

On the influence of the coupling between mechanical and thermal processes in inelastic bifurcation

TH. LEHMANN (BOCHUM)

IF INELASTIC deformation processes of a body run with finite rate, the heat produced by the dissipation of mechanical work leads to changes of the temperature field within the body. Since the material properties depend on temperature, a coupling between mechanical and thermal processes arises in such cases. At first, a general frame for the description of such coupled processes is outlined. The respective considerations are restricted to the classification of dissipative and nondissipative processes and to the formulation of balance equations of specific free enthalpy and specific entropy. From these balance equations some strong restrictions follow for the formulation of constitutive laws for material behaviour. Starting from some phenomena which can be observed in simple experiments, some possible formulations of constitutive laws for the material behaviour in large nonisothermic inelastic deformations are considered. The consequences arising from the coupling of mechanical and thermal processes with respect to bifurcation and localization problems are discussed. Due to the large gradients of temperature and stresses which may occur in such processes, the influence of such coupling becomes particularly large. As an example the necking problem in the tension test will be discussed in some details.

W procesach niesprężystego odkształcania ze skończonymi prędkościami ciepło wytworzone przez dysypację pracy mechanicznej zmienia temperaturę ciała. Ponieważ własności ciała zależą od temperatury, to pojawiają się w takim przypadku sprzężenia termomechaniczne. W pracy naszkicowano najpierw ogólne ramy opisu takich procesów; ograniczono się przy tym do klasyfikacji procesów dysypatywnych i niedysypatywnych i do sformułowania równań zachowania swobodnej entalpii właściwej oraz entropii właściwej. Z równań tych wynikają pewne silne ograniczenia dotyczące sformułowania praw konstytutywnych. Opierając się na pewnych zjawiskach obserwowanych w prostych eksperymentach, rozważono możliwość sformułowania praw konstytutywnych dla materiałów poddanych dużym niesprężystym i nieizotermicznym odkształceniom. Omówiono konsekwencje wynikające ze sprzężenia procesów i dotyczące problemów lokalizacji i bifurkacji. Konsekwencje te są szczególnie duże wobec gradientów temperatury i naprężeń w omawianych procesach. Bardziej szczegółowo omówiono przykład dotyczący szyjkowania w próbie rozciągania.

В процессах неупругого деформирования с конечными скоростями тепло, образованное путем диссипации механической работы, изменяет температуру тела. Т. к. свойства тела зависят от температуры, в таком случае появляются термо-механические сопряжения. В работе сначала начерчены общие рамки описания таких процессов; при этом ограничили к классификации диссипативных и недиссипативных процессов и к формулировке уравнений сохранения удельной свободной энтальпии и удельной энтропии. Из этих уравнений вытекают некоторые сильные ограничения, касающиеся формулировки определяющих законов. Опираясь на некоторые явления, наблюдаемые в простых экспериментах, обсуждена возможность формулировки определяющих законов для материалов, подвергнутых большому неупругому и неизоотермическому деформациям. Обсуждены последствия, вытекающие из сопряжения процессов и касающиеся проблем локализации и бифуркации. Эти последствия особенно большие из-за значительных градиентов температуры и напряжений в обсуждаемых процессах. Более подробно обсужден пример касающийся шейкообразования в испытании растяжения.

1. Introduction

DURING DEFORMATION processes of solid bodies temperature changes of material elements result from a) dissipation of mechanical work in inelastic deformations, b) dissipation of energy in connection with internal processes, c) divergence of heat flux, d) coupling effects with nondissipative processes (thermo-elastic coupling etc.).

Only at a very low deformation rate does the process become isothermal approximately. In this case the divergence of heat flux counterbalances the other influences. At a very high deformation rate on the other hand the process becomes locally adiabatic approximately, i.e. the heat flux can be disregarded. Inbetween the interaction of the different phenomena has to be treated carefully. Certainly in many cases the coupling effects listed under d) remain negligibly small in comparison to the other effects, particularly at large inelastic deformations.

Since the material properties depend on temperature, a coupling of mechanical and thermal processes arises in nonisothermal processes. These coupling effects can change the feature of a deformation process fundamentally in comparison to a corresponding isothermal process. This is particularly true with respect to bifurcation problems. This will be demonstrated by an example later.

