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Homogenization of fissured Reissner-like plates
Part II. Convergence

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THE PURPOSE of the second part of the paper is to prove the results obtained in the first part by using the method of two-scale asymptotic expansions. The convergence is proved by means of an appropriate generalization of the results due to ATTOUCH and MURAT [1].

W drugiej części pracy zostały udowodnione rezultaty, które w części pierwszej wyprowadzono, stosując metodę rozwinięć asymptotycznych. W celu wykazania zbieżności odpowiednio rozszerzono podejście ATTOUCHA i MURATA [1].

Целью второй части работы является доказательство результатов, полученных асимптотическим методом в первой части. Сходимость доказана путем соответственного расширения подхода Атуша и Мюра [1].

1. Introduction

IN THE FIRST part of the paper [7] we have formulated and solved the problem of the homogenization of elastic Reissner-like plates damaged by periodically distributed fissures. Our attention has been focussed on unilateral fissures. The homogenization problem is non-trivial since it means the homogenization of a variational inequality posed on a domain dependent explicitly on a small and variable parameter $\varepsilon > 0$.

The method of two-scale asymptotic expansions has been used to derive the equations of effective or homogenized plates. Unfortunately, such an approach, though effective as a method of averaging, is formal and requires rigorous mathematical justification. Exactly such a justification has been proposed in the present part of the paper. Toward this end we follow an ingenious approach proposed by ATTOUCH and MURAT [1] in a purely scalar case. Our problem is more complicated due to the presence of one scalar and two vector kinematical fields. The study of convergence is restricted to the passage to a limit in an appropriate variational inequality. The method of Γ-convergence, or rather epi-convergence [1], is left apart as more complicated and leading to the same results.

Roman numerals refer to the relevant sections, equations and references of the first part of the paper. The same notations as in Part I will be used here.

2. The study of convergence

In our case we must pass to the limit with $\{w^\varepsilon\}_{\varepsilon\to 0}$ and $\{\mathbf{v}^\varepsilon, \boldsymbol{\varphi}^\varepsilon\}_{\varepsilon\to 0}$, see Section I.3. The results obtained by ATTOUCH and MURAT [1] enable us to pass to the limit with the scalar sequence $\{w^\varepsilon\}_{\varepsilon\to 0}$. The results of these authors are not directly applicable to the vector case. The latter case has been solved in the present section.

2.1. Preliminaries

In the case under study the cracked domain Ω^ε depends explicitly on ε. Thus it varies as $\varepsilon \to 0$. To pass to a limit as $\varepsilon \to 0$ we shall construct a sequence $\{Q_1^\varepsilon w^\varepsilon, Q_2^\varepsilon \mathbf{v}^\varepsilon, Q_2^\varepsilon \boldsymbol{\varphi}^\varepsilon\}_{\varepsilon\to 0}$ such that $Q_1^\varepsilon w^\varepsilon \in H_0^1(\Omega)$, $Q_2 \mathbf{v}^\varepsilon \in [H_0^1(\Omega)]^2$ and $Q_2^\varepsilon \boldsymbol{\varphi}^\varepsilon \in [H_0^1(\Omega)]^2$. The linear and continuous operator Q_1^ε has been constructed in [1]. Its main property has been formulated by Attouch and Murat as Proposition 4.2. Here we reformulate it as

LEMMA 2.1. For any sequence $\{w^\varepsilon\}_{\varepsilon\to 0}$ satisfying $\sup_{\varepsilon>0} \|w^\varepsilon\|_{1,\Omega^\varepsilon} < \infty$ there exists a sequence $\{Q_1^\varepsilon w^\varepsilon\}_{\varepsilon\to 0}$ bounded in $H_0^1(\Omega)$ and such that

$$\|Q_1^\varepsilon w^\varepsilon - w^\varepsilon\|_{0,\Omega} \to 0 \quad \text{as} \quad \varepsilon \to 0. \qquad \blacksquare$$

Let us note that the operator Q_1^ε is obtained from the operator Q_1 similarly as the operator Q_2^ε from Q_2, see Remark 2.1. below.

2.2. Extension operators Q_2 and Q_2^ε

An essential idea of Attouch and Murat [1] consists in working with a hole F_η instead of the fissure F, see Fig. 1.

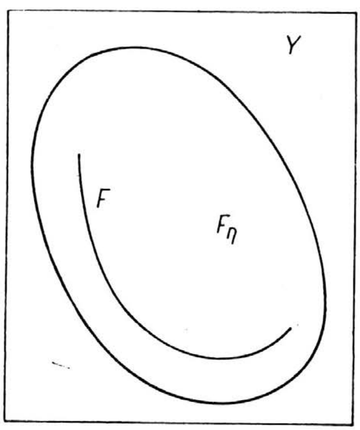

FIG. 1. Subdomain F_η, $F = \bar{F} \subset F_\eta$.

The parameter $\eta > 0$ is kept fixed, $\eta = \eta_0$ and $F \subset F_\eta$. Moreover it is assumed that the boundary of F_η is sufficiently smooth.

We shall first construct the extension operator Q_2.

LEMMA 2.2. There exists an extension operator
$\mathsf{P}_2 : [H^1(Y \setminus F_\eta)]^2 \to [H^1(Y)]^2$ such that

(a) $\|\mathsf{P}_2 \mathbf{v}\|_{0,Y} \leqslant c \|\mathbf{v}\|_{0,Y\setminus F\eta}$,

(b) $\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\mathsf{P}_2 \mathbf{v})\|_{0,Y} \leqslant c \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\mathbf{v})\|_{0,Y\setminus F\eta} \leqslant c \|\boldsymbol{\gamma}(\mathbf{v})\|_{0,YF}$.

P r o o f. Such an operator may be constructed as follows. Let $\mathscr{R}$ denote the space of rigid displacements. Then each $\mathbf{v} \in [H^1(Y \setminus F_\eta)]^2$ can be decomposed according to

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{r}, \tag{2.1}$$

where $\mathbf{r} \in \mathscr{R}$ and $\mathbf{v}_1 \perp \mathscr{R}$ in $[L^2(Y \setminus F_\eta)]^2$.

To extend $\mathbf{v}$ on Y we extend $\mathbf{v}_1$ continuously, see [3, 9]. This linear and continuous operator is denoted by P_2. Thus

$$\mathsf{P}_2 \mathbf{v} = \mathsf{P}_2 \mathbf{v}_1 + \mathbf{r}. \tag{2.2}$$

Let us prove (a). We have

$$\|\mathsf{P}_2 \mathbf{v}\|_{0,Y} = \|\mathsf{P}_2 \mathbf{v}_1 + \mathbf{r}\|_{0,Y} = \|\mathsf{P}_2(\mathbf{v}_1 + \mathbf{r})\|_{0,Y} \leqslant c \|\mathbf{v}_1 + \mathbf{r}\|_{0,Y\setminus F\eta} = c \|\mathbf{v}\|_{0,Y\setminus F\eta}.$$

The first inequality in (b) has been proved by Léné [5, 6]. The second one is obvious since $Y \setminus F_\eta \subset YF$.

REMARK 2.1. For the scalar case an extension operator P_1 may be defined as follows

$$\begin{aligned} &\mathsf{P}_1 : H^1(Y \setminus F_\eta) \to H^1(Y), \\ &w \to \mathsf{P}_1(w) = \mathsf{P}_1^0(w - \langle w \rangle_{Y\setminus F\eta}) + \langle w \rangle_{Y\setminus F\eta}, \end{aligned} \tag{2.3}$$

where

$$\langle w \rangle_{Y\setminus F\eta} = \frac{1}{|Y \setminus F_\eta|} \int_{Y\setminus F\eta} w(y)\, dy. \tag{2.4}$$

P_1^0 is the linear and continuous extension operator, $\mathsf{P}_1^0 : H^1(Y \setminus F_\eta) \to H^1(Y)$, see [3, 9] and

$$\|\mathsf{P}_1^0 w\|_{1,Y} \leqslant c \|w\|_{1,Y\setminus F\eta},$$
$$\|\mathsf{P}_1^0 w\|_{0,Y} \leqslant c \|w\|_{0,Y\setminus F\eta}.$$

The extension operator $\mathsf{Q}_1 : H^1(YF) \to H^1(Y)$ is defined by

$$\mathsf{Q}_1 = \mathsf{P}_1 \mathsf{R}_1, \tag{2.5}$$

where $\mathbf{R}_1$ is the restriction operator

$$\mathsf{R}_1 : H^1(YF) \to H^1(Y \setminus F_\eta).$$

■

The operator Q_2 is defined similarly.

DEFINITION 5.1. *The extension operator*

$$\mathsf{Q}_2 : [H^1(YF)]^2 \to [H^1(Y)]^2$$

is equal to

$$\mathsf{Q}_2 = \mathsf{P}_2 \mathsf{R}_2, \tag{2.6}$$

where $\mathsf{R}_2 : [H^1(YF)]^2 \to [H^1(Y \setminus F_\eta)]^2$ *is the restriction operator.*

The operator $\mathbf{Q}_2$ is characterized by

LEMMA 2.3. The operator $\mathbf{Q}_2$ has the following properties

(i) $\mathbf{Q}_2\mathbf{v} = \mathbf{v}$ on $Y \setminus F_\eta$.

(ii) $\|\mathbf{Q}_2\mathbf{v}\|_{0,Y} \leqslant c\|\mathbf{v}\|_{0,Y\setminus F\eta} \leqslant c\|\mathbf{v}\|_{0,YF} = c\|\mathbf{v}\|_{0,Y}$.

