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Viscoelastic boundary layer: the stagnation point flows as flows with dominating extensions

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THE PLANE laminar boundary layer in the case of an incompressible viscoelastic (simple) fluid in a stagnation point flow is considered in great detail. The scaling procedure characterising the "elastic-type" boundary layer and the constitutive equations valid for thin-layer flows with dominating extensions (FDE) are applied. It is shown that the layer thickness as well as the velocity profiles essentially depend on the Weissenberg number and the extensional viscosity function.

Rozważono szczegółowo płaską, laminarną warstwę przyścienną w przypadku nieściśliwej cieczy lepkosprężystej (prostej) w przepływie stagnacyjnym. Zastosowano procedurę skalowania charakteryzującą warstwę przyścienną „typu sprężystego" oraz równania konstytutywne dla cienkowarstwowych przepływów z dominującymi rozciąganiami (FDE). Pokazano, że zarówno grubość warstwy jak i profile prędkości zależą istotnie od liczby Weissenberga i funkcji lepkości przy rozciąganiu.

Рассмотрен детально плоский, ламинарный пограничный слой в случае несжимаемой вязкоупругой (простой) жидкости в стагнационном течении. Применена процедура масштабного преобразования, характеризующая пограничный слой „упругого типа", а также определяющие уравнения для тонкослоистых течений с доминирующими растяжениями (FDE). Показано, что так толщина слоя, как и профили скорости существенно зависят от числа Вейссенберга и функции вязкости при растяжении.

1. Introduction

IN OUR PREVIOUS papers [1, 2] we discussed some aspects of flows in viscoelastic boundary layers of the "viscous-type", the scaling procedure for which was based on the assumption of high Reynolds numbers like in the classical Prandtl theory. The corresponding governing equations could be derived under the assumption of nearly viscometric or quasi-viscometric approximations. It is noteworthy that other known solutions of the stagnation point flows of various viscoelastic fluids were obtained under similar assumptions (cf. [3, 4, 5, 6]). All the above analyses are not expected to be valid near the leading edge of a bluff body where the Deborah or Weissenberg numbers may be appreciable, and where the customary concept of a boundary layer is open to question (cf. [7]). Therefore, for plane stagnation point flows we use the notion of the "elastic-type" boundary layer introduced by G. ASTARITA and G. MARRUCCI [8]. The corresponding scaling procedure is based on small values of the so-called elasticity number, i.e. the ratio of Weissenberg to Reynolds numbers.

Since in the stagnation point flow or near the leading edge of a bluff body, the Deborah number may be pretty high and the extensional effects are of greater importance as compared with the shearing effects at the wall, we use the constitutive equations valid for the so-called flows with dominating extensions (FDE approximations) defined elsewhere [9, 10]. It is shown that under the above assumptions the boundary-layer thickness as well as the velocity profiles essentially depend on the extensional viscosity function and its derivative with respect to the extension rate. Some particular cases of solutions can be discussed in greater detail without solving the corresponding two-parameter differential equations.

2. The "elastic-type" boundary-layer equations

In plane steady flows of an arbitrary incompressible fluid, the dynamic equations of equilibrium and the continuity condition take the following form:

$$
\begin{aligned}
&\varrho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}+\frac{\partial T_E^{12}}{\partial y}+\frac{\partial T_E^{11}}{\partial x},\\
&\varrho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}+\frac{\partial T_E^{12}}{\partial x}+\frac{\partial T_E^{22}}{\partial y},\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0.
\end{aligned}
\tag{2.1}
$$

where u and v denote the velocity components in the direction of x and y-axes, respectively, T_E^{ij} $(i, j = 1, 2)$ are the extra-stress components, ϱ is the constant mass density, and p — the hydrostatic pressure.