In Chapter 2, first of all, we shall define a general frame for the description of coupled thermo-mechanical processes occurring in large, nonisothermal, inelastic deformations of solid bodies. In Chapter 3 we shall discuss some general aspects which have to be observed in the formulation of constitutive laws for elastic-plastic and elastic-viscoplastic material behaviour. One example of a constitutive law will be discussed in Chapter 4 briefly. Some general consequences of the coupling of mechanical and thermal processes concerning the onset of bifurcation and the post-critical behaviour will be considered in Chapter 5. The necking problem in the uniaxial tension test will be treated as an example in Chapter 6 with some more details. Finally some open questions will be addressed in Chapter 7.

2. General frame for the phenomenological description of coupled thermo-mechanical processes

The intended general frame for the phenomenological description of coupled thermo-mechanical processes is based on two fundamental assumptions:

(I) the body can be considered as a classical continuum;

(II) the thermodynamical state of each material element is uniquely determined by the values of a finite set of (external and internal) thermodynamical state variables (defined in a so-called small state space) even if the body as a whole is not in thermodynamical equilibrium (assumption of local thermodynamical state).

From these assumptions, of course, some restrictions result with respect to the capability of such a theory (for more details see [1, 2]). For many technical problems with moderate deformation rate, however, such an approach turns out to be satisfactory. The real thermo-mechanical process carries the body from the initial state $\overset{\circ}{B}$ (characterized by its initial

geometrical configuration and the initial thermodynamical state of each material element) into the actual state B . We attach to the actual state B an accompanying fictitious reference state $\overset{\circ}{B}$ by a fictitious reversible process with frozen internal variables which carries each material element from its actual thermodynamical state into an unstressed state at reference temperature $\overset{\circ}{T}$ (see Fig. 1). By this procedure we obtain a unique decomposition of the

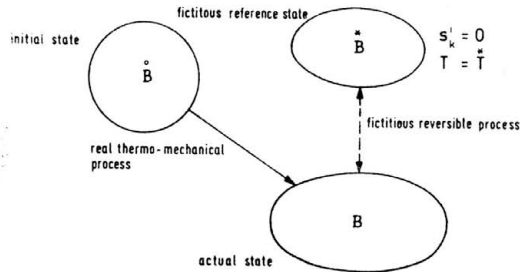


FIG. 1. Thermo-mechanical process.

total work rate \dot{w} into the strictly reversible (elastic) part $\dot{w}_{(r)}$ and the remaining (inelastic) part $\dot{w}_{(i)}$ according to

$$(2.1) \quad \dot{w} = \frac{1}{\rho} s_k^i d_t^k = \dot{w}_{(r)} + \dot{w}_{(i)}$$

where s_k^i denotes the weighted Cauchy stress tensor

$$(2.2) \quad s_k^i = \frac{\hat{\rho}}{\rho} \sigma_k^i$$

and d_k^i designates the total deformation rate. ρ is the mass density. All quantities are related to the actual configuration of the body. A superscripted \circ refers to the initial state.

In general $\dot{w}_{(i)}$ is not identical with the dissipative part $\dot{w}_{(d)}$ of the work rate, since one part $\dot{w}_{(h)}$ of $\dot{w}_{(i)}$ is interacting with the energy stored in the microstress fields resulting from lattice defects. Therefore we have to write

$$(2.3) \quad \dot{w}_{(i)} = \dot{w}_{(h)} + \dot{w}_{(d)}$$

where the dissipated part

$$(2.4) \quad \dot{w}_{(d)} \geq 0$$

according to the second law of thermodynamics, whereas $\dot{w}_{(h)}$ is not restricted in this way, i.e.

$$(2.5) \quad \dot{w}_{(h)} \geq 0.$$

Using a material model as sketched in Fig. 2, from Eq. (2.1) we derive a corresponding additive decomposition of the total deformation rate:

$$(2.6) \quad \dot{w} = \frac{1}{\varrho} s_k^i d_t^k = \dot{w}_{(r)} + \dot{w}_{(i)} = \frac{1}{\varrho} s_k^i d_{(r)}^k + \frac{1}{\varrho} s_k^i d_{(i)}^k$$

with $d_k^i = d_{(r)}^i + d_{(i)}^i$.

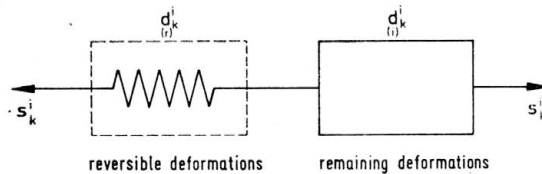


FIG. 2. Material model.

