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Homogenization of fissured Reissner-like plates
Part I. Method of two-scale asymptotic expansion

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THE FIRST part of the paper deals with the determination of effective properties of Reissner-like plates damaged by periodically distributed microfissures. To approximate better the real behaviour of the fissured plate we perform a transformation of the basic equations changing the origin of the coordinate normal to the mid-plane. The model studied admits various unilateral conditions to be satisfied along microfissures. The presence of unilateral conditions leads quite naturally to the homogenization of variational inequalities. The homogenized or effective constitutive equations are derived using the method of two-scale asymptotic expansion.

W pierwszej części pracy rozpatrzono zagadnienie wyznaczenia zastępczych sztywności płyt typu Reissnera osłabionych periodycznie rozłożonymi mikroszczelinami. W celu lepszej aproksymacji rzeczywistego zachowania się płyty zarysowanej dokonano transformacji równań wyjściowych, zmieniając współrzędną normalną do płaszczyzny środkowej. Rozpatrywany model dopuszcza różnorodne warunki jednostronne zachodzące wzdłuż mikroszczelin. Obecność warunków jednostronnych prowadzi w naturalny sposób do homogenizacji nierówności wariacyjnych. Zhomogenizowane czyli efektywne związki konstytutywne wyprowadzono stosując metodę rozwinięć asymptotycznych.

Первая часть работы посвящена определению эффективных свойств пластин типа Рейсснера ослабленных периодически расположенными микротрещинами. Чтобы лучше аппроксимировать действительное поведение такой пластины произведена трансформация основных уравнений с помощью перемены координаты нормальной к срединной плоскости. Рассматриваемая модель допускает разнообразные одностронние условия вдоль трещин. Наличие односторонних связей приводит естественным образом к гомогенизации вариационных неравенств. Гомогенизированные или эффективные определяющие уравнения выведены асимптотическим методом.

1. Introduction

IN THE CASE of nonhomogeneous solids and structures, like fissured elastic plates, it is often desirable to know the respective overall properties. To determine such properties, mathematically elegant methods of homogenization have proved to be very useful. Yet these methods are most effective in the case of periodicity or non-uniform periodicity [4, 7, 8, 12, 13, 15, 38, 39]. The review paper [39] summarizes the progress achieved in the domain of applications of the method of homogenization to various problems of mechanics, see also [1, 10, 12, 17, 38, 39].

Mechanical and physical aspects of the fracture and damage mechanics are reviewed in [6] and [29], see also [18, 34, 40] and the excellent book by KACHANOV [23]. For references related to the homogenization of fissured elastic solids the reader should be referred to [4, 14, 26, 30, 31].

In our previous papers [30, 31] we have studied the homogenization of a fissured Kirchhoff plate in bending. On account of the presence of internal unilateral conditions of Signorini's type we had to deal in fact with five independent problems. This is in contrast with the three-dimensional case [38], obviously involving only one convex set of unilateral constraints. Variational formulations of all such unilateral problems result in variational inequalities defined in a variable domain. To carry out homogenization we have employed the two-scale asymptotic method. The homogenization of fissured Kirchhoff plates yields five different, physically nonlinear hyperelastic plates without fissures. It is worth noting that, as far as we know, fissured plates have not yet been studied by using the methods of variational inequalities since, quite surprisingly, unilateral conditions are usually not taken into account, see [9, 41].

The present contribution is concerned with the homogenization of Reissner-like plates weakened by periodically distributed microfissures. Though the requirement of periodicity is certainly an idealization of real behaviour of the plate yet in such a case the homogenization leads to effective formulae. In the non-periodic case the homogenization might result in, for instance, the effective plate weakened by a global fissure of a Christmas tree type.

Inspired by HELLAN's paper [20] we assume an occurance of a quantity e characterizing the "hinge" behaviour of the plate, see Section 2 below. Reissner-like plates considered in this paper are described by three independent kinematical fields. This implies a large number of kinematically admissible cracking mechanisms (modes). They are examined in Sects. 2 and 3. The model considered is based on a kinematical hypothesis of plane sections and is more general than the Kirchhoff's model. Unfortunately, under the hypothesis of plane section, lips of unilateral fissures may interpenetrate, see Fig. 3 below. Though such interpenetration is physically not plausible, yet it may occur within the model studied provided that unilaterally behaving fissures are admissible. Obviously not every unilateral mode results in interpenetration but such situations are rather typical for flexural fissures.

As we have already noted, our interest lies exclusively in the study of overall properties of the fissured plates. It seems therefore that for such case interpenetration is plausible.

We feel that more exact modelling of fissured plates would require more elaborate and complicated plate models. In particular the hypothesis of plane sections should be relaxed and replaced by the hypothesis of only piecewise plane sections. However, then the number of unknowns rapidly grows and a relative simplicity of even more elaborate and already existing models based on the hypothesis of plane sections is lost.

The first part of the paper contains only the results of application of the method of asymptotic expansions to the problem considered. A rather detailed study of problems of convergence is presented in the second part of the paper. The third part deals with some particular cases. An illustrative example of homogenization is also given.

2. Simultaneous bending and stretching of a plate weakened by a single unilateral fissure

2.1.

Let us consider an anisotropic elastic plate of thickness h, cf. Fig. 1.

The material of the plate is characterized by the elasticity tensor (c_{ijkl}); $i, j, k, l = 1, 2, 3$, and $c_{333\alpha} = c_{3\alpha\beta\gamma} = 0$. Thus planes $\bar{x}_3 = \text{const}$ are the planes of material

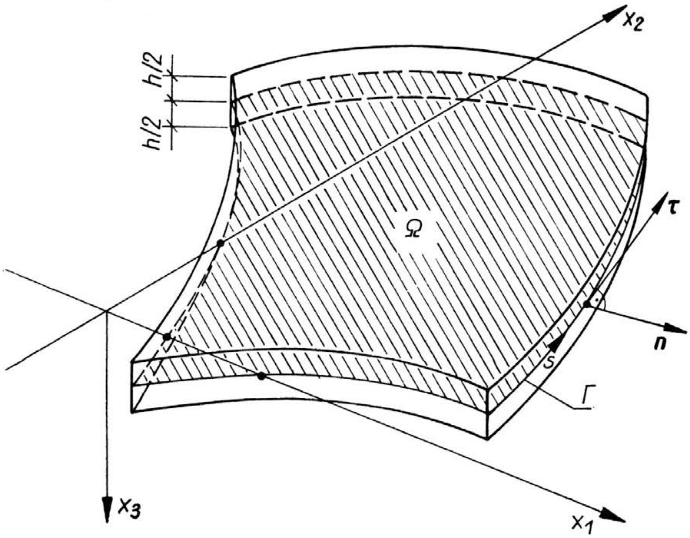

FIG. 1. Geometry of the plate.

symmetry. Let $\Omega^0 \subset \mathrm{R}^2$ be a bounded sufficiently regular plane domain; $\Gamma^0 = \partial\Omega^0$ denotes its boundary. The plate considered occupies the closed domain $\Omega^0 \times [-h/2, h/2]$. The Cartesian coordinates parametrize the domain Ω^0 or the mid-plane of the plate. Throughout this paper Greek indices take values 1 and 2. By $\boldsymbol{\sigma} = (\sigma_{ij})$ and $p = (p_i^\pm)$. we denote the stress tensor and density of surface forces, respectively, where

$$\sigma_{\alpha 3}\left(\bar{x}, \pm\frac{h}{2}\right) = p_\alpha^\pm, \quad \sigma_{33}\left(\bar{x}, \pm\frac{h}{2}\right) = p_3^\pm, \quad \bar{x} = (\bar{x}_\alpha). \tag{2.1}$$

We assume that the plate is clamped along the boundary; moreover, the body forces are omitted.

Suppose that $\bar{\mathbf{w}} = (\bar{w}_i) = (\bar{w}_i(\bar{x}, \bar{x}_3))$ is the displacement vector of the plate. According to the Hencky kinematical assumptions we have (see Remark 2.1 below)

$$\begin{aligned} \bar{w}_\alpha(\bar{x}, \bar{x}_3) &= \bar{u}_\alpha(\bar{x}) + \bar{x}_3\bar{\varphi}_\alpha(\bar{x}), \\ \bar{w}_3(\bar{x}, \bar{x}_3) &= \bar{w}(\bar{x}). \end{aligned} \tag{2.2}$$

Here $\bar{\mathbf{u}} = (\bar{u}_\alpha)$ is the in-plane displacement vector whereas $\bar{\varphi}_\alpha$ is the bending slope along the $\bar{x}_\alpha$-axis. In our derivations the influence of stress σ_{33} is omitted.

The constitutive equations have the form

$$\begin{aligned} \bar{N}_{\alpha\beta} &= \bar{A}_{\alpha\beta\lambda\mu}\bar{\eta}_{\lambda\mu}, \\ \bar{M}_{\alpha\beta} &= \bar{D}_{\alpha\beta\lambda\mu}\bar{\varrho}_{\lambda\mu}, \\ \bar{Q}_\alpha &= \bar{H}_{\alpha\beta}\bar{d}_\beta. \end{aligned} \tag{2.3}$$

Here $\bar{\mathbf{N}} = (\bar{N}_{\alpha\beta})$, $\bar{\mathbf{M}} = (\bar{M}_{\alpha\beta})$ and $\bar{\mathbf{Q}} = (\bar{Q}_\alpha)$ denote the membrane force tensor, the bending moment tensor and the shear force vector, respectively. Moreover, we have

$$\begin{aligned} &\bar{A}_{\alpha\beta\lambda\mu} = hC_{\alpha\beta\lambda\mu}, \quad \bar{D}_{\alpha\beta\lambda\mu} = h^3 C_{\alpha\beta\lambda\mu}/12, \\ &C_{\alpha\beta\lambda\mu} = c_{\alpha\beta\lambda\mu} - c_{\alpha\beta 33}c_{33\lambda\mu}c_{3333}^{-1}, \quad \bar{H}_{\alpha\beta} = 5hc_{\alpha 3\beta 3}/6. \end{aligned} \tag{2.4}$$

