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On well-posed mixed problems for ideal incompressible magnetohydrodynamics

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IN THIS PAPER well-posed initial boundary value problems are found for equations of ideal magnetohydrodynamics describing a motion of an ideal infinitely conductive fluid with constant density in a bounded domain in R^3.

W pracy znaleziono dobrze postawione problemy początkowo brzegowe dla równań idealnej magnetohydrodynamiki opisującej ruch idealnej nieskończenie przewodzącej cieczy ze stałą gęstością w ograniczonym obszarze w R^3.

В работе найдены корректно поставленные начально-краевые задачи для уравнений идеальной магнитогидродинамики, описывающей движение идеальной, бесконечно-проводящей жидкости с постоянной плотностью в ограниченной области в R^3.

1. Introduction

THE AIM of this paper is to present well-posed initial boundary value problems for equations of magnetohydrodynamics describing a motion of an ideal incompressible and infinitely conductive fluid in a bounded domain $\Omega \subset \mathsf{R}^3$ [1]:

$$B_t + v \cdot \nabla B - B \cdot \nabla v = 0, \tag{1.1}$$

$$v_t + v \cdot \nabla v + \nabla p + \frac{1}{4\pi\varrho_0} B \times \operatorname{rot} B = f, \tag{1.2}$$

$$\operatorname{div} v = 0, \tag{1.3}$$

$$\operatorname{div} B = 0, \tag{1.4}$$

where $B = B(x,t)$ is the magnetic induction, $v = v(x,t)$ is the velocity, $p = p(x,t)$ is the pressure, $f = f(x,t)$ is the external force and ϱ_0 is the constant density.

By well-posed initial boundary value problems to Eqs. (1.1)–(1.4) we mean such a problem for which the existence, uniqueness and continuous dependence on data can be proved.

At first we assume the initial conditions

$$v|_{t=0} = v_0(x), \tag{1.5}$$

$$B|_{t=0} = B_0(x), \tag{1.6}$$

which, in view of Eqs. (1.3) and (1.4), satisfy,

$$\operatorname{div} v_0 = 0, \quad \operatorname{div} B_0 = 0. \tag{1.7}$$

To find boundary data for the Cauchy problem (1.1)–(1.6) we replace Eqs. (1.1)–(1.4) by a system of two such problems for which well-posed initial boundary value problems are well known. First we introduce new variables

$$\omega = \frac{B}{\sqrt{4\pi\varrho_0}}, \quad \omega_0 = \frac{B_0}{\sqrt{4\pi\varrho_0}}, \tag{1.8}$$

$$q = p+\frac{B^2}{8\pi\varrho_0}, \tag{1.9}$$

where q is the total pressure because $\boldsymbol{B}^2/(8\pi\varrho_0)$ is the pressure of the magnetic field. Next, using the identity $\boldsymbol{B}\times\operatorname{rot}\boldsymbol{B} = \frac{1}{2}\nabla\boldsymbol{B}^2-\boldsymbol{B}\cdot\nabla\boldsymbol{B}$ in Eq. (1.2) and then adding and subtracting Eqs. (1.1) and (1.2) give

$$\alpha_t+\beta\cdot\nabla\alpha = f-\nabla q, \tag{1.10}$$

$$\beta_t+\alpha\cdot\nabla\beta = f-\nabla q, \tag{1.11}$$

where

$$\alpha = v+\omega, \quad \beta = v-\omega. \tag{1.12}$$

Moreover, using Eqs. (1.5), (1.6) and (1.8) implies

$$\alpha|_{t=0} = \alpha_0 = v_0+\omega_0, \quad \beta|_{t=0} = \beta_0 = v_0-\omega_0. \tag{1.13}$$

Therefore we have obtained the evolution problem (1.10)–(1.13) where the unknown functions are α, β and q is assumed as given. To obtain an equation for q, we apply the divergence operator to Eqs. (1.10) and (1.11) and using Eqs. (1.3) and (1.4), we get

$$\Delta q = \operatorname{div} f-\sum_{i,j=1}^{3}\nabla_i\alpha_j\nabla_j\beta_i, \tag{1.14}$$

where α, β are treated as given and $\nabla_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$.