For thermodynamical reasons it must be possible to express the rate of reversible work in the form

$$(2.7) \quad \dot{w}_{(r)} = \frac{1}{\varrho} s_k^i d_{(r)}^k = \frac{1}{\varrho} \tilde{s}_k^i \overset{\nabla}{\varepsilon}_k^i,$$

where \tilde{s}_k^i and $\tilde{\varepsilon}_k^i$ represent a conjugated pair of stress and reversible strain [1, 3, 4] and (∇) denotes a suitably chosen objective time derivative. If the reversible behaviour is isotropic as can be assumed in many cases, the weighted Cauchy stress tensor and the (logarithmic) Hencky strain tensor ε_k^i fulfill this requirement. (∇) denotes in this case the Zaremba–Jaumann time derivative [1, 4].

The first law of thermodynamics states

$$(2.8) \quad \dot{u} = \dot{w} - \frac{1}{\varrho} q^i|_i + r = \dot{w}_{(r)} + \dot{w}_{(h)} + \dot{w}_{(d)} - \frac{1}{\varrho} q^i|_i + r,$$

where u denotes the specific internal energy, q^i the resulting energy flux, r the sum of specific energy sources, $|_i$ means the covariant derivative in the actual configuration. According to our basic assumption (II), u must be expressible as a function of a set of state variables, i.e.

$$(2.9) \quad u = u(\varepsilon_k^i, s, a, \alpha_k^i),$$

where s means the specific entropy and a, α_k^i represent a set of internal state variables which can be extended arbitrarily. It is convenient to introduce the specific free enthalpy as a state function by a double Legendre-transformation, i.e.

$$(2.10) \quad \psi = u - \frac{1}{\varrho} s_k^i \varepsilon_k^i - Ts = \psi(s_k^i, T, a, \alpha_k^i),$$

where T denotes the absolute temperature.

From Eq. (2.10) we derive the following two expressions:

$$(2.11) \quad \dot{\psi} = \dot{u} - \frac{1}{\varrho} \frac{\nabla}{s_k^i} \varepsilon_k^i - \frac{1}{\varrho} \frac{\nabla}{s_k^i} \varepsilon_k^i - \dot{T}s - T\dot{s} = \frac{\partial \psi}{\partial s_k^i} \frac{\nabla}{s_k^i} + \frac{\partial \psi}{\partial T} \dot{T} + \frac{\partial \psi}{\partial a} \dot{a} + \frac{\partial \psi}{\partial \alpha_k^i} \frac{\nabla}{\alpha_k^i}.$$

From the properties of the Legendre-transformation we obtain immediately

$$(2.12) \quad \begin{aligned} \varepsilon_k^i &= -\varrho \frac{\partial \psi}{\partial s_k^i} = \varepsilon_k^i(s_k^i, T, a, \alpha_k^i) \quad (\text{thermic state equation}), \\ s &= -\frac{\partial \psi}{\partial T} = s(s_k^i, T, a, \alpha_k^i) \quad (\text{caloric state equation}). \end{aligned}$$

Using the relations (2.8) and (2.12), we derive from Eq. (2.11) the balance equation for the remaining energy supply

$$(2.13) \quad \begin{aligned} \dot{w} + \dot{w} - \frac{1}{\varrho} q^i|_i + r &= -T \left\{ \frac{\partial^2 \psi}{\partial s_k^i \partial T} \frac{\nabla}{s_k^i} + \frac{\partial^2 \psi}{\partial T^2} \dot{T} \right\} \\ &+ \frac{\partial}{\partial a} \left\{ \psi - T \frac{\partial \psi}{\partial T} \right\} \dot{a} + \frac{\partial}{\partial \alpha_k^i} \left\{ \psi - T \frac{\partial \psi}{\partial T} \right\} \frac{\nabla}{\alpha_k^i} \end{aligned}$$

and the evolution equation for specific entropy (Gibbs-equation)

$$(2.14) \quad T\dot{s} = \dot{w} + w - \frac{1}{\varrho} q^i|_i + r - \frac{\partial \psi}{\partial a} \dot{a} - \frac{\partial \psi}{\partial \alpha_k^i} \frac{\nabla}{\alpha_k^i}.$$

