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Exact solutions for the uniaxial extension of a mixture slab

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RAJAGOPAL and WINEMAN [1] have recently established several new exact solutions to boundary-value problems in nonlinear elasticity. They have shown, for example, that a nonlinearly elastic slab can exhibit nonuniform uniaxial extension solutions, in addition to the classical uniform uniaxial extension solution. In this paper, a slab consisting of a mixture of a nonlinearly elastic solid and an ideal fluid is modelled in the context of Mixture Theory. The problem of uniaxial extension of this slab mixture is considered, and the possibility of an infinity of exact solutions is demonstrated.

RAJAGOPAL i WINEMAN [1] otrzymali ostatnio szereg nowych rozwiązań ścisłych zagadnień brzegowych nieliniowej teorii sprężystości. Pokazali oni np., że rozważenie warstwy nieliniowo sprężystej prowadzić może do nierównomiernego rozciągania jednoosiowego, niezależnie od klasycznego stanu równomiernego rozciągania. W obecnej pracy rozważa się w ramach ogólnej teorii mieszanin warstwę składającą się z mieszaniny stałego ciała sprężystego oraz cieczy doskonałej. Bada się zagadnienie jednoosiowego rozciągania takiej warstwy oraz przedstawia się możliwość istnienia nieskończonej liczby rozwiązań ścisłych.

РАЯГОПАЛЬ и ВИНЕМАН [1] получили в последнее время ряд новых точных решений краевых задач нелинейной теории упругости. Они показали например, что рассмотрение нелинейно упругого слоя может привести к неравномерному одноосному растяжению, независимо от классического состояния равномерного растяжения. В настоящей работе рассматривается, в рамках общей теории смесей, слой, состоящий из смеси твердого упругого тела и идеальной жидкости. Исследуется задача одноосного растяжения такого слоя, а также представляется возможность существования бесконечного количества точных решений.

1. Introduction

RECENTLY, RAJAGOPAL and WINEMAN (1] have presented new exact solutions for boundary-value problems in nonlinear elasticity. In particular, for the problem of uniaxial extension they have demonstrated that an axial variation of the stretch ratio is possible for nonlinearly elastic materials. In addition, they have obtained an infinite class of exact solutions for the uniaxial extension of a neo-Hookean slab of finite thickness whose other dimensions are infinite, the top and bottom surfaces of the slab being bonded to rigid plates. In this work, the same boundary-value problem is studied for a slab which is a mixture of a nonlinearly elastic solid and an ideal fluid. Boundary-value problems for solid-fluid mixtures of this kind have been studied previously [2-5] in the context of Mixture Theory [6, 7]. The same approach is adopted in this paper to study the uniaxial extension of a slab mixture, and the possibility of an infinity of exact solutions is demonstrated. The preliminaries and the basic equations of Mixture Theory are presented in Sect. 2. The constitutive equations for a mixture of a nonlinearly elastic solid and an ideal fluid are discussed in Sect. 3. The problem of non-uniform uniaxial extension of a mixture of a nonlinearly elastic solid and an ideal fluid is presented in Sect. 4.

2. Preliminaries: notations and basic equations

A brief review of the notations and basic equations of Mixture Theory (also known as the Theory of Interacting Continua) are presented in this section for completeness and continuity. The historical development and a detailed exposition of the theory are succinctly presented in the comprehensive review articles by ATKIN and CRAINE [6] and BOWEN [7].

Let Ω and Ω_t denote the reference configuration and the configuration of the body at time t, respectively. For a function defined on $\Omega \times \mathbf{R}$ and $\Omega_t \times \mathbf{R}$, ∇ and grad are used to represent the partial derivative with respect to Ω and Ω_t, respectively. Also d/dt and $\partial/\partial t$ denote the total and partial derivative with respect to t, respectively. The divergence operator related to grad is denoted by div.