The functions $\bar{A}_{\alpha\beta\lambda\mu}$, $\bar{D}_{\alpha\beta\lambda\mu}$ and $\bar{H}_{\alpha\beta}$ may depend on $\bar{x} = (\bar{x}_\alpha)$. The kinematical relations are given by

$$\bar{\eta}_{\alpha\beta}(\bar{\mathbf{u}}) = \left(\frac{\partial \bar{u}_\alpha}{\partial \bar{x}_\beta} + \frac{\partial \bar{u}_\beta}{\partial \bar{x}_\alpha}\right)\Big/2, \tag{2.5}$$

$$\bar{\varrho}_{\alpha\beta}(\bar{\varphi}) = \left(\frac{\partial \bar{\varphi}_\alpha}{\partial \bar{x}_\beta} + \frac{\partial \bar{\varphi}_\beta}{\partial \bar{x}_\alpha}\right)\Big/2, \tag{2.6}$$

$$\bar{d}_\alpha(\bar{w}, \bar{\boldsymbol{\varphi}}) = \bar{w}_{,\alpha} + \bar{\varphi}_\alpha. \tag{2.7}$$

The equilibrium equations read

$$\frac{\partial \bar{M}_{\alpha\beta}}{\partial \bar{x}_\beta} - \bar{Q}_\alpha + \bar{m}_\alpha = 0 \quad \text{in } \Omega^0, \tag{2.8}$$

$$\frac{\partial \bar{N}_{\alpha\beta}}{\partial \bar{x}_\beta} + \bar{p}_\alpha = 0 \quad \text{in } \Omega^0, \tag{2.9}$$

$$\frac{\partial \bar{Q}_\alpha}{\partial \bar{x}_\alpha} + \bar{p} = 0 \quad \text{in } \Omega^0, \tag{2.10}$$

where

$$\bar{p} = p_3^+ + p_3^-, \quad \bar{p}_\alpha = p_\alpha^+ + p_\alpha^-, \quad \bar{m}_\alpha = (hp_\alpha^+ - p_\alpha^-)/2. \tag{2.11}$$

We assume that

$$\bar{p} \in L^2(\Omega^0), \quad \bar{p}_\alpha \in L^2(\Omega^0), \quad \bar{m}_\alpha \in L^2(\Omega^0). \tag{2.12}$$

Kinematically admissible fields $(\bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}})$ are elements of the space

$$V(\Omega^0) = [H_0^1(\Omega^0)]^2 \times H_0^1(\Omega^0) \times [H_0^1(\Omega^0)]^2, \tag{2.13}$$

where $H_0^1(\Omega^0)$ is the usual Sobolev space of functions equal to zero at Γ^0 in the sense of a trace, cf. [2].

Multiplying the equilibrium equations by $\bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}}$, respectively, adding them and performing the integration by parts we obtain the variational formulation or the principle of virtual displacements. It reads:

$$\left.\begin{array}{l}\text{find } (\bar{\mathbf{u}}, \bar{w}, \bar{\boldsymbol{\varphi}}) \in V(\Omega^0) \quad \text{such that} \\ \bar{a}(\bar{\mathbf{u}}, \bar{w}, \bar{\boldsymbol{\varphi}}; \bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}}) = \bar{f}(\bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}}), \forall (\bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}}) \in V(\Omega^0)\end{array}\right| (\mathscr{P}_{\Omega^0}), \tag{2.14}$$

where

$$\bar{a}(\bar{\mathbf{u}}, \bar{w}, \bar{\boldsymbol{\varphi}}; \bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}}) = \int_{\Omega^0} \{\bar{N}_{\alpha\beta}(\bar{\mathbf{u}})\bar{\eta}_{\alpha\beta}(\bar{\mathbf{z}}) + \bar{M}_{\alpha\beta}(\bar{\boldsymbol{\varphi}})\bar{\varrho}_{\alpha\beta}(\bar{\boldsymbol{\psi}}) + \bar{Q}_\alpha(\bar{w}, \bar{\boldsymbol{\varphi}})(\bar{u}_{,\alpha} + \bar{\psi}_\alpha)\} d\bar{x}, \tag{2.15}$$

$$f(\bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}}) = \int_{\Omega^0} (\bar{p}\bar{u} + \bar{p}_\alpha \bar{z}_\alpha + \bar{m}_\alpha \bar{\psi}_\alpha) d\bar{x}, \quad (\bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}}) \in V(\Omega^0) \tag{2.16}$$

and

$$\bar{N}_{\alpha\beta}(\bar{\mathbf{u}}) = \bar{A}_{\alpha\beta\lambda\mu}\bar{\eta}_{\lambda\mu}(\bar{\mathbf{u}}), \; \bar{M}_{\alpha\beta}(\bar{\boldsymbol{\varphi}}) = \bar{D}_{\alpha\beta\lambda\mu}\bar{\varrho}_{\lambda\mu}(\bar{\boldsymbol{\varphi}}), \quad \bar{Q}_\alpha(\bar{w}, \bar{\boldsymbol{\varphi}}) = \bar{H}_{\alpha\beta}\bar{d}_\beta.$$

The bilinear form $\bar{a}(\cdot, \cdot)$ is continuous and coercive [11], that is

(2.17) $$\bar{a}(\bar{\mathbf{u}}, \bar{w}, \bar{\boldsymbol{\varphi}}; \bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}}) \leqslant c\|(\bar{\mathbf{u}}, \bar{w}, \bar{\boldsymbol{\varphi}})\|_{V(\Omega^0)}\|(\bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}})\|_{V(\Omega^0)} \quad \forall(\bar{\mathbf{u}}, \bar{w}, \bar{\boldsymbol{\varphi}}), (\bar{\mathbf{z}}, \bar{u}, \bar{\boldsymbol{\psi}}) \in V(\Omega^0),$$

(2.18) $$\bar{a}(\bar{\mathbf{u}}, \bar{w}, \bar{\boldsymbol{\varphi}}; \bar{\mathbf{u}}, \bar{w}, \bar{\boldsymbol{\varphi}}) \geqslant c_1(\|\bar{w}\|^2_{1,\Omega^0} + \|\bar{\mathbf{u}}\|^2_{1,\Omega^0} + \|\bar{\boldsymbol{\varphi}}\|^2_{1,\Omega^0}) \forall(\bar{\mathbf{u}}, \bar{w}, \bar{\boldsymbol{\varphi}}) \in V(\Omega^0)$$

provided that

(2.19) $$\bar{A}_{\alpha\beta\lambda\mu} a_{\alpha\beta} a_{\lambda\mu} \geqslant c_2 a_{\alpha\beta} a_{\alpha\beta}, \quad \forall \mathbf{a} = (a_{\alpha\beta}) \in \mathsf{M}_s(\mathsf{R}),$$

(2.20) $$\bar{D}_{\alpha\beta\lambda\mu} a_{\alpha\beta} a_{\lambda\mu} \geqslant c_3 a_{\alpha\beta} a_{\alpha\beta} \forall \mathbf{a} = (a_{\alpha\beta}) \in \mathsf{M}_s(\mathsf{R}),$$

(2.21) $$\bar{H}_{\alpha\beta} b_\alpha b_\beta \geqslant c_4 b_\alpha b_\alpha \forall \mathbf{b} = (b_\alpha) \in \mathsf{R}^2.$$

Here and in the sequel c, c_1 etc. denote positive constants and M_s (R) stands for the space of symmetric 2×2 real matrices. We also assume that

(2.22) $$\bar{A}_{\alpha\beta\lambda\mu} \in L^\infty(\Omega^0), \quad \bar{D}_{\alpha\beta\lambda\mu} \in L^\infty(\Omega^0), \quad \bar{H}_{\alpha\beta} \in L^\infty(\Omega^0).$$

Hence

(2.23) $$\bar{A}_{\alpha\beta\lambda\mu} a_{\alpha\beta} a_{\lambda\mu} \leqslant c a_{\alpha\beta} a_{\alpha\beta}, \quad \bar{D}_{\alpha\beta\lambda\mu} a_{\alpha\beta} a_{\lambda\mu} \leqslant c_1 a_{\alpha\beta} a_{\alpha\beta}, \quad \bar{H}_{\alpha\beta} b_\alpha b_\beta \leqslant c_2 b_\alpha b_\alpha.$$

The norm of a function $u \in H^1(\Omega^0)$ is expressed as follows:

(2.24) $$\|u\|^2_{1,\Omega^0} = \int_{\Omega^0} u^2 d\bar{x} + \int_{\Omega^0} \frac{\partial u}{\partial \bar{x}_\alpha} \frac{\partial u}{\partial \bar{x}_\alpha} d\bar{x},$$

while

(2.25) $$\|u\|^2_{0,\Omega^0} = \int_{\Omega^0} u^2 d\bar{x}.$$

Obviously, the function space $H^1(\Omega)$ used below is defined similarly.

Having in view the homogenization of the fissured Reissner-like plate we first perform the transformation of the origin of $\bar{x}_3$-axis. It is also convenient to represent all the relevant quantities in a dimensionless form. For this purpose we set

(2.26) $$\tilde{x}_3 = \bar{e} + \bar{x}_3.$$

Thus the new origin is shifted to $\bar{x}_3 = -\bar{e}$; here $\bar{e}$ may be a constant or a function of $\bar{} = (\bar{x}_\alpha)$. The kinematical hypothesis (2.2) takes the form

(2.27) $$\bar{w}_\alpha(\bar{x}, \tilde{x}_3) = \bar{v}_\alpha(\bar{x}) + \tilde{x}_3 \bar{\varphi}_\alpha(\bar{x}), \quad \bar{w}_3(\bar{x}, \tilde{x}_3) = \bar{w}(\bar{x}).$$

The vector field

(2.28) $$\bar{v}_\alpha(\bar{x}) = \bar{u}_\alpha(\bar{x}) - \bar{e}\bar{\varphi}_\alpha(\bar{x})$$

stands for the in-plane $\bar{x}_3 = -\bar{e}$ displacement.

Let us assume that L, H represent characteristic dimensions of Ω^0 and of the plate thickness, respectively. We set

(2.29) $$x_\alpha = \bar{x}_\alpha/L, \quad x_3 = \bar{x}_3/H, \quad u_\alpha = \bar{u}_\alpha/H, \quad w = \bar{w}/L, \quad e = \bar{e}/H, \quad \varphi_\alpha = \bar{\varphi}_\alpha, \quad v_\alpha = \bar{v}_\alpha/H,$$

(2.29) [cont.]
$$A_{\alpha\beta\lambda\mu} = H^2\bar{A}_{\alpha\beta\lambda\mu}/L^2Eh, \quad D_{\alpha\beta\lambda\mu} = \bar{D}_{\alpha\beta\lambda\mu}/L^2Eh,$$
$$H_{\alpha\beta} = \bar{H}_{\alpha\beta}/Eh, \quad E_{\alpha\beta\lambda\mu} = eA_{\alpha\beta\lambda\mu},$$
$$G_{\alpha\beta\lambda\mu} = e^2A_{\alpha\beta\lambda\mu}+D_{\alpha\beta\lambda\mu}, \quad p = \bar{p}L/Eh, \quad p_\alpha = H\bar{p}_\alpha/Eh,$$
$$m_\alpha = (\bar{m}_\alpha/Eh)+ep_\alpha.$$

Here $E = (c^{-1}_{1111})^{-1}$ (1). In the case of isotropy E is equal to the Young modulus.