The evolution of entropy has to be decomposed into the reversible evolution $\dot{s}^{(r)}$ and into the entropy production $\dot{s}^{(d)}$ according to

$$(2.15) \quad \dot{s} = \dot{s}^{(r)} + \dot{s}^{(d)}$$

with

$$(2.16)_1 \quad \dot{s}^{(r)} = \frac{1}{T} \left\{ \dot{w} + r - \frac{\partial \psi}{\partial a} \dot{a} - \frac{\partial \psi}{\partial \alpha_k^i} \frac{\nabla}{\alpha_k^i} \right\} - \frac{1}{\varrho} \left(\frac{q^i}{T} \right)|_i - \dot{\eta}$$

and

$$(2.16)_2 \quad \dot{s}^{(d)} = \frac{1}{T} \frac{\dot{w}}{(d)} - \frac{1}{\varrho T^2} q^i T|_i + \dot{\eta} \geq 0.$$

$\dot{\eta}$ is the entropy production involved in internal processes. The balance equation (2.13) for the remaining energy supply can be decomposed finally into

$$(2.17)_1 \quad \dot{w} + r = \frac{\partial \psi}{\partial a} \dot{a} + \frac{\partial \psi}{\partial \alpha_k^i} \frac{\nabla}{\alpha_k^i} = h\dot{a} + H_i^k \frac{\nabla}{\alpha_k^i}$$

and

$$(2.17)_2 \quad c_p \dot{T} = \dot{w} - \frac{1}{\varrho} q^i|_i + T\dot{\eta} - B_i^k \frac{\nabla}{s_k^i} - g\dot{a} - G_i^k \frac{\nabla}{\alpha_k^i}$$

where

$$\begin{aligned}
 h &= \frac{\partial \psi}{\partial a}, & H_i^k &= \frac{\partial \psi}{\partial \alpha_k^i} \\
 c_p &= -T \frac{\partial^2 \psi}{\partial T^2} & & \text{(heat capacity at constant } s_k^i, a, \alpha_k^i), \\
 B_i^k &= -T \frac{\partial^2 \psi}{\partial s_k^i \partial T} & & \text{(thermo-elastic coupling),} \\
 g &= -T \frac{\partial^2 \psi}{\partial a \partial T} = -T \frac{\partial h}{\partial T} \\
 G_i^k &= -T \frac{\partial^2 \psi}{\partial \alpha_k^i \partial T} = -T \frac{\partial H_i^k}{\partial T}
 \end{aligned}
 \left. \vphantom{\begin{aligned} g \\ G_i^k \end{aligned}} \right\} \text{(other couple terms).}$$

From the definition (2.18) we see that the coefficients entering the balance equations (2.13), and (2.17) have to obey the following integrability conditions among others:

$$\begin{aligned}
 \frac{\partial}{\partial s_k^i} \left(\frac{c_p}{T} \right) &= \frac{\partial}{\partial T} \left(\frac{B_i^k}{T} \right), & \frac{\partial h}{\partial \alpha_k^i} &= \frac{\partial H_i^k}{\partial a}, \\
 \frac{\partial}{\partial a} \left(\frac{c_p}{T} \right) &= -\frac{\partial^2 h}{\partial T^2}, & \frac{\partial}{\partial \alpha_k^i} \left(\frac{c_p}{T} \right) &= -\frac{\partial^2 H_i^k}{\partial T^2}.
 \end{aligned}
 \tag{2.19}$$

Within this frame the constitutive law of the respective material has to be defined.

3. Some aspects concerning the formulation of constitutive laws for elastic-plastic and elastic-viscoplastic material behaviour

In inelastic bodies different yield mechanisms are interacting. We restrict ourselves to elastic-plastic and elastic-viscoplastic behaviour in the narrow sense. The corresponding inelastic deformations are mainly stress-induced but also depending on temperature. Processes which are mainly thermally activated (but also influenced by stresses) like high-temperature creep or relaxation phenomena, recrystallization, recovery etc. are disregarded here. The general frame developed in Chapter 2, however, covers such processes, too.

Stress-induced deformation processes are characterized by the existence of accompanying local constraint equilibrium states which are bounded by a yield condition in the space of stress and temperature. The yield condition embraces the range of elastic behaviour. Therefore the material model sketched in Fig. 2 is applicable to the kind of processes under consideration.