Considering plane flows in a thin layer of thickness δ, in a neighbourhood of a rigid impermeable wall, we introduce the dimensionless quantities (with overbars) through the following relations:

$$
x = L\bar{x}, \quad y = \delta\bar{y}, \quad p = \varrho U_0^2\bar{p}, \quad u = U_0\bar{u}, \quad v = \varepsilon U_0\bar{v},
\tag{2.2}
$$

where L denotes some characteristic length, U_0 — some characteristic velocity, and we assume, moreover, that

$$
\varepsilon = \frac{\delta}{L} \ll 1.
\tag{2.3}
$$

The dimensionless extra-stress components result from

$$
T_E^{12} = \frac{\eta_0 U_0}{\delta}\bar{T}_E^{12}, \qquad T_E^{ii} = \frac{N_{10}U_0^2}{\delta^2}\bar{T}_E^{ii}, \qquad ii = 11, 22,
\tag{2.4}
$$

where

$$
\begin{aligned}
&\eta_0 = \lim_{\varkappa\to 0}\eta = \lim_{\varkappa\to\varkappa}\left(\frac{T_E^{12}}{\varkappa}\right)_w,\\
&N_{10} = \lim_{\varkappa\to 0}N_1 = \lim_{\varkappa\to 0}\left(\frac{T_E^{11}-T_E^{22}}{\varkappa^2}\right)_w\\
&\varkappa \equiv \frac{\partial u}{\partial y}
\end{aligned}
\tag{2.5}
$$

define the material parameters with dimensions of viscosity [Nsm^{-2}] and normal-stress coefficient [Ns^2m^{-2}], respectively. These quantities are well defined since at the wall (w—denote values at the wall) any flow of an incompressible fluid is locally viscometric.

Hence, Eqs. (2.1) can be expressed in the dimensionless form:

$$\begin{aligned}
&\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\varepsilon^{-2}}{\mathrm{Re}}\frac{\partial \bar{T}_E^{12}}{\partial \bar{y}} + \varepsilon^{-2}\mathbf{El}\frac{\partial \bar{T}_E^{11}}{\partial \bar{x}},\\
&\varepsilon\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) = -\varepsilon^{-1}\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\varepsilon^{-1}}{\mathrm{Re}}\frac{\partial \bar{T}_E^{12}}{\partial \bar{x}} + \varepsilon^{-3}\mathbf{El}\frac{\partial \bar{T}_E^{22}}{\partial \bar{y}},\\
&\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0,
\end{aligned} \tag{2.6}$$

where

$$\mathrm{Re} = \frac{\varrho U_0 L}{\eta_0}, \quad \mathrm{Ws} = \frac{N_{10} U_0}{\eta_0 L}, \quad \mathbf{El} = \frac{\mathrm{Ws}}{\mathrm{Re}} = \frac{N_{10}}{\varrho L^2}, \tag{2.7}$$

define the corresponding Reynolds, Weissenberg and elasticity numbers (cf. [8]).

The concept of a viscoelastic boundary layer is based on rather intuitive than physical assumption that in many situations there exists some thin layer close to the wall in which not only viscous but also viscoelastic (normal-stress) effects are meaningful, and the outside flow is exactly an inviscid one, governed by the Euler equation (cf. [1, 7]).

Thus, for sufficiently high Reynolds numbers, finite Weissenberg numbers and sufficiently small elasticity numbers, viz.

$$\mathbf{El} = \frac{\mathrm{Ws}}{\mathrm{Re}} = O(\varepsilon^2), \quad \mathrm{Ws} = O(1) > 1, \tag{2.8}$$

we arrive at the following equations (zero-order approximations with respect to ε):

$$\begin{aligned}
&\varrho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp^*}{dx} + \frac{\partial T_E^{12}}{\partial y} + \frac{\partial}{\partial x}(T_E^{11} - T_E^{22}),\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\end{aligned} \tag{2.9}$$

where

$$p^* = p - T_E^{22}, \quad \frac{\partial p^*}{\partial y} = 0. \tag{2.10}$$

The external flow is described by the Euler-type equation, viz.

$$U\frac{dU}{dx} = -\frac{1}{\varrho}\frac{dp^*}{dx}, \tag{2.11}$$

where $U(x)$ denotes the velocity of an inviscid solution at the wall, i.e. for $y = 0$.

For the "elastic-type" boundary layer defined by Eqs. (2.9), the inertia and pressure forces are mainly equilibrated by elastic (normal) stresses. Thus we have

$$\varepsilon = \frac{\delta}{L} = O(\sqrt{\mathbf{El}}) = O\left(\sqrt{\frac{N_{10}}{\varrho L^2}}\right), \tag{2.12}$$

and also

$$\delta = O\left(\sqrt{\frac{N_{10}}{\varrho}}\right). \tag{2.13}$$

This result means that the thickness δ of the "elastic-type" boundary layer is constant independent of the velocity U_0 and length L.