Therefore we replaced the Cauchy problem (1.1)–(1.6) by two problems: the evolution problem (1.10), (1.11), (1.13) and the Poisson equation (1.14). Firstly there appears the question of equivalence. In Sect. 2 it is shown that Eqs. (1.1)–(1.6) and Eqs. (1.10)–(1.14) are equivalent Cauchy problems ($\Omega = R^3$, see Lemma 1). However this equivalence exists in a class of sufficiently smooth functions because Eq. (1.14) is the second order equation whereas Eqs. (1.1)–(1.4) are only of the first order.

Going back to mixed problems we know what kinds of boundary data may be imposed on the Poisson equation (1.14) and also on the evolution equations (1.10) and (1.11), for which the existence of solutions follows from the method of characteristics. However we cannot prescribe the boundary data arbitrarily. They must be assumed in such a way that the obtained initial boundary value problems to Eqs. (1.10) and (1.11) and the boundary data to Eq. (1.4) imply Eqs. (1.3) and (1.4) (because Eqs. (1.3) and (1.4) are not explicitly included in Eqs. (1.10), (1.11), and (1.14)). In Theorem 1 the proper boundary data to Eqs. (1.10), (1.11) and (1.14) are formulated.

We must emphasize that the obtained boundary data depend on replacing the basic equations (1.1)–(1.4) by Eqs. (1.10), (1.11) and (1.14). In [3] well-posed mixed problems to Eqs. (1.1)–(1.4) are found by replacing Eqs. (1.1)–(1.4) by the symmetric hyperbolic system for v, $\boldsymbol{B}$ and Eq. (1.14), where Eqs. (1.8) and (1.12) must be used. Considering noncharacteristic boundary value problems for the hyperbolic system, we found in [3] a narrower class of boundary conditions than in this paper (see Theorem 2).

However, the problem of finding all suitable boundary data to Eqs. (1.1)–(.14) is still open because we do not know if Eqs. (1.1)–(1.4) can be replaced (or not) by other equations which admit other boundary data.

The idea of finding boundary data was used in [4, 5] for Euler equations describing a motion of an ideal, incompressible fluid.

Equations (1.10) and (1.11) are not known in literature. They describe an evolution of invariants of Alfvén waves which are one kind of Riemann simple waves for equations of ideal magnetohydrodynamics. For Alfvén waves we have q = const, so $\alpha = v+\frac{B}{\sqrt{4\pi\varrho_0}}$, $\beta = v-\frac{B}{4\pi\varrho_0}$ are also constants (where equations with $f = 0$ are considered) (see [6, 7]).

Equations (1.1)–(1.4) contain both elipticity and hyperbolicity, therefore it is understandable that they are replaced by hyperbolic and elliptic equations.

The main result of this paper is formulated in Theorem 2 where three kinds of the simplest well-posed initial boundary value problems to Eqs. (1.1)–(1.4) are described.

2. Formulation of mixed problems

In this section we assume that a solution of the problem (1.1)–(1.7) is as smooth as we need.

First we show the equivalence of Cauchy problems (1.1)–(1.6) and (1.10)–(1.14). Applying the divergence operator to Eqs. (1.10) and (1.11), one gets respectively

$$\Delta q = \operatorname{div} f-\sum_{i,j=1}^{3}\nabla_i\alpha_j\nabla_j\beta_i-(\operatorname{div}\alpha_t+\beta\cdot\nabla\operatorname{div}\alpha), \tag{2.1}$$

$$\Delta q = \operatorname{div} f-\sum_{i,j=1}^{3}\nabla_i\alpha_j\nabla_j\beta_i-(\operatorname{div}\beta_t+\alpha\cdot\nabla\operatorname{div}\beta). \tag{2.2}$$

Therefore by Eq. (1.14) one has

$$(\operatorname{div}\alpha)_{,t}+\beta\cdot\nabla\operatorname{div}\alpha = 0, \tag{2.3}$$

$$(\operatorname{div}\beta)_{,t}+\alpha\cdot\nabla\operatorname{div}\beta = 0, \tag{2.4}$$

so in view of Eqs. (1.7) and (1.13), Eqs. (1.3) and (1.4) are satisfied. The converse implication is also valid because from Eqs. (2.1) and (2.2) and the relations (1.3) and (1.4) we get Eq. (1.14). Hence we have proved

LEMMA 1

The problems (1.1)–(1.6) and (1.10)–(1.14) are equivalent for $\Omega = \mathrm{R}^3$.

