Arch. Mech., 40, 2–3, pp. 265–274, Warszawa 1988

Existence and uniqueness of solutions of some mixed problems for ideal incompressible magnetohydrodynamics
Part I. The case of impermeable boundary

W. M. ZAJĄCZKOWSKI (WARSZAWA)

EQUATIONS of magnetohydrodynamics which describe the motion of an ideal incompressible fluid with infinite conductivity in a bounded domain are considered. Vanishing of normal components of velocity and magnetic induction on the boundary are assumed as boundary conditions. Existence and uniqueness of classical solutions (local in time) are proved.

Rozpatrzono równania magnetohydrodynamiki opisujące ruch idealnej nieściśliwej cieczy z nieskończoną przewodnością w obszarze ograniczonym. Jako warunki brzegowe przyjmujemy znikanie składowych normalnych prędkości i indukcji magnetycznej na brzegu. Pokazano istnienie i jednoznaczność klasycznych rozwiązań lokalnych w czasie.

Рассмотрены уравнения магнетогидродинамики, описывающие движение идеальной несжимаемой жидкости с бесконечной электропроводностью в ограниченной области. Как граничные условия принимаем исчезновение нормальных составляющих скорости и магнитной индукции на границе. Показано существование и единственность классических, локальных во времени решений.

1. Introduction

IN THIS PAPER the following systems of equations

$$v_t+v\cdot\nabla v+\nabla\varrho+\frac{1}{4\pi\varrho_0}B\times\operatorname{rot}B=f \quad \text{in } \Omega\times]0,T[\,\equiv\Omega^T, \tag{1.1}$$

$$B_t+v\cdot\nabla B-B\cdot\nabla v=0 \quad \text{in } \Omega^T, \tag{1.2}$$

$$\operatorname{div}v=0,\quad \operatorname{div}B=0 \quad \text{in } \Omega^T, \tag{1.3}$$

$$v|_{t=0}=v_0,\quad B|_{t=0}=B_0 \quad \text{in } \Omega, \tag{1.4}$$

$$v_n|_S=0,\quad B_n|_S=0 \quad \text{on } S\times]0,T[\,\equiv S^T \tag{1.5}$$

is considered in a bounded domain $\Omega\subset\mathbf{R}^3$ with a boundary S. Here $v_n\equiv v\cdot\bar{n}$, $B_n\equiv B\cdot\bar{n}$ and $\bar{n}=\bar{n}(x)$ is the unit outward vector normal to the boundary. The Eqs. (1.1)–(1.3) describe a motion of ideal incompressible infinitely conductive fluid in a magnetic field where $v=v(x,t)$ is the velocity, $p=p(x,t)$ the pressure, ϱ_0 the constant density, $B=B(x,t)$ the magnetic inducton, $f=f(x,t)$ the external force field.

From Eqs. (1.3)–(1.5) the following compatibility conditions are found

$$\operatorname{div}v_0=0,\quad \operatorname{div}B_0=0 \quad \text{in } \Omega, \tag{1.6}$$

$$v_0\cdot\bar{n}=0,\quad B_0\cdot\bar{n}=0 \quad \text{on } S. \tag{1.7}$$

It is shown in [4] that the problem (1.1)–(1.5) is well-posed. Hence, in order to prove the existence of solutions of the problem, there are introduced new quantities (see [4])

$$(1.8)\qquad \omega = \frac{B}{\sqrt{4\pi\varrho_0}}, \quad \alpha = v+\omega, \quad \beta = v-\omega, \quad q = p+\frac{B^2}{8\pi\varrho_0},$$

so that from the initial conditions (1.4) we define

$$(1.9)\qquad \omega_0 = \frac{B_0}{\sqrt{4\pi\varrho_0}}, \quad \alpha_0 = v_0+\omega_0, \quad \beta_0 = v_0-\omega_0.$$

Using the quantities (1.8) we replace the problem (1.1)–(1.5) by an equivalent system of problems (see [4])

$$(A)\qquad \begin{aligned} &\alpha_t+\beta\cdot\nabla\alpha = f-\nabla q && \text{in} \quad \Omega^T,\\ &\alpha|_{t=0} = \alpha_0 && \text{in} \quad \Omega,\\ &\beta_n = 0 && \text{on} \quad S^T, \end{aligned}$$

where β and q are considered to be known functions,

$$(B)\qquad \begin{aligned} &\beta_t+\alpha\cdot\nabla\beta = f-\nabla q && \text{in } \Omega^T,\\ &\beta|_{t0} = \beta_0 && \text{in } \Omega,\\ &\alpha_n = 0 && \text{on } S^T; \end{aligned}$$

Here α and q are treated as given quantities; finally,

$$(E)\qquad \begin{aligned} &\Delta q = \operatorname{div} f-\nabla_j\alpha_i\nabla_i\beta_j && \text{in} \quad \Omega\times\{t\},\\ &\frac{\partial q}{\partial n} = f_n+n_{i,x_j}\alpha_i\beta_j && \text{on} \quad S\times(t\},\\ &\int_\Omega q = 0, \end{aligned}$$

where α, β are prescribed.