(iii) $\|\boldsymbol{\gamma}(\mathbf{Q}_2\mathbf{v})\|_{0,Y} \leqslant c\|\boldsymbol{\gamma}(\mathbf{v})\|_{0,Y\setminus F\eta} \leqslant c\|\boldsymbol{\gamma}(\mathbf{v})\|_{0,YF}$.

(iv) $\|\mathbf{Q}_2\mathbf{v}-\mathbf{v}\|_{1,YF} \leqslant c\|\boldsymbol{\gamma}(\mathbf{v})\|_{0,YF}$.

Proof. The property (i) follows immediately from the definition of $\mathbf{Q}_2$.

(ii) $\|\mathbf{Q}_2\mathbf{v}\|_{0,Y} = \|\mathbf{P}_2(\mathbf{R}_2\mathbf{v})\|_{0,Y} \leqslant c\|\mathbf{R}_2\mathbf{v}\|_{0,Y\setminus F\eta} = c\|\mathbf{v}\|_{0,Y\setminus F\eta} \leqslant c\|\mathbf{v}\|_{0,YF}$,

(iii) $\|\boldsymbol{\gamma}(\mathbf{Q}_2\mathbf{v})\|_{0,Y} = \|\boldsymbol{\gamma}(\mathbf{P}_2(\mathbf{R}_2\mathbf{v}))\|_{0,Y} \leqslant c\|\boldsymbol{\gamma}(\mathbf{R}_2\mathbf{v})\|_{0,Y\setminus F\eta} = c\|\boldsymbol{\gamma}(\mathbf{v})\|_{0,Y\setminus F\eta} \leqslant c\|\boldsymbol{\gamma}(\mathbf{v})\|_{0,YF}$,

by using the property (b) of the Lemma 2.2.

(iv) Korn's inequality implies

$$\|\mathbf{Q}_2\mathbf{v}-\mathbf{v}\|_{1,F_\eta^\alpha} \leqslant c\|\boldsymbol{\gamma}(\mathbf{Q}_2\mathbf{v}-\mathbf{v})\|_{0,F_\eta^\alpha}; \qquad \alpha = 1, 2,$$

since $\mathbf{Q}_2\mathbf{v}-\mathbf{v} = 0$ on a part of the boundary of F_η^α of strictly positive length, see Fig. 2.

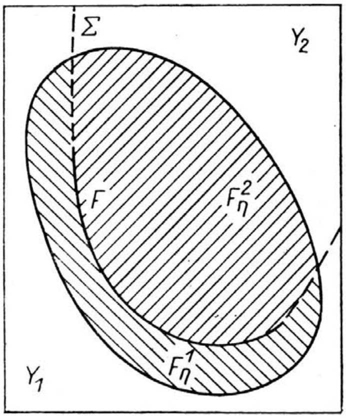

FIG. 2. Extension of F

$Y = Y_1 \cup Y_2 \cup \Sigma, \quad Y_1 \cap Y_2 = \emptyset,$

$F = F_\eta^1 \cup F_\eta^2 \cup (\Sigma \cap F_\eta), \quad F_\eta^\alpha = F_\eta \cap Y_\alpha, \quad \alpha = 1, 2.$

Hence by using the properties (i) and (iii) we obtain

$$\|\mathbf{Q}_2\mathbf{v}-\mathbf{v}\|_{1,YF} \leqslant c\sum_\alpha \|\boldsymbol{\gamma}(\mathbf{Q}_2\mathbf{v}-\mathbf{v})\|_{0,F_\eta^\alpha} \leqslant c\sum_\alpha (\|\boldsymbol{\gamma}(\mathbf{Q}_2\mathbf{v})\|_{0,Y_\alpha} + \|\boldsymbol{\gamma}(\mathbf{v})\|_{0,Y_\alpha})$$

$$\leqslant c(\|\boldsymbol{\gamma}(\mathbf{Q}_2\mathbf{v})\|_{0,Y} + \|\boldsymbol{\gamma}(\mathbf{v})\|_{0,YF}) \leqslant cc_1\|\boldsymbol{\gamma}(\mathbf{v})\|_{0,YF} + c\|\boldsymbol{\gamma}(\mathbf{v})\|_{0,YF} = c(c_1+1)\|\boldsymbol{\gamma}(\mathbf{v})\|_{0,YF}.$$

Thus the lemma is proved. ■

We observe that the property (iv) implies

$$\|\mathbf{Q}_2\mathbf{v}-\mathbf{v}\|_{0,Y} \leqslant c\|\boldsymbol{\gamma}(\mathbf{v})\|_{0,YF} \tag{2.7}$$

and

$$\|\nabla(\mathbf{Q}_2\mathbf{v}-\mathbf{v})\|_{0,YF} \leqslant c\|\boldsymbol{\gamma}(\mathbf{v})\|_{0,YF}, \tag{2.8}$$

where

$$\nabla \mathbf{z} = (z_{\alpha,\beta}).$$

Having defined and examined the operator Q_2 we pass to the extension operator Q_2^ε acting on functions determined on Ω^ε. From the formula (I.3.1) we know that $F^\varepsilon = \bigcup_{i \in I(\varepsilon)} F_{\varepsilon,i}$ where $F_{\varepsilon,i} \subset Y_{\varepsilon,i}$ for every $i \in I(\varepsilon)$ and $Y_{\varepsilon,i}$ is the εY cell corresponding to i or

$$Y_{\varepsilon,i} = \varepsilon Y + \xi_{i,\varepsilon}, \quad \xi_{i,\varepsilon} \in \mathbb{R}^2.$$

Next operators $Q_2^{\varepsilon,i}$ are constructed, see the scheme below. The global operator Q_2^ε is derived from $Q_2^{\varepsilon,i}$ by the method of stitching the operators $Q_2^{\varepsilon,i}$. The global operator Q_2^ε is obtained in the following way

$\mathbf{z} \in [H^1(\Omega^\varepsilon)]^2$

↓ restriction

$\mathbf{z}^{\varepsilon,i} \in [H^1(Y_{\varepsilon,i} \setminus F_{\varepsilon,i})]^2 = [H^1(\varepsilon YF + \xi_{i,\varepsilon})]^2$

↓ translation and change of scale

$\mathbf{z}_1 \in [H^1(YF)]^2$ or $\mathbf{z}_1(y) = \mathbf{z}^{\varepsilon,i}(\varepsilon y + \xi_{i,\varepsilon})$ (i not summed !)

↓ restriction and extension

$\mathbf{z}_2 = Q_2 \mathbf{z}_1 \in [H^1(Y)]^2$

↓ translation and change of scale

$\mathbf{z}_3 = Q_2^{\varepsilon,i} \mathbf{z}_2 \in [H^1(Y_{\varepsilon,i})]^2$ or $\mathbf{z}_3(x) = \mathbf{z}_2\left(\dfrac{x - \xi_{i,\varepsilon}}{\varepsilon}\right)$(1)

↓ stitching with respect to $i \in I(\varepsilon)$

$\mathbf{z}_4 = Q_2^\varepsilon \mathbf{z}$

According to our earlier assumptions the operator Q_2^ε may be set equal to the identity near the boundary Γ of Ω, see the property (2.9) below. Then

$$\mathbf{z} \in [H_1^1(\Omega^\varepsilon)]^2 \Rightarrow Q_2^\varepsilon \mathbf{z} \in [H_0^1(\Omega)]^2.$$

The basic properties of the operator Q_2^ε are given by

LEMMA 2.4. For each $\varepsilon > 0$ the operator $Q_2^\varepsilon : [H^1(\Omega^\varepsilon)]^2 \to [H^1(\Omega)]^2$ is linear and continuous. Moreover we have

(2.9) $Q_2^\varepsilon \mathbf{z} = \mathbf{z}$ on $\Omega \setminus F_\varepsilon^\eta$, $F_\varepsilon^\eta = \cup F_{\varepsilon,i}^\eta$,

(2.10) $\|Q_2^\varepsilon \mathbf{z}\|_{0,\Omega} \leqslant c \|\mathbf{z}\|_{0,\Omega}$,

(2.11) $\|\boldsymbol{\gamma}(Q_2^\varepsilon \mathbf{z})\|_{0,\Omega} \leqslant c \|\boldsymbol{\gamma}(\mathbf{z})\|_{0,\Omega^\varepsilon}$,

(2.12) $\|Q_2^\varepsilon \mathbf{z} - \mathbf{z}\|_{0,\Omega} \leqslant c\varepsilon \|\boldsymbol{\gamma}(\mathbf{z})\|_{0,\Omega^\varepsilon}$,

(2.13) $\|\nabla(Q_2^\varepsilon \mathbf{z} - \mathbf{z})\|_{0,\Omega^\varepsilon} \leqslant c \|\boldsymbol{\gamma}(\mathbf{z})\|_{0,\Omega^\varepsilon}$.

P r o o f. The scheme preceding the lemma shows that Q_2^ε is a composition of linear and continuous operators.

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(1) $x = \varepsilon y + \xi_{i,\varepsilon} \Rightarrow y = (x - \xi_{i,\varepsilon})/\varepsilon$

The properties (2.9), (2.10) and (2.11) result immediately from the properties (i), (ii) and (iii) specified by the Lemma 2.3.