Now we have to underline that, contrary to the problem (1.1)–(1.6), one can easily prove the existence of solutions of the problem (1.10)–(1.14).

To formulate the boundary conditions we introduce curvilinear coordinates. Let $\bar{n}(x)$, $\bar{\tau}_1(x)$, $\bar{\tau}_2(x)$ be an orthonormal system of vectors defined in the neighbourhood of a boundary $\partial\Omega = S$, such that $\bar{n}(x)$, $x \in S$ is the unit outward vector normal to the boundary and $\bar{\tau}_1(x)$, $\bar{\tau}_2(x)$, $x \in S$ are tangent to S. Moreover, by $n(x)$, $\tau_1(x)$, $\tau_2(x)$, we denote orthonormal curvilinear coordinates corresponding to the above system of vectors such that $n(x) = 0$ describes the boundary locally and $\tau_1(x)$, $\tau_2(x)$ are tangent coordinates on it.

Now we use the results for the Euler equations [4, 5]. First we introduce characteristics to Eqs. (1.10) and (1.11), respectively,

$$
\begin{aligned}
&\frac{d\gamma_i(x,t;s)}{ds} = \delta_i(\gamma_i(x,t;s),s),\\
&\gamma_i(x,t;t) = x,
\end{aligned}
\tag{2.5}
$$

$i = 1, 2$, $\delta_1(x,t) = \alpha(x,t)$, $\delta_2(x,t) = \beta(x,t)$. Therefore Eqs. (1.10) and (1.11) can be written in the form

$$
\frac{d}{ds}\delta_i(\gamma_j(x,t;s),s) = f(\gamma_j(x,t,s),s) - \nabla_{\gamma_j} q(\gamma_j(x,t;s),s),
\tag{2.6}
$$

where $i \neq j$; $i, j = 1, 2$, and $i = 1, j = 2$, $i = 2, j = 1$ are obtained from Eqs. (1.10) and (1.11), respectively. Note that the evolution of α is given along the characteristic, the velocity of which is equal to β and vice versa.

To integrate Eq. (2.6) we distinguish the following kinds of characteristic curves (2.5). Let γ, δ be either γ_1, δ_1 or γ_2, δ_2.

a) $\gamma(x,t,s) \in \Omega$ for any $s \in [0,t]$, $x \in \Omega$.

b) There exists a moment $t^*(x,t) \in (0,t]$ such that $\gamma(x,t,t^*(x,t)) \in S$, $\gamma(x,t;s) \in \Omega$ for $s > t^*(x,t)$ and $\gamma(x,t,s) \notin \Omega$ for $s < t^*(x,t)$. Moreover $x \in \Omega$ and $\delta_n(\gamma(x,t;t^*(x,t))) < 0$ (where $\delta_n = \delta \cdot \bar{n}$ and $\bar{n}$ is the unit outward vector normal to the boundary).

c) There exists a moment $\tau^*(x,t) \in (0,t]$ such that $\gamma(x,t;\tau^*(x,t)) \in S$, $\gamma(x,t;s) \in \Omega$ for $s < \tau^*(x,t)$ and $\gamma(x,t,s) \notin \Omega$ for $s > \tau^*(x,t)$. Moreover $x \notin \Omega$ and $\delta_n(\gamma(x,t;\tau^*(x,t))) > 0$.

In the sequel we shall use (a_1), (a_2), t_1^*, t_2^* and so on for $\gamma_1(x,t;s)$ and $\gamma_2(x,t;s)$, respectively.

Integrating Eqs. (2.6) along the characteristics (2.5), respectively, we get

$$
\begin{aligned}
\delta_i(\gamma_j(x,t;s),s) &= \delta_i(\gamma_j(x,t;\bar{t}(x,t)),\bar{t}(x,t))\\
&\quad + \int\limits_{\bar{t}(x,t)}^{s} [f(\gamma_j(x,t;\tau),\tau) - \nabla_{\gamma_j} q(\gamma_j(x,t;\tau),\tau)]\,d\tau.
\end{aligned}
\tag{2.7}
$$

where we have two cases $i = 1$, $j = 2$ and $i = 2$, $j = 1$.