For plane stagnation point flows, an inviscid solution is of the form:

$$U = cx, \quad V = -cy, \tag{2.14}$$

where the extension rate

$$c = U_0/L. \tag{2.15}$$

Hence it appears that U_0 may be interpreted as the velocity at the distance L from the rigid wall. Equation (2.11) leads to the solution

$$p_0^* - p^* = \frac{1}{2}\varrho c^2(x^2+y^2), \quad p_0^* = \frac{1}{2}\varrho U_0^2. \tag{2.16}$$

Introducing the corresponding kinematic quantities, viz.

$$\nu_0 = \eta_0/\varrho, \quad \nu_{10} = N_{10}/\varrho, \tag{2.17}$$

we can also write

$$\mathrm{Re} = \frac{U_0 L}{\nu_0} = \frac{U_0^2}{\nu_0 c}, \quad \mathrm{Ws} = \frac{\nu_{10} U_0}{\nu_0 L} = \frac{\nu_{10}}{\nu_0} c, \quad \mathrm{El} = \frac{\nu_{10}}{L^2}, \tag{2.18}$$

and, on the basis of Eq. (2.13)

$$\delta = O(\sqrt{\nu_{10}}) = O\left(\sqrt{\frac{\nu_0}{c}}\sqrt{\mathrm{Ws}}\right). \tag{2.19}$$

It is seen from the above expression that for higher Weissenberg numbers the thickness of the "elastic-type" boundary layer may be greater as compared with that for purely viscous flows. On the other hand, it is noteworthy that for $\mathrm{Ws} \equiv 0$ the above definition oses its sense.

3. The stagnation point flows as flows with dominating extensions

The plane flows with dominating extensions (FDE) have been defined and discussed elsewhere [9]. In what follows, we repeat only the most important properties relevant for the "elastic-type" boundary layer.

Many plane steady-state flows can be presented in the following form:

$$\begin{aligned} u^* &= cx + u(x, y), \\ v^* &= -cy + v(x, y), \end{aligned} \tag{3.1}$$

where c is a constant and u, v denote additional velocity components along the axes x, y, respectively. If, moreover, the above flows are realised in a thin layer, in which one of the characteristic dimensions L is much greater than the dimension δ describing the layer thickness, i.e. Eq. (2.3) is satisfied, we can write the velocity gradients as

(3.2)
$$\frac{\partial u^*}{\partial x} = c\left(1+\varepsilon\frac{\partial \bar{u}}{\partial \bar{x}}\right), \qquad \frac{\partial u^*}{\partial y} = c\frac{\partial \bar{u}}{\partial \bar{y}},$$
$$\frac{\partial v^*}{\partial x} = c\varepsilon^2\frac{\partial \bar{v}}{\partial \bar{x}}, \qquad \frac{\partial v^*}{\partial y} = c\left(-1+\varepsilon\frac{\partial \bar{v}}{\partial \bar{y}}\right),$$
$$\omega^* = \frac{1}{2}c\left(\frac{\partial \bar{u}}{\partial \bar{y}} - \varepsilon^2\frac{\partial \bar{v}}{\partial \bar{x}}\right),$$

where Eqs. (2.2) have been taken into account. It is seen that for small vorticities ω^* the first terms in diagonal components (3.2) may be more meaningful than the remaining terms (they are at least of order $0(\varepsilon)$).

On the basis of Eqs. (3.1), we arrive at the following Rivlin–Ericksen kinematic tensor (cf. [11]):

$$(3.3) \qquad [\mathbf{A}_1^*] = [\mathbf{A}_1]+[\mathbf{A}_1'] = \begin{bmatrix} 2c & 0 \\ 0 & -2c \end{bmatrix} + \begin{bmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} & 2\dfrac{\partial v}{\partial y} \end{bmatrix},$$

and its invariants

(3.4)
$$\mathrm{tr}\,\mathbf{A}_1^* = 0, \qquad \mathrm{tr}\,\mathbf{A}_1^{*3} = 0,$$
$$\mathrm{tr}\,\mathbf{A}_1^{*2} = \mathrm{tr}\,\mathbf{A}_1 + (\mathrm{tr}\,\mathbf{A}_1)' = 8c^2+16c\frac{\partial u}{\partial x}+8\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2,$$

where the primed part refers to the additional velocity field introduced in Eqs. (3.1).