The solid-fluid aggregate will be considered a mixture with S_1 representing the solid and S_2 representing the fluid. At any instant of time t, it is assumed that each place in the space is occupied by particles belonging to both S_1 and S_2. Let $\mathbf{X}$ and $\mathbf{Y}$ denote the reference positions of typical particles of S_1 and S_2. The motion of the solid and the fluid is represented by

$$\mathbf{x} = \mathbf{x}_1(\mathbf{X}, t) \quad \text{and} \quad \mathbf{y} = \mathbf{x}_2(\mathbf{Y}, t). \tag{2.1}$$

These motions are assumed to be one-to-one, continuous and invertible. The various kinematical quantities associated with the solid S_1 and the fluid S_2 are

velocity:

$$\mathbf{u} = \frac{d\mathbf{x}_1}{dt}, \qquad \mathbf{v} = \frac{d\mathbf{x}_2}{dt}, \tag{2.2}$$

acceleration:

$$\mathbf{f} = \frac{d^2\mathbf{x}_1}{dt^2}, \qquad \mathbf{g} = \frac{d^2\mathbf{x}_2}{dt^2}, \tag{2.3}$$

velocity gradient:

$$\mathbf{L} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \qquad \mathbf{M} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}},$$

rate of deformation tensor:

$$\mathbf{D} = \frac{1}{2}(\mathbf{L}+\mathbf{L}^T), \qquad \mathbf{N} = \frac{1}{2}(\mathbf{M}+\mathbf{M}^T). \tag{2.4}$$

The deformation gradient $\mathbf{F}$ associated with the solid is given by

$$\mathbf{F} = \frac{\partial \mathbf{x}_1}{\partial \mathbf{X}}. \tag{2.5}$$

The total density of the mixture ϱ and the mean velocity of the mixture $\mathbf{w}$ are defined by

$$\varrho = \varrho_1 + \varrho_2, \tag{2.6}$$

$$\varrho \mathbf{w} = \varrho_1 \mathbf{u} + \varrho_2 \mathbf{v}, \tag{2.7}$$

where ϱ_1 and ϱ_2 are the densities of S_1 and S_2 in the mixed state, repectively, defined per unit volume of the mixture at time t.

The basic equations of the Theory of Interacting Continua are presented next.

2.1. Conservation of mass

Assuming no interconversion of mass between the two interacting continua, the appropropriate forms for the conservation of mass for the solid and the fluid are

$$\varrho_1 |\det \mathbf{F}|| = \varrho_{10}, \tag{2.8}$$

and

$$\frac{\partial \varrho_2}{\partial t} + \operatorname{div}(\varrho_2 \mathbf{v}) = 0, \tag{2.9}$$

where ϱ_{10} is the mass density of the solid in the reference state.

2.2. Conservation of linear momentum

Let $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ denote the partial stress tensors associated with the solid S_1 and the fluid S_2, respectively. Then, assuming that there are no external body forces, the balance of linear momentum for the solid and fluid are given by

$$\operatorname{div} \boldsymbol{\sigma} - \mathbf{b} = \varrho_1 \mathbf{f}, \tag{2.10}$$

$$\operatorname{div} \boldsymbol{\pi} + \mathbf{b} = \varrho_2 \mathbf{g}. \tag{2.11}$$

In equations (2.10) and (2.11), $\mathbf{b}$ denotes the interaction body force vector, which accounts for the mechanical interaction between the solid and the fluid.

2.3. Conservation of angular momentum

This condition states that

$$\boldsymbol{\sigma} + \boldsymbol{\pi} = \boldsymbol{\sigma}^T + \boldsymbol{\pi}^T. \tag{2.12}$$

However, the partial stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ need not be symmetric.

2.4. Surface conditions

Let $\mathbf{t}$ and $\mathbf{p}$ denote the surface traction vectors taken by S_1 and S_2, respectively, and let $\mathbf{n}$ denote the unit outer normal vector at a point on the surface of the mixture region. Then the partial surface tractions are related to the partial stress tensors by

$$\mathbf{t} = \boldsymbol{\sigma}^T \mathbf{n}, \quad \text{and} \quad \mathbf{p} = \boldsymbol{\pi}^T \mathbf{n}. \tag{2.13}$$

2.5. Thermodynamical considerations

The laws of conservation of energy and the entropy production inequality are not explicitly mentioned here for brevity. However, the relevant results are quoted. A complete discussion of these issues is presented in [8].