The aim of this paper is to prove the existence and uniqueness of solutions to the problem (1.1)–(1.5), which is replaced by the equivalent problem (A, B, E). The existence of classical local solutions is proved (see Theorem 1). Uniqueness is stated in Theorem 2. This paper is based mostly on [2].

2. Notations

We assume

$$\|u\|_{W_p^l(\Omega)} \equiv \|u\|_{l,p}, \quad \|u\|_{L_p(0,T,W_r^l(\Omega))} = \|u\|_{l,r,p,\Omega^T}, \quad l\in \mathbf{N},$$
$$r,p\in R, \quad 1\leqslant r,p\leqslant\infty.$$

For non-integer l we set

$$\|u\|_{W_p^l(S)} \equiv \|u\|_{l,p,S}.$$

Finally, the summation convention over repeated indices is assumed.

3. Existence of solutions

To prove the existence of solutions of the problem (1.1)–(1.5) we shall use the following method of successive approximations (see [2])

$$\binom{m+1}{A}\qquad \begin{aligned} &\overset{m+1}{\alpha'_t} + \overset{m}{\beta}\cdot\nabla\overset{m+1}{\alpha'} = f-\nabla\overset{m}{q} &&\text{in } \Omega^T,\\ &\overset{m+1}{\alpha'}|_{t=0} = \alpha_0 &&\text{in } \Omega, \end{aligned}$$

where $\overset{m}{q}$ and $\overset{m}{\beta}$ are given and

$$\overset{m}{\beta_p}|_S = 0 \qquad \text{on } S^T, \tag{3.1}$$

$$\binom{m+1}{B}\qquad \begin{aligned} &\overset{m+1}{\beta'_t} + \overset{m}{\alpha}\cdot\nabla\overset{m+1}{\beta} = f-\nabla\overset{m}{q} &&\text{in } \Omega^T,\\ &\overset{m+1}{\beta}|_{t=0} = \beta_0 &&\text{in } \Omega, \end{aligned}$$

where $\overset{m}{q}$ and $\overset{m}{\alpha}$ are given and

$$\overset{m}{\alpha_n}|_S = 0 \quad \text{on } S^T. \tag{3.2}$$

Finally, for given $\overset{m}{\alpha}$ and $\overset{m}{\beta}$, total pressure $\overset{m}{q}$ is determined for the Neumann problem

$$\binom{m}{E}\qquad \begin{aligned} &\Delta\overset{m}{q} = \operatorname{div} f - \nabla_i\overset{m}{\alpha_j}\nabla_j\overset{m}{\beta_i} \quad \text{in } \Omega\times\{t\}, \quad t\in\,]0, T[,\\ &\frac{\partial\overset{m}{q}}{\partial n} = f_n + n_{i,x_j}\overset{m}{\alpha_i}\overset{m}{\beta_j} \quad \text{on } S\times\{t\},\\ &\int_\Omega \overset{m}{q} = 0. \end{aligned}$$

In the above formulations we assume $m = 0, 1, \ldots,$ and $\overset{0}{\alpha} = \alpha_0$, $\overset{0}{\beta} = \beta_0$ are such that

$$\alpha_0\cdot\bar{n}|_S = 0, \quad \beta_0\cdot\bar{n}|_S = 0, \quad \operatorname{div}\alpha_0 = 0, \quad \operatorname{div}\beta_0 = 0. \tag{3.3}$$

Now let us explain why the quantities $\overset{m}{\gamma'}$ (where from now on α and β will be replaced by γ) are introduced and find the relations between $\overset{m}{\gamma'}$ and $\overset{m}{\gamma}$. The functions $\overset{m+}{\alpha'}$ $\overset{m+1}{\beta'}$ determined by the problems $\binom{m+1}{A}$, $\binom{m+1}{B}$, respectively, are such that, in general, div $\overset{m+1}{\gamma'} \neq 0$ and $\overset{m+1}{\gamma'_n}|_S = 0$. But the problem $(\overset{m}{E})$ will have solution if the compatibility condition for the Neumann problem is satisfied what can be fulfilled if

$$\operatorname{div}\overset{m}{\gamma} = 0 \quad \text{in } \Omega^T, \quad \overset{m}{\gamma_n} = 0 \quad \text{on } S^T, \quad m = 1, 2, \ldots. \tag{3.4}$$