To prove (2.12) we first consider the cell $Y_{\varepsilon,i}$. We have

$$(2.14)\qquad \int\limits_{Y_{\varepsilon,i}} |\mathbf{Q}_2^{\varepsilon,i}\mathbf{z}(x)-\mathbf{z}(x)|^2 dx = \int\limits_{Y} |\mathbf{Q}_2\mathbf{z}(\varepsilon y+\boldsymbol{\xi}_{i,\varepsilon})-\mathbf{z}(\varepsilon y+\boldsymbol{\xi}_{i,\varepsilon})|^2\varepsilon^2 dy$$

since $y = (x-\boldsymbol{\xi}_{i,\varepsilon})/\varepsilon$. Further, the properties (2.7) and (2.14) yield

$$\int\limits_{Y_{\varepsilon,i}} |\mathbf{Q}_2^{\varepsilon,i}\mathbf{z}(x)-\mathbf{z}(x)|^2 dx \leqslant c \int\limits_{YF} |\boldsymbol{\gamma}^y(\mathbf{z})(\varepsilon y+\boldsymbol{\xi}_{i,\varepsilon})|^2\varepsilon^2 dy \leqslant c \int\limits_{YF} |\boldsymbol{\gamma}^x(\mathbf{z})(\varepsilon y+\boldsymbol{\xi}_{i,\varepsilon})|^2\varepsilon^4 dy.$$

Now we set $x = \varepsilon y+\boldsymbol{\xi}_{i,\varepsilon}$. Hence

$$\int\limits_{Y_{\varepsilon,i}} |\mathbf{Q}_2^{\varepsilon,i}\mathbf{z}(x)-\mathbf{z}(x)|^2 dx \leqslant c\varepsilon^2 \int\limits_{Y_{\varepsilon,i}\setminus F_{\varepsilon,i}} |\boldsymbol{\gamma}(\mathbf{z}(\mathbf{x}))|^2 dx.$$

Adding over all cells $Y_{\varepsilon,i}$ we arrive at

$$\int\limits_{\Omega} |\mathbf{Q}_2^{\varepsilon}\mathbf{z}-\mathbf{z}|^2 dx \leqslant c\varepsilon^2 \int\limits_{\Omega^\varepsilon} |\boldsymbol{\gamma}(\mathbf{z}(x))|^2 dx.$$

From the last inequality follows the required result or (2.12).

Finally, since the change of scale equally affects both sides of (2.8), the inequality (2.13) results immediately. Thus the proof is complete.

■

By noting that (2.12) and (2.13) yield

$$(2.15)\qquad ||\mathbf{Q}_2^\varepsilon\mathbf{z}-\mathbf{z}||_{1,\Omega^\varepsilon} \leqslant (c_1\varepsilon+c)||\boldsymbol{\gamma}(\mathbf{z})||_{0,\Omega^\varepsilon}$$

we can formulate

THEOREM 2.1. (*Korn's inequality for Ω^ε*). *For each $\mathbf{z}\in[H_1^1(\Omega^\varepsilon)]^2$ Korn's inequality*

$$(2.16)\qquad ||\mathbf{z}||_{1,\Omega^\varepsilon} \leqslant (c\varepsilon+c_1)||\boldsymbol{\gamma}(\mathbf{z})||_{0,\Omega^\varepsilon}$$

is satisfied. Here $0<\varepsilon<\varepsilon_0$ and ε_0 is held fixed.

Proof. The assumption implies that $\mathbf{Q}_2^\varepsilon\mathbf{z}\in[H_0^1(\Omega)]^2$. Korn's inequality applied to $\mathbf{Q}_2^\varepsilon\mathbf{z}$ yields

$$||\mathbf{Q}_2^\varepsilon\mathbf{z}||_{1,\Omega} \leqslant c||\boldsymbol{\gamma}(\mathbf{Q}_2^\varepsilon\mathbf{z})||_{0,\Omega}.$$

Hence by using the property (2.11) we obtain

$$(2.17)\qquad ||\mathbf{Q}_2^\varepsilon\mathbf{z}||_{1,\Omega} \leqslant c||\boldsymbol{\gamma}(\mathbf{Q}_2^\varepsilon\mathbf{z})||_{0,\Omega} \leqslant c_1||\boldsymbol{\gamma}(\mathbf{z})||_{0,\Omega^\varepsilon}.$$

The triangle inequality furnishes

$$(2.18)\qquad ||\mathbf{z}||_{1,\Omega^\varepsilon} \leqslant ||\mathbf{z}-\mathbf{Q}_2^\varepsilon\mathbf{z}||_{1,\Omega^\varepsilon}+||\mathbf{Q}_2^\varepsilon\mathbf{z}||_{1,\Omega},$$

since the Lebesgue measure of Ω^ε is equal to that of Ω or $|\Omega^\varepsilon| = |\Omega|$ and $||\mathbf{Q}_2^\varepsilon\mathbf{z}||_{1,\Omega^\varepsilon} = ||\mathbf{Q}_2^\varepsilon\mathbf{z}||_{1,\Omega}$.

Substituting (2.15) and (2.17) into (2.18) we get

$$||\mathbf{z}||_{1,\Omega^\varepsilon} \leqslant (c_2\varepsilon+c_3)||\boldsymbol{\gamma}(\mathbf{z})||_{0,\Omega^\varepsilon}+c_1||\boldsymbol{\gamma}(\mathbf{z})||_{0,\Omega^\varepsilon} = (c_2\varepsilon+\tilde{c})||\boldsymbol{\gamma}(\mathbf{z})||_{0,\Omega^\varepsilon}.$$

Hence we infer that for $0<\varepsilon<\varepsilon_0$ (ε_0-fixed) Korn's inequality is satisfied.

■

REMARK 2.2. For the scalar case a similar role is played by the Poincaré inequality in which the parameter ε also enters explicitly [1]. For perforated domains the proof of the Korn inequality is straightforward (see CONCA [I.13]).

■

Now we can formulate the vector counterpart of Lemma 2.1.

THEOREM 2.2. *For any sequence* $\{\mathbf{v}^\varepsilon\}_{\varepsilon\to 0}$ *satisfying* $\sup_{\varepsilon>0}||\mathbf{v}^\varepsilon||_{1,\Omega^\varepsilon} < \infty$ *there exists a sequence* $\{\mathbf{Q}_2^\varepsilon \mathbf{v}^\varepsilon\}_{\varepsilon\to 0}$ *bounded in* $[H^1(\Omega)]^2$ *and such that*

$$||\mathbf{Q}_2^\varepsilon \mathbf{v}^\varepsilon - \mathbf{v}^\varepsilon||_{0,\Omega} \to 0 \quad as \quad \varepsilon \to 0.$$

Proof. The assumption gives $||\boldsymbol{\gamma}(\mathbf{v}^\varepsilon)||_{0,\Omega^\varepsilon} < \infty$. Hence by using the inequality (2.12) we deduce

$$||\mathbf{Q}_2^\varepsilon \mathbf{v}^\varepsilon - \mathbf{v}^\varepsilon||_{0,\Omega} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

The theorem is proved.

■

2.3. Boundedness

In the sequel we shall restrict ourselves to the constraint set K_ε^Δ defined by

$$K_\varepsilon := K_\varepsilon^{\mathfrak{bi}} \times K_\varepsilon^{\mathfrak{b}} \times K_\varepsilon^{\mathfrak{bi}},$$

The index Δ is dropped.

We observe that the remaining cases can be studied by similar techniques and are usually simpler.

For $0 < \varepsilon < \varepsilon_0$ the variational problem $\mathscr{P}_e^\varepsilon$ is equivalent to, see (I.3.6)

$$\min\left\{\frac{1}{2}a_e^\varepsilon(\mathbf{z}, u, \boldsymbol{\psi} : \mathbf{z}, u, \boldsymbol{\psi}) - f_e(\mathbf{z}, u, \boldsymbol{\psi}) | (\mathbf{z}, u, \boldsymbol{\psi}) \in K_\varepsilon\right\} \tag{2.19}$$

or

$$\frac{1}{2}a_e^\varepsilon(\mathbf{v}^\varepsilon, w^\varepsilon, \boldsymbol{\varphi}^\varepsilon; \mathbf{v}^\varepsilon, w^\varepsilon, \boldsymbol{\varphi}^\varepsilon) - f_e(\mathbf{v}^\varepsilon, w^\varepsilon, \boldsymbol{\varphi}^\varepsilon) \leqslant \frac{1}{2}a_e^\varepsilon(\mathbf{z}, u, \boldsymbol{\psi}; \mathbf{z}, u, \boldsymbol{\psi}) - f_e(\mathbf{z}, u, \boldsymbol{\psi}), \ (\mathbf{z}, u, \boldsymbol{\psi}) \in K_\varepsilon. \tag{2.20}$$

From the definition of the set considered K_ε we deduce that $(\mathbf{z}, u, \boldsymbol{\psi}) = (\mathbf{0}, 0, \mathbf{0}) \in K_\varepsilon$. Then (I.2.50) and (2.20) in conjunction with the Cauchy–Schwarz inequality result in

$$\begin{aligned}\int_{\Omega^\varepsilon} (c_1|\boldsymbol{\gamma}(\mathbf{v}^\varepsilon)|^2 + c_2|\boldsymbol{\rho}(\boldsymbol{\varphi}^\varepsilon)|^2 + c_3|\operatorname{grad} w^\varepsilon + \boldsymbol{\varphi}^\varepsilon|^2)dx &\leqslant \int_\Omega (pw^\varepsilon + p_\alpha v_\alpha^\varepsilon + m_\alpha \varphi_\alpha^\varepsilon)dx \\ &= \Big(\int_\Omega p^2 dx\Big)^{1/2}\Big(\int_\Omega (w^\varepsilon)^2 dx\Big)^{1/2} + \Big(\int_\Omega p_\alpha p_\alpha dx\Big)^{1/2}\Big(\int_\Omega v_\alpha^\varepsilon v_\alpha^\varepsilon dx\Big)^{1/2} \\ &\quad + \Big(\int_\Omega m_\alpha m_\alpha dx\Big)^{1/2}\Big(\int_\Omega \varphi_\alpha^\varepsilon \varphi_\alpha^\varepsilon dx\Big)^{1/2}.\end{aligned} \tag{2.21}$$

The Poincaré inequality implies

$$||w^\varepsilon||_{0,\Omega}^2 \leqslant c_4||\operatorname{grad} w^\varepsilon||_{0,\Omega^\varepsilon}^2. \tag{2.22}$$

Note that the constants c_2 and c_3 in (2.21) can be chosen such that $\frac{c_2}{c\varepsilon+c_1}-c_3>0$, where c and c_1 are constants entering (2.16). Taking account of (2.16), (2.22) and (I.2.12) we arrive at

$$\text{(2.23)}\qquad \|\mathbf{v}^\varepsilon\|^2_{1,\Omega^\varepsilon}+\|\boldsymbol{\varphi}^\varepsilon\|^2_{1,\Omega^\varepsilon}+\|w^\varepsilon\|^2_{1,\Omega^\varepsilon}\leqslant c(\|\mathbf{v}^\varepsilon\|_{0,\Omega}+\|\boldsymbol{\varphi}^\varepsilon\|_{0,\Omega}+\|w^\varepsilon\|_{0,\Omega}),$$

where c is a new constant.