Let the Helmholtz free energy per unit mass of S_1 and S_2 be denoted by A_1 and A_2, respectively. The Helmholtz free energy per unit mass of the mixture is defined by

$$\varrho A = \varrho_1 A_1 + \varrho_2 A_2. \tag{2.14}$$

Note that by setting

(2.15) $$\mathbf{b} = \operatorname{grad}\phi_1 + \bar{\mathbf{b}} = -\operatorname{grad}\phi_2 + \bar{\mathbf{b}},$$

(2.16) $$\boldsymbol{\sigma} = \phi_1 \mathbf{I} + \bar{\boldsymbol{\sigma}},$$

(2.17) $$\boldsymbol{\pi} = \phi_2 \mathbf{I} + \bar{\boldsymbol{\pi}},$$

where

(2.18) $$\phi_1 = \varrho_1(A_1 - A), \quad \phi_2 = \varrho_2(A_2 - A), \quad \phi_1 + \phi_2 = 0,$$

Equations (2.10)–(2.12) become

(2.19 $$\operatorname{div}\bar{\boldsymbol{\sigma}} - \bar{\mathbf{b}} = \varrho_1 \mathbf{f},$$

(2.20) $$\operatorname{div}\bar{\boldsymbol{\pi}} + \bar{\mathbf{b}} = \varrho_2 \mathbf{f},$$

(2.21) $$\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\sigma}}^T + \bar{\boldsymbol{\pi}}^T.$$

The terms in $\boldsymbol{\sigma}$, $\boldsymbol{\pi}$ and $\mathbf{b}$ which depend on ϕ_1 and ϕ_2 do not contribute to the equations of motion or the total stress. This was first pointed out by GREEN and NAGHDI [8].

2.6. Volume additivity constraint

Attention is restricted to a mixture of incompressible materials. It is assumed that the volume of the mixture is the sum of the volumes occupied by the solid and fluid constituents in their reference states. This implies that the motion of the ineracting continua is such that it meets the following volume additivity constraint (cf. [9]),

(2.22) $$\frac{\varrho_1}{\varrho_{10}} + \frac{\varrho_2}{\varrho_{20}} = 1,$$

where ϱ_{20} is the mass density of the fluid in the reference state.

3. Constitutive equations

A mixture of an elastic solid and a fluid is considered. The solid is assumed to be nonlinearly elastic, and the fluid is assumed to be ideal. Thus all the constitutive functions are required to depend on the following variables:

$$\mathbf{F}, \nabla\mathbf{F}, \varrho_2, \operatorname{grad}\varrho_2, T, \operatorname{grad} T, \mathbf{u} \text{ and } \mathbf{v},$$

where T denotes the common absolute temperature of the solid and the fluid.

A lengthy but standard argument, based on the balance of energy, entropy production inequality, restrictions due to material frame indifference and the assumption that the solid is isotropic in its reference state, leads to the following results (cf [2]).

The constitutive equations are written in terms of the Helmholtz free energy function A per unit mass of the mixture, and the form of this function is given by

(3.1) $$A = \hat{A}(I_1, I_2, I_3, \varrho_2, T),$$

where I_1, I_2, I_3 are the principal invariants of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ defined through

(3.2) $$I_1 = \operatorname{tr}\mathbf{B},$$

(3.3) $$I_2 = \frac{1}{2}[(\mathrm{tr}\,\mathbf{B})^2 - \mathrm{tr}\,\mathbf{B}^2],$$

(3.4) $$I_3 = \det\mathbf{B} = (\det\mathbf{F})^2.$$

Using Eqs. (2.8), (2.22) and (3.4), I_3 can be expressed in terms of ϱ_2 by the relation

(3.5) $$I_3^{1/2} = \det\mathbf{F} = (1-\varrho_2/\varrho_{20})^{-1}.$$

Furthermore, on restricting attention to isothermal conditions, Eq. (3.1) reduces to

(3.6) $$A = A(I_1, I_2, \varrho_2).$$

The components of the partial stresses in the solid and fluid, and the interaction body force for isothermal conditions are given by

(3.7) $$\bar{\sigma}_{ki} = -P\frac{\varrho_1}{\varrho_{10}}\delta_{ki} + 2\varrho\left\{\left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}I_1\right)B_{ki} - \frac{\partial A}{\partial I_2}B_{km}B_{mi}\right\},$$

(3.8) $$\bar{\pi}_{ki} = -P\frac{\varrho_2}{\varrho_{20}}\delta_{ki} - \varrho\varrho_2\frac{\partial A}{\partial \varrho_2}\delta_{ki},$$

(3.9) $$\bar{b}_k = -\frac{P}{\varrho_{10}}\frac{\partial \varrho_1}{\partial x_k} + \varrho_1\frac{\partial A}{\partial \varrho_2}\frac{\partial \varrho_2}{\partial x_k} - \varrho_2\left\{\left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}I_1\right)\delta_{il} - \frac{\partial A}{\partial I_2}B_{il}\right\}B_{il,k} + \alpha\frac{\varrho_1}{\varrho_{10}}\frac{\varrho_2}{\varrho_{20}}(u_k - v_k).$$