Therefore, in order to satisfy (3.4) we introduce the projections

$$\overset{m+1}{\pi_\gamma}\overset{m+1}{\gamma'} = \overset{m+1}{\gamma'} - \nabla\overset{m+1}{\varphi_\gamma} \tag{3.5}$$

such that

$$(3.6)\quad \begin{aligned} &\Delta \overset{m+1}{\varphi_\gamma} = \operatorname{div} \overset{m+1}{\gamma'} \quad \text{in } \Omega^T, \\ &\frac{\partial \overset{m+1}{\varphi_\gamma}}{\partial n} = \overset{m+1}{\gamma'} \cdot \bar{n} \quad \text{on } S^T, \\ &\int_\Omega \overset{m+1}{\varphi_\gamma} = 0, \end{aligned}$$

and then we assume

$$(3.7)\quad \overset{m+1}{\gamma} = \overset{m+1}{\pi_\gamma} \overset{m+1}{\gamma'}.$$

Now let us determine the existence of the presented sequence and the necessary a priori estimates to prove the existence of solutions of the problem (A, B, E). Assume that $\overset{m}{\alpha}$, $\overset{m}{\beta} \in C^0([0, T]; W_r^2(\Omega))$, $r > 3$, α_0, $\beta_0 \in W_r^2(\Omega)$ and satisfy Eq. (3.4). Moreover, $\nabla \overset{m}{q}, f \in L_1(0, T, W_r^2(\Omega))$.

Then the existence of solution to problems $\binom{m+1}{A}$, $\binom{m+1}{B}$ is proved by means of the method of characteristics. Moreover, by applying D_x^σ to $\binom{m+1}{A}_1$ $(|\sigma| \leqslant 2)$, by multiplying it by $D_x^\sigma \overset{m+1}{\alpha'} |D_x^\sigma \overset{m+1}{\alpha'}|^{r-2}$, integrating over Ω and repeating the same procedure also for $\binom{m+1}{B}_1$, one obtains

$$(3.8)\quad \frac{d}{dt} \|\overset{m+1}{\alpha'}\|_{2,r}^r \leqslant c\|\overset{m}{\beta}\|_{2,r}^r \|\overset{m+1}{\alpha'}\|_{2,r}^r + c(\|f\|_{2,r} + \|\nabla \overset{m}{q}_{2,r})\| \overset{m+1}{\alpha'}\|_{2,r}^{r-1},$$

$$(3.9)\quad \frac{d}{dt} \|\overset{m+1}{\beta'}\|_{2,r}^r \leqslant c\|\overset{m}{\alpha}\|_{2,r} \|\overset{m+1}{\beta'}\|_{2,r}^r + c(\|f\|_{2,r} + \|\nabla \overset{m}{q}\|_{2,r})\| \overset{m+1}{\beta'}\|_{2,r}^{r-1}$$

(here and in the sequel each constant depends at most on r and Ω).

Integrating the equations (3.8) and (3.9) with respect to time one obtains

$$(3.10)\quad \begin{aligned} &\|\overset{m+1}{\alpha'}(t)\|_{2,r} \leqslant [\|\alpha_0\|_{2,r} + c\int_0^t (\|f\|_{2,r} + \|\nabla \overset{m}{q}\|_{2,r}) dt'] \exp(ct\|\overset{m}{\beta}\|_{2,r,\infty,\Omega^t}), \\ &\|\overset{m+1}{\beta'}(t)\|_{2,r} \leqslant [\|\beta_0\|_{2,r} + c\int_0^t (\|f\|_{2,r} + \|\nabla \overset{m}{q}\|_{2,r}) dt'] \exp(ct\|\overset{m}{\alpha}\|_{2,r,\infty,\Omega^t}), \end{aligned}$$

so $\overset{m+1}{\alpha'}$, $\overset{m=1}{\beta'} \in L_\infty(0, T, W_r^2\Omega)$. Now let us prove that $\overset{m+1}{\alpha'}$, $\overset{m+1}{\beta'} \in C^0([0, T]; W_r^2(\Omega))$. Applying once more D_x^σ to $\binom{m+1}{A}_1(|\sigma| \leqslant 2)$, multiplying by $D_x^\sigma \overset{m+1}{\alpha'} |D_x^\sigma \overset{m+1}{\alpha'}|^{r-2}$, integrating the result over Ω and, next, with respect to time from t' to t (and repeating the same procedure for $\binom{m+1}{B}_1$) one obtains

$$(3.11)\qquad \|\overset{m+1}{\alpha'}(t)\|_{2,r}^{r}-\|\overset{m+1}{\alpha'}(t')\|_{2,r}^{r} = \sum_{|\sigma|\leqslant 2}\int_{t'}^{t} D_x^{\sigma}[-\beta\cdot\nabla\overset{m+1}{\alpha'} +f-\nabla\overset{m}{q}]D_x^{\sigma}\overset{m+1}{\alpha'}|D_x^{\sigma}\overset{m+1}{\alpha'}|^{r-2}\,dx\,dt,\qquad t\geqslant t',$$

and

$$(3.12)\qquad \|\overset{m+1}{\beta'}(t)\|_{2,r}^{r}-\|\overset{m+1}{\beta'}(t')\|_{2,r}^{r} = \sum_{\sigma\leqslant 2}\int_{t'}^{t} D_x^{\sigma}[-\alpha\cdot\nabla\overset{m+1}{\beta'} +f-\nabla\overset{m}{q}]\cdot D_x^{\sigma}\overset{m+1}{\beta'}|D_x^{\sigma}\overset{m+1}{\beta'}|^{r-2}\,dx\,dt,\qquad t\geqslant t'.$$

From (3.11) and (3.12), using Eqs. (3.1), (3.2), estimates (3.10) and the fact that f, $\nabla\overset{m}{q}\in L_1(0,T;W_r^2(\Omega))$ it follows that

$$(3.13)\qquad \left|\|\overset{m+1}{\alpha'}(t)\|_{2,r}^{r}-\|\overset{m+1}{\alpha'}(t')\|_{2,r}^{r}\right| \leqslant \Big[|t-t'|\,\|\beta\|_{2,r,\infty,\Omega^T}\cdot\|\overset{m+1}{\alpha'}\|_{2,r,\infty,\Omega^T}+\int_{t'}^{t}(\|f\|_{2,r}+\|\nabla\overset{m}{q}\|_{2,r})dt\Big]\|\overset{m+1}{\alpha'}\|_{2,r,\infty,\Omega^T}^{r-2},$$

and

$$(3.14)\qquad \left|\|\overset{m+1}{\beta'}(t)\|_{2,r}^{r}-\|\overset{m+1}{\beta'}(t')\|_{2,r}^{r}\right| \leqslant \Big[|t-t'|\,\|\overset{m}{\alpha}\|_{2,r,\infty,\Omega^T}\cdot\|\overset{m+1}{\beta'}\|_{2,r,\infty,\Omega^T}+\int_{t'}^{t}(\|f\|_{2,r}+\|\nabla\overset{m}{q}\|_{2,r})dt\Big]\|\overset{m+1}{\beta'}\|_{2,r,\infty,\Omega^T}^{r-1}.$$

So, by the theorem on the absolute continuity of the integral (see for instance [1], p. 63) we have proved the theorem.

Due to the properties of $\overset{m+1}{\alpha'}$, $\overset{m+1}{\beta'}$ shown above we conclude that solutions of the problem (3.6) belong to $C^0([0,T];W_r^2(\Omega))$ (the existence easily follows because the necessary compatibility condition for the Neumann problem is trivially satisfied); hence $\overset{m+1}{\alpha}$, $\overset{m+1}{\beta}$ also belong to $C^0([0,T;W_r^2(\Omega))$, and

$$(3.15)\qquad \|\overset{m+1}{\gamma}\|_{2,r,\infty,\Omega^T}\leqslant c\|\overset{m+1}{\gamma'}\|_{2,r,\infty,\Omega^T}.$$

Similarly, from the problem $(\overset{m}{\mathrm{E}})$ for $S\in C^4$ follows the existence of $\overset{m}{q}\in L_1(0,T;W_r^3(\Omega))$ such that

$$(3.16)\qquad \|\overset{m}{q}\|_{3,r,1,\Omega^T}\leqslant c[\|f\|_{2,r,1,\Omega^T}+t\|\overset{m}{\alpha}\|_{2,r,\infty,\Omega^T}\|\overset{m}{\beta}\|_{2,r,\infty,\Omega^T}].$$

Introducing the quantity

$$(3.17)\qquad \overset{m}{y}(t)=\|\overset{m}{\alpha}\|_{2,r,\infty,\Omega^t}+\|\overset{m}{\beta}\|_{2,r,\infty,\Omega^t}$$

and

$$y_0=\|\alpha_0\|_{2,r}+\|\beta_0\|_{2,r}$$

from Eqs. (3.10), (3.15), (3.16) one gets

$$\overset{m+1}{y}(t) \leqslant y_0 + c\left[\int_0^t \|f(t')\|_{2,r} dt' + t\overset{m}{y}{}^2(t)\right] e^{ct\overset{m}{y}(t)}. \tag{3.18}$$

Let $\varrho > 1$ and let $\overset{m}{y}(t) \leqslant \varrho y_0$. Then there exists time $t_1(\varrho)$ such that the following inequality

$$y_0 + c\left[\int_0^t \|f(t')\|_{2,r} dt' + t\varrho^2 y_0^2\right] e^{ct\varrho y_0} \leqslant \varrho y_0 \tag{3.19}$$

is valid for $t \leqslant t_1(\varrho)$. Hence we have obtained the estimate

$$\overset{m}{y}(t) \leqslant \varrho y_0, \quad \text{for} \quad m = 0, 1, \ldots, \quad \text{and} \quad t \leqslant t_1(\varrho). \tag{3.20}$$

Therefore we have proved

LEMMA 1

Let $S \in C^4$, α_0, $\beta_0 \in W_r^2(\Omega)$, $f \in L_1(0, t; W_r^2(\Omega))$, $r > 3$. Let $\varrho > 1$. Then for $t \leqslant t_1(\varrho)$, where $t_1(\varrho)$ is a solution of the equality in (3.19), the estimate (3.20) is valid, $\overset{m}{y}(t)$ being determined by (3.17).

REMARK 1

To obtain the estimates (3.8), (3.9), (3.13), (3.14), the third derivatives of $\overset{m+1}{\alpha'}$, $\overset{m+1}{\beta'}$ are required to belong to suitable spaces; this, however, is not so important because they do not enter the final estimates (here density-type arguments must be used).

To prove the convergence of the sequences $\{\overset{m}{\gamma}, \overset{m}{\gamma'}, \overset{m}{q}, \overset{m}{\varphi_\gamma}\}$, the following problems must be considered:

$$\binom{m+1}{a} \qquad \begin{gathered} \overset{m+1}{A'_t} + \overset{m}{\beta} \cdot \nabla \overset{m+1}{A'} + \overset{m}{B} \cdot \nabla \overset{m}{\alpha'} = -\nabla \overset{m}{Q}, \\ \overset{m+1}{A'}|_{t=0} = 0, \\ \overset{m+1}{\beta_n}|_S = 0, \end{gathered}$$

where

$$\overset{m}{A'} = \overset{m}{\alpha'} - \overset{m-1}{\alpha'}, \quad \overset{m}{A} = \overset{m}{\alpha} - \overset{m-1}{\alpha}, \quad \overset{m}{B} = \overset{m}{\beta} - \overset{m-1}{\beta},$$

$$\overset{m}{Q} = \overset{m}{q} - \overset{m-1}{q}, \quad \overset{0}{A} = \alpha_0, \quad \overset{0}{B} = \beta_0,$$

$$\binom{m+1}{b} \qquad \begin{gathered} \overset{m+1}{B'_t} + \overset{m}{\alpha} \cdot \nabla \overset{m+1}{B'} + \overset{m}{A} \cdot \nabla \overset{m}{\beta'} = -\nabla \overset{m}{Q}, \\ \overset{m+1}{B'}|_{t=0} = 0, \\ \overset{m}{\alpha_n}|_S = 0, \end{gathered}$$

where

$$\overset{m}{B'} = \overset{m}{\beta'} - \overset{m-1}{\beta'},$$

$$\binom{m}{e} \qquad \begin{aligned} \Delta \overset{m}{Q} &= -(\nabla_i \overset{m}{A_j} \nabla_j \overset{m}{\beta} + \nabla_i \overset{m-1}{\alpha_j} \nabla_j \overset{m}{B_i}), \\ \left.\frac{\partial \overset{m}{Q}}{\partial n}\right|_S &= n_{i,x_j}(\overset{m}{A_i} \overset{m}{\beta_j} + \overset{m-1}{\alpha_i} \overset{m}{B_j}), \\ \int_\Omega \overset{m}{Q} &= 0, \end{aligned}$$

$$(3.21) \qquad \begin{aligned} \Delta \overset{m+1}{\Phi_\gamma} &= \operatorname{div} \overset{m+1}{\Gamma'}, \\ \left.\frac{\partial \overset{m+1}{\Phi_\gamma}}{\partial n}\right|_S &= \overset{m+1}{\Gamma'} \cdot \bar{n}|_S, \\ \int_\Omega \overset{m+1}{\Phi} &= 0, \end{aligned}$$

where

$$\overset{m}{\Phi_\gamma} = \overset{m}{\varphi_\gamma} - \overset{m-1}{\varphi_\gamma}, \quad \overset{m}{\Gamma'} = \overset{m}{\gamma'} - \overset{m-1}{\gamma'}.$$

Finally, we have the relations

$$(3.22) \qquad \overset{m+1}{\Gamma} = \overset{m+1}{\Gamma'} - \nabla \overset{m+1}{\Phi_\gamma},$$

were

$$\overset{m}{\Gamma} = \overset{m}{\gamma} - \overset{m-1}{\gamma}.$$

From the problems $\binom{m+1}{a}$ and $\binom{m+1}{b}$ it follows that

$$(3.23) \qquad \frac{d}{dt}(\|\overset{m+1}{A'}\|^r_{1,r} + \|\overset{m+1}{B'}\|^r_{1,r}) \leqslant c(\|\overset{m}{\alpha}\|_{2,r} + \|\overset{m}{\beta}\|_{2,r}) \cdot (\|\overset{m+1}{A'}\|^r_{1,r} + \|\overset{m+1}{B'}\|^r_{1,r}) + c\|\overset{m}{\alpha'}\|_{2,r}\|\overset{m}{B}\|_{1,r}\|\overset{m+1}{A'}\|^{r-1}_{1,r} + c\|\overset{m}{\beta'}\|_{2,r} \cdot \|\overset{m}{A}\|_{1,r}\|\overset{m+1}{B'}\|^{r-1}_{1,r} + c\|\overset{m}{Q}\|_{2,r}(\|\overset{m+1}{A'}\|^{r-1}_{1,r} + \|\overset{m+1}{B'}\|^{r-1}_{1,r}).$$

Introducing the new quantity

$$(3.24) \qquad \overset{m}{Y'}(t) = \|\overset{m}{A'}\|_{1,r,\infty,\Omega^t} + \|\overset{m}{B'}\|_{1,r,\infty,\Omega^t}$$

and $\overset{m}{Y}(t)$ for $\overset{m}{A}$ and $\overset{m}{B}$, after integration of Eq. (3.23) with respect to time, we obtain

$$(3.25) \qquad \overset{m+1}{Y'}(t) \leqslant e^{\tilde{c}t}\Big[\tilde{c}t\overset{m}{Y}(t) + c\int_0^t \|\overset{m}{Q}(t')\|_{2,r}dt'\Big], \quad m = 0, 1, \ldots,$$

where $\tilde{c} = c\varrho y_0$, use being made of the equality

$$(\|\overset{m+1}{A'}\|_{1,r} + \|\overset{m+1}{B'}\|_{1,r})|_{t=0} = 0.$$

The problem $\binom{m+1}{e}$ implies

$$(3.26) \qquad \|\overset{m}{Q}\|_{2,r,1,\Omega^t} \leqslant \tilde{c}t\overset{m}{Y}(t)$$

and from (3.21) and (3.22) we have

$$\overset{m+1}{Y}(t) \leqslant c\overset{m+1}{Y'}(t). \tag{3.27}$$

Using (3.26) and (3.27) in (3.25) we get

$$\overset{m+1}{Y}(t) < \tilde{c}te^{\tilde{c}t}\overset{m}{Y}(t), \quad m = 0, 1, \ldots. \tag{3.28}$$

Knowing that $\overset{0}{Y} = ||\alpha_0||_{1,r}+||\beta_0||_{1,r} \equiv Y_0$, from the inequality (3.28) and sufficiently small $t \leqslant t_2$ $(\tilde{c}, Y_0)$, it follows that the sequence $\{\overset{m}{\gamma}, \overset{m}{\gamma'}, \overset{m}{q}, \overset{m}{\varphi_\gamma}\}$ converges to a solution of the problems (A', B', E) and (3.6) for the limit function γ, γ', q, φ_γ and $\gamma' = \pi_\gamma\gamma$ (where (A'), (B') denote problems $(\overset{m}{A'})$, $(\overset{m}{B'})$ for the limit functions).

It remains to show that for the limit functions we have $\gamma = \gamma'$, so that div $\gamma' = 0$, $\gamma'\cdot\bar{n}|_S = 0$. Then $\varphi_\gamma = 0$ and γ, q are solutions of the problem (A, B, E). To show it let us use [2]. Taking the divergence of (A') and (B') and using (E), (3.5), (3.7) one obtains

$$\begin{aligned}(\operatorname{div}\alpha')_t+\beta\cdot\nabla\operatorname{div}\alpha'+\nabla_i\beta_j\nabla_i\nabla_j\varphi_\alpha &= 0 \quad \text{in} \quad \Omega^T,\\ (\operatorname{div}\beta')_t+\alpha\cdot\nabla\operatorname{div}\beta'+\nabla_i\alpha_j\nabla_i\nabla_j\varphi_\beta &= 0 \quad \text{in} \quad \Omega^T.\end{aligned} \tag{3.29}$$