Hence

$$\|\mathbf{v}^\varepsilon\|^2_{0,\Omega}+\|\boldsymbol{\varphi}^\varepsilon\|^2_{0,\Omega}+\|w^\varepsilon\|^2_{0,\Omega}\leqslant c(\|\mathbf{v}^\varepsilon\|_{0,\Omega}+\|\boldsymbol{\varphi}^\varepsilon\|_{0,\Omega}+\|w^\varepsilon\|_{0,\Omega}).$$

Thus

$$\text{(2.24)}\qquad \|\mathbf{v}^\varepsilon\|_{0,\Omega}+\|\boldsymbol{\varphi}^\varepsilon\|_{0,\Omega}+\|w^\varepsilon\|_{0,\Omega}\leqslant \text{const}<\infty.$$

Now taking account of (2.24) in (2.21) and (2.22) we obtain

$$\text{(2.25)}\qquad \sup_{\varepsilon>0}(\|\boldsymbol{\gamma}(\mathbf{v}^\varepsilon)^2_{0,\Omega^\varepsilon}\|+\|\boldsymbol{\rho}(\boldsymbol{\varphi}^\varepsilon)\|^2_{0,\Omega^\varepsilon}+\|\operatorname{grad} w^\varepsilon\|^2_{0,\Omega^\varepsilon})\leqslant \text{const}<\infty,$$

and

$$\text{(2.26)}\qquad \sup_{\varepsilon>0}(\|\mathbf{v}^\varepsilon\|^2_{1,\Omega^\varepsilon}+\|\boldsymbol{\varphi}^\varepsilon\|^2_{1,\Omega^\varepsilon}+\|w^\varepsilon\|^2_{1,\Omega^\varepsilon})\leqslant \text{const}<\infty,$$

respectively.

The estimate (2.25) implies that the sequences $\{N^\varepsilon_{\alpha\beta}\}_{\varepsilon\to 0}$, $\{M^\varepsilon_{\alpha\beta}\}_{\varepsilon\to 0}$ and $\{Q^\varepsilon_\alpha\}_{\varepsilon\to 0}$ are bounded in the space $L^2(\Omega)$.

Here

$$\text{(2.27)}\qquad N^\varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon,\boldsymbol{\varphi}^\varepsilon)=A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon)+E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon),$$

$$\text{(2.28)}\qquad M^\varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon,\boldsymbol{\varphi}^\varepsilon)=E_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon)+G_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon),$$

$$\text{(2.29)}\qquad Q^\varepsilon_\alpha(w^\varepsilon,\boldsymbol{\varphi}^\varepsilon)=H_{\alpha\beta}(w^\varepsilon_{,\beta}+\varphi^\varepsilon_\beta).$$

To proceed further we recall two basic properties of the extension operator $\mathbf{Q}^\varepsilon_1$ [1]

$$\text{(2.30)}\qquad \|\mathbf{Q}^\varepsilon_1 w\|_{0,\Omega}\leqslant c\|w\|_{0,\Omega},\quad w\in H^1(\Omega^\varepsilon),$$

$$\text{(2.31)}\qquad \|\operatorname{grad}\mathbf{Q}^\varepsilon_1 w\|_{0,\Omega}\leqslant c\|\operatorname{grad} w\|_{0,\Omega^\varepsilon},\quad w\in H^1(\Omega^\varepsilon).$$

The estimates (2.25), (2.26) and the inequalities (2.30), (2.31) imply that $\{\mathbf{Q}^\varepsilon_1 w^\varepsilon\}_{\varepsilon\to 0}$ is bounded in $H^1(\Omega)$. Similarly, using the estimates (2.25), (2.26), the inequalities (2.11), (2.12) and the Korn inequality applied to the domain Ω we deduce that the sequences $\{\mathbf{Q}^\varepsilon_2\mathbf{v}^\varepsilon\}_{\varepsilon\to 0}$ and $\{\mathbf{Q}^\varepsilon_2\boldsymbol{\varphi}^\varepsilon\}_{\varepsilon\to 0}$ are bounded in the norm $\|\cdot\|_{1,\Omega}$. Thus we have

$$\text{(2.32)}\qquad \mathbf{Q}^\varepsilon_1 w^\varepsilon\to w\quad\text{strongly in } L^2(\Omega),$$

$$\text{(2.33)}\qquad \mathbf{Q}^\varepsilon_2\mathbf{v}^\varepsilon\to\mathbf{v},\quad \mathbf{Q}^\varepsilon_2\boldsymbol{\varphi}^\varepsilon\to\boldsymbol{\varphi}\quad\text{strongly in } [L^2(\Omega)]^2,$$

for subsequences, still indexed with ε.

Using Lemma 2.1, the inequality (2.13) and (2.32), (2.33) we deduce

$$\text{(2.34)}\qquad w^\varepsilon\to w,\quad v^\varepsilon_\alpha\to v_\alpha,\quad \varphi^\varepsilon_\alpha\to\varphi_\alpha\quad\text{strongly in } L^2(\Omega)$$

$$\text{(2.35)}\qquad M^\varepsilon_{\alpha\beta}\rightharpoonup M_{\alpha\beta},\quad N^\varepsilon_{\alpha\beta}\rightharpoonup N_{\alpha\beta},\quad Q^\varepsilon_\alpha\rightharpoonup Q_\alpha\quad\text{weakly in } L^2(\Omega)$$

since, for instance

$$\|\mathbf{Q}^\varepsilon_2\mathbf{v}^\varepsilon-\mathbf{v}^\varepsilon\|_{0,\Omega}=\|(\mathbf{Q}^\varepsilon_2\mathbf{v}^\varepsilon-\mathbf{v})-(\mathbf{v}^\varepsilon-\mathbf{v})\|_{0,\Omega}\geqslant\big|\|\mathbf{Q}^\varepsilon_2\mathbf{v}^\varepsilon-\mathbf{v}\|_{0,\Omega}-\|\mathbf{v}^\varepsilon-\mathbf{v}\|_{0,\Omega}\big|$$

and

$$||Q_2^\varepsilon \mathbf{v}^\varepsilon - \mathbf{v}^\varepsilon||_{0,\Omega} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

2.4. Localization

Before proving the convergence we shall first localize the variational inequality (I.3.6), which now can be written in the form

find $(\mathbf{v}^\varepsilon, w^\varepsilon, \boldsymbol{\varphi}^\varepsilon) \in K_\varepsilon$ such that

$$(2.36)\quad \int_{\Omega^\varepsilon} \{N^\varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon, \boldsymbol{\varphi}^\varepsilon)\gamma_{\alpha\beta}(\mathbf{z}-\mathbf{v}^\varepsilon) + M^\varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon, \boldsymbol{\varphi}^\varepsilon)\varrho_{\alpha\beta}(\boldsymbol{\psi}-\boldsymbol{\varphi}^\varepsilon) + Q^\varepsilon_\alpha(\boldsymbol{\varphi}^\varepsilon, w^\varepsilon)\big((u-w^\varepsilon)_{,\alpha}$$

$$+\psi_\alpha - \varphi^\varepsilon_\alpha)\}dx \geqslant \int_\Omega \{p(u-w^\varepsilon) + p_\alpha(z_\alpha - v^\varepsilon_\alpha) + m_\alpha(\psi_\alpha - \varphi^\varepsilon_\alpha)\}dx, \forall(\mathbf{z}, u, \boldsymbol{\psi}) \in K_\varepsilon.$$