For the characteristic curves of the family (a) and (c), $\bar{t}(x,t) = 0$ and $\delta(\gamma(x,t;0),0)$ is determined by the initial data (1.13) only because $\gamma(x,t;0) \in \Omega$. For the characteristic curves of the family (b) — $\bar{t}(x,t) = t^*(x,t) \leqslant s$ and to determine solution (2.7) uniquely $\delta|_S = \delta(\gamma, t; t^*(x,t)), t^*(x,t))$ must be prescribed.

The above considerations imply different types of boundary data. Assume that the boundary of Ω may consist of parts S_ν, $\nu = 0, \ldots, 4$, with the following types of boundary conditions.

Let S_1 be such that

$$\alpha_n|_{S_1} < 0, \quad \beta_n|_{S_1} < 0, \tag{2.8}$$

then Eqs. (2.7) and (2.6) imply that all quantities α, β have to be prescribed on S_1, so

$$\alpha|_{S_1} = \eta, \quad \beta|_{S_1} = \vartheta, \tag{2.9}$$

such that the relation (2.8) is satisfied.

Let S_2 be such that either

$$\alpha_n|_{S_2} < 0, \quad \beta_n|_{S_2} > 0, \tag{2.10}$$

or

$$\alpha_n|_{S_2} < 0, \quad \beta_n|_{S_2} = 0, \tag{2.11}$$

be satisfied. Then Eqs. (2.7) and (2.6) imply that β has to be prescribed only, so

$$\beta|_{S_2} = \vartheta, \tag{2.12}$$

where $\vartheta_n \geqslant 0$.

Similarly let S_3 be such that either

$$\alpha_n|_{S_3} > 0, \quad \beta_n|_{S_3} < 0 \tag{2.13}$$

or

$$\alpha_n|_{S_3} = 0, \quad \beta_n|_{S_3} < 0 \tag{2.14}$$

be satisfied. Then Eqs. (2.7) and (2.6) imply that α has to be given only, so

$$\alpha|_{S_3} = \eta, \tag{2.15}$$

where $\eta_n \geqslant 0$. Let S_4 be such that

$$\alpha_n|_{S_4} > 0, \quad \beta_n|_{S_4} > 0, \tag{2.16}$$

$$\begin{aligned} \alpha_n|_{S_4} = 0, \quad \beta_n|_{S_4} > 0, \\ \alpha_n|_{S_4} > 0, \quad \beta_n|_{S_4} = 0. \end{aligned} \tag{2.17}$$

The above boundary conditions are valid for some interval of time determined by theorems of existence. Finally by S_0 we assume

$$\alpha_n|_{S_0} = \beta_n|_{S_0} = 0. \tag{2.18}$$

Then no quantity must be given on S_4 and S_0.

Therefore the above considerations imply that for S_1 curves $(a_i)(b_i)$, $i = 1, 2$, for S_2 curves (a_1), (a_2), (b_1), (c_2), for S_3 curves (a_1), (a_2), (c_1), (b_2) and for S_4 curves (a_i), (c_i), $i = 1, 2$, appear. In the end, for S_0 the curves (a_i), $i = 1, 2$ appear only.

Now we formulate the boundary conditions for Eq. (1.14) in such a way that they imply

$$\operatorname{div}\alpha| = \operatorname{div}\beta| = 0,$$
$$\operatorname{div}\beta|_{S_2} = 0, \tag{2.19}$$
$$\operatorname{div}\alpha|_{S_3} = 0,$$

because then by the definition of S_i, $i = 1, 2, 3$, and Eqs. (2.3)–(2.4) it follows that in the neighbourhood of S_i in Ω, $i = 1, 2, 3$, $\operatorname{div}\alpha$ and $\operatorname{div}\beta$ vanish.