On the other hand, the constitutive equation of an incompressible simple fluid in plane extensional flows can be presented as (cf. [11])

$$(3.5) \qquad \mathbf{T} = -p\mathbf{1}+\beta(I_2)\mathbf{A}_1, \qquad I_2 = \mathrm{tr}\,\mathbf{A}_1^2,$$

where the function β depends only on the second invariant. In our paper [9] we defined the plane "flows with dominating extensions" (FDE) as such thin-layer flows in which the constitutive equations (3.5) may be used in a form linearly perturbed with respect to additional velocity gradients (3.2). This means that for FDE we have

$$(3.6) \qquad \mathbf{T}^* = -p\mathbf{1}+\beta\mathbf{A}_1+\beta\mathbf{A}_1'+\frac{d\beta}{dc}c'\mathbf{A}_1,$$

where the linear increment of the extension rate c, denoted by c', amounts to

$$(3.7) \qquad c' = \frac{\partial u}{\partial x}+\frac{1}{2c}\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{8c}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2.$$

Thus we have the stress components:

(3.8)
$$T^{*11} = -p+2\beta c+2\beta\frac{\partial u}{\partial x}+\frac{1}{4}\frac{d\beta}{ac}\left[8c\frac{\partial u}{\partial x}+4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right],$$
$$T^{*22} = -p-2\beta c-2\beta\frac{\partial u}{\partial x}-\frac{1}{4}\frac{d\beta}{dc}\left[8c\frac{\partial u}{\partial x}+4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right],$$
$$T^{*12} = \beta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right).$$

The terms involved in Eqs. (3.8) are of different orders of magnitude with respect to small parameter $\varepsilon = \delta/L$. Substituting the above relations into the boundary-layer equations $(2.9)_1$ and retaining only terms of the highest order of magnitude with respect to ε, we obtain the equation:

$$\varrho\left(u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y}\right) = -\frac{dp^*}{dx} + \beta \frac{\partial^2 u}{\partial y^2} + \frac{d\beta}{dc}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} \tag{3.9}$$

or, after taking into account Eqs. (3.1), (2.11), (2.14),

$$cx \frac{\partial u}{\partial x} + cu - cy \frac{\partial u}{\partial y} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu_0 b(c) \frac{\partial^2 u}{\partial y^2} + \nu_0 \frac{db}{dc}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}, \tag{3.10}$$

where we have denoted

$$\beta/\varrho = \nu = \nu_0 b(c), \quad \nu_0 = \eta_0/\varrho. \tag{3.11}$$

It is also noteworthy that the simplified constitutive equations, leading immediately to Eqs. (3.9) or (3.10), can be written as follows (cf. [9]):

$$\begin{aligned} T^{*11} &= -p + \frac{1}{4}\frac{d\beta}{dc}\left(\frac{\partial u}{\partial y}\right)^2, \\ T^{*22} &= -p - \frac{1}{4}\frac{d\beta}{dc}\left(\frac{\partial u}{\partial y}\right)^2, \\ T^{*12} &= \beta \frac{\partial u}{\partial y}. \end{aligned} \tag{3.12}$$

The function $\beta(c)$ (or the dimensionless function $b(c)$) can be simply related to the corresponding planar elongational viscosity $\eta^*(c)$, viz.

$$\eta^*(c) = \frac{1}{c}(T^{11} - T^{22}) = 4\beta(c). \tag{3.13}$$

In what follows, we shall call the function $\beta(c)$ the extensional viscosity function.

According to Eqs. $(2.5)_2$ and (3.12), we can define the Weissenberg number for the flows considered in the form:

$$\begin{aligned} N_{10} &= \lim_{\varkappa \to 0}\left(\frac{T^{*11} - T^{*22}}{\varkappa^2}\right)_w = \frac{1}{2}\frac{d\beta}{dc}, \\ \mathrm{Ws} &= \frac{1}{2}\frac{db}{dc}c. \end{aligned} \tag{3.14}$$

This number is proportional to the derivative of the extensional viscosity function with respect to the extension rate c.