For this purpose we take $\mathbf{z} = \mathbf{v}^\varepsilon \pm \boldsymbol{\theta}$, $\boldsymbol{\psi} = \boldsymbol{\varphi}^\varepsilon \pm \boldsymbol{\eta}$, $u = w^\varepsilon \pm \xi$, where ξ, θ_α, $\eta_\alpha \in \mathscr{D}(\Omega)$. Here $\mathscr{D}(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω. Noting that $(\mathbf{z}, u, \boldsymbol{\psi}) \in K_\varepsilon$ and applying the Green formula one readily obtains

$$(2.37)\quad \int_{\Omega^\varepsilon} (-N^\varepsilon_{\alpha\beta,\beta}\theta_\alpha - M^\varepsilon_{\alpha\beta,\beta}\eta_\alpha - Q^\varepsilon_{\alpha,\alpha}\xi + Q^\varepsilon_\alpha \eta_\alpha)dx + \int_{F^\varepsilon} \{(\overset{1}{N^\varepsilon_{\alpha\beta}} - \overset{2}{N^\varepsilon_{\alpha\beta}})n_\beta\theta_\alpha$$

$$+(\overset{1}{M^\varepsilon_{\alpha\beta}} - \overset{2}{M^\varepsilon_{\alpha\beta}})n_\beta\eta_\alpha + (\overset{1}{Q^\varepsilon} - \overset{2}{Q^\varepsilon})\xi\}ds = \int_\Omega (p\xi + p_\alpha\theta_\alpha + m_\alpha\eta_\alpha)dx \quad \forall \xi, \theta_\alpha, \eta_\alpha \in \mathscr{D}(\Omega),$$

where

$$\overset{\sigma}{N}_{\alpha\beta} = N_{\alpha\beta|\sigma}, \quad \text{etc.}$$

Hence

$$(2.38)\quad M^\varepsilon_{\alpha\beta,\beta} - Q^\varepsilon_\alpha + m_\alpha = 0,$$

$$(2.39)\quad N^\varepsilon_{\alpha\beta,\beta} + p_\alpha = 0, \quad Q^\varepsilon_{\alpha,\alpha} + p = 0.$$

Obviously, the equilibrium equations (2.38) and (2.39) are to be understood in the sense of distributions or $\mathscr{D}'(\Omega)$. From (2.37) we also have

$$(2.40)\quad \overset{1}{N^\varepsilon_{\alpha\beta}}n_\beta = \overset{2}{N^\varepsilon_{\alpha\beta}}n_\beta, \quad \overset{1}{M^\varepsilon_{\alpha\beta}}n_\beta = \overset{2}{M^\varepsilon_{\alpha\beta}}n_\beta, \quad \overset{1}{Q^\varepsilon} = \overset{2}{Q^\varepsilon} \quad \text{on } F^\varepsilon.$$

In the last relations we recognize the principle of action and reaction on F^ε.

Let us return to the variational inequality (2.36). Performing the integration by parts and taking account of (2.38), (2.39) and (2.40) we arrive at

$$(2.41)\quad \int_{F^\varepsilon} \{-N^\varepsilon_{\alpha\beta}n_\beta[z_\alpha - v^\varepsilon_\alpha] - M^\varepsilon_{\alpha\beta}n_\beta[\psi_\alpha - \varphi^\varepsilon_\alpha] - Q^\varepsilon[u-w^\varepsilon]\}ds \geqslant 0 \;\forall(\mathbf{z}, u, \boldsymbol{\psi}) \in K_\varepsilon,$$

where

$$N^\varepsilon_{\alpha\beta} := \overset{1}{N^\varepsilon_{\alpha\beta}}n_\beta = \overset{2}{N^\varepsilon_{\alpha\beta}}n_\beta, \quad \text{etc.}$$

The localization of (2.41) is performed as follows. It can be written in the equivalent form of three inequalities

$$(2.42)\quad -\int_{F^\varepsilon} N^\varepsilon_{\alpha\beta}n_\beta[z_\alpha - v^\varepsilon_\alpha]ds = -\int_{F^\varepsilon} (N^\varepsilon_n[z_n - v^\varepsilon_n] + N^\varepsilon_\tau[z_\tau - v^\varepsilon_\tau])ds \geqslant 0 \;\forall\, \mathbf{z} \in K^{\mathrm{bi}}_\varepsilon,$$

$$(2.43)\qquad -\int_{F^\varepsilon} M^\varepsilon_{\alpha\beta} n_\beta[\psi_\alpha-\varphi^\varepsilon_\alpha]ds = -\int_{F^\varepsilon}(M^\varepsilon_n[\psi_n-\varphi^\varepsilon_n]+M^\varepsilon_\tau[\psi_\tau-\varphi^\varepsilon_\tau])ds \geqslant 0,\ \forall \boldsymbol{\psi}\in K^{\mathfrak{bi}}_\varepsilon,$$

$$(2.44)\qquad -\int_{F^\varepsilon} Q^\varepsilon[u-w^\varepsilon]ds \geqslant 0 \qquad \forall u\in K^{\mathfrak{b}}_\varepsilon.$$

Since for the case considered no constraints are imposed on z_τ and ψ_τ therefore, by taking $z_n = v^\varepsilon_n$, $\psi_n = \varphi^\varepsilon_n$, we deduce that $N^\varepsilon_\tau = 0$, $M^\varepsilon_\tau = 0$. Then the inequalities (2.42) and(2.43) reduce to

$$(2.45)\qquad -\int_{F^\varepsilon} N^\varepsilon_n[z_n-v^\varepsilon_n]ds \geqslant 0 \qquad \forall \mathbf{z}\in K^{\mathfrak{bi}}_\varepsilon,$$

$$(2.46)\qquad -\int_{F^\varepsilon} M^\varepsilon_n[\psi_n-\varphi^\varepsilon_n]ds \geqslant 0 \qquad \forall \boldsymbol{\psi}\in K^{\mathfrak{bi}}_\varepsilon,$$

respectively. It is thus sufficient to localize one of the inequalities (2.44)–(2.46), for instance the second one. For this purpose we take $z_n = (1-\theta)\,v^\varepsilon_n+\theta\eta$, where $\theta\in\mathscr{D}(\Omega)$, $0\leqslant\theta\leqslant 1$ and $[\eta]_{F^\varepsilon}\geqslant 0$. Noting that these inequalities are positively homogeneous we readily obtain

$$(2.47)\qquad \int_{F^\varepsilon}\theta N^\varepsilon_n[\eta-v^\varepsilon_n]ds \leqslant 0 \qquad \forall\theta\in\mathscr{D}^+(\Omega)\ \forall\eta,[\eta]_{F^\varepsilon}\geqslant 0,$$

where

$$\mathscr{D}^+(\Omega) = \{\theta\in\mathscr{D}(\Omega)|\theta(x)\geqslant 0,\, x\in\Omega\}.$$

Now we take $\eta = 0$ and next $\eta = 2v^\varepsilon_n$. Hence

$$(2.48)\qquad N^\varepsilon_n[v^\varepsilon_n] = 0 \quad \text{on } F^\varepsilon.$$

By taking $\eta = v^\varepsilon_n+\zeta$, $[\zeta]_{F^\varepsilon} = 1$, from (2.47) we obtain

$$(2.49)\qquad N^\varepsilon_n\leqslant 0,$$

since $\theta\in\mathscr{D}^+(\Omega)$. The unilateral conditions satisfied on F^ε are of the Signorini-type. Their final form is

$$(2.50)\qquad [v^\varepsilon_n]\geqslant 0,\quad N^\varepsilon_n\leqslant 0,\quad N^\varepsilon_\tau = 0,\quad N^\varepsilon_n[v^\varepsilon_n] = 0 \quad \text{on } F^\varepsilon,$$

$$(2.51)\qquad [\varphi^\varepsilon_n]\geqslant 0,\quad M^\varepsilon_n\leqslant 0,\quad M^\varepsilon_\tau = 0,\quad M^\varepsilon_n[\varphi^\varepsilon_n] = 0 \quad \text{on } F^\varepsilon,$$

$$(2.52)\qquad [w^\varepsilon]\geqslant 0,\quad Q^\varepsilon\leqslant 0,\quad Q^\varepsilon[w^\varepsilon] = 0 \quad \text{on } F^\varepsilon.$$

Having in mind a later application let us return to (2.42)–(2.44) and take $\mathbf{z} = (1-\theta)\mathbf{v}^\varepsilon+ +\theta\boldsymbol{\eta}$, $\boldsymbol{\psi} = (1-\theta)\boldsymbol{\varphi}^\varepsilon+\theta\boldsymbol{\eta}$, $u = (1-\theta)w^\varepsilon+\theta\xi$, $[\eta_n]_{F^\varepsilon}\geqslant 0$, $[\xi]_{F^\varepsilon}\geqslant 0$ and θ as previously. We obtain

$$(2.53)\qquad -\int_{F^\varepsilon}\theta N^\varepsilon_{\alpha\beta}n_\beta[z_\alpha-v^\varepsilon_\alpha]ds\geqslant 0,\ \forall\theta\in\mathscr{D}^+(\Omega),\ \forall\mathbf{z},\,[z_n]_{F^\varepsilon}\geqslant 0$$

$$(2.54)\qquad -\int_{F^\varepsilon}\theta M^\varepsilon_{\alpha\beta}n_\beta[\psi_\alpha-\varphi^\varepsilon_\alpha]ds\geqslant 0,\ \forall\theta\in\mathscr{D}^+(\Omega),\ \forall\boldsymbol{\psi},[\psi_n]_{F^\varepsilon}\geqslant 0$$

$$(2.55)\qquad -\int_{F^\varepsilon}\theta Q^\varepsilon[u-w^\varepsilon]ds\geqslant 0,\ \forall\theta\in\mathscr{D}^+(\Omega),\ \forall u,\,[u]_{F^\varepsilon}\geqslant 0.$$

The variational inequality (I.3.17) gives

$$\boldsymbol{\varkappa} \in \mathsf{M}_s(\mathsf{R}), \quad \boldsymbol{\epsilon} \in \mathsf{M}_s(\mathsf{R}),$$

$$(2.56) \quad \int_{F^\varepsilon} \{A_{\alpha\beta\lambda\mu}(\varepsilon_{\lambda\mu}+\gamma^y_{\lambda\mu}(\mathbf{v}^1))+E_{\alpha\beta\lambda\mu}(\varkappa_{\lambda\mu}+\varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1))\}\gamma^y_{\alpha\beta}(\mathbf{z}-\mathbf{v}^1)\,dy \geqslant 0 \;\; \forall \mathbf{z} \in K^{\mathrm{bi}}_{YF}.$$