Let us consider the case of S_1 boundary. Projecting the normal components of Eqs. (1.10) and (1.11) on S_1, summing and subtracting the results one gets

$$\bar{n}\cdot\nabla q|_{S_1} = -\frac{1}{2}(\alpha_{n,t}+\beta_{n,t}+\alpha\cdot\nabla\beta\cdot\bar{n}+\beta\cdot\nabla\alpha\cdot\bar{n}-2f_n)|_{S_1} \tag{2.20}$$

and

$$(\alpha_{n,t}-\beta_{n,t}+\beta\cdot\nabla\alpha\cdot\bar{n}-\alpha\cdot\nabla\beta\cdot\bar{n})|_{S_1} = 0. \tag{2.21}$$

Using the curvilinear coordinates the equation $\operatorname{div}\delta|_S = 0$ can be written in the form

$$\Big[\bar{n}\cdot\nabla\delta_n+\delta_n\operatorname{div}\bar{n}+\sum_{i=1}^{2}(\bar{\tau}_i\cdot\nabla\delta_{\tau_i}+\delta_{\tau_i}\operatorname{div}\bar{\tau}_i)\Big]\Big|_S = 0, \tag{2.22}$$

where $\delta_{\tau_i} = \delta\cdot\bar{\tau}_i$, $i = 1, 2$, and then assuming that $S = S_1$, $\delta = \alpha$ and $\delta = \beta$ instead of Eqs. (2.20), (2.21) and (2.9) we have

$$\begin{aligned}
\frac{1}{\varkappa_n}\frac{\partial q}{\partial n}\Big|_{S_1} &= -\frac{1}{2}(\eta_{n,t}+\vartheta_{n,t})+\sum_{i,j=1}^{3} n_{i,x_j}(\eta_i\vartheta_j+\eta_j\vartheta_i)\\
&-\sum_{\mu=1}^{2}(\eta_\mu\bar{\tau}_\mu\cdot\nabla\vartheta_n+\vartheta_\mu\bar{\tau}_\mu\cdot\nabla\eta_n)+\eta_n\Big[\vartheta_n\operatorname{div}\bar{n}+\sum_{i=1}^{2}(\bar{\tau}_i\cdot\nabla\vartheta_{\tau_i}+\vartheta_{\tau_i}\operatorname{div}\bar{\tau}_i)\Big]\\
&+\vartheta_n\Big[\eta_n\operatorname{div}\bar{n}+\sum_{i=1}^{2}(\bar{\tau}_i\cdot\nabla\eta_{\tau_i}+\eta_{\tau_i}\operatorname{div}\bar{\tau}_i)\Big]-\eta_n\operatorname{div}\beta|_{S_1}-\vartheta_n\operatorname{div}\alpha|_{S_1}+f_n|_{S_1}\\
&\equiv g_1(\eta,\vartheta,f,S_1)-\eta_n\operatorname{div}\beta|_{S_1}-\vartheta_n\operatorname{div}\alpha|_{S_1},
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&\eta_{n,t}-\vartheta_{n,t}+\sum_{i=1}^{2}(\vartheta_{\tau_i}\bar{\tau}_i\cdot\nabla\eta_n-\eta_{\tau_i}\bar{\tau}_i\cdot\nabla\vartheta_n+\sum_{i,j=1}^{3}(\eta_j\vartheta_i-\vartheta_j\eta_i)n_{i,x_j}\\
&+\sum_{i=1}^{2}[\eta_n(\bar{\tau}_i\cdot\nabla\vartheta_{\tau_i}+\vartheta_{\tau_i}\operatorname{div}\bar{\tau}_i)-\vartheta_n(\bar{\tau}_i\cdot\nabla\eta_{\tau_i}+\eta_{\tau_i}\operatorname{div}\bar{\tau}_i)]+\vartheta_n\operatorname{div}\alpha|_{S_1}-\eta_n\operatorname{div}\beta|_{S_1}\\
&\equiv g_0(\eta,\vartheta,S_1)+\vartheta_n\operatorname{div}\alpha|_{S_1}-\eta_n\operatorname{div}\beta|_{S_1} = 0,
\end{aligned} \tag{2.24}$$

where $\varkappa_n$ is the Lamé's coefficient. To satisfy the relation $(2.19)_1$ instead of Eqs. (2.23) and (2.24), we assume

$$\frac{1}{\varkappa_n}\frac{\partial q}{\partial n}\Big|_{S_1} = g_1(\eta,\vartheta,f,S_1), \tag{2.25}$$

$$g_0(\eta,\vartheta,S_1) = 0 \tag{2.26}$$

so comparing Eqs. (2.25) and (2.26) with Eqs. (2.23) and (2.24) we have

$$\tag{2.27} \begin{pmatrix} \vartheta_n, & \eta_n \\ \vartheta_n, & -\eta_n \end{pmatrix} \begin{pmatrix} \operatorname{div}\alpha|_{S_1} \\ \operatorname{div}\beta|_{S_1} \end{pmatrix} = 0.$$