The boundary conditions in the case considered are exactly the same as those for purely viscous flow (cf. [1, 12]), viz.

$$\begin{aligned} u^* = v^* &= 0 && \text{at} \quad y = 0, \\ u^* = U(x) &= cx && \text{at} \quad y \to \infty. \end{aligned} \tag{3.15}$$

It can be verified that T^{*12} as well as $T^{*11} - T^{*22}$ tend to zero for $y \to \infty$, if also $\partial u/\partial y \to 0$ at $y \to \infty$. The latter condition directly results from the asymptotic properties of Eqs. $(3.15)_2$.

4. The similarity transformations and governing equations

For plane stagnation point flows there exists neither characteristic length L nor velocity U_0, but only the ratio $c = U_0/L$. A dimensional analysis of Eq. (3.10) proves that it can be expressed in a dimensionless form, if

$$u = cx(f'-1), \quad v = -\sqrt{\nu_0 c}\,(f-\eta), \quad y = \eta\sqrt{\frac{\nu_0}{c}}, \tag{4.1}$$

where f is a function of η only, and the condition of continuity is satisfied automatically. This also implies that

$$u^* = cxf'(\eta), \quad v^* = -\sqrt{\nu_0 c}\, f(\eta). \tag{4.2}$$

Calculating the corresponding derivatives, viz.

$$\begin{aligned} &\frac{\partial u}{\partial y} = \sqrt{\frac{c^3}{\nu_0}}\, xf''(\eta), && \frac{\partial u}{\partial x} = c(f'(\eta)-1), \\ &\frac{\partial^2 u}{\partial y^2} = \frac{c^2}{\nu_0} xf'''(\eta), && \frac{\partial^2 u}{\partial x \partial y} = \sqrt{\frac{c^3}{\nu_0}}\, f''(\eta), \end{aligned} \tag{4.3}$$

where primes denote differentiation with respect to η, and substituting Eqs. (4.1), (4.3) into Eq. (3.10), we arrive at

$$bf''' + f''f - f'^2 + 1 = -kf''^2, \tag{4.4}$$

where according to Eq. $(3.14)_2$

$$k = 2\,\mathrm{Ws} = \frac{db}{dc}c. \tag{4.5}$$

Similarly, on the basis of Eqs. (4.2), the boundary conditions (3.15) lead to

$$f(0) = f'(0) = 0, \quad \lim_{\eta\to\infty} f'(\eta) = 1. \tag{4.6}$$

The governing equation (4.4) is a nonlinear ordinary differential equation of the third order. Its solution satisfying the boundary conditions (4.6) depends on the following two parameters: the dimensionless extensional viscosity function $b(c)$ and the Weissenberg number $\mathrm{Ws} = \frac{1}{2}k$.

For $b = 1$ and $\mathrm{Ws} = 0$, Eq. (4.4) takes the form well known for Newtonian fluids

$$f''' + f''f - f'^2 + 1 = 0. \tag{4.7}$$

The numerical solutions of the above equation can be found in numerous references (cf. [12]). The corresponding values of f, f', f'' etc. are usually tabulated; at the wall, i.e. for $\eta = 0$, we have: $f(0) = 0$, $f'(0) = 0$, $f''(0) = 1.2326$, $f'''(0) = -1$.

Some particular solutions of Eq. (4.4) with boundary conditions (4.6) can be discussed in greater detail without solving the differential equation in a numerical way. To this end, we shall consider certain relations resulting from Eq. (4.4), especially those satified at the wall ($\eta = 0$).

5. The boundary-layer thickness and possible velocity profiles

According to Eqs. (2.19) and $(4.1)_3$, we obtain for the "elastic-type" boundary-layer thickness the following expressions:

$$\delta_{e1} = \eta_{e1}\sqrt{\frac{\nu_0}{c}}\sqrt{\mathrm{Ws}} = \eta_\delta\sqrt{\frac{\nu_0}{c}}, \tag{5.1}$$

where both η_{e1} and η_δ denote the values at which the velocities reach 99% of the maximum values corresponding to $\eta \to \infty$ (cf.Eq.(4.6)). It results from Eq.(5.1) that

$$\eta_{e1} = \frac{\eta_\delta}{\sqrt{\mathrm{Ws}}}; \tag{5.2}$$

hence, $\eta_{e1} = \eta_\delta$ only for Ws $= 1$. At this place it is noteworthy to repeat once more that the concept of the "elastic-type" boundary layer has been introduced for Ws $= O(1) > 1$; for Ws $= 0$ it loses any physical sense. On the other hand, for the "viscous-type" boundary layer we have (cf.[12])

$$\delta_N = 2.4\sqrt{\frac{\nu_0}{c}}. \tag{5.3}$$

It can also be verified that at the wall ($\eta = 0$) the Newtonian solution substituted into Eq. (4.4) leads to

$$\Delta b f_N'''(0) = -k f_N''^2(0), \quad \Delta b = b-1. \tag{5.4}$$

Taking into account the values of $f_N'''(0)$ and $f_N''(0)$ resulting from the solution of Eq. (4.7). we arrive at

$$\frac{\Delta b}{k} = \frac{\Delta b}{2\,\mathrm{Ws}} = 1.52. \tag{5.5}$$

The above result means that the inclination of the velocity profile at $\eta = 0$ is exactly the same as for the Newtonian case, if the ratio $\Delta b/k$ takes the value (5.5).

On the other hand, an approximate solution of Eq. (4.4) for $\eta \to \infty$ ($f'(0) \to 1$, $f''^2(0) \to 0$) leads to

$$f''(\eta) \sim C\exp\left[-\frac{1}{2b}(\eta-0.62)^2\right], \tag{5.6}$$

where C is an integration constant. Comparing the above result with the corresponding Newtonian expression (for $b \equiv 1$, $k = 0$), it is seen that for $b \neq 1$ and $\Delta b/k = 1.52$ the velocity profile is steeper as compared with its Newtonian counterpart. This also means that for Ws $= 1$, $\eta_{e1} = \eta_\delta$ is essentially greater than 2.4 (η_δ can be estimated as 5.6),

The velocity profiles in the "elastic-type" boundary layer are shown in Fig. 1 for $\Delta b/k = 1.52$ and various values of the Weissenberg numbers Ws. For comparison the approximate Newtonian profile is shown by a broken line. Thus we see that for $\Delta b/k = 1.52$, the increasing Weissenberg numbers make the velocity profiles fuller as compared with those for Ws $\leqslant 1$.

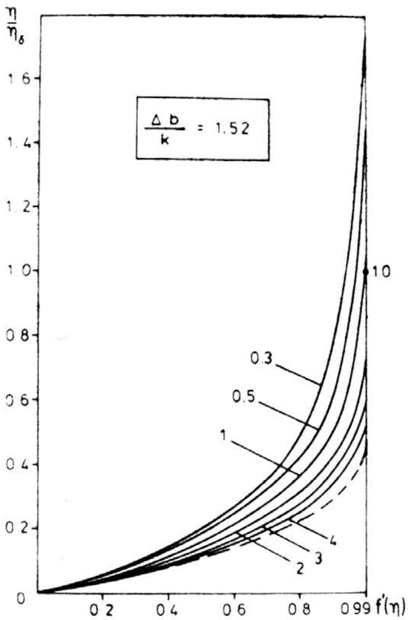

FIG. 1.

It also results from Eqs. (4.4), (4.6) that at the wall ($\eta = 0$), we have

(5.7) $$b(c)f'''(0)+1 = -kf''^2(0).$$

Denoting by $f'''(0) = -1+\Delta f'''$, we arrive at

(5.8) $$f''^2(0) = \frac{\Delta b}{k} - \frac{b}{k}\Delta f''',$$

and

(5.9) $$f''^2(0) > \frac{\Delta b}{k} \quad \text{for} \quad \Delta f''' < 0,$$
$$f''^2(0) < \frac{\Delta b}{k} \quad \text{for} \quad \Delta f''' > 0.$$

Taking into account the result (5.4), we note that $\Delta f''' = 0$ if $\Delta b/k = 1.52$. For other values of $\Delta b/k$ and for small values of $\Delta f'''$, and this is the case at $\eta = 0$ where $f'''(0)$ does not differ essentially from $f_N'''(0) = -1$, we can assume that approximately

(5.10) $$f''^2(0) \simeq \frac{\Delta b}{k} \neq 1.52.$$

The corresponding velocity profiles in the "elastic-type" boundary layer are schematically shown in Fig. 2 for Ws = 1 and various values of the ratios $\Delta b/k$. For comparison the Newtonian profile is shown by a broken line. Thus we see that for constant Ws = 1, the increasing ratios $\Delta b/k$ make the velocity profiles fuller as compared with those for smaller $\Delta b/k$.

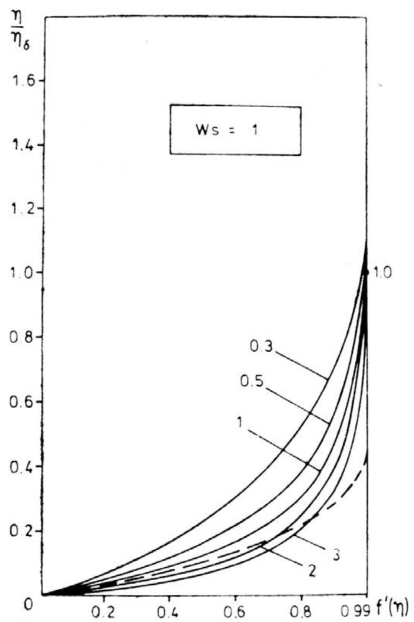

FIG. 2.

Another meaning of the ratio $\Delta b/k$ results from Eq. (4.5). For the dimensionless extensional viscosity, we can write the expression:

$$b(c) = 1+\alpha c^{\frac{k}{\Delta b}}, \tag{5.11}$$

where α is a constant. Therefore, various values of the ratio $\Delta b/k$ determine possible variability of the extensional viscosity function with respect to the extension rate. For instance, $\Delta b/k = 1$ corresponds to a linear function $b(c)$, $\Delta b/k = 2$ gives a square-root function $b(c)$, while $\Delta b/k = 0.5$ expresses a parabolic dependence of $b(c)$. For $\Delta b/k = 1.52$ in particular, we obtain

$$b(c) = 1+\alpha c^{0.66}. \tag{5.12}$$

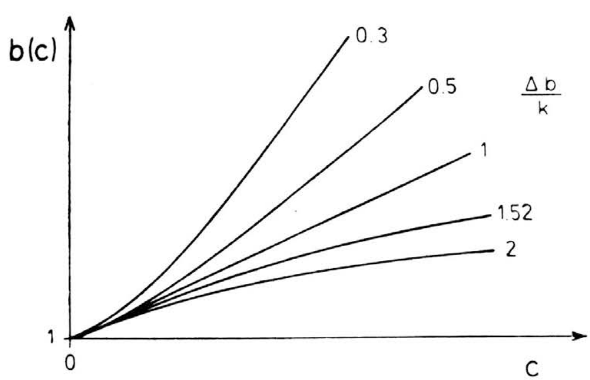

FIG. 3.

It is clearly seen from Fig. 3 that smaller values of the ratio $\Delta b/k$ (or smaller values of Δb for constant $k = 2$ Ws) determine faster increases of the extensional viscosity function. According to Fig. 1 very small $\Delta b/k$ may correspond to high increases of $b(c)$ and, in consequence, to very steep velocity profiles.

6. Conclusions

At the end of the present considerations the following conclusions can be formulated:

1) the thickness of the "elastic-type" boundary layer does not depend either on the characteristic length or velocity; it is constant for constant normal-stress coefficients;

2) the thickness of the "elastic-type" boundary layer may be much greater than that for purely Newtonian flows (of the "viscous-type");

3) the increasing Weissenberg numbers for established variability of the extensional viscosity function with respect to the extension rate (e.g. linear, parabolic etc.) give the velocity profiles fuller as compared with smaller numbers;

4) the increasing ratios responsible for variability of the extensional viscosity function with respect to the extension rate (higher ratios correspond to weaker variabilities) give the velocity profiles fuller as compared with smaller ratios;

5) the concept of the "elastic-type" boundary layer seems to lead to reliable results or Weissenberg numbers greater than unity.

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