The shape of the elastic range depends on the previous loading or deformation history which determines the actual values of the internal variables. The elastic range does not necessarily include the stress-free state $s_k^i = 0$ (see [5 to 8]). If this stress-free state is not included into the elastic range, the inelastic work rate can become negative as Fig. 3 shows. In this case a physically meaningful decomposition of total work rate according to Eqs. (2.3) to (2.5) becomes very important in order to avoid violation of the second law of thermodynamics. In this decomposition the balance law (2.17)₁ with the integrability

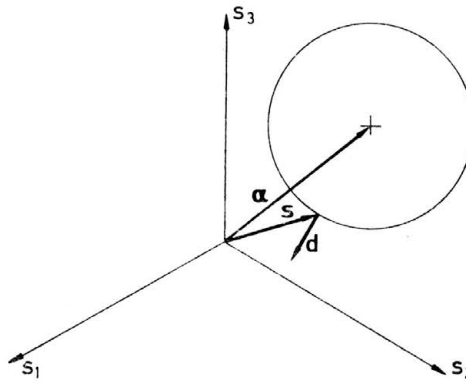


FIG. 3. Yield condition not embracing the point $s_k^i = 0$; projection into the π -plane.

conditions (2.19) plays an important role. It describes the interaction between the external non-dissipative energy supply and changes of the internal variables.

The values of the internal variables determine together with the external variables s_k^i and T the actual state of the material elements. They describe the hardening (or softening) of the material as well as the state of damage. It may be useful or convenient to introduce separate internal variables for the state of hardening and damage [9 to 11]. Formally, however, there is no distinction between hardening parameters and damage parameters in principle.

If we disregard energy fluxes apart from heat, the changes of both kinds of internal variables are governed by evolution laws, i.e. first order ordinary differential equations with respect to time of the general form

$$(3.1) \quad \begin{aligned} \dot{a} &= \dot{a}(s_k^i, T, a, \alpha_k^i, \overset{\nabla}{s}_k^i, \dot{T}), \\ \overset{\nabla}{\alpha}_k^i &= \overset{\nabla}{\alpha}_k^i(s_k^i, T, a, \alpha_k^i, \overset{\nabla}{s}_k^i, \dot{T}). \end{aligned}$$

They can also depend on certain side conditions apart or distinct from the yield condition and we may distinguish different types [1, 4]. The key role of the balance equation (2.17)₁ with respect to the formulation of the evolution laws (3.1) etc. has to be emphasized once more.

This becomes clear when we consider the total scheme of the constitutive law. It comprises:

1) definition of the state function for the specific free enthalpy ψ which governs at the same time the reversible deformations by the thermic state equation (2.12)₁, which can be converted into an evolution law of the form

$$(3.2) \quad d_{(r)}^i = d_{(r)}^i(s_k^i, T, a, \alpha_k^i, \overset{\nabla}{s}_k^i, \dot{T});$$

2) definition of the evolution law for the inelastic deformations

$$(3.3) \quad d_{(i)}^i = d_{(i)}^i(s_k^i, T, a, \alpha_k^i, \overset{\nabla}{s}_k^i, \dot{T})$$

(possibly consisting of different terms [1, 2, 4, 7, 8]) with corresponding side conditions;

3) definition of the evolution laws for internal variables (hardening and damage parameters) of the form (3.1);

4) definition of the entropy production (2.16) by defining \dot{w} and $\dot{\eta}$ (see also (2.17)₂;^(d)

5) definition of the law for the heat flux q^i .

The items 2) and 3) are coupled because the evolution of the internal variables depends on inelastic deformations. By the way of expressing the inelastic deformation rate by their evolution law (3.3), we reduce the evolution laws for the internal variables to the general form (3.1). The definition of entropy production (item 4)) determines at the same time that part of the inelastic work rate which interacts with the evolution of internal variables. Therefore the items 2), 3), and 4) are immediately coupled. The law of heat flux (item 5) can be defined separately. It enters the process description only by the balance law (2.17)₂ which determines the local changes of temperature.

Therefore we can distinguish two separate influences on the coupling of mechanical and thermal processes (neglecting thermo-elastic effects):

(a) coupling effects which already occur in homogeneous (for instance adiabatic) deformation processes with homogeneous temperature fields resulting from local entropy production due to dissipation;

(b) coupling effects resulting from heat flux in inhomogeneous temperature fields.

These different influences can also be observed in bifurcation problems as will be shown in the following. Before doing this, one example of a constitutive law defined in the given frame will be demonstrated in the next chapter.