Therefore the relation $(2.19)_1$ is satisfied if $\eta_n \vartheta_n \neq 0$, what follows from the relation (2.8). Then we have shown

LEMMA 2

For Ω with the boundary S_1 the problems (1.10), (1.11) (1.13), (2.8), (2.9) and (1.14), (2.25), (2.26) are equivalent to Eqs. (1.1)–(1.6), (2.8), (2.9), (2.25), (2.26).

For solutions of class $C^1(\bar{\Omega}^T)$, $\bar{\Omega}^T = \bar{\Omega} \times [0, T]$, we have the following compatibility conditions:

$$\tag{2.28} \begin{aligned} (\bar{\tau} \cdot \nabla)^s (\alpha_0|_{S_1} - \eta|_{t=0}) &= 0, \\ (\bar{\tau}_\mu \cdot \nabla)^s (\beta_0|_{S_1} - \vartheta|_{t=0}) &= 0, \end{aligned}$$

where $s = 0, 1$, $\mu = 1, 2$ and $\bar{\tau}_\mu \in TS_1$ which is the tangent space to S_1.

Let us consider the boundary S_2. Since the curves (b_1) appear only, we have

$$\tag{2.29} \operatorname{div}\alpha = 0 \quad \text{in } \Omega.$$

Similarly as above, to guarantee $\operatorname{div}\beta = 0$ in Ω we shall obtain boundary conditions for q in the form which implies the relation $(2.19)_2$. Projecting the normal component of Eq. (1.11) on S_2, using Eqs. (2.12) and (2.22) for $\delta = \beta$ and $S = S_2$, we obtain

$$\tag{2.30} \begin{aligned} \frac{1}{\varkappa_n} \frac{\partial q}{\partial n}\Big|_{S_2} &= f_n|_{S_2} - \vartheta_{n,t} + \Big(\sum_{i,j=1}^{3} n_{i,x_j} \alpha_j \vartheta_i - \sum_{i=1}^{2} \alpha_{\tau_i} \bar{\tau}_i \cdot \nabla \vartheta_n \Big)\Big|_{S_2} \\ &\quad + \alpha_n \Big[\vartheta_n \operatorname{div}\bar{n} + \sum_{i=1}^{2} (\bar{\tau}_i \cdot \nabla \vartheta_{\tau_i} + \vartheta_{\tau_i} \operatorname{div}\bar{\tau}_i) \Big]\Big|_{S_2} - \alpha_n \operatorname{div}\beta|_{S_2} \\ &\equiv g_2(\alpha, \vartheta, f, S_2)|_{S_2} - \alpha_n \operatorname{div}\beta|_{S_2}. \end{aligned}$$

Assuming the boundary condition in the form

$$\tag{2.31} \frac{1}{\varkappa_n} \frac{\partial q}{\partial n}\Big|_{S_2} = g_2(\alpha, \vartheta, f, S_2)|_{S_2}$$

by comparing Eq. (2.31) with Eq. (2.30) and using Eqs. (2.10) and (2.11), we get the relation $(2.19)_2$. In the case of C^1 functions the condition $(2.28)_2$ on S_2 must be satisfied.

Hence we can formulate

LEMMA 3

For Ω with the boundary S_2 problems (1.10), (1.11), (1.13), (2.10), (2.11), (2.12) and (1.14), (2.31) are equivalent to Eqs. (1.1)–(1.6), (2.10)–(2.12).