In Eqs. (3.7)–(3.9), P is a scalar which arises due to the volume additivity constraint. The constitutive parameter α accounts for a contribution to the interaction body force due to relative motion between the solid and the fluid. The interaction between the solid and the fluid is evident in these equations, where the partial stress of each constituent is affected by the deformed state of both the constituents. It is also useful to record the representation for the total stress

(3.10) $$T_{ki} = \bar{\sigma}_{ki} + \bar{\pi}_{ki} = -P\delta_{ki} - \varrho\varrho_2\frac{\partial A}{\partial \varrho_2}\delta_{ki} + 2\varrho\left\{\left(\frac{\partial A}{\partial I_1}\frac{\partial A}{\partial I_2}I_1\right)B_{ki} - \frac{\partial A}{\partial I_2}B_{km}B_{mi}\right\}.$$

In the remainder of this paper, only $\bar{\boldsymbol{\sigma}}$, and $\bar{\boldsymbol{\pi}}$ and $\bar{\mathbf{b}}$, will be used. Hence, for notational convenience, the superposed bars are dropped.

The results in the subsequent section are derived by assuming [2–5, 10] that the functional form for the Helmholtz free energy function A for the mixture is given by

(3.11) $$A = \tilde{A}(I_1) + \tilde{\tilde{A}}(\varrho_2).$$

Furthermore, the mixture is assumed to be of a "neo-Hookean-type", that is, $\tilde{A}$ is a linear function of I_1.

4. Non-uniform uniaxial extension

Consider a mixture slab of a nonlinearly elastic solid and an ideal fluid. The slab is assumed to have finite thickness, and the other two dimensions of the slab are assumed to be infinite. Let (X, Y, Z) denote the coordinates of a particle in the reference configuration and (x, y, z), the coordinates of the same particle in the deformed configuration. Consider the deformation

$$x = f(Z)X, \quad y = f(Z)Y, \quad z = \lambda(Z). \tag{4.1}$$

The deformation gradient $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{bmatrix} f & 0 & Xf' \\ 0 & f & Yf' \\ 0 & 0 & \lambda' \end{bmatrix}. \tag{4.2}$$

The prime denotes differentiation with respect to Z. The Cauchy–Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ can now be represented as

$$\mathbf{B} = \begin{bmatrix} f^2+X^2f'^2 & XYf'^2 & Xf'\lambda' \\ XYf'^2 & f^2+Y^2f'^2 & Yf'\lambda' \\ Xf'\lambda' & Yf'\lambda' & \lambda' \end{bmatrix}. \tag{4.3}$$

The equilibrium equations are expressed in terms of the reference configuration for computational convenience. Assuming no external body forces, the equations of equilibrium for the mixture take the form

$$\frac{\partial T_{ij}}{\partial X_p} F_{pj}^{-1} = 0. \tag{4.4}$$

The tensor $\mathbf{F}^{-1}$ that appears in these equations has the form

$$\mathbf{F}^{-1} = \begin{bmatrix} \dfrac{1}{f} & 0 & -\dfrac{Xf'}{f\lambda'} \\ 0 & \dfrac{1}{f} & -\dfrac{Yf'}{f\lambda'} \\ 0 & 0 & \dfrac{1}{\lambda'} \end{bmatrix}. \tag{4.5}$$

The equilibrium equations for the mixture reduce to

$$-\frac{\partial P}{\partial X} + 4A_1 \varrho X f'^2 + 2A_1 X \frac{f}{\lambda'} \frac{\partial}{\partial Z}(\varrho f'\lambda') = 0, \tag{4.6}$$

$$-\frac{\partial P}{\partial Y} + 4A_1 \varrho Y f'^2 + 2A_1 Y \frac{f}{\lambda'} \frac{\partial}{\partial Z}(\varrho f'\lambda') = 0, \tag{4.7}$$

and

$$-\frac{f}{\lambda'}\frac{\partial P}{\partial Z} + 4A_1 \varrho f'\lambda' + 2A_1 \frac{f}{\lambda'}\frac{\partial}{\partial Z}(\varrho\lambda'^2) + \frac{f'}{\lambda'}\left[X\frac{\partial P}{\partial X} + Y\frac{\partial P}{\partial Y}\right] - \frac{f}{\lambda'}\frac{\partial}{\partial Z}\left(\varrho_2 \varrho \frac{\partial A}{\partial \varrho_2}\right) = 0. \tag{4.8}$$