4. An example for a constitutive law for elastic-plastic and elastic-viscoplastic behaviour

In the following example we do not take into account the damage problem. This means we restrict ourselves to internal parameters describing the hardening state of the material before the onset of void nucleation. Furthermore we assume that the elastic behaviour is isotropic and unaffected by accompanying inelastic deformations [7, 8, 12]. In this case we can write

$$(4.1) \quad \psi = \psi^*(s_k^i, T) + \psi^{**}(T, a, \alpha_k^i).$$

If the elastic behaviour is tensor-linear, we obtain

$$(4.2)_1 \quad \psi^*(s_k^i, T) = -\frac{1}{\varrho} \left\{ \frac{1}{2} s_i^k C_{ks}^{ir} s_r^s + \alpha(T - \overset{\circ}{T}) s_i^k \delta_k^i \right\} - c_p T \ln \left(\frac{T}{\overset{\circ}{T}} \right)$$

leading to

$$(4.2)_2 \quad \varepsilon_k^i = -\varrho \frac{\partial \psi^*}{\partial s_i^k} = C_{ks}^{ir} s_r^s + \alpha(T - \overset{\circ}{T}) \delta_k^i,$$

where C_{ks}^{ir} denotes the compliance tensor and α is the coefficient of thermal expansion.

Furthermore the specific heat capacity is

$$(4.2)_3 \quad c_p = -T \frac{\partial^2 \psi^*}{\partial T^2} = \text{const}$$

if it does not depend on the internal variables.

From Eq. (4.2)₂ we can derive the evolution law for the reversible deformations (3.2). In many cases we can replace the hyper-elastic law by an hypo-elastic incremental law of the linear form

$$(4.3) \quad d_{(r)}^i = \frac{1}{2G} t_k^i + \left\{ \frac{1}{9K} \dot{s}_r + \alpha \dot{T} \right\} \delta_k^i,$$

with the following notations: t_k^i — weighted stress deviator, G — shear modulus, K — bulk modulus.

The following constitutive relations for the inelastic behaviour are based on the fact that inelastic deformations result from local processes on the one hand and from global processes on the other hand. Local processes correspond to generation, dissolution and redistribution of lattice defects within the single crystal grains. They are influenced essentially by stress increments and are mainly nondissipative. Global processes are due to slip processes running through the whole body across the grain boundaries. They can be assumed to be dissipative totally.

Basing on this physical background, the following constitutive relations for the inelastic behaviour can be established, restricting ourselves to plastic behaviour for the present (for more details see [1, 2, 7]):

specific free enthalpy:

$$(4.4) \quad \psi^{**} = \psi^{**}(T, a, A) \quad \text{with} \quad A = \alpha_k^i \alpha_i^k;$$

yield condition:

$$(4.5)_1 \quad F = (t_k^i - \alpha_k^i)(t_i^k - \alpha_i^k) - k^2(a, T) = 0;$$

consistency condition:

$$(4.5)_2 \quad \dot{F} = \frac{\partial F}{\partial s_k^i} \overset{\nabla}{s}_k^i + \frac{\partial F}{\partial T} \dot{T} + \frac{\partial F}{\partial a} \dot{a} + \frac{\partial F}{\partial \alpha_k^i} \overset{\nabla}{\alpha}_k^i \\ = 2(t_k^i - \alpha_k^i)(t_i^k - \alpha_i^k) - \frac{\partial k^2}{\partial a} \dot{a} - \frac{\partial k^2}{\partial T} \dot{T} = 0;$$

evolution law for inelastic deformations (local and global processes):

$$(4.6) \quad d_k^i = d_{(l)}^i + d_{(g)}^i, \\ d_{(l)}^i = \begin{cases} \lambda \frac{\partial F}{\partial s_k^i} + \alpha t_k^i & \text{if } F = 0 \quad \text{and} \quad \dot{F} = 0, \\ 0 & \text{otherwise,} \end{cases} \\ d_{(g)}^i = \begin{cases} \mu \frac{\partial F}{\partial s_k^i} & \text{if } F = 0, \dot{F} = 0 \quad \text{and} \quad s_k^i \frac{\partial F}{\partial s_k^i} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

where the nonnegative quantities λ , κ , μ still have to be defined; evolution laws for internal variables:

$$(4.7) \quad \begin{aligned} \dot{a} &= \frac{\delta}{\varrho} (t_k^i - \alpha_k^i) d_{(i)}^k = \frac{\delta}{\varrho} \{2\lambda k^2 + \kappa (t_k^i - \alpha_k^i) t_i^k\}, \\ \alpha_k^i &= \frac{\zeta d_{(i)}^k}{\varrho} - 2\lambda \gamma k \alpha_k^i = 2\lambda \{\zeta (t_k^i - \alpha_k^i) - \gamma k \alpha_k^i\} + \kappa \zeta t_i^k, \end{aligned}$$

entropy production:

$$(4.8) \quad \begin{aligned} \dot{w}_{(a)} &= \frac{1}{\varrho} \xi t_k^i d_{(i)}^k + \frac{1}{\varrho} t_k^i d_{(s)}^k, \\ \dot{\eta} &= 0. \end{aligned}$$