Now we take $\mathbf{z} = \mathbf{v}^1+\boldsymbol{\theta}$, $\boldsymbol{\theta} \in K^{\mathrm{bi}}_{YF}$. Hence

$$(2.57) \quad \int_{YF} \{A_{\alpha\beta\lambda\mu}(\varepsilon_{\lambda\mu}+\gamma^y_{\lambda\mu}(\mathbf{v}^1))+E_{\alpha\beta\lambda\mu}(\varkappa_{\lambda\mu}+\varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1))\}\gamma^y_{\alpha\beta}(\boldsymbol{\theta})\,dy \geqslant 0 \;\; \forall \boldsymbol{\theta} \in K^{\mathrm{bi}}_{YF}.$$

Let us take $\theta_\alpha \in \mathscr{D}(YF)$, that is θ_α equals zero in a neighbourhood of $\partial(YF) = \partial Y \cup F$. Hence

$$(2.58) \quad -[A_{\alpha\beta\lambda\mu}(\varepsilon_{\lambda\mu}+\gamma^y_{\lambda\mu}(\mathbf{v}^1))+E_{\alpha\beta\lambda\mu}(\varkappa_{\lambda\mu}+\varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1))]_{,\beta} = 0 \quad \text{in } \mathscr{D}'(YF)$$

or taking account of (I.3.12)

$$(2.59) \quad -n_{\alpha\beta,\beta} = 0 \quad \text{in } \mathscr{D}'(YF).$$

In a quite similar manner the variational inequality (I.3.18) leads up to

$$(2.60) \quad \int_{YF} \{E_{\alpha\beta\lambda\mu}(\varepsilon_{\lambda\mu}+\gamma^y_{\lambda\mu}(\mathbf{v}^1))+G_{\alpha\beta\lambda\mu}(\varkappa_{\lambda\mu}+\varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1))\}\varrho_{\alpha\beta}(\boldsymbol{\eta})\,dy \geqslant 0 \;\forall \boldsymbol{\eta} \in K^{\mathrm{bi}}_{YF}.$$

Hence

$$(2.61) \quad -[E_{\alpha\beta\lambda u}(\varepsilon_{\lambda\mu}+\gamma^y_{\lambda\mu}(\mathbf{v}^1))+G_{\alpha\beta\lambda\mu}(\varkappa_{\lambda\mu}+\varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1))]_{,\beta} = 0 \quad \text{in } \mathscr{D}'(YF),$$

or from (I.3.13)

$$(2.62) \quad -m_{\alpha\beta,\beta} = 0 \quad \text{in } \mathscr{D}'(YF).$$

Finally, the variational inequality (I.3.19) gives

$$(2.63) \quad \int_{YF} H_{\alpha\beta}\left(\omega_\beta+\frac{\partial w^1}{\partial y_\beta}\right)\frac{\partial \xi}{\partial y_\alpha}\,dy \geqslant 0 \;\; \forall \xi \in K^{\mathrm{b}}_{YF}.$$

Thus

$$(2.64) \quad -\left(H_{\alpha\beta}\left(\omega_\beta+\frac{\partial w^1}{\partial y_\beta}\right)\right)_{,\alpha} = 0 \quad \text{in } \mathscr{D}'(YF)$$

or taking account of (I.3.14)

$$(2.65) \quad -q_{\alpha,\alpha} = 0 \quad \text{in } \mathscr{D}'(YF).$$

Let us return to the inequality (2.57) and take $\boldsymbol{\theta} \in K^{\mathrm{bi}}_{YF}$ equal to zero in a neighbourhood of F. By using the Green formula and taking account of (2.59) we infer

(2.66) $|n_{\alpha\beta}n_\beta$ takes opposite values on the opposite sides of the basic cell Y.

Here (n_α) is the outward unit normal to ∂Y.

Similarly, from (2.60) and (2.63) we get

(2.67) $|m_{\alpha\beta}n_\beta$ and $q_\alpha n_\alpha$ take opposite values on the opposite sides of Y.

Due to (2.59), (2.62), (2.65) and (2.66), (2.67), we can write

$$(2.68) \quad -n_{\alpha\beta,\beta} = 0 \quad \text{in} \quad \mathscr{D}'\big(\mathsf{R}^2 \setminus \cup(F+(\mathfrak{n}_1 y_1, \mathfrak{n}_2 y_2))\big),$$

(2.69) $$-m_{\alpha\beta,\beta} = 0 \quad \text{in} \quad \mathscr{D}'\big(\mathsf{R}^2 \setminus \cup (F + (\mathfrak{n}_1 y_1, \mathfrak{n}_2 y_2))\big),$$

(2.70) $$-q_{\alpha,\alpha} = 0 \quad \text{in} \quad \mathscr{D}'\big(\mathsf{R}^2 \setminus \cup (F + (\mathfrak{n}_1 y_1, \mathfrak{n}_2 y_2))\big)$$

where $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathsf{Z}$ and $(y_1, y_2) \in F$. Here Z stands for the set of integers.

Let us examine the variational inequality (2.56). Taking $\mathbf{z}$ such that $\mathbf{z} = \mathbf{v}^1$ in a neighbourhood of ∂Y and performing the integration by parts we obtain

(2.71) $$\int_F \{\overset{1}{n}_{\alpha\beta} N_\beta (z_\alpha - v^1_\alpha)_{|1} - \overset{2}{n}_{\alpha\beta} N_\beta (z_\alpha - v^1_\alpha)_{|2}\} ds \geqslant 0$$

for any $\mathbf{z}$ such that $[z_N] \geqslant 0$ on F, since (2.59) is satisfied. Now $n_{\alpha\beta} N_\beta = n_N N_\alpha + n_T T_\alpha$, $n_N = n_{\alpha\beta} N_\alpha N_\beta$, $n_T = n_{\alpha\beta} N_\alpha T_\beta$, $n_{\alpha\beta} N_\beta z_\alpha = n_N z_N + n_T z_T$.

Hence

(2.72) $$\int_F \{\overset{1}{n}_N (z_N - \overset{1}{v}_N)_{|1} + \overset{1}{n}_T (z_T - v^1_T)_{|1} - \overset{2}{n}_N (z_N - v^1_N)_{|2} - \overset{2}{n}_T (z_T - v^1_T)_{|2}\} ds \geqslant 0,$$

for any $\mathbf{z}$ such that $[z_N] \geqslant 0$ on F.

By a reasoning similar to that which resulted in (2.47) we obtain

(2.73) $$\int_F \{\overset{1}{n}_N (z_N - v^1_N)_{|1} - \overset{2}{n}_N (z_N - v^1_N)_{|2}\} ds \geqslant 0,$$

for any $\mathbf{z}$ such that $[z_N] \geqslant 0$ on F.

Next, the variational inequality (I.3.18) gives

(2.74) $$\int_F \{\overset{1}{m}_N (\psi_N - \varphi^1_N)_{|1} - \overset{2}{m}_N (\psi_N - v^1_N)_{|2}\} ds \geqslant 0,$$

for any $\boldsymbol{\psi}$ such that $[\psi_N] \geqslant 0$ on F.

From (I.3.19) we obtain

(2.75) $$\int_F \{\overset{1}{q} (u - w^1)_{|1} - \overset{2}{q} (u - w^1)_{|2}\} ds \geqslant 0,$$

for any u such that $[u] \geqslant 0$ on F.

Let us set

(2.76) $$w_{\boldsymbol{\omega}}(y) = w^1(y) + \langle \boldsymbol{\omega}, y \rangle = w^1(y) + \omega_\alpha y_\alpha,$$

(2.77) $$\mathbf{v}_\epsilon(y) = \mathbf{v}^1(y) + \mathbf{P}^1(y),$$

(2.78) $$\boldsymbol{\varphi}_\varkappa(y) = \boldsymbol{\varphi}^1(y) + \mathbf{P}^2(y),$$

where $P^1_\alpha(y) = \varepsilon_{\alpha\beta} y_\beta$, $P^2_\alpha(y) = \varkappa_{\alpha\beta} y_\beta$. By using the localization technique similar to that which resulted in (2.47) and replacing $\mathbf{z}$, $\boldsymbol{\psi}$, u in (2.73), (2.74) and (2.75) by $\mathbf{z} - \mathbf{P}^1$, $\boldsymbol{\psi} - \mathbf{P}^2$ $u - \langle \boldsymbol{\omega}, \cdot \rangle$, respectively, we eventually arrive at

(2.79) $$\int_F \theta \{(z_\alpha - v_{\epsilon\alpha})_{|1} [A_{\alpha\beta\lambda\mu} \gamma^y_{\lambda\mu}(\mathbf{v}_\epsilon) + E_{\alpha\beta\lambda\mu} \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}_\varkappa)]_{|1} N_\beta - (z_\alpha - v_{\epsilon\alpha})_{|2} [A_{\alpha\beta\lambda\mu} \gamma^y_{\lambda\mu}(\mathbf{v}_\epsilon) + E_{\alpha\beta\lambda\mu} \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}_\varkappa)]_{|2} N_\beta\} ds \geqslant 0,$$

(2.80) $$\int_F \theta \{(\psi_\alpha - \varphi_{\varkappa\alpha})_{|1} [E_{\alpha\beta\lambda\mu} \gamma^y_{\lambda\mu}(\mathbf{v}_\epsilon) + G_{\alpha\beta\lambda\mu} \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}_\varkappa)]_{|1} N_\beta - (\psi_\alpha - \varphi_{\varkappa\alpha})_{|2} [E_{\alpha\beta\lambda\mu} \gamma^y_{\lambda\mu}(\mathbf{v}_\epsilon) + G_{\alpha\beta\lambda\mu} \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}_\varkappa)]_{|2} N_\beta\} ds \geqslant 0,$$

$$(2.81)\qquad \int_F \theta\left\{(u-w_\omega)_{|1}\left(H_{\alpha\beta}\frac{\partial w_\omega}{\partial y_\beta}\right)_{|1} N_\alpha-(u-w_\omega)_{|2}\left(H_{\alpha\beta}\frac{\partial w_\omega}{\partial y_\beta}\right)_{|2} N_\alpha\right\} ds \geqslant 0$$

for any $\mathbf{z}$, $\boldsymbol{\psi}$, u such that $[z_N]_F \geqslant 0$, $[\psi_N]_F \geqslant 0$, $[u]_F \geqslant 0$; $\theta \in \mathscr{D}^+(\Omega)$.

Now we shall change the scale knowing that $y = x/\varepsilon$. Toward this end we define

$$(2.82)\qquad w^\varepsilon_\omega(x) = \varepsilon w_\omega(x/\varepsilon) = \langle \boldsymbol{\omega}, x\rangle + \varepsilon w^1(x/\varepsilon),$$

$$(2.83)\qquad \mathbf{v}^\varepsilon_\epsilon(x) = \varepsilon \mathbf{v}_\epsilon(x/\varepsilon) = \mathbf{P}^1(x) + \varepsilon \mathbf{v}^1(x/\varepsilon),$$

$$(2.84)\qquad \boldsymbol{\varphi}^\varepsilon_\varkappa(x) = \varepsilon \boldsymbol{\varphi}_\varkappa(x/\varepsilon) = \mathbf{P}^2(x) + \varepsilon \boldsymbol{\varphi}^1(x/\varepsilon).$$

We see that $[w^\varepsilon_\omega]_{F^\varepsilon} \geqslant 0$, $[v^\varepsilon_{\epsilon n}]_{F^\varepsilon} \geqslant 0$, and $[\varphi^\varepsilon_{\varkappa n}]_{F^\varepsilon} \geqslant 0$.

The equations (2.68), (2.69) and (2.70) give, respectively,

$$(2.85)\qquad -\left(A_{\alpha\beta\lambda\mu}\gamma^y_{\lambda\mu}(\mathbf{v}^\varepsilon_\epsilon) + E_{\alpha\beta\lambda\mu}\varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa)\right)_{,\beta} = 0 \quad \text{in } \mathscr{D}'(\Omega^\varepsilon),$$

$$(2.86)\qquad -\left(E_{\alpha\beta\lambda\mu}\gamma^y_{\lambda\mu}(\mathbf{v}^\varepsilon_\epsilon) + G_{\alpha\beta\lambda\mu}\varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa)\right)_{,\beta}, = 0 \quad \text{in } \mathscr{D}'(\Omega^\varepsilon),$$

$$(2.87)\qquad -(H_{\alpha\beta} w^\varepsilon_{\omega,\beta})_{,\alpha} = 0 \quad \text{in } \mathscr{D}'(\Omega^\varepsilon).$$

Further, the inequalities (2.79), (2.80) and (2.81) transform, respectively, into

$$(2.88)\qquad \int_{F^\varepsilon} \theta\{(z_\alpha - v^\varepsilon_{\epsilon\alpha})_{|1}[A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon_\epsilon) + E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa)]_{|1} n_\beta$$
$$- (z_\alpha - v^\varepsilon_{\epsilon\alpha})_{|2}[A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon_\epsilon) + E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa)]_{|2} n_\beta\} ds \geqslant 0,$$

$$(2.89)\qquad \int_{F^\varepsilon} \theta\{(\psi_\alpha - \varphi^\varepsilon_{\varkappa\alpha})_{|1}[E_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon_\epsilon) + G_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa)]_{|1} n_\beta$$
$$- (\psi_\alpha - \varphi^\varepsilon_{\varkappa\alpha})_{|2}[E_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon) + G_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa)]_{|2} n_\beta) ds \geqslant 0,$$

$$(2.90)\qquad \int_{F^\varepsilon} \theta\{(u - w^\varepsilon_\omega)_{|1}(H_{\alpha\beta} w^\varepsilon_{\omega,\beta})_{|1} n_\alpha - (u - w^\varepsilon_\omega)_{|2}(H_{\alpha\beta} w^\varepsilon_{\omega,\beta})_{|2} n_\alpha\} ds \geqslant 0,$$

for any $\mathbf{z} \in K^{\mathfrak{bi}}_\varepsilon$, $\boldsymbol{\psi} \in K^{\mathfrak{bi}}_\varepsilon$, $u \in K^{\mathfrak{b}}_\varepsilon$, $\theta \in \mathscr{D}^+(\Omega^\varepsilon)$.

2.5. The last step: identification of v, w and φ

The final step consists in proving that $\mathbf{v} = \mathbf{v}^0$, $w = w^0$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}^0$, see (I.3.7)–(I.3.9) and (2.34).

As we know, the stored energy function g given by (I.2.38) is convex and differentiable. This implies subdifferentiability and maximal monotonicity of the subdifferential ∂g [2, I.37]. The latter property results in

$$(2.91)\qquad J^\varepsilon_1 := \int_{\Omega^\varepsilon} \theta(x)\{A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon) + E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon)) - \left(A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon_\epsilon)\right.$$
$$\left. + E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa)\right)\}\gamma_{\alpha\beta}(\mathbf{v}^\varepsilon - \mathbf{v}^\varepsilon_\epsilon) dx \geqslant 0,$$

$$(2.92)\qquad J^\varepsilon_2 := \int_{\Omega^\varepsilon} \theta(x)\{(E_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon) + G_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon)) - \left(E_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon_\epsilon)\right.$$
$$\left. + G_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa)\right)\}\varrho_{\alpha\beta}(\boldsymbol{\varphi}^\varepsilon - \boldsymbol{\varphi}^\varepsilon_\varkappa) dx \geqslant 0,$$

$$(2.93)\qquad J^\varepsilon_3 := \int_{\Omega^\varepsilon} \theta(x)\{H_{\alpha\beta}(w^\varepsilon_{,\beta} + \varphi^\varepsilon_\beta) - H_{\alpha\beta} w^\varepsilon_{\omega,\beta}\}\left((w^\varepsilon_{,\alpha} + \varphi^\varepsilon_\alpha) - w^\varepsilon_{\omega,\alpha}\right) dx \geqslant 0.$$

Here $\boldsymbol{\omega} \in \mathsf{R}^2$, $\boldsymbol{\epsilon} \in \mathsf{M}_s(\mathsf{R}^1)$, $\boldsymbol{\varkappa} \in \mathsf{M}_s(\mathsf{R}^1)$, $\theta \in \mathscr{D}^+(\Omega)$ and the test functions w^ε_ω, $\mathbf{v}^\varepsilon_\epsilon$ and $\boldsymbol{\varphi}_\varkappa$ have been defined earlier.

Let us now pass to the limit in (2.91). To this end perform the integration by parts and next take account of (2.27), $(2.39)_1$ and (2.85). Then

$$(2.94)\quad J^\varepsilon_1 = -\int_{\Omega^\varepsilon} \{N^\varepsilon_{\alpha\beta} - (A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon_\epsilon) + E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa))\}\theta_{,\beta}(v^\varepsilon_\alpha - v^\varepsilon_{\epsilon\alpha})dx$$

$$+\int_{\Omega^\varepsilon}\theta p_\alpha(v^\varepsilon_\alpha - v^\varepsilon_{\epsilon\alpha})dx + \int_{F^\varepsilon}\theta\{(v^\varepsilon_\alpha - v^\varepsilon_{\epsilon\alpha})_{|1}(A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon) + E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon))_{|1}$$

$$-(v^\varepsilon_\alpha - v^\varepsilon_{\epsilon\alpha})_{|2}(A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon) + E_{\alpha\beta\lambda u}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon))_{|2}\}ds - \int_{F\varepsilon}\theta\{(v^\varepsilon_\alpha - v^\varepsilon_{\epsilon\alpha})_{|1}(A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(v^\varepsilon_\epsilon).$$

$$+E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa))_{|1} - (v^\varepsilon_\alpha - v^\varepsilon_{\epsilon\alpha})_{|2}(A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon_\epsilon) + E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa))_{|2}\}ds \geqslant 0$$

On account of (2.53) and (2.88) the integrals over F^ε are non-positive.

Hence

$$(2.95)\quad -\int_{\Omega^\varepsilon}\{N^\varepsilon_{\alpha\beta} - (A_{\alpha\beta\lambda\mu}\gamma_{\lambda u}(\mathbf{v}^\varepsilon_\epsilon) + E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa))\}\theta_{,\beta}(v^\varepsilon_\alpha - v^\varepsilon_{\epsilon\alpha})dx + \int_{\Omega^\varepsilon}\theta p_\alpha(v^\varepsilon_\alpha - v^\varepsilon_{\epsilon\alpha})dx \geqslant 0.$$

Further, we have, cf. [4], p. 268 and [I.38], p. 77

$$(2.96)\qquad \mathbf{v}^\varepsilon - \mathbf{v}^\varepsilon_\epsilon \to \mathbf{v} - \mathbf{P}^1 \quad \text{strongly in } [L^2(\Omega)]^2 \quad \text{as } \varepsilon \to 0,$$

$$(2.97)\quad A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v}^\varepsilon_\epsilon) + E_{\alpha\beta\lambda u}\varrho_{\lambda\mu}(\boldsymbol{\varphi}^\varepsilon_\varkappa) \rightharpoonup \frac{1}{|Y|}\int_{YF}(A_{\alpha\beta\lambda\mu}\gamma^y_{\lambda\mu}(\mathbf{v}_\epsilon) + E_{\alpha\beta\lambda\mu}\varrho^y_{\lambda\mu}(\boldsymbol{\varphi}_\varkappa))dy$$

$$= \frac{\partial W}{\partial \varepsilon_{\alpha\beta}} \quad \text{weakly in } L^2(\Omega) \quad \text{as } \varepsilon \to 0$$

We recall that $W(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega})$, see Part I.

For $\varepsilon \to 0$ the inequality (2.95) in conjunction with (2.96) and (2.97) gives

$$(2.98)\qquad -\int_\Omega\left(N_{\alpha\beta} - \frac{\partial W}{\partial \varepsilon_{\alpha\beta}}\right)\theta_{,\beta}(v_\alpha - P^1_\alpha)dx + \int_\Omega \theta p_\alpha(v_\alpha - P^1_\alpha)dx \geqslant 0.$$

Integrating by parts we obtain

$$(2.99)\quad \int_\Omega \theta N_{\alpha\beta,\beta}(v_\alpha - P^1_\alpha)dx + \int_\Omega \theta\left(N_{\alpha\beta} - \frac{\partial W}{\partial \varepsilon_{\alpha\beta}}\right)\gamma_{\alpha\beta}(\mathbf{v} - \mathbf{P}^1)dx + \int_\Omega \theta p_\alpha(v_\alpha - P^1_\alpha)dx \geqslant 0.$$

The relation $(2.35)_2$ and Eq. $(2.39)_1$ result in

$$(2.100)\qquad N_{\alpha\beta,\beta} + p_\alpha = 0 \quad \text{in } \mathscr{D}'(\Omega).$$

Substituting (2.100) into (2.99) we get

$$(2.101)\qquad \int_\Omega \theta\left(N_{\alpha\beta} - \frac{\partial W}{\partial \varepsilon_{\alpha\beta}}\right)\gamma_{\alpha\beta}(\mathbf{v} - \mathbf{P}^1)dx \geqslant 0\,\forall\theta \in \mathscr{D}^+(\Omega).$$

Hence

$$(2.102)\qquad \left(N_{\alpha\beta}(x) - \frac{\partial W}{\partial \varepsilon_{\alpha\beta}}\right)(\gamma_{\alpha\beta}(\mathbf{v}(x)) - \varepsilon_{\alpha\beta}) \geqslant 0,\ \forall \boldsymbol{\epsilon} \in \mathsf{M}_s(\mathsf{R}^1)$$

for almost every $(=\text{a.e.})\, x \in \Omega$.

The functional J_2^ε, given by (2.92), can be studied in a similar way. Now Eq. (2.38) and $(2.35)_1$, $(2.35)_3$ give

$$M_{\alpha\beta,\beta}-Q_\alpha+m_\alpha = 0 \quad \text{in } \mathscr{D}'(\Omega). \tag{2.103}$$

Further, we arrive at

$$\int_\Omega \theta\left(M_{\alpha\beta}-\frac{\partial W}{\partial \varkappa_{\alpha\beta}}\right)\varrho_{\alpha\beta}(\boldsymbol{\varphi}-\mathbf{P}^2)\,dx \geqslant 0 \;\; \forall\theta\in\mathscr{D}^+(\Omega). \tag{2.104}$$

Thus

$$\left(M_{\alpha\beta}(x)-\frac{\partial W}{\partial \varkappa_{\alpha\beta}}\right)\left(\varrho_{\alpha\beta}(\boldsymbol{\varphi}(x))-\varkappa_{\alpha\beta}\right) \geqslant 0 \;\; \forall\boldsymbol{\varkappa}\in \mathsf{M}_s(\mathsf{R}^1), \text{ a.e. } x\in\Omega. \tag{2.105}$$

From $(2.35)_3$ and $(2.39)_2$ we infer

$$Q_{\alpha,\alpha}+p = 0 \quad \text{in } \mathscr{D}'(\Omega). \tag{2.106}$$

To pass to the limit as $\varepsilon\to 0$ with J_3^ε we write it in the form

$$\int_{\Omega^\varepsilon} \theta(Q_\alpha^\varepsilon-H_{\alpha\beta}w_{\omega,\beta}^\varepsilon)(w^\varepsilon-w_\omega^\varepsilon)_{,\alpha}\,dx+\int_{\Omega^\varepsilon}\theta(Q_\alpha^\varepsilon-H_{\alpha\beta}w_{\omega,\beta}^\varepsilon)\varphi_\alpha^\varepsilon\,dx \geqslant 0. \tag{2.107}$$

The passage to the limit as $\varepsilon\to 0$ in the second integral is straightforward since $\varphi_\alpha^\varepsilon\to\varphi_\alpha$ strongly in $L^2(\Omega)$. Thus we obtain

$$\int_{\Omega^\varepsilon} \theta(Q_\alpha^\varepsilon-H_{\alpha\beta}w_{\omega,\beta}^\varepsilon)\varphi_\alpha^\varepsilon\,dx \to \int_\Omega \theta\left(Q_\alpha-\frac{\partial W}{\partial\omega_\alpha}\right)\varphi_\alpha\,dx \tag{2.108}$$

since

$$\int_{\Omega^\varepsilon} H_{\alpha\beta}w_{\omega,\beta}^\varepsilon \rightharpoonup \frac{1}{|Y|}\int_{YF} H_{\alpha\beta}\left(\omega_\beta+\frac{\partial w^1}{\partial y_\beta}\right)dy = \frac{\partial W}{\partial\omega_\alpha} \qquad \text{weakly in } L^2(\Omega). \tag{2.109}$$

The passage to the limit in the first integral entering (2.107) is carried out similarly as previously. Finally, from (2.90), (2.106), (2.107) and (2.108) we get

$$\int_\Omega \theta\left(Q_\alpha-\frac{\partial W}{\partial\omega_\alpha}\right)(w_{,\alpha}+\varphi_\alpha-\omega_\alpha)\,dx \geqslant 0, \;\; \forall\boldsymbol{\omega}\in\mathsf{R}^2, \;\; \forall\theta\in\mathscr{D}^+(\Omega). \tag{2.110}$$

Hence

$$\left(Q_\alpha(x)-\frac{\partial W}{\partial\omega_\alpha}\right)(\varphi_\alpha(x)+w_{,\alpha}^{\xi}(x)-\omega_\alpha^{\xi}) \geqslant 0, \;\; \forall\boldsymbol{\omega}\in\mathsf{R}^2, \quad \text{a.e.} \quad x\in\Omega. \tag{2.111}$$

The maximal monotonicity of the subdifferential ∂W (see Part I) and the relations (2.102), (2.105) and (2.111) imply, cf. [2, p. 22]

$$N_{\alpha\beta}(x) = \partial W/\partial\gamma_{\alpha\beta}, \qquad \text{a.e.} \quad x\in\Omega, \tag{2.112}$$

$$M_{\alpha\beta}(x) = \partial W/\partial\varrho_{\alpha\beta}, \qquad \text{a.e.} \quad x\in\Omega, \tag{2.113}$$

$$Q_\alpha(x) = \partial W/\partial(w_{,\alpha}+\varphi_\alpha), \qquad \text{a.e.} \quad x\in\Omega, \tag{2.114}$$

where $W = W(\boldsymbol{\gamma}(\mathbf{v}(x)),\, \boldsymbol{\rho}(\boldsymbol{\varphi}(x)),\, \operatorname{grad} w(x)+\boldsymbol{\varphi}(x))$.

Taking account of (2.112)–(2.114) in Eqs. (2.100), (2.103) and (2.106) we arrive at the equilibrium equations (I.3.32)–(I.3.34) where $\mathfrak{M} = \mathbf{M}$, $\mathfrak{N} = \mathbf{N}$, $\mathfrak{Q} = \mathbf{Q}$ and $\mathbf{v}^0 = \mathbf{v}$, $w^0 = w$, $\boldsymbol{\varphi}^0 = \boldsymbol{\varphi}$. Thus the proof of the convergence is complete. ■

REMARK 2.3. The above proof of convergence is based on the energy method of the homogenization [10, 11] originally proposed for scalar equations. The same result can be achieved by using the method of the so called epi-convergence [1, I.7]. However, in our case, the proof would still be longer and more complicated. On the other hand, the epi-convergence results in the convergence of the total potential energy of the fissured plate to the total potential energy of the homogenized plate, that is

$$(2.115)\qquad \frac{1}{2}a_e^{\varepsilon}(\mathbf{v}^{\varepsilon}, w^{\varepsilon}, \boldsymbol{\varphi}^{\varepsilon}; \mathbf{v}^{\varepsilon}, w^{\varepsilon}, \boldsymbol{\varphi}^{\varepsilon}) - f_e(\mathbf{v}^{\varepsilon}, w^{\varepsilon}, \boldsymbol{\varphi}^{\varepsilon}) \to \int_{\Omega} W(\boldsymbol{\gamma}(\mathbf{v}), \boldsymbol{\rho}(\boldsymbol{\varphi}), \operatorname{grad} w + \boldsymbol{\varphi})\,dx - f_e(\mathbf{v}, w, \boldsymbol{\varphi}).$$

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