In Eqs. (4.6)–(4.8),

$$A_1 = \frac{\partial A}{\partial I_1}.$$

Let

$$g(Z) = 4A_1 \varrho f'^2 + 2A_1 \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\varrho f' \lambda'). \tag{4.9}$$

Then, Eqs. (4.6)–(4.8) may be written as

$$-\frac{\partial P}{\partial X} + Xg(Z) = 0, \tag{4.10}$$

$$-\frac{\partial P}{\partial Y} + Yg(Z) = 0, \tag{4.11}$$

and

$$-\frac{\partial P}{\partial Z} + \frac{4A_1 \varrho f' \lambda'^2}{f} + 2A_1 \frac{\partial}{\partial Z} (\varrho \lambda'^2) + [X^2+Y^2] g(Z) - \frac{\partial}{\partial Z} \left(\varrho_2 \varrho \frac{\partial A}{\partial \varrho_2} \right) = 0. \tag{4.12}$$

The scalar P is eliminated by the standard procedure of cross-differentiation to obtain

$$g'(Z) = 2 \frac{f'}{f} g(Z). \tag{4.13}$$

The equilibrium equations for the solid take the form

$$\frac{\partial \sigma_{ij}}{\partial X_p} F_{pj}^{-1} - b_i = 0. \tag{4.14}$$

The equilibrium equations for the solid reduce to

$$-\frac{\partial P}{\partial X} + 2A_1 \frac{\varrho_{10}}{\varrho_1} X f'^2 (2\varrho + \varrho_2) + 2A_1 \frac{\varrho_{10}}{\varrho_1} X \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\varrho f' \lambda') = 0, \tag{4.15}$$

$$-\frac{\partial P}{\partial Y} + 2A_1 \frac{\varrho_{10}}{\varrho_1} Y f'^2 (2\varrho + \varrho_2) + 2A_1 \frac{\varrho_{10}}{\varrho_1} Y \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\varrho f' \lambda') = 0, \tag{4.16}$$

and

$$\begin{aligned} -\left[\frac{\varrho_1}{\varrho_{10}} \frac{\partial P}{\partial Z} + 2A_1 \frac{\partial}{\partial Z} (\varrho \lambda') \right] \frac{f}{\lambda'} + 4\varrho A_1 f' \lambda' + \frac{\varrho_1}{\varrho_{10}} \frac{f'}{\lambda'} \left[X \frac{\partial P}{\partial X} + Y \frac{\partial P}{\partial Y} \right] \\ - \varrho_1 \frac{\partial A}{\partial \varrho_2} \frac{\varrho_2' f}{\lambda'} - 2A_1 \varrho_2 \frac{f'^3}{\lambda'} [X^2+Y^2] + A_1 \frac{\varrho_2 f}{\lambda'} [4ff' \\ + 2(X^2+Y^2) f' f'' + 2\lambda' \lambda''] = 0. \end{aligned} \tag{4.17}$$

Let

$$h(Z) = 2A_1 \frac{\varrho_{10}}{\varrho_1} f'^2 (2\varrho + \varrho_2) + 2A_1 \frac{\varrho_{10}}{\varrho_1} \frac{f}{\lambda'} \frac{\partial}{\partial Z} (\varrho f' \lambda'). \tag{4.18}$$

Then, Eqs. (4.15)–(4.17) may be written as

$$-\frac{\partial P}{\partial X} + Xh(Z) = 0, \tag{4.19}$$

$$-\frac{\partial P}{\partial Y} + Yh(Z) = 0, \tag{4.20}$$

and

$$(4.21)\quad -\left[\frac{\varrho_1}{\varrho_{10}}\frac{\partial P}{\partial Z}+2A_1\frac{\partial}{\partial Z}(\varrho\lambda')\right]\frac{f}{\lambda'}+4\varrho A_1 f'\lambda'+\frac{\varrho_1}{\varrho_{10}}\frac{f'}{\lambda'}[X^2+Y^2]h(Z)$$
$$-\varrho_1\frac{\partial A}{\partial\varrho_2}\frac{\varrho_2' f}{\lambda'}-2A_1\varrho_2\frac{f'^3}{\lambda'}[X^2+Y^2]+A_1\frac{\varrho_2 f}{\lambda'}[4ff'$$
$$+2(X^2+Y^2)f'f''+2\lambda'\lambda'']=0.$$

Again, the scalar P is eliminated using the procedure of cross-differentiation to obtain

$$(4.22)\quad h'(Z)=2\frac{f'}{f}h(Z)+\hat{h}(Z),$$

where

$$(4.23)\quad \hat{h}(Z)=4A_1\varrho_2\frac{\varrho_{10}}{\varrho_1}f'\left[f''-\frac{f'^2}{f}\right].$$

A simple integration of Eq. (4.13) yields

$$(4.24)\quad g(Z)=C_1 f_2,$$

where C_1 is a constant.