Hence

(2.30)
$$\frac{\partial\bar{u}_\alpha}{\partial\bar{x}_\beta} = \frac{H}{L}\frac{\partial u_\alpha}{\partial x_\beta} = \frac{H}{L}u_{\alpha,\beta}, \quad \frac{\partial\bar{w}}{\partial\bar{x}_\alpha} = \frac{\partial w}{\partial x_\alpha} = w_{,\alpha},$$
$$\frac{\partial\bar{\varphi}_\alpha}{\partial\bar{x}_\beta} = \frac{1}{L}\frac{\partial\varphi_\alpha}{\partial x_\beta} = \frac{1}{L}\varphi_{\alpha,\beta}, \quad \bar{\eta}_{\alpha\beta}(\bar{\mathbf{u}}) = \frac{H}{L}\tilde{\gamma}_{\alpha\beta}(\mathbf{u}),$$
$$\bar{\varrho}_{\alpha\beta}(\bar{\boldsymbol{\psi}}) = \frac{1}{L}\varrho_{\alpha\beta}(\boldsymbol{\psi}),$$

where

(2.31)
$$\tilde{\gamma}_{\alpha\beta}(\mathbf{u}) = u_{(\alpha,\beta)} = (u_{\alpha,\beta}+u_{\beta,\alpha})/2, \quad \varrho_{\alpha\beta}(\boldsymbol{\psi}) = \psi_{(\alpha,\beta)}.$$

The domain Ω^0 transforms onto

(2.32)
$$\Omega = \{(x, -e)|x = \bar{x}/L, \quad \bar{x} \in \Omega^0\}.$$

Taking account of (2.28)–(2.32) in (2.14) and knowing that test functions $\bar{\mathbf{z}}$ are expressed by $\bar{\mathbf{z}} = \bar{\boldsymbol{\xi}}+\bar{e}\bar{\boldsymbol{\psi}}$, the principle of virtual displacements $\mathscr{P}_{\Omega^0}$ takes the form

(2.33)
$$\left.\begin{array}{l}\text{find}(\mathbf{v}, w, \boldsymbol{\varphi}) \in V \quad \text{such that}\\ a_e(\mathbf{v}, w, \boldsymbol{\varphi}; \mathbf{z}, u, \boldsymbol{\psi}) = f_e(\mathbf{z}, u, \boldsymbol{\psi}), \quad \forall\, (\mathbf{z}, u, \boldsymbol{\psi}) \in V\end{array}\right| (\mathscr{P}_e),$$

where

(2.34)
$$V = [H^1_0(\Omega)]^2 \times H^1_0(\Omega) \times [H^1_0(\Omega)]^2,$$

and

(2.35)
$$a_e(\mathbf{v}, w, \boldsymbol{\varphi}; \mathbf{z}, u, \boldsymbol{\psi}) = \int_\Omega \{(A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v})+E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}))\gamma_{\alpha\beta}(\mathbf{z})$$
$$+(E_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{v})+G_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}(\boldsymbol{\varphi}))\varrho_{\alpha\beta}(\boldsymbol{\psi})+H_{\alpha\beta}d_\alpha(w, \boldsymbol{\varphi})d_\beta(u, \boldsymbol{\psi})\}dx,$$

(2.36)
$$f_e(\mathbf{z}, u, \boldsymbol{\psi}) = \int_\Omega (pu+p_\alpha z_\alpha+m_\alpha\psi_\alpha)dx.$$

Here

(2.37)
$$d_\alpha(w, \boldsymbol{\varphi}) = w_{,\alpha}+\varphi_\alpha, \quad \gamma_{\alpha\beta}(\mathbf{v}) = v_{(\alpha,\beta)}.$$

The constitutive relations result immediately from the form of the density of the stored elastic energy g of the plate or the integrand of the bilinear form a_e. We have

(2.38)
$$g(\boldsymbol{\gamma}, \boldsymbol{\rho}, \mathbf{d}) = \frac{1}{2}(A_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}+2E_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\varrho_{\lambda\mu}+G_{\alpha\beta\lambda\mu}\varrho_{\alpha\beta}\varrho_{\lambda\mu}+H_{\alpha\beta}d_\alpha d_\beta).$$

(1) Here $\mathbf{c}^{-1}\mathbf{c} = \mathbf{I}$, $\mathbf{I}$ — identity tensor.

The constitutive equations are now expressed as follows:

$$N_{\alpha\beta} = \partial g/\partial \gamma_{\alpha\beta} = A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}, \tag{2.39}$$

$$M_{\alpha\beta} = \partial g/\partial \varrho_{\alpha\beta} = E_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + G_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}, \tag{2.40}$$

$$Q_\alpha = \partial g/\partial d_\alpha = H_{\alpha\beta} d_\beta. \tag{2.41}$$

We see that

$$M_{\alpha\beta} = \tilde{M}_{\alpha\beta} + eN_{\alpha\beta}, \tag{2.42}$$

where

$$\tilde{M}_{\alpha\beta} = D_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}.$$

We note that the potential $g_1 = g - \frac{1}{2} H_{\alpha\beta} d_\alpha d_\beta$ resembles that of a two-dimensional Cosserat continuum, see [22].

Let us now pass to an examination of the convexity and the boundedness from below of the function g.

The relation (2.28) implies

$$\mathbf{v} = \mathbf{u} - e\boldsymbol{\varphi}, \quad \gamma_{\alpha\beta}(\mathbf{v}) = \tilde{\gamma}_{\alpha\beta}(\mathbf{u}) - e\varrho_{\alpha\beta}(\boldsymbol{\varphi}) \tag{2.43}$$

or

$$\gamma_{\alpha\beta} = \tilde{\gamma}_{\alpha\beta} - e\varrho_{\alpha\beta}. \tag{2.44}$$

Here $\tilde{\boldsymbol{\gamma}} = (\tilde{\gamma}_{\alpha\beta})$ denotes the strain tensor referred to the mid-plane $e = 0$ of the plate. The stored energy function $\tilde{g}$ referred to the same mid-plane is given by

$$\tilde{g}(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\rho}, \mathbf{d}) = \frac{1}{2}(A_{\alpha\beta\lambda\mu}\tilde{\gamma}_{\alpha\beta}\tilde{\gamma}_{\lambda\mu} + D_{\alpha\beta\lambda\mu}\varrho_{\alpha\beta}\varrho_{\lambda\mu} + H_{\alpha\beta}d_\alpha d_\beta). \tag{2.45}$$

The usual symmetry properties of $(A_{\alpha\beta\lambda\mu})$, $(D_{\alpha\beta\lambda\mu})$ and $(H_{\alpha\beta})$ and (2.19)–(2.21) render the function g strictly convex. Obviously we have

$$\tilde{g}(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\rho}, \mathbf{d}) \geqslant c_1|\tilde{\boldsymbol{\gamma}}|^2 + c_2|\boldsymbol{\rho}|^2 + c_3|\mathbf{d}|^2 \geqslant c(|\tilde{\boldsymbol{\gamma}}|^2 + |\boldsymbol{\rho}|^2 + |\mathbf{d}|^2) \quad \forall\, \tilde{\boldsymbol{\gamma}} \in \mathsf{M}_s(\mathsf{R}), \quad \forall\, \boldsymbol{\rho} \in \mathsf{M}_s(\mathsf{R}), \quad \forall\, \mathbf{d} \in \mathsf{R}^2, \tag{2.46}$$

where

$$|\tilde{\boldsymbol{\gamma}}|^2 = \tilde{\gamma}_{\alpha\beta}\tilde{\gamma}_{\alpha\beta}, \quad |\mathbf{d}|^2 = d_\alpha d_\alpha \quad \text{and} \quad c = \min(c_1, c_2, c_3).$$

The transformations $(\mathbf{u}, \boldsymbol{\varphi}) \to (\mathbf{v}, \boldsymbol{\varphi})$ and $(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\rho}) \to (\boldsymbol{\gamma}, \boldsymbol{\rho})$ are linear and invertible. Specifically we can write

$$\begin{bmatrix} \tilde{\boldsymbol{\gamma}} \\ \boldsymbol{\rho} \end{bmatrix} = \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\rho} \end{bmatrix} \quad \text{or} \quad \tilde{\mathbf{U}} = \mathbf{BU}, \tag{2.47}$$

where

$$\tilde{\mathbf{U}} = (\tilde{\boldsymbol{\gamma}}, \boldsymbol{\rho}) \quad \text{and} \quad \mathbf{U} = (\boldsymbol{\gamma}, \boldsymbol{\rho}).$$

Under the transformation (2.47) the function

$$g(\boldsymbol{\gamma}, \boldsymbol{\rho}, \mathbf{d}) = \tilde{g}(\mathbf{BU}, \mathbf{d}), \tag{2.48}$$

where g is given by (2.38), preserves strict convexity (see [37] Theorem 5.7).

Using the elementary inequality

$$|\boldsymbol{\gamma}+e\boldsymbol{\rho}|^2 \geqslant \frac{1}{2}|\boldsymbol{\gamma}|^2-e^2|\boldsymbol{\rho}|^2, \tag{2.49}$$

as well as (2.46) and (2.48), we obtain the ellipticity condition

$$g(\boldsymbol{\gamma}, \boldsymbol{\rho}, \mathbf{d}) \geqslant c_1|\boldsymbol{\gamma}|^2+c_2|\boldsymbol{\rho}|^2+c_3|\mathbf{d}|^2 \geqslant c(|\boldsymbol{\gamma}|^2+|\boldsymbol{\rho}|^2+|\mathbf{d}|^2). \tag{2.50}$$

provided that

$$e^2 < 1. \tag{2.51}$$

We observe that the constants c_1, c_2 and c_3 entering (2.50) are independent. This remark will be important for the study of convergence in the second part of the paper. Next, we see that it is just the condition (2.51) which naturally requires the dimensionless form of all the relevant relations. Throughout this paper we assume that the inequality (2.51) holds true.

By virtue of the Lax–Milgram theorem [38] it is evident that a solution of the problem $(\mathscr{P}_e)$ exists and is unique since the bilinear form $a_e(\cdot, \cdot)$ is coercive and the functional f_e is linear and continuous.