Considering the boundary S_3, we have that $\operatorname{div}\beta = 0$ in Ω. To satisfy the relation $(2.19)_3$ and then the relation (2.29), we obtain the boundary condition for q from Eq. (1.10), similarly as above, in the following form:

$$\tag{2.32} \frac{1}{\varkappa_n} \frac{\partial q}{\partial n}\Big|_{S_3} = f_n|_{S_3} - \eta_{n,t} + \Big(\sum_{i,j=1}^{3} n_{i,x_j} \eta_i \beta_j - \sum_{i=1}^{2} \beta_{\tau_i} \bar{\tau}_i \cdot \nabla \eta_n \Big)\Big|_{S_3}$$

(2.32) [cont.]
$$+\beta_n\left[\eta_n \operatorname{div}\bar{n}+\sum_{i=1}^{2}(\bar{\tau}_i\cdot\nabla\eta_{\tau_i}+\eta_{\tau_i}\operatorname{div}\bar{\tau}_i)\right]\Bigg|_{S_3},$$

where we used Eqs. (2.13) and (2.14) to satisfy the relation $(2.19)_3$. We have

LEMMA 4

For Ω with the boundary S_3 the problems (1.10), (1.11) (1.13), (2.13)-(2.15) and (1.14), (2.32) are equivalent to Eqs. (1.1)–(1.6), (2.13)–(2.15).

Now we consider the boundary S_4. In this case the curves of families (a_i), (c_i), $i = 1, 2$, appear only. Since Eqs. (1.3) and (1.4) are satisfied, we have an arbitrariness in assuming boundary data for Eq. (1.14). We prescribe the Dirichlet condition

$$q|_{S_4} = \pi. \tag{2.33}$$

However, the case (2.17) cannot be considered because we cannot guarantee that $\alpha_n|_{S_4} = 0$ or $\beta_n|_{S_4} = 0$ for $t > 0$ if they are satisfied for $t = 0$. This fact follows from that we have not any relation describing a behaviour of $\alpha_n = \alpha_n(t)$, $\beta_n = \beta_n(t)$ on S_4. For the same reasons the cases (2.11) and (2.14) cannot be considered. Therefore we have

LEMMA 5

For Ω with the boundary S_4 the problems (1.10), (1.11) (1.13), (2.16) and (1.14), (2.33) are equivalent to (1.1)–(1.6), (2.16).

In the case of the boundary S_0 the curves (a_i) $i = 1, 2$, appear only. Therefore Eqs. (1.3) and (1.4) are satisfied, too. However, we have to show that the conditions (2.18) are stable, what means that they would be satisfied for any finite time if they were satisfied in $t = 0$. To show this we take the normal components of Eqs. (1.10) and (1.11) and project the results on S_0. Using

$$\sum_{i,j=1}^{3}\alpha_j\beta_i n_{i,x^j}\Big|_{S_0} = \sum_{i,j=1}^{3}\alpha_j\beta_i n_{j,x^i}\Big|_{S_0},$$

one gets

$$\bar{n}\cdot\nabla q\Big|_{S_0} = \Big(f_n+\sum_{i,j=1}^{3}\alpha_i\beta_j n_{i,x^j}\Big)\Big|_{S_0} \tag{2.34}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\beta\cdot\nabla\right)\alpha_n &= 0 \quad \text{on } S_0,\\ \left(\frac{\partial}{\partial t}+\alpha\cdot\nabla\right)\beta_n &= 0 \quad \text{on } S_0.\end{aligned} \tag{2.35}$$

Equations (2.34) and (2.35) are disjoined because at first Eq. (2.34) gives a boundary datum for Eq. (1.14) and at second Eq. (2.35) implies that the conditions (2.18) are stable in time. Equations (2.35) show noncontradiction of the conditions (2.18) with the considered problem. Conversely, if Eq. (2.34) is satisfied, then Eqs. (2.35) are satisfied, too, and the problem with the boundary S_0 is well posed.

This problem is very interesting because no quantity has to be assumed on a boundary. Therefore the problem can be considered as a free boundary problem. Moreover this problem corresponds to the problem for an ideal incompressible fluid with a vanishing normal component of a velocity on a boundary, where also no quantity on the boundary has to be prescribed.

We have shown:

LEMMA 6

For a domain Ω with S_0 boundary the problems (1.10), (1.11), (1.13), (2.18) and (1.14), (2.34) are equivalent to Eqs. (1.1)–(1.6), (2.18).