By virtue of (4.9), Eq. (4.24) may be written as

$$(4.25)\quad f''+\frac{f'\lambda''}{\lambda'}+\frac{\varrho'}{\varrho}f'+\frac{2f'^2}{f}-\frac{C_1 f}{2A_1\varrho}=0.$$

Similarly, Eqs. (4.18), (4.22), and (4.23) may be combined to yield

$$(4.26)\quad \varrho_2' f'^2-\frac{\varrho_1'}{\varrho_1}\left[\varrho_2 f'^2+\frac{C_1}{2A_1}f^2\right]=0.$$

Exact solutions to Eqs. (4.25) and (4.26) are presented next.

First, consider the case when the density of the solid remains constant. That is

$$(4.27)\quad \frac{\varrho_1}{\varrho_{10}}=\text{const}.$$

Using Eqs. (4.27) and (2.22), Eq. (4.26) is identically satisfied. By virtue of Eq. (2.8)

$$(4.28)\quad \frac{\varrho_1}{\varrho_{10}}=\frac{1}{\lambda' f^2},$$

which reduces Eq. (4.25) to

$$(4.29)\quad \lambda'''-\frac{3}{2}\frac{\lambda''^2}{\lambda'}+\frac{\lambda' C_1}{2\varrho A_1}=0.$$

In Eq. (4.29), A_1 is a constant when the Helmholtz free energy function A for the mixture is linear in I_1 (a "neo-Hookean-type" mixture). Then,

$$(4.30)\quad \lambda'''-\frac{3}{2}\frac{\lambda''^2}{\lambda'}=C\lambda',$$

where

$$C = \frac{-C_1}{2\varrho A_1}.$$

When $C > 0$, it can be shown that

$$\lambda'(Z) = \frac{1}{\left[A_1 \sin\sqrt{\frac{C}{2}}\,Z + B_1 \cos\sqrt{\frac{C}{2}}\,Z\right]^2} = \eta(A_1, B_1, C, Z). \tag{4.31}$$

Consider a mixture layer of thickness H, fixed at the bottom, whose deformed thickness s h. The appropriate boundary conditions are

$$z(0) = 0, \tag{4.32}$$

$$z(H) = h. \tag{4.33}$$

Then,

$$z(Z) = \lambda(Z) = \int_0^Z \eta(A_1, B_1, C, \bar{Z})\,d\bar{Z}. \tag{4.34}$$

When $C < 0$,

$$\lambda'(Z) = \frac{1}{\left[A_2 e^{\sqrt{\frac{C'}{2}}\,Z} + B_2 e^{-\sqrt{\frac{C'}{2}}\,Z}\right]^2}, \tag{4.35}$$

where $C' = -C$, and $C' > 0$.

When $C = 0$,

$$\lambda'(Z) = \text{const}, \tag{4.36}$$

is a solution to Eq. (4.30), which corresponds to the classical solution.

Now, consider the general case where the density of the solid is a function of the space coordinates. For this case the equilibrium equations for the mixture (4.25) and the solid (4.26) reduce to

$$f^2 f' \lambda'' + (f^2 f'' + 2ff'^2 - Cf^3)\lambda' + \left(\frac{\varrho_{10} - \varrho_{20}}{\varrho_{20}}\right) f'' = 0, \tag{4.37}$$

and

$$f'^2 + Kf^2 = 0, \tag{4.38}$$

respectively, where $K = \dfrac{C_1}{2\varrho_{20} A_1}$. Equation (4.38) can be solved independently of Eq. (4.37) to obtain $f(Z)$.

When $K < 0$, let $K = -a^2$, $a > 0$. Equation (4.38) has solutions given by

$$f_1 = \beta_1 e^{az}, \tag{4.39}$$

$$f_2 = \beta_2 e^{-az}. \tag{4.40}$$

When $K > 0$, Eq. (4.38) has imaginary solutions, which are not physically meaningful.

When $K = 0$, Eq. (4.38) admits the classical solution,

$$f(Z) = \text{const}. \tag{4.41}$$

Equation (4.37) can be used to obtain the transverse stretch ratio $\lambda(Z)$ corresponding to $f_1(Z)$ and $f_2(Z)$ given by Eq. (4.39) and (4.40). Substituting Eq. (4.39) in Eq. (4.37) gives

$$\lambda'' + 4a\lambda' - \gamma_1 e^{-2aZ} = 0, \tag{4.42}$$

where

$$\gamma_1 = \frac{\varrho_{10} - \varrho_{20}}{\varrho_{20}} \frac{a}{\beta_1^2},$$

which admits a one-parameter family of solutions

$$\lambda(Z) = -\frac{\gamma_1}{4a^2} e^{-2aZ} + L_1 e^{-4aZ}. \tag{4.43}$$

Substituting Eq. (4.40) in Eqs. (4.37) gives

$$\lambda'' - 4a\lambda' - \gamma_2 e^{2aZ} = 0,$$

where

$$\gamma_2 = \frac{\varrho_{10} - \varrho_{20}}{\varrho_{20}} \frac{a}{\beta_2},$$

which admits a one parameter family of solutions

$$\lambda(Z) = -\frac{\gamma_2}{4a^2} e^{2aZ} + L_1 e^{4aZ}. \tag{4.44}$$

Corresponding to the classical solution for $f(Z)$ given by (4.41), Eq. (4.37) admits the classical solution given by (4.36). Figure 1 shows the variation in the deformation along the thickness of the layer with respect to the reference coordinate Z for various values

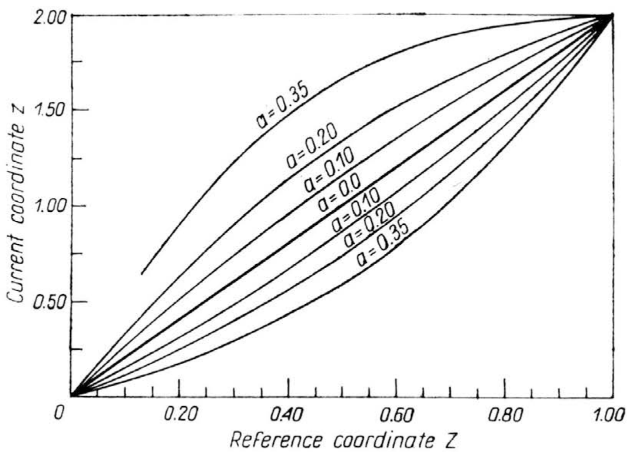

FIG. 1. Variation of deformation along the thickness of the layer.

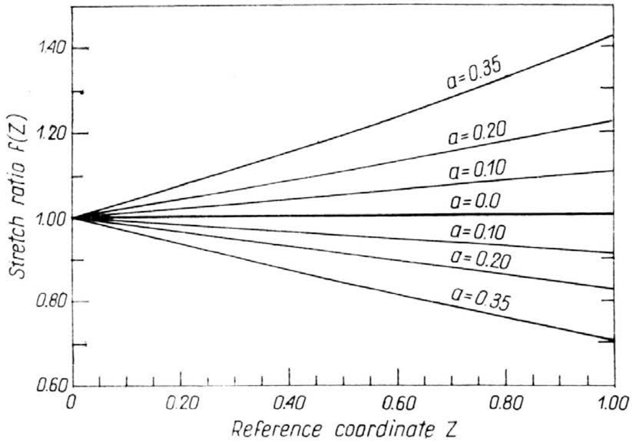

FIG. 2. Variation of stretch ratio $f(\acute{C})$ as a function of reference coordinate Z.

of the parameter a. The appropriate boundary conditions used in obtaining the results presented in Fig. 1 by using equation (4.43) are:

$$z(1) = 2 \quad \text{and} \quad z(0) = 0.$$

Figure 2 shows the corresponding variation in lateral stretch ratio $f(Z)$ with respect to the reference coordinate Z for various values of the parameter a.

5. Conclusion

In this work, the possibility of an axial variation of the stretch ratio for the problem of uniaxial extension of a mixture of a nonlinearly elastic solid and an ideal fluid has been demonstrated. In addition to the classical solution, a one-parameter family of solutions has also been presented.

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