Projecting the normal components of (A') and (B') on S and using (E), (3.5), (3.7) one obtains

$$\begin{aligned}\alpha'_{n,t}+\beta\cdot\nabla\alpha'_n &= \beta\cdot\nabla n_i\nabla_i\varphi_\alpha \quad \text{on} \quad S^T,\\ \beta'_{n,t}+\alpha\cdot\nabla\beta'_n &= \alpha\cdot\nabla n_i\nabla_i\varphi_\beta \quad \text{on} \quad S^T.\end{aligned} \tag{3.30}$$

Using the problem (3.6), we obtain from (3.29)

$$\begin{aligned}\frac{d}{dt}||\operatorname{div}\alpha'||_{0,2} &\leqslant c||\nabla\beta||_{0,\infty}||\varphi_\alpha||_{2,2} \leqslant c||\nabla\beta||_{0,\infty}\cdot(||\operatorname{div}\gamma'||_{0,2}+||\alpha'_n||_{1/2,2,S}),\\ \frac{d}{dt}||\operatorname{div}\beta'||_{0,2} &\leqslant c||\nabla\alpha||_{0,\infty}||\varphi_\beta||_{2,2} \leqslant c||\nabla\alpha||_{0,\infty}\cdot(||\operatorname{div}\beta'||_{0,2}+||\beta'_n||_{1/2,2,S}).\end{aligned} \tag{3.31}$$

To obtain the estimates for α'_n, β'_n on S, we introduce the following curves on S

$$\begin{aligned}\frac{d}{ds}y(x,t;s) &= \alpha\big(y(x,t;s),s\big) \quad \text{on} \quad S^T, \quad y(x,t;t) = x \quad \text{on} \quad S,\\ \frac{d}{ds}z(x,t;s) &= \beta\big(z(x,t;s),s\big) \quad \text{on} \quad S^T, \quad z(x,t;t) = x \quad \text{on} \quad S.\end{aligned} \tag{3.32}$$

Therefore Eq. (3.30) can be written in the form

$$\begin{aligned}\frac{d}{ds}\alpha'_n\big(z(x,t;s),s\big) &= \beta_i\big(z(x,t;s),s\big)\nabla_{z_i}n_j\big(z(x,t;s)\big)\nabla_{z_j}\varphi_\alpha\big(z(x,t;s),s\big),\\ \frac{d}{ds}\beta'_n\big(y(x,t;s),s\big) &= \alpha_i\big(y(x,t;s),s\big)\nabla_{y_i}n_j\big(y(x,t;s)\big)\nabla_{y_j}\varphi_\beta\big(y(x,t;s),s\big).\end{aligned} \tag{3.33}$$

Setting $\tilde{\alpha}'(x,\tau;t) = \alpha'\big(z(x,\tau;t),t\big)$, $\tilde{\beta}'(x,\tau;t) = \beta'\big(y(x,\tau;t),t\big)$ and proceeding similarly with respect to other functions from (3.33), one gets

(3.34)
$$\frac{d}{dt}||\tilde{\alpha}'_n||_{1/2,2,S} \leqslant c||\beta||_{2,r}||\tilde{\varphi}_\alpha||_{3/2,2,S},$$
$$\frac{d}{dt}||\tilde{\beta}'_n||_{1/2,2,S} \leqslant c||\alpha||_{2,r}||\tilde{\varphi}_\beta||_{3/2,2,S}.$$

Using the inequalities

(3.35)
$$c\exp(-ct||\alpha||_{2,r,\infty,\Omega^t})||\beta'_n||_{1/2,2,S} \leqslant ||\beta'_n||_{1/2,2,S} \leqslant c\exp(ct||\alpha||_{2,r,\infty,\Omega})||\tilde{\beta}'_n||_{1/2,2,S};$$
$$c\exp(-ct||\beta||_{2,r,\infty,\Omega^t})||\alpha'_n||_{1/2,2,S} \leqslant ||\alpha'_n||_{1/2,2,S} \leqslant c\exp(ct||\beta||_{2,r,\infty,\Omega^t})||\tilde{\alpha}'_n||_{1/2,2,S}$$

and similar inequalities for φ_α and φ_β one obtains from Eqs. (3.34), (3.6)

(3.36)
$$\frac{d}{dt}||\tilde{\alpha}'_n||_{1/2,2,S} \leqslant c_0(\varrho y_0)(||\operatorname{div}\alpha'||_{0,2}+||\tilde{\alpha}'_n||_{1/2,2,S}),$$
$$\frac{d}{dt}||\tilde{\beta}'_n||_{1/2,2,S} \leqslant c_0(\varrho y_0)(||\operatorname{div}\beta'||_{0,2}+||\tilde{\beta}'_n||_{1/2,2,S}).$$

Here condition (3.20) has been used and c_0 denotes a certain function. Using (3.35) at the right-hand side of (3.31) and knowing that $\operatorname{div}\alpha'|_{t=0} = \operatorname{div}\beta'|_{t=0} = \alpha'_n|s|_{t=0} = \beta'_n|s|_{t=0} = 0$, we see that equations (3.31) and (3.36) imply $\operatorname{div}\alpha' = \operatorname{div}\beta' = 0$, $\alpha'_n|_S = \beta'_n|_S = 0$. Therefore it follows that $\varphi_\alpha = \varphi_\beta = 0$, so that $\alpha' = \alpha$, $\beta' = \beta$.