2.2.

We pass now to the study of the Reissner-like plate obeying the constitutive equations (2.39)–(2.41) and damaged by a single fissure $F \subset \Omega$, see Fig. 2.

F is closed as a set, $\overline{F} = F$, of class C^1 and is strictly contained in Ω. The normal and tangent vectors $\mathbf{n}$ and $\boldsymbol{\tau}$ are shown in the Fig. 2. Obviously we have $\tau_1 = -n_2$, $\tau_2 = n_1$.

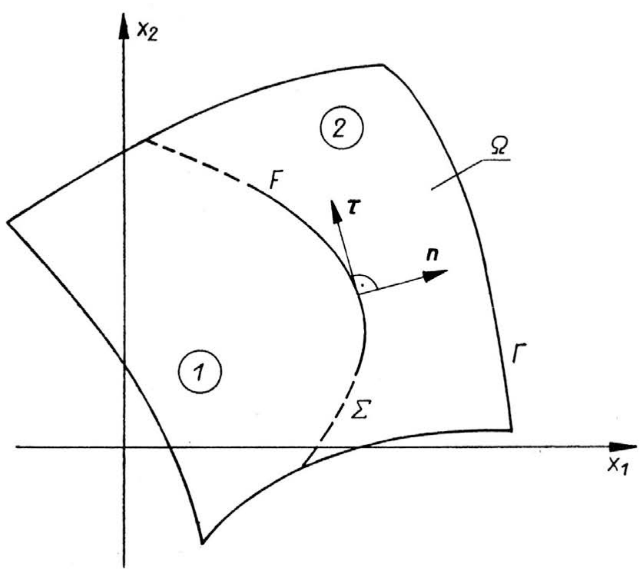

FIG. 2.

Let us consider all cracking modes admitted by the Reissner-like plate theory considered here. The following kinematical quantities may experience jumps along F:

$$\varphi_n = \varphi_\alpha n_\alpha, \quad \varphi_\tau = \varphi_\alpha \tau_\alpha, \quad v_n = v_\alpha n_\alpha, \quad v_\tau = v_\alpha \tau_\alpha, \quad w.$$

The corresponding moments and forces

$$M_n = M_{\alpha\beta} n_\alpha n_\beta, \quad M_\tau = M_{\alpha\beta} n_\alpha \tau_\beta, \quad N_n = N_{\alpha\beta} n_\alpha n_\beta, \quad N_\tau = N_{\alpha\beta} n_\alpha \tau_\beta, \quad Q = Q_\alpha n_\alpha$$

behave according to the principle of action and reaction.

We set

$$\mathbf{\Phi} = (\Phi_A) = (\varphi_n, \varphi_\tau, v_n, v_\tau, w), \quad S = (S_A) = (M_n, M_\tau, N_n, N_\tau, Q), \quad A = 1, 2, \ldots, 5.$$

A jump of a function g across F will be denoted by $[\![g]\!]$ and is expressed as follows

$$[\![g]\!] = g|_2 - g|_1 \quad \text{on } F, \tag{2.52}$$

where $g|_\alpha$ stands for the value of g at the α-side of the fissure, see Fig. 2.

Three situations can occur:

(a) no constraint is imposed on $[\![\Phi_A]\!]$ on F,

(b) $[\![\Phi_A]\!] = 0$ on F,

(c) $[\![\Phi_A]\!] \geqslant 0$ on F.

Then we have, in conformity with the principle of action and reaction,

(a₁) $\overset{1}{S}_A = \overset{2}{S}_A = 0$ on F,

(b₁) $\overset{1}{S}_A = \overset{2}{S}_A$ on F,

(c₁) $S_A = \overset{1}{S}_A = \overset{2}{S}_A, \quad S_A \leqslant 0, \quad S_A[\![\Phi_A]\!] = 0$ on F (no summation!),

where

$$\overset{\sigma}{M}_n = M_{\alpha\beta}|_\sigma n_\alpha n_\beta, \quad \overset{\sigma}{M}_\tau = M_{\alpha\beta}|_\sigma n_\alpha \tau_\beta,$$
$$\overset{\sigma}{N}_n = N_{\alpha\beta}|_\sigma n_\alpha n_\beta, \quad \overset{\sigma}{N}_\tau = N_{\alpha\beta}|_\sigma n_\alpha \tau_\beta, \quad \overset{\sigma}{Q} = Q_\alpha|_\sigma n_\alpha, \quad \sigma = 1, 2.$$

Hence we infer that the plate considered admits $3^5 = 243$ cracking modes (mechanisms). We observe that now the equilibrium equations are given by

$$M_{\alpha\beta,\beta} - Q_\alpha + m_\alpha = 0 \quad \text{in} \quad \Omega \diagdown F, \tag{2.53}$$

$$N_{\alpha\beta,\beta} + p_\alpha = 0, \quad \text{in } \Omega \diagdown F, \tag{2.54}$$

$$Q_{\alpha,\alpha} + p = 0, \quad \text{in } \Omega \diagdown F. \tag{2.55}$$

Let us proceed to the variational formulation of the equilibrium problem of the plate weakened by the fissure F. Toward this end we set

$$H^1_1(\Omega \diagdown F) = \{v \in H^1(\Omega \diagdown F) | v = 0 \text{ on } \Gamma\}, \tag{2.56}$$

where (see [19])

$$H^1(\Omega \diagdown F) = \{v \in L^2(\Omega \diagdown F) | v_{,\alpha}, \in L^2(\Omega \diagdown F)\} \tag{2.57}$$

or equivalently

$$H^1(\Omega \diagdown F) = \{v \in H^1(\Omega_1) \cup H^1(\Omega_2) | \gamma_1(v) = \gamma_2(v) \text{ on } \Sigma \diagdown F\}. \tag{2.58}$$

Here γ_α is the trace operator for functions defined on Ω_α, $\alpha = 1, 2$.

Kinematically admissible in-plane displacements and bending slopes will be elements of appropriate closed and convex sets. These sets are defined as follows:

(2.59)
$$\begin{aligned}
C_{\mathfrak{dc}} &= \{\mathbf{v} = (v_\alpha) \in [H^1_1(\Omega\setminus F)]^2 | [\![v_n]\!] \geqslant 0, [\![v_\tau]\!] = 0 \text{ on } F\},\\
C_{\mathfrak{cd}} &= \{\mathbf{v} \in [H^1_1(\Omega\setminus F)]^2 | [\![v_n]\!] = 0,\ [\![v_\tau]\!] \geqslant 0 \text{ on } F\},\\
C_{\mathfrak{dd}} &= \{\mathbf{v} \in [H^1_1(\Omega\setminus F)]^2 |]\!]v_n]\!] \geqslant 0, [\![v_\tau]\!] \geqslant 0 \text{ on } F\},\\
C_{\mathfrak{di}} &= \{\mathbf{v} \in [H^1_1(\Omega\setminus F)]^2 | [\![v_n]\!] \geqslant 0 \text{ on } F\},\\
C_{\mathfrak{id}} &= \{\mathbf{v} \in [H^1_1(\Omega\setminus F)]^2 | [\![v_\tau]\!] \geqslant 0 \text{ on } F\},\\
C_{\mathfrak{ci}} &= \{\mathbf{v} \in [H^1_1(\Omega\setminus F)]^2 | [\![v_n]\!] = 0 \text{ on } F\},\\
C_{\mathfrak{ic}} &= \{\mathbf{v} \in [H^1_1(\Omega\setminus F)]^2 | [\![[v_\tau]\!] = 0 \text{ on } F\},\\
C_{\mathfrak{cc}} &= [H^1_0(\Omega)]^2, \quad C_{\mathfrak{ii}} = [H^1_1(\Omega\setminus F)]^2
\end{aligned}$$

($\mathfrak{c}$ — continuous, $\mathfrak{d}$ — discontinuous, $\mathfrak{i}$ — indeterminate though not necessarily continuous).

Kinematically admissible transverse displacements will be elements of one of the following closed and convex sets

(2.60)
$$\begin{aligned}
C_{\mathfrak{d}} &= \{w \in H^1_1(\Omega\setminus F) | [\![w]\!] \geqslant 0 \text{ on } F\},\\
C_{\mathfrak{c}} &= H^1_0(\Omega), \quad C_{\mathfrak{i}} = H^1_1(\Omega\setminus F).
\end{aligned}$$

By

(2.61)
$$V_{abcdh} = C_{ab} \times C_c \times C_{dh}, \quad a, b, c, d, h \in (\mathfrak{d}, \mathfrak{c}, \mathfrak{i})$$

we denote the set of kinematically admissible fields $(\mathbf{z}, u, \boldsymbol{\psi})$. From (2.59)–(2.61) we readily infer that among the sets V_{abcdh} there are $2 \cdot 4 \cdot 4 = 32$ spaces. The remaining closed and convex sets will be denoted, for the sake of simplicity, by K^{Δ}, $\Delta = 1, 2, \ldots, 211$. Hence the Reissner-like plate considered admits 211 modes of unilateral fissures. In the sequel our attention will be focussed on unilateral fissures since this case is more difficult than the bilateral cases. In the latter case inequalities are absent.

Of particular interest is the bending cracking mode (flexural fissure) for which $[\![\varphi_n]\!] \geqslant 0$ of F, whereas φ_τ, v_n, v_τ and w do not experience jumps, see Fig. 3.

This case corresponds to the following set of kinematically admissible fields

$$K^{\text{ben}} = V_{\mathfrak{ccc}\mathfrak{dc}} = C_{\mathfrak{cc}} \times C_{\mathfrak{c}} \times C_{\mathfrak{dc}}.$$

Fig. 4 illustrates the case when $[\![w]\!] \geqslant 0$ on F, while the remaining fields are continuous. Thus now

$$K^{\text{shear}} = V_{\mathfrak{cc}\mathfrak{d}\mathfrak{cc}} = C_{\mathfrak{cc}} \times C_{\mathfrak{d}} \times C_{\mathfrak{cc}}.$$

The next simple cracking mode corresponds to tension, see Fig. 5.

Now $[\![v_n]\!] = [\![u_n]\!] \geqslant 0$ on F, and the remaining kinematical fields are continuous. Here $\mathbf{u}$ and $\mathbf{v}$ are interrelated by $(2.43)_1$. Obviously, we have

$$K^{\text{ten}} = V_{\mathfrak{dcccc}} = C_{\mathfrak{dc}} \times C_{\mathfrak{c}} \times C_{\mathfrak{cc}}.$$

We pass to the variational formulation. Multiplying Eqs. (2.53)–(2.55) by $(\psi_\alpha - \varphi^{\Delta}_\alpha)$, $(z_\alpha - v^{\Delta}_\alpha)$ and $(u - w^{\Delta})$ respectively, integrating over $\Omega\setminus F$ and adding we arrive at

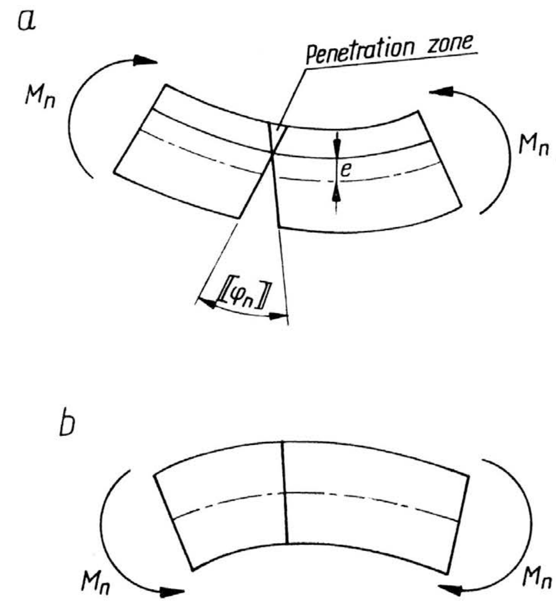

FIG. 3. Behaviour of the flexural fissure; a) the fissure is open, $M_n = 0$, $[\![\varphi_n]\!] \geqslant 0$; b) the fissure is closed, $M_n \leqslant 0$, $[\![\varphi_n]\!] = 0$.

$$
\begin{aligned}
(2.62) \qquad & a_e(\mathbf{v}^{\Delta}, w^{\Delta}, \boldsymbol{\varphi}^{\Delta}; \mathbf{z}-\mathbf{v}^{\Delta}, u-w^{\Delta}, \boldsymbol{\psi}-\boldsymbol{\varphi}^{\Delta}) \\
& = f_e(\mathbf{z}-\mathbf{v}^{\Delta}, u-w^{\Delta}, \boldsymbol{\psi}-\boldsymbol{\varphi}^{\Delta}) + \int_F \{[\overset{1}{N}_n(\overset{1}{z}_n-\overset{1}{v}{}^{\Delta}_n) - \overset{2}{N}_n(\overset{2}{z}_n-\overset{2}{v}{}^{\Delta}_n)] \\
& + [\overset{1}{N}_\tau(\overset{1}{z}_\tau-\overset{1}{v}{}^{\Delta}_\tau) - \overset{2}{N}_\tau(\overset{2}{z}_\tau-\overset{2}{v}{}^{\Delta}_\tau)] + [\overset{1}{M}_n(\overset{1}{\psi}_n-\overset{1}{\varphi}{}^{\Delta}_n) - \overset{2}{M}_n(\overset{2}{\psi}_n-\overset{2}{\varphi}{}^{\Delta}_n)] \\
& + [\overset{1}{M}_\tau(\overset{1}{\psi}_\tau-\overset{1}{\varphi}{}^{\Delta}_\tau) - \overset{2}{M}_\tau(\overset{2}{\psi}_\tau-\overset{2}{\varphi}{}^{\Delta}_\tau)] + [\overset{1}{Q}(\overset{1}{u}-\overset{1}{w}{}^{\Delta}) - \overset{2}{Q}(\overset{2}{u}-\overset{2}{w}{}^{\Delta})]\}\, ds, \qquad (\mathbf{z}, u, \boldsymbol{\psi}) \in K^{\Delta}
\end{aligned}
$$

where $\overset{\alpha}{u} = u|_\alpha$, $\overset{\alpha}{w}{}^{\Delta} = w^{\Delta}|_\alpha$, etc. Taking account of the definition of the set K^{Δ} we readily infer that in (2.62) the integrals along the fissure F are non-negative. For the sake of simplicity, from now on we shall write $(\mathbf{v}, w, \boldsymbol{\varphi})$ instead of $(\mathbf{v}^{\Delta}, w^{\Delta}, \boldsymbol{\varphi}^{\Delta})$. Thus the variational formulation reads

$$
(2.63) \qquad \left.
\begin{aligned}
& \text{find} \quad (\mathbf{v}, w, \boldsymbol{\varphi}) \in K^{\Delta} \quad \text{such that} \\
& a_e(\mathbf{v}, w, \boldsymbol{\varphi}; \mathbf{z}-\mathbf{v}, u-w, \boldsymbol{\psi}-\boldsymbol{\varphi}) \\
& \geqslant f_e(\mathbf{z}-\mathbf{v}, u-w, \boldsymbol{\psi}-\boldsymbol{\varphi}) \,\forall\, (\mathbf{z}, u, \boldsymbol{\psi}) \in K^{\Delta}
\end{aligned}
\right| (\mathscr{P}^{\Delta}_e).
$$

Similarly as in [31] it can be shown that the variational inequality (2.63) yields Eqs. (2.53)–(2.55) and the conditions corresponding to K^{Δ}, provided that functions $\mathbf{v}$, w and $\boldsymbol{\varphi}$ are sufficiently regular.

The variational problem $\mathscr{P}^{\Delta}_e$ is equivalent to the minimization problem

$$
(2.64) \qquad \inf\left\{\frac{1}{2}\, a_e(\mathbf{z}, u, \boldsymbol{\psi}; \mathbf{z}, u, \boldsymbol{\psi}) - f_e(\mathbf{z}, u, \boldsymbol{\psi}) \,|\, (\mathbf{z}, u, \boldsymbol{\psi}) \in K^{\Delta}\right\} (\mathscr{P}^{e,\Delta}_m)
$$

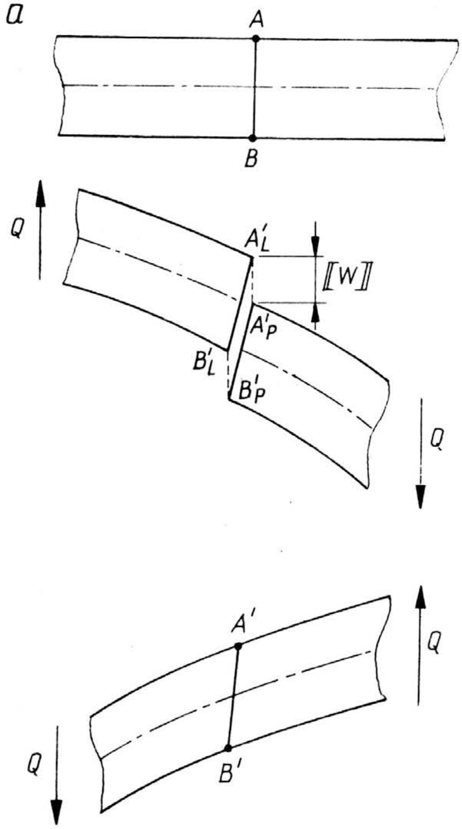

FIG. 4. Behaviour of the shear fissure; a) the fissure is developed, $Q = 0$, $[\![w]\!] \geqslant 0$; b) the plate is unfissured, $Q \leqslant 0$, $[\![w]\!] = 0$.

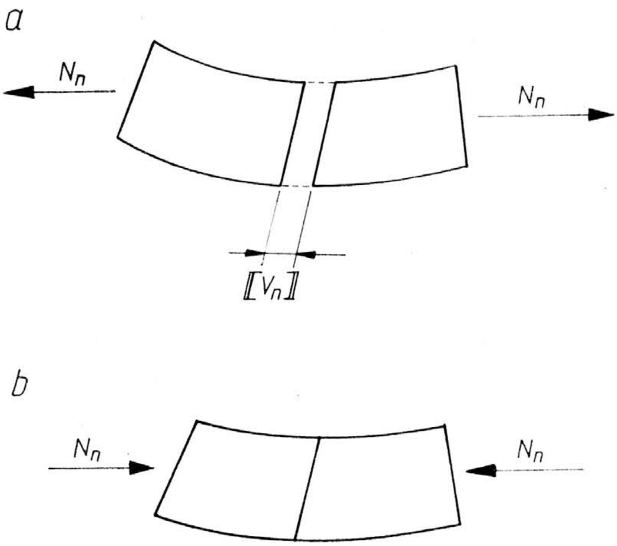

FIG. 5. Behaviour of the extensional fissure; a) the fissure is open. $N_n = 0$, $[\![v_n]\!] \geqslant 0$; b) the fissure is closed, $N_n \leqslant 0$, $[\![v_n]\!] = 0$.

Due to the coercivity of the bilinear form $a_\varepsilon(\cdot, \cdot)$, a solution $(\mathbf{v}, w, \boldsymbol{\varphi})$ exists and is unique.

REMARK 2.1

In the original paper [35] Reissner cosidered the influence of stress σ_{33} on the strain energy of the isotropic plate. Thus the model studied in the present paper does not reduce to the Reissner model in the isotropic case. It reduces to HENCKY's model [21] proposed later on with modified shearing stiffness. The latter model is usually [5, 16] (but not very rightly) associated with the name of Mindlin, see the comments by REISSNER [36]. In this paper we consider a natural generalization of the modified Hencky's approach to the case of plates made of anisotropic material, see [3, 25, 32].

3. Homogenization of a Reissner-like plate damaged by periodically distributed microfissures

Hitherto we have examined the variational approach to the unilateral boundary value problem for the Reissner-like plate weakened by one fissure only. In the present section we study the problem of determination of effective properties of a Reissner-like plate weakened by many microfissures. At the actual stage of the development of the homogenization theory one can effectively solve only periodic problems cf. Refs. [7, 38, 39]. Therefore we assume that microfissures are distributed in a periodic way.

3.1. Asymptotic analysis

Let us assume that the plate clamped at the boundary is weakened by microfissures εF distributed εY-periodically, see Fig. 6.

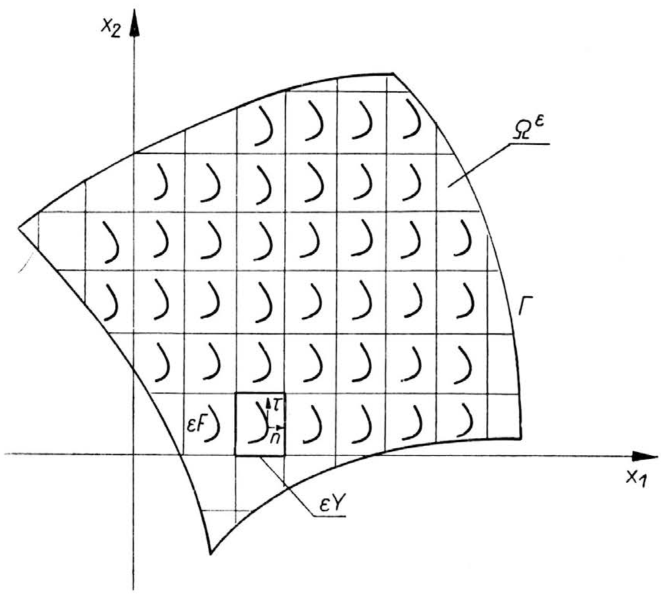

FIG. 6. Plate with fissures.

We see that every εY-cell, homothetic to the so-called basic cell Y, is damaged by the microfissure εF, $\varepsilon > 0$. Fig. 7 represents the basic cell.

As previously, we assume that F is of class C^1 and $\bar{F} = F \subset Y$. We note that F may be a sum of disjoint fissures. Moreover we assume that the domain $YF = Y \setminus F$ is connected. This means that F does not intersect the boundary $\partial Y = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3 \cup \bar{\Gamma}_4$ of Y.

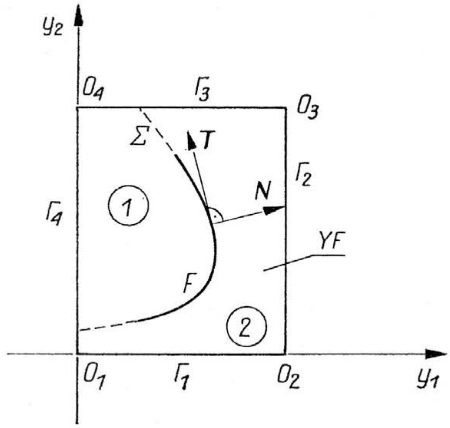

FIG. 7. Geometry of the basic cell YF.

Here $\Gamma_1 = 0_1 0_2, \ldots, \Gamma_4 = 0_4 0_1$, and $0_1, \ldots, 0_4$ are vertices of Y, see Fig. 7. The following notation is introduced for the sum of microfissures such that the corresponding εY-cells are contained in the domain Ω

(3.1) $$F^{\varepsilon} = \bigcup_{i \in I(\varepsilon)} F_{\varepsilon, i}.$$

Further we set $\Omega^{\varepsilon}_{\sharp} = \Omega \setminus F^{\varepsilon}$. The space of Y-periodic functions

(3.2) $$H^1_{\mathrm{per}}(YF) = \{v \in H^1(YF) | v \quad \text{takes equal values at the opposite sides of } Y\}$$

is of fundamental importance.

Now we can define closed and convex sets of kinematically admissible microscopical displacements

(3.3)
$$\begin{aligned}
K^{\mathfrak{bc}}_{YF} &= \{\mathbf{v} = (v_\alpha) \in [H^1_{\mathrm{per}}(YF)]^2 | \llbracket v_N \rrbracket \geqslant 0, \llbracket v_T \rrbracket = 0 \text{ on } F\},\\
K^{\mathfrak{cb}}_{YF} &= \{\mathbf{v} \in [H^1_{\mathrm{per}}(YF)]^2 | \llbracket v_N \rrbracket = 0, \llbracket v_T \rrbracket \geqslant 0 \text{ on } F\},\\
K^{\mathfrak{bb}}_{YF} &= \{\mathbf{v} \in [H^1_{\mathrm{per}}(YF)]^2 | \llbracket v_N \rrbracket \geqslant 0, \llbracket v_T \rrbracket \geqslant 0 \text{ on } F\},\\
K^{\mathfrak{bi}}_{YF} &= \{\mathbf{v} \in [H^1_{\mathrm{per}}(YF)]^2 | \llbracket v_N \rrbracket \geqslant 0 \text{ on } F\},\\
K^{\mathfrak{ib}}_{YF} &= \{\mathbf{v} \in [H^1_{\mathrm{per}}(YF)]^2 | \llbracket v_T \rrbracket \geqslant 0 \text{ on } F\},\\
K^{\mathfrak{ci}}_{YF} &= \{\mathbf{v} \in [H^1_{\mathrm{per}}(YF)]^2 | \rrbracket v_N \rrbracket = 0 \text{ on } F\},\\
K^{\mathfrak{ic}}_{YF} &= \{\mathbf{v} \in [H^1_{\mathrm{per}}(YF)]^2 | \llbracket v_T \rrbracket = 0 \text{ on } F\},\\
K^{\mathfrak{cc}}_{YF} &= [H^1_{\mathrm{per}}(Y)]^2, \qquad K^{\mathfrak{ii}}_{YF} = [H^1_{\mathrm{per}}(YF)]^2,\\
K^{\mathfrak{b}}_{ZF} &= \{w \in H^1_{\mathrm{per}}(YF) | \llbracket w \rrbracket \geqslant 0 \text{ on } F\},\\
K^{\mathfrak{c}}_{YF} &= H^1_{\mathrm{per}}(Y), \qquad K^{\mathfrak{i}}_{YF} = H^1_{\mathrm{per}}(YF).
\end{aligned}$$

We introduce the bilinear form $a_e^\varepsilon(\cdot, \cdot)$ defined for functions determined over the domain Ω^ε

$$(3.4)\quad a_e^\varepsilon(\mathbf{v}, w, \boldsymbol{\varphi}; \mathbf{z}, u, \boldsymbol{\psi}]) = \int_{\Omega^\varepsilon} \{N_{\alpha\beta}(\mathbf{v}, \boldsymbol{\varphi})\gamma_{\alpha\beta}(\mathbf{z}) + M_{\alpha\beta}(\mathbf{v}, \boldsymbol{\varphi})\varrho_{\alpha\beta}(\boldsymbol{\psi}) + Q_\alpha(w, \boldsymbol{\varphi})(u_{,\alpha}+\psi_\alpha)\}dx,$$

$$(\mathbf{v}, w, \boldsymbol{\varphi}), (\mathbf{z}, u, \boldsymbol{\psi}) \in V_\varepsilon,$$

where

$$V_\varepsilon = [H^1(\Omega^\varepsilon)]^2 \times H^1(\Omega^\varepsilon) \times [H^1(\Omega^\varepsilon)]^2.$$

Further, for $\varepsilon > 0$ we define the closed and convex sets of kinematically admissible displacement fields

$$(3.5)\quad \begin{aligned}
K_\varepsilon^{bc} &= \{\mathbf{v} \in [H_1^1(\Omega^\varepsilon)]^2 | [\![v_n]\!] \geqslant 0, [\![v_\tau]\!] = 0 \text{ on } F^\varepsilon\},\\
K_\varepsilon^{cb} &= \{\mathbf{v} \in [H_1^1(\Omega^\varepsilon)]^2 | [\![v_n]\!] = 0, [\![v_\tau]\!] \geqslant 0 \text{ on } F^\varepsilon\},\\
K_\varepsilon^{bb} &= \{\mathbf{v} \in [H_1^1(\Omega^\varepsilon)]^2 | [\![v_n]\!] \geqslant 0, [\![v_\tau]\!] \geqslant 0 \text{ on } F^\varepsilon\},\\
K_\varepsilon^{bi} &= \{\mathbf{v} \in [H_1^1(\Omega^\varepsilon)]^2 | [\![v_n]\!] \geqslant 0 \text{ on } F^\varepsilon\},\\
K_\varepsilon^{ib} &= \{\mathbf{v} \in [H_1^1(\Omega^\varepsilon)]^2 | [\![v_\tau]\!] \geqslant 0 \text{ on } F^\varepsilon\},\\
K_\varepsilon^{ci} &= \{\mathbf{v} \in [H_1^1(\Omega^\varepsilon)]^2 | [\![v_n]\!] = 0 \text{ on } F^\varepsilon\},\\
K_\varepsilon^{ic} &= \{\mathbf{v} \in [H_1^1(\Omega^\varepsilon)]^2 | [\![v_\tau]\!] = 0 \text{ on } F^\varepsilon\},\\
K_\varepsilon^{cc} &= \{H_0^1(\Omega)\}^2, \quad K_\varepsilon^{ii} = [H_1^1(\Omega^\varepsilon)]^2,\\
K_\varepsilon^{b} &= \{w \in H_1^1(\Omega^\varepsilon) | [\![w]\!] \geqslant 0 \text{ on } F^\varepsilon\},\\
K_\varepsilon^{c} &= H_0^1(\Omega), \quad K_\varepsilon^{i} = H_1^1(\Omega^\varepsilon),
\end{aligned}$$

where $\mathbf{n}$, $\boldsymbol{\tau}$ denote the unit normal and tangent vectors to F^ε, respectively, see Fig. 6.

Bearing in mind the considerations of the preceding section it is evident that for $\varepsilon > 0$ the unilateral cracking modes are determined by the sets K_ε^Δ, $\Delta = 1, 2, \ldots, 211$. The sets K_ε^Δ are Cartesian products of the sets specified by (3.5), see (2.61).

For a fixed $\varepsilon > 0$ a solution corresponding to the microcracking mode defined by the set of unilateral constraints K_ε^Δ is denoted by $(\mathbf{v}_\Delta^\varepsilon, w_\Delta^\varepsilon, \boldsymbol{\varphi}_\Delta^\varepsilon)$. However, for the sake of simplicity of notations, the subscript Δ may be dropped. The functions $(\mathbf{v}^\varepsilon, w^\varepsilon, \boldsymbol{\varphi}^\varepsilon) \in K_\varepsilon^\Delta$ are solutions of the following variational inequality

$$(3.6)\quad \left.\begin{aligned}
&\text{find } (\mathbf{v}^\varepsilon, w^\varepsilon, \boldsymbol{\varphi}^\varepsilon) \in K_\varepsilon^\Delta \quad \text{such that}\\
&a_e^\varepsilon(\mathbf{v}^\varepsilon, w^\varepsilon, \boldsymbol{\varphi}^\varepsilon; \mathbf{z}-\mathbf{v}^\varepsilon, u-w^\varepsilon, \boldsymbol{\psi}-\boldsymbol{\varphi}^\varepsilon)\\
&\geqslant f_e(\mathbf{z}-\mathbf{v}^\varepsilon, u-w^\varepsilon, \boldsymbol{\psi}-\boldsymbol{\varphi}^\varepsilon)\ \forall\ (\mathbf{z}, u, \boldsymbol{\psi}) \in K_\varepsilon^\Delta
\end{aligned}\right| (\mathscr{P}_e^{\varepsilon,\Delta}).$$

Due to the coerciveness of the bilinear form $a_e^\varepsilon(\cdot, \cdot)$ a solution of the problem $\mathscr{P}_e^{\varepsilon,\Delta}$ exists and is unique. The coerciveness is a direct consequence of the Korn and Poincaré inequalities. On account of high irregularity of the domain Ω^ε the known formulations of these inequalities are not applicable. ATTOUCH and MURAT [8] extended the Poincaré inequality to domains like Ω^ε. In the second part of the paper the Korn inequality is generalized.

From the mathematical point of view homogenization means a passage to zero with the parameter ε. To carry out this process we postulate here the two-scale expansions similar for $(\mathbf{v}^\varepsilon, w^\varepsilon, \boldsymbol{\varphi}^\varepsilon)$ and virtual fields $(\mathbf{z}, u, \boldsymbol{\psi})$

$$w^\varepsilon(x) = w^0(x) + \varepsilon w^1(x, y) + \ldots, \qquad y = x/\varepsilon, \tag{3.7}$$
$$u(x) = u^0(x) + \varepsilon u^1(x, y) + \ldots;$$

$$v^\varepsilon_\alpha(x) = v^0_\alpha(x) + \varepsilon v^1_\alpha(x, y) + \ldots, \tag{3.8}$$
$$z_\alpha(x) = z^0_\alpha(x) + \varepsilon z^1_\alpha(x, y) + \ldots;$$

$$\varphi^\varepsilon_\alpha(x) = \varphi^0_\alpha(x) + \varepsilon \varphi^1_\alpha(x, y) + \ldots, \tag{3.9}$$
$$\psi_\alpha(x) = \psi^0_\alpha(x) + \varepsilon \psi^1_\alpha(x, y) + \ldots,$$

where the functions w^1, u^1, $\mathbf{z}^1$, $\boldsymbol{\varphi}^1$, $\boldsymbol{\psi}^1$ are defined on $\Omega \times YF$. Moreover we assume that w^0, u^0, v^0_α, z^0_α, φ^0_α and ψ^0_α belong to $H^1_0(\Omega)$ and

$$w^1(x, \cdot), u^1(x, \cdot) \in K^c_{YF},$$
$$\mathbf{v}^1(x, \cdot), \mathbf{z}^1(x, \cdot) \in K^{ab}_{YF},$$
$$\boldsymbol{\varphi}^1(x, \cdot), \boldsymbol{\psi}^1(x, \cdot) \in K^{dh}_{YF},$$

where $a, b, c, d, h \in (\mathfrak{c}, \mathfrak{d}, \mathfrak{i})$.

Now we substitute (3.7), (3.8) and (3.9) into the variational inequality (3.6). Performing the passage to the limit ($\varepsilon \to 0$) in the standard manner [31, 38] we finally obtain the variational formulation of the equilibrium problem of the homogenized Reissner-like plate

$$\left.\begin{array}{l} \text{find} \quad (\mathbf{v}^0, w^0, \boldsymbol{\varphi}^0) \in V \quad \text{such that} \\ b(\mathbf{v}^0, w^0, \boldsymbol{\varphi}^0; \mathbf{z}^0, u^0, \boldsymbol{\psi}^0) = f_e(\mathbf{z}^0, u^0, \boldsymbol{\psi}^0) \\ \hfill \forall (\mathbf{z}^0, u^0, \boldsymbol{\psi}^0) \in V \end{array}\right| (\mathscr{P}^h_\Delta), \tag{3.10}$$

where

$$b(\mathbf{v}^0, w^0, \boldsymbol{\varphi}^0; \mathbf{z}^0, u^0, \boldsymbol{\psi}^0) = \int_\Omega \{\langle n_{\alpha\beta}\rangle \gamma_{\alpha\beta}(\mathbf{z}^0) + \langle m_{\alpha\beta}\rangle \varrho_{\alpha\beta}(\boldsymbol{\psi}^0) + \langle q_\alpha\rangle (u^0_{,\alpha} + \psi^0_\alpha)\} dx \tag{3.11}$$

and

$$n_{\alpha\beta} = A_{\alpha\beta\lambda\mu}[\gamma_{\lambda\mu}(\mathbf{v}^0) + \gamma^y_{\lambda\mu}(\mathbf{v}^1)] + E_{\alpha\beta\lambda\mu}[\varrho_{\lambda\mu}(\boldsymbol{\varphi}^0) + \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1)], \tag{3.12}$$

$$m_{\alpha\beta} = E_{\alpha\beta\lambda\mu}[\gamma_{\lambda\mu}(\mathbf{v}^0) + \gamma^y_{\lambda\mu}(\mathbf{v}^1)] + G_{\alpha\beta\lambda\mu}[\varrho_{\lambda\mu}(\boldsymbol{\varphi}^0) + \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1)], \tag{3.13}$$

$$q_\alpha = H_{\alpha\beta}\left(\varphi^0_\beta + w^0_{,\beta} + \frac{\partial w^1}{\partial y_\beta}\right). \tag{3.14}$$

Here $\langle Z\rangle$ stands for the mean value of a function $Z(x, y)$ with respect to the local variable y

$$\langle Z\rangle = \frac{1}{|Y|}\int_{YF} Z(x, y)\, dy, \tag{3.15}$$

and

$$\gamma^y_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_\alpha}{\partial y_\beta} + \frac{\partial v_\beta}{\partial y_\alpha}\right). \tag{3.16}$$

$\varrho^y_{\lambda\mu}(\varphi^1)$ is defined similarly.

The local or microscopic fields $\mathbf{v}^1$, w^1 and $\boldsymbol{\varphi}^1$ are solutions of the local problems posed on the basic cell where the independent variable x is treated as a parameter. These problems read

$$\left.\begin{array}{ll} & \text{find} \quad \mathbf{v}^1 \in K_{YF}^{ab} \quad \text{and} \quad \boldsymbol{\varphi}^1 \in K_{YF}^{dh} \quad \text{such that} \\ (3.17) & a_A(\mathbf{v}^1, \mathbf{z}-\mathbf{v}^1)+a_E(\boldsymbol{\varphi}^1, \mathbf{z}-\mathbf{v}^1) \geqslant L_1(\mathbf{z}-\mathbf{v}^1)\ \forall\, \mathbf{z} \in K_{YF}^{ab} \\ (3.18) & a_E(\mathbf{v}^1, \boldsymbol{\psi}-\boldsymbol{\varphi}^1)+a_G(\boldsymbol{\varphi}^1, \boldsymbol{\psi}-\boldsymbol{\varphi}^1) \geqslant L_2(\boldsymbol{\psi}-\boldsymbol{\varphi}^1)\ \forall\, \boldsymbol{\psi} \in K_{YF}^{dh} \end{array}\right| (\mathscr{P}_{\text{loc}}^1).$$

$$(3.19) \qquad \left.\begin{array}{l} \text{find } w^1 \in K_{YF}^{c} \quad \text{such that} \\ a_H(w^1, u-w^1) \geqslant L_3(u-w^1)\ \forall\, u \in K_{YF}^{c} \end{array}\right| (\mathscr{P}_{\text{loc}}^2),$$

where

$$(3.20) \qquad a_A(\mathbf{v}, \mathbf{z}) = \int_{YF} A_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}^{y}(\mathbf{v})\gamma_{\alpha\beta}^{y}(\mathbf{z})\,dy;\ \forall\, \mathbf{v}, \mathbf{z} \in [H^1(YF)]^2,$$

$$(3.21) \qquad a_E(\mathbf{v}, \boldsymbol{\psi}) = \int_{YF} E_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}^{y}(\mathbf{v})\varrho_{\alpha\beta}^{y}(\boldsymbol{\psi})\,dy;\ \forall\, \mathbf{v}, \boldsymbol{\psi} \in [H^1(YF)]^2$$

$$\left(\text{or } a_E(\boldsymbol{\psi}, \mathbf{v}) = \int_{YF} E_{\alpha\beta\lambda\mu}\varrho_{\lambda\mu}^{y}(\boldsymbol{\psi})\gamma_{\alpha\beta}^{y}(\mathbf{v})\,dy\right),$$

$$(3.22) \qquad a_G(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \int_{YF} G_{\alpha\beta\lambda\mu}\varrho_{\alpha\beta}^{y}(\boldsymbol{\varphi})\varrho_{\lambda\mu}^{y}(\boldsymbol{\psi})\,dy;\ \boldsymbol{\varphi}, \boldsymbol{\psi} \in [H^1(YF)]^2,$$

$$(3.23) \qquad a_H(w, u) = \int_{YE} H_{\alpha\beta}\frac{\partial w}{\partial y_\beta}\frac{\partial u}{\partial y_\alpha}\,dy;\ \forall\, u, w \in H^1(YF).$$

The linear forms L_1, L_2 and L_3 are defined as follows

$$(3.24) \qquad L_1(\mathbf{z}) = -\int_{YF} (E_{\alpha\beta\lambda\mu}\varkappa_{\lambda\mu}+A_{\alpha\beta\lambda\mu}\,\varepsilon_{\lambda u})\gamma_{\alpha\beta}^{y}(\mathbf{z})\,dy,$$

$$(3.25) \qquad L_2(\boldsymbol{\psi}) = -\int_{YF} (E_{\alpha\beta\lambda\mu}\,\varepsilon_{\lambda\mu}+G_{\alpha\beta\lambda\mu}\varkappa_{\lambda u})\varrho_{\alpha\beta}^{y}(\boldsymbol{\psi})\,dy,$$

$$(3.26) \qquad L_3(u) = -\int_{YF} H_{\alpha\beta}\omega_\beta\frac{\partial u}{\partial y_\alpha}\,dy.$$

Here we have set

$$(3.27) \qquad \varkappa_{\alpha\beta} = \varrho_{\alpha\beta}(\boldsymbol{\varphi}^0), \qquad \varepsilon_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{v}^0),$$

$$(3.28) \qquad \omega_\alpha = \varphi_\alpha^0+w_{,\alpha}^0.$$

Macroscopic quantities $\boldsymbol{\epsilon}$, $\boldsymbol{\varkappa}$ and $\boldsymbol{\omega}$ are assumed to be given when the local problems are considered.

Observe that the variational inequalities (3.17) and (3.18) are coupled unless $e = 0$. Some specific cases are studied in the next section.

A solution w^1 of the local problem $\mathscr{P}_{\text{loc}}^2$ exists and is unique up to an additive constant. Functions $\mathbf{v}^1$ and $\boldsymbol{\varphi}^1$ solving $\mathscr{P}_{\text{loc}}^1$ exist and are unique up to constant additive vectors. These statements result directly from the form of the variational inequalities considered, (3.20)–(3.23) and general existence theorems concerning variational inequalities in Hilbert spaces [27]. Observe also that the problem $\mathscr{P}_{\text{loc}}^1$ is constituted by the system of two variational inequalities [24]. Our local problems may also be formulated as minimization problems over closed convex sets.

It is obvious that $\mathbf{v}^1$ and $\boldsymbol{\varphi}^1$ depend upon $\boldsymbol{\epsilon}$ and $\boldsymbol{\varkappa}$ whereas w^1 depends on $\boldsymbol{\omega}$.

3.2. Homogenized plate

By $\mathfrak{N} = (\mathfrak{N}_{\alpha\beta})$, $\mathfrak{M} = (\mathfrak{M}_{\alpha\beta})$, $\mathfrak{Q} = (\mathfrak{Q}_\alpha)$ we denote the membrane force tensor, the bending moment tensor and the shear force vector, respectively, of the homogenized plate (the superscript Δ is omitted).

The homogenized constitutive relation are, see Eqs. (3.12), (3.13) and (3.14)

$$\mathfrak{N}_{\alpha\beta} = \frac{1}{|Y|} \int\limits_{YF} \left\{A_{\alpha\beta\lambda\mu}\left(\varepsilon_{\lambda\mu} + \gamma^y_{\lambda\mu}(\mathbf{v}^1)\right) + E_{\alpha\beta\lambda\mu}\left(\varkappa_{\lambda\mu} + \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1)\right)\right\} dy, \tag{3.29}$$

$$\mathfrak{M}_{\alpha\beta} = \frac{1}{|Y|} \int\limits_{YF} \left\{E_{\alpha\beta\lambda\mu}\left(\varepsilon_{\lambda\mu} + \gamma^y_{\mu\lambda}(\mathbf{v}^1)\right) + G_{\alpha\beta\lambda\mu}\left(\varkappa_{\lambda\mu} + \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1)\right)\right\} dy, \tag{3.30}$$

$$\mathfrak{Q}_\alpha = \frac{1}{|Y|} \int\limits_{YF} H_{\alpha\beta}\left(\omega_\beta + \frac{\partial w^1}{\partial y_\beta}\right) dy. \tag{3.31}$$

Taking account of the problem $\mathscr{P}^h_\Delta$ and of Eqs. (3.29), (3.30) and (3.31) we infer that the equilibrium equations of the homogenized plate are given by

$$\mathfrak{M}_{\alpha\beta,\beta} - \mathfrak{Q}_\alpha + m_\alpha = 0 \quad \text{in } \Omega, \tag{3.32}$$

$$\mathfrak{N}_{\alpha\beta,\beta} + p_\alpha = 0 \quad \text{in } \Omega, \tag{3.33}$$

$$\mathfrak{Q}_{\alpha,\alpha} + p = 0 \quad \text{in } \Omega. \tag{3.34}$$

The constitutive equations (3.29), (3.30) and (3.31) suggest that the homogenized plate is hyperelastic and the elastic potential W has the form (see Theorem 3.1 below and the second part of the paper)

$$\begin{aligned} W(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega}) = \frac{1}{2|Y|} \int\limits_{YF} \Big\{ & A_{\alpha\beta\lambda\mu}\left(\varepsilon_{\lambda\mu} + \gamma^y_{\lambda\mu}(\mathbf{v}^1)\right)\left(\varepsilon_{\alpha\beta} + \gamma^y_{\alpha\beta}(\mathbf{v}^1)\right) \\ & + 2E_{\alpha\beta\lambda\mu}\left(\varkappa_{\lambda\mu} + \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1)\right)\left(\varepsilon_{\alpha\beta} + \gamma^y_{\alpha\beta}(\mathbf{v}^1)\right) + G_{\alpha\beta\lambda\mu}\left(\varkappa_{\lambda\mu} + \varrho^y_{\lambda\mu}(\boldsymbol{\varphi}^1)\right)\left(\varkappa_{\alpha\beta} + \varrho^y_{\alpha\beta}(\boldsymbol{\varphi}^1)\right) \\ & + H_{\alpha\beta}\left(\omega_\beta + \frac{\partial w^1}{\partial y_\beta}\right)\left(\omega_\alpha + \frac{\partial w^1}{\partial y_\alpha}\right)\Big\} dy. \end{aligned} \tag{3.35}$$

An important property of the function W follows from

THEOREM 3.1. *The potential W is of class C^1, positive and strictly convex. Moreover, we have*

$$\mathfrak{N}_{\alpha\beta} = \partial W/\partial \varepsilon_{\alpha\beta}, \quad \mathfrak{M}_{\alpha\beta} = \partial W/\partial \varkappa_{\alpha\beta}, \quad \mathfrak{Q}_\alpha = \partial W/\partial \omega_\alpha. \tag{3.36}$$

P r o o f. Taking account of (2.38) and (2.50) we obtain

$$\begin{aligned} W(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega}) &= \frac{1}{|Y|} \int\limits_{YF} g\left(\boldsymbol{\epsilon} + \boldsymbol{\gamma}^y(\mathbf{v}^1), \boldsymbol{\varkappa} + \boldsymbol{\rho}^y(\boldsymbol{\varphi}^1), \boldsymbol{\omega} + \operatorname{grad}_y w^1\right) dy \\ &\geqslant \frac{c}{|Y|} \int\limits_{YF} \left(|\boldsymbol{\epsilon} + \boldsymbol{\gamma}^y(\mathbf{v}^1)|^2 + |\boldsymbol{\varkappa} + \boldsymbol{\rho}^y(\boldsymbol{\varphi}^1)|^2 + |\boldsymbol{\omega} + \operatorname{grad}_y w^1|^2\right) dy. \end{aligned}$$

Thus W is positive.

Let $\boldsymbol{\epsilon}_\alpha \in \mathsf{M}_s(\mathsf{R})$, $\boldsymbol{\varkappa}_\alpha \in \mathsf{M}_s(\mathsf{R})$, $\omega_\alpha \subset \mathsf{R}$ and let $\mathbf{v}^1_\alpha$, $\boldsymbol{\varphi}^1_\alpha$ and w^1_α be the corresponding solutions of the local problems. Then we can write

$$(3.37)\quad W[(\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2)/2, (\boldsymbol{\varkappa}_1+\boldsymbol{\varkappa}_2)/2, (\boldsymbol{\omega}_1+\boldsymbol{\omega}_2)/2]$$

$$= \frac{1}{|Y|}\int\limits_{YF} g\Bigg[\frac{1}{2}(\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2)+\frac{1}{2}\left(\boldsymbol{\gamma}^y(\mathbf{v}^1_1)+\boldsymbol{\gamma}^y(\mathbf{v}^1_2)\right), \frac{1}{2}(\boldsymbol{\varkappa}_1+\boldsymbol{\varkappa}_2)$$

$$+\frac{1}{2}\left(\boldsymbol{\rho}^y(\boldsymbol{\varphi}^1_1)+\boldsymbol{\rho}^y(\boldsymbol{\varphi}^1_2)\right), \frac{1}{2}(\boldsymbol{\omega}_1+\boldsymbol{\omega}_2)+\frac{1}{2}(\mathrm{grad}_y w^1_1+\mathrm{grad}_y w^1_2)\Bigg]dy$$

$$\leqslant \frac{1}{|Y|}\int\limits_{YF}\left\{\frac{1}{2}g(\boldsymbol{\epsilon}_1, \boldsymbol{\varkappa}_1, \boldsymbol{\omega}_1)+\frac{1}{2}g(\boldsymbol{\epsilon}_2, \boldsymbol{\varkappa}_2, \boldsymbol{\omega}_2)\right\}dy$$

$$= \frac{1}{2}W(\boldsymbol{\epsilon}_1, \boldsymbol{\varkappa}_1, \boldsymbol{\omega}_1)+\frac{1}{2}W(\boldsymbol{\epsilon}_2, \boldsymbol{\varkappa}_2, \boldsymbol{\omega}_2).$$

Hence the potential W is convex. Moreover, in (3.37) the equality holds if and only if $\boldsymbol{\epsilon}_1 = \boldsymbol{\epsilon}_2$, $\boldsymbol{\varkappa}_1 = \boldsymbol{\varkappa}_2$, $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$; thus W is strictly convex.

Since W is convex and finite, it is of class C^0 and subdifferentiable at each point $(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega})$ (see [37], Corollary 10.1.1 and Th. 23.4). Hence

$$(3.38)\qquad (\mathfrak{N}, \mathfrak{M}, \mathfrak{Q}) \in \partial W(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega}),$$

where ∂W stands for the subdifferential of the potential W. With the help of Sanchez-Palencia's Lemma 7.2 [38] the subdifferential constitutive law (3.38) becomes the hyperelastic law (3.36) and the proof is complete.

Elementary properties of W are a direct consequence of (2.19), (2.23) and (3.35) and are given by

(a) $W(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega})$ is positively homogeneous of order 2

$$W(\lambda\boldsymbol{\epsilon}, \lambda\boldsymbol{\varkappa}, \lambda\boldsymbol{\omega}) = \lambda^2 W(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega}); \qquad \lambda \geqslant 0,$$

(b) There exist positive constants c_1 and c_2 such that

$$c_1(|\boldsymbol{\epsilon}|^2+|\boldsymbol{\varkappa}|^2+|\boldsymbol{\omega}|^2) \leqslant W(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}, \boldsymbol{\omega}) \leqslant c_2(|\boldsymbol{\epsilon}|^2+|\boldsymbol{\varkappa}|^2+|\boldsymbol{\omega}|^2).$$

As we already know the potential W is strictly convex. Hence solutions of boundary value problems for homogenized plates exist and are unique provided that coercivity holds in the closed subspace $\mathscr{V}$ such that $V \subset \mathscr{V} \subset [H^1(\Omega)]^2 \times H^1(\Omega) \times [H^1(\Omega)]^2$. Thus mixed boundary conditions are admissible.

4. Final remark

Since our considerations start from the two-dimensional plate model, the resulting effective model with smeared-out fissures is applicable only to such plates with periodically distributed fissures for which the three-dimensional cells of periodicity are plates themself, viz. they are thin. Thus we do not consider the second extreme case when the in-plane dimensions of the cells are much less than the plate thickness nor the intermediate case when the in-plane and transverse dimensions of the cells are comparable. Thus in the next part of the paper we shall study the convergence when $\varepsilon \to 0$ while h is held fixed.

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