nu)(\varkappa_{12}\,\varepsilon_{12}+\varkappa_{21}\,\varepsilon_{21})] + B\left[\left(1-\frac{\nu^2 e}{R}\right)\varepsilon_{11}^2+2\nu\left(1-\frac{e}{R}\right)\varepsilon_{11}\,\varepsilon_{22}+\left(1-\frac{e}{R}\right)\varepsilon_{22}^2 + (1-\nu)(\varepsilon_{12}^2+\varepsilon_{21}^2)\right]+S(\omega_1^2+\omega_2^2)\right\},$$

$$S = 5/12(1+\nu).$$

The following relationships hold

$$\mathfrak{M}_{\alpha\beta} = \frac{\partial W}{\partial\varkappa_{\alpha\beta}}, \quad \mathfrak{N}_{\alpha\beta} = \frac{\partial W}{\partial\varepsilon_{\alpha\beta}}, \quad \mathfrak{Q}_\alpha = \frac{\partial W}{\partial\omega_\alpha}. \tag{3.2}$$

The potential W given by Eq. (3.1) is not strictly convex since det $[\partial^2 W_2/\partial\varkappa_{\alpha\beta}\partial\varkappa_{\lambda\mu}] = 0$. The lack of the strict convexity is a result of the violation of the assumption concerning connectedness of the basic cell Y.

4. Final remarks

More general unilateral conditions imposed on F or F^ε can also be considered. For instance, friction or friction-like conditions can be studied. However, such problems lead up to homogenization of implicit variational inequalities which seem to be intractable by asymptotic methods.

Problems involving no unilateral constraints are simpler to treat since then the local problem $\mathscr{P}^1_{loc}$ and $\mathscr{P}^2_{loc}$ reduce to variational equations.

In our developments the functions $A_{\alpha\beta\lambda\mu}$, $D_{\alpha\beta\lambda\mu}$, and $H_{\alpha\beta}$ have been assumed to be independent of the local variable y. A generalization to the case when these functions are Y-periodic is straightforward. On the other hand, it would be interesting to study the case when these functions are X-periodic, say, and $X \neq Y$. Such case seems to be non-trivial.

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