The assumptions concerning entropy production mean that only one part of the work corresponding to the local processes interacts with the changes of the internal parameters whereas the global processes are dissipative totally. In this case from the balance equation (2.17)₁ and from the requirements of the second law of thermodynamics, we conclude

$$(4.9) \quad \begin{aligned} \frac{1}{\delta} &= \frac{\partial \psi^{**}}{\partial a} = h(T, a, A), \\ \frac{1}{2\varrho \zeta} &= \frac{\partial \psi^{**}}{\partial A} = 2H_i^k \alpha_k^i = H(T, a, A) \end{aligned}$$

with the integrability conditions

$$(4.9) \quad \begin{aligned} \frac{1}{2\varrho} \frac{\partial}{\partial a} \left(\frac{1}{\zeta} \right) &= \frac{\partial}{\partial A} \left(\frac{1}{\delta} \right), \\ \frac{\partial}{\partial a} \left(\frac{c_p}{T} \right) &= - \frac{\partial^2}{\partial T^2} \left(\frac{1}{\delta} \right), \\ \frac{\partial}{\partial A} \left(\frac{c_p}{T} \right) &= - \frac{1}{2\varrho} \frac{\partial^2}{\partial T^2} \left(\frac{1}{\zeta} \right). \end{aligned}$$

From Eqs. (4.9)₂ and (4.9)₃ we see that δ and ζ have to be independent of the temperature, if c_p does not depend on the internal variables. Furthermore we derive

$$(4.10) \quad \frac{\xi}{\varrho} t_k^i d_{(i)}^k = \frac{2\gamma}{\varrho \zeta} \lambda k A.$$

Therefore the total entropy production reads

$$(4.11) \quad \dot{w}_{(a)} = \lambda \frac{2\gamma}{\varrho \zeta} k A + \frac{2\mu}{\varrho} t_k^i (t_i^k - \alpha_i^k).$$

Whereas the choice of the coefficients δ and ζ in the evolution laws for internal variables is strongly restricted by the integrability conditions (4.9'), the coefficient γ in Eq. (4.7)₂ is not restricted in this way. The same is true for the coefficients κ and μ occurring in the evolution law for inelastic deformations.

When the quantities $\delta, \zeta, \gamma, \varkappa$ are fixed within the given frame, the as yet undetermined quantity λ can now be calculated from the consistency condition (4.5)₂.

We obtain

$$(4.12) \quad \lambda = \frac{\left\{ 1 - \varkappa \zeta - \frac{1}{2} \frac{\varkappa \delta}{\dot{\rho}} \frac{\partial k^2}{\partial a} \right\} (t_k^i - \alpha_k^i) \dot{t}_i^k - \frac{1}{2} \frac{\partial k^2}{\partial T} \dot{T}}{k^2 \left\{ 2\zeta + \frac{\delta}{\dot{\rho}} \frac{\partial k^2}{\partial a} - 2 \frac{\gamma}{k} (t_s^r - \alpha_s^r) \alpha_r^s \right\}}$$

From Eq. (4.12) we learn subsequently that nevertheless a certain restriction exists with respect to the otherwise unrestricted choice of quantities \varkappa, γ . They result from the requirement that λ has to be non-negative. However, these restrictions are rather weak as can be shown by adjusting the material functions

$$k^2(a, T), \delta(a, A, T), \zeta(a, A, T), \gamma(s_k^i, T, a, \alpha_k^i), \varkappa(s_k^i, T, a, \alpha_k^i), \text{ and } \mu(s_k^i, T, a, \alpha_k^i)$$

to experimental results. It may be mentioned that the term containing \varkappa in the evolution law (4.6)₂ for the inelastic deformations plays an important role in nonproportional loading or deformation paths, particularly in such with abrupt changes [13 to 15]. On the other hand the evolution for anisotropic hardening (4.7)₂ plays the key role in cyclic processes. The approach (4.7)₂ is based on ideas introduced in connection with cyclic deformations [16 to 18].