Summarizing the above considerations one can formulate:

THEOREM 1. *For equations of an ideal incompressible magnetohydrodynamics* (1.1)–(1.4) *with the initial conditions* (1.5) *and* (1.6) *one can distinguish the following types of boundary conditions*:

(2.36)
$$\begin{aligned} &0. \quad (2.18),\\ &1. \quad (2.8),\ (2.9),\\ &2. \quad (2.10)\div(2.12),\\ &3. \quad (2.13)\div(2.15),\\ &4. \quad (2.16),\ (2.33).\end{aligned}$$

Hence the boundary condition $(2.36)_\nu$ *corresponds to the* S_ν *boundary*, $\nu = 0, \ldots, 4$. *Let us denote by* (L_ν) *the problem* (1.1)–(1.6), $(2.36)_\nu$, $\nu = 0, \ldots, 4$.

REMARK 1

Equations (1.3) and (1.4) imply the following integral compatibility conditions:

(2.37)
$$\int\limits_{\partial\Omega} v_n = 0, \qquad \int\limits_{\partial\Omega} B_n = 0.$$

Therefore the problem (L_0) can be considered in Ω with $\partial\Omega = S_0$. For the other problems $((L_i),\ i = 1, \ldots, 4)$ we have to assume that $\partial\Omega$ consists of at least two disjoined parts on each of them different boundary conditions are imposed. For example we have the following possibilities:

(2.38)
$$\begin{aligned} &\partial\Omega = S_1\cup S_4, \quad S_1\cap S_4 = \emptyset,\\ &\partial\Omega = S_2\cup S_3, \quad S_2\cap S_3 = \emptyset,\\ &\partial\Omega = S_1\cup S_2\cup S_3, \quad S_i\cap S_j = \emptyset, \quad i,j = 1,2,3,\\ &\partial\Omega = S_2\cup S_3\cup S_4, \quad S_i\cap S_j = \emptyset, \quad i,j = 2,3,4, \text{ etc.}\end{aligned}$$

Let (A_ν), (B_ν) be the evolution problems (1.10), (1.11), (1.13) for α and β, respectively, with boundary data $(2.36)_\nu$ for a given q, $\nu = 0, \ldots, 4$. Let (E_ν) be the elliptic problem (1.14), (D_ν) for q, $\nu = 0, \ldots, 4$, where α, β are given and $(D_0) = (2.34)$, $(D_1) = (2.25)$, $(D_2) = (2.31)$, $(D_3) = (2.32)$, $(D_4) = (2.32)$.

REMARK 2

We cannot consider Neumann problems (E_i), $i = 1, 2, 3$, only because we do not know how to satisfy the compatibility condition $\int\limits_{\Omega} \Delta u = \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}$. Therefore we shall consider combinations of Neumann problems (E_i), $i = 1, 2, 3$ and the Dirichlet problem (E_4) in such a way that (2.37) is satisfied (see [3]).

From Theorem 1 and Remarks 1, 2 we have

THEOREM 2. *Instead of initial boundary value problems described in Theorem* 1 *we have the following examples of well posed systems of problems*

$$(\mathrm{P}_1) \quad (A_1, B_1, E_1)+(A_4, B_4, E_4),$$

$$(\mathrm{P}_2) \quad (A_2, B_2, E_2)+(A_3, B_3, E_3)+A_4, B_4, E_4),$$

$$(\mathrm{P}_0) \quad (A_0, B_0, E_0).$$

To formulate the result, suitable compatibility conditions (see Eqs. (2.28)) *are assumed.*

REMARK 3

The above representation suggests that the existence of solutions of the problems (P_0), (P_1), (P_2) can be proved by the method of successive approximations.

REMARK 4

Let us generalize the boundary conditions (2.12) and (2.15). Assume that

$$\beta|_{S_2} = \Phi(\alpha, \vartheta), \tag{2.39}$$

where Φ is a sufficiently smooth function. Then, considering the boundary data (2.31), we see that $\left.\frac{\partial q}{\partial n}\right|_{S_2}$ depends on the first derivatives on α on S_2. However, from Eq. (1.10) it follows that to estimate the k-derivative of α we need the $k+1$-derivative of q. This contradicts the claim that the boundary condition (2.39) can be assumed. The same considerations can be applicable to the relation (2.15).

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