Hence we have proved

THEOREM 1. *Let* $S \in C^4$, α_0, $\beta_0 \in W^2_r(\Omega)$, $\operatorname{div}\alpha_0 = \operatorname{div}\beta_0 = 0$, $\alpha_0\cdot\bar{n}|_S = \beta_0\cdot\bar{n}|_S = 0$, $r > 3$. *Then there exists such* $T = \min\{t_1(\varrho), t_2(c, Y_0)\}$, *sufficiently small* (see (3.19), (3.28)) *that for* $t \in]0, T[$ and $f \in L_1(0; t; W^2_r(\Omega))$ *we have* α, $\beta \in C^0([0;t]; W^2_r(\Omega)) \cap W^1_1(0, t; W^1_r(\Omega))$, $q \in L_1(0, t; W^3_r(\Omega))$, *which are solutions of the problem* (A, B, E) *in* Ω^t.

If $f \in C^0([0, t]; W^2_r(\Omega))$, *then* α, $\beta \in C^0([0, t]; W^2_r(\Omega)) \cap C^1([0, t]; W^1_r(\Omega))$ *and* $q \in C^0([0, t]; W^3_r(\Omega))$, *hence* α, β, q *are classical solutions of* (A, B, E).

From Eq. (1.8) we find the classes to which belong v, B and p.

Finally, let us prove the uniqueness. Let (α_i, β_i, q_i), $i = 1, 2$, be two solutions of the problem (A, B, E). Let $A = \alpha_1-\alpha_2$, $B = \beta_1-\beta_2$, $Q = q_1-q_2$. Then from problems (A), (B), (E) we get

(3.37)
$$A_t+\beta_1\cdot\nabla A+B\cdot\nabla\alpha_2 = -\nabla Q, \quad A|_{t=0} = 0, \quad A_n|_S = 0,$$

(3.38)
$$B_t+\alpha_1\cdot\nabla B+A\cdot\nabla\beta_2 = -\nabla Q, \quad B|_{t=0} = 0, \quad B_n|_S = 0,$$

(3.39)
$$\Delta Q = -\nabla A\nabla\beta_1-\nabla\alpha_2\nabla B \equiv -\operatorname{div}g,$$
$$\frac{\partial Q}{\partial n} = n_{i,x_j}(A_i\beta_{1j}+\alpha_{2i}B_j) \equiv -g\cdot\bar{n},$$
$$\int_\Omega Q = 0,$$

where $g = A\cdot\nabla\beta_1-B\cdot\nabla\alpha_2$. Multiplying $(3.39)_1$ by Q, integrating over Ω and using $(3.39)_2$ one gets

(3.40)
$$||\nabla Q||_{0,2} \leqslant ||g||_{0,2} \leqslant \sup_\Omega(|\nabla\alpha_2|+|\nabla\beta_1|)(||A||_{0,2}+||B||_{0,2}).$$

Multiplying (3.37) by A and (3.38) by B, adding these results, integrating over Ω and assuming

$$z^2(t) = ||A(t)||^2_{0,2} + ||B(t)||^2_{0,2}, \tag{3.41}$$

one gets

$$\frac{d}{dt} z^2 \leqslant 2 \sum_{i=1}^{2} \sup_{\Omega}(|\nabla \alpha_i| + |\nabla \beta_i|) z^2 + 4||\nabla Q||_{0,2} z. \tag{3.42}$$

Using (3.40) in (3.42) and integrating the result with respect to time one obtains

$$z^2(t) \leqslant z^2(0) \exp c \int_0^t \sum_{i=1}^{2} \sup_{\Omega}(|\nabla \alpha_i| + |\nabla \beta_i|) dt. \tag{3.43}$$

Hence we have proved

THEOREM 2. *Let* $\alpha, \beta \in L_1(0, T; W^1_\infty(\Omega))$, $q \in L_1(0, T; W^2_r(\Omega))$, $\forall r > 1$ *be solutions to the problem* (A, B, E). *Then the solutions to the problem* (A, B, E) *are unique in the class* $L_\infty(0, T; L_2(\Omega)) \times L_\infty(0, T; L_2(\Omega)) \times L_\infty(0, T; W^1_2(\Omega))$, *respectively.*

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Received June 30, 1987.