The preceding considerations can be extended to elastic-viscoplastic materials very easily adopting the so-called overstress approach for the description of viscous damping. In this case we decompose the stress according to

$$(4.13) \quad s_k^i = (s_k^i - \bar{s}_k^i) + \bar{s}_k^i,$$

where \bar{s}_k^i characterizes the accompanying constraint equilibrium state and $s_k^i - \bar{s}_k^i$ represents the viscous overstress.

Then we can establish the following constitutive relations for the inelastic behaviour: yield condition:

$$(4.14)_1 \quad F = (t_k^i - \alpha_k^i) (t_i^k - \alpha_i^k) - k^2(a, T) = f^2 - k^2 > 0;$$

accompanying constraint equilibrium state:

$$(4.14)_2 \quad \bar{F} = (\bar{t}_k^i - \alpha_k^i) (\bar{t}_i^k - \alpha_i^k) - k^2(a, T) = \bar{f}^2 - k^2 = 0,$$

evolution laws for inelastic deformations:

$$(4.15) \quad d_k^i = \begin{cases} A \left(\frac{F}{k^2} \right) \frac{\partial F}{\partial s_k^i} + \varkappa t_k^i = \lambda \frac{\partial \bar{F}}{\partial s_k^i} + \varkappa t_k^i & \text{if } F > 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_k^i = \begin{cases} M \frac{\partial F}{\partial s_k^i} & \text{if } F > 0 \text{ and } s_k^i \frac{\partial F}{\partial s_k^i} > 0, \\ 0 & \text{otherwise,} \end{cases}$$

evolution laws for internal variables:

$$(4.16) \quad \begin{aligned} \dot{a} &= \frac{\delta}{\varrho} (\bar{t}_k^i - \alpha_k^i) d_i^k = \frac{\delta}{\varrho} \left\{ 2kf\lambda + \varkappa \frac{k}{f} (t_k^i - \alpha_k^i) t_i^k \right\}, \\ \alpha_k^i &= \zeta d_k^i - 2\lambda \gamma f \alpha_k^i = 2\lambda \{ \zeta (t_k^i - \alpha_k^i) - \gamma f \alpha_k^i \} + \varkappa \zeta t_k^i; \end{aligned}$$

entropy production:

$$(4.17) \quad \begin{aligned} \dot{w}_{(d)} &= \frac{1}{\varrho} \xi \bar{t}_k^i d_i^k + \frac{1}{\varrho} (t_k^i - \bar{t}_k^i) d_i^k + \frac{1}{\varrho} t_k^i d_i^k, \\ \dot{\eta} &= 0. \end{aligned}$$

In this case $\lambda(s_k^i, T, a, \alpha_k^i)$ is unrestricted. Concerning λ we derive immediately from equation (4.15)₁

$$(4.18) \quad \lambda = \lambda \frac{f}{k}.$$

With respect to the other material functions $\delta, \zeta, \varkappa, \gamma, M$ analogous statements are valid as in the case of plastic behaviour.

5. Some general consequences resulting from the coupling of mechanical and thermal processes concerning bifurcation problems

As already mentioned in Chapter 3, coupling effects between mechanical and thermal processes enter in two different ways (neglecting thermo-elastic effects)

(a) by entropy production due to dissipative mechanical and internal processes ($w_{(d)}$ and $\dot{\eta}$),

(b) by divergence of heat flux ($q^i|_i$).

This is true in bifurcation problems, too.

At a very high deformation rate the influence of entropy production is prevailing since the process becomes locally adiabatic approximately. At a moderate deformation rate the two effects are interacting. As soon as heat flux affects the process, it becomes rate sensitive even if the evolution laws for inelastic deformations and for the internal variables are rate independent.

Bifurcation is generally (always?) connected with the breakdown of any symmetry of the process. At least the process becomes more inhomogeneous after bifurcation. This means the gradients of the state variables increase at least in certain regions of the body. Therefore the influence of thermo-mechanical coupling increases with bifurcation. However, also before bifurcation occurs, coupling of mechanical and thermal processes can change the feature of the process essentially. The symmetry of a process, for instance, depends on the symmetry of the initial state of the body and the corresponding symmetry of the independent process variables. In order to keep the initial symmetry, it is necessary (but not sufficient) that the mechanical boundary conditions as well as the thermal boundary conditions fulfill the respective symmetry requirements.

Thus any symmetry of a process can break down before bifurcation if only the mechanical boundary conditions satisfy the respective symmetry conditions but the thermal boundary conditions do not. This will be demonstrated with respect to the necking problem in the uniaxial tensile test in the following chapter.

6. The coupled thermo-mechanical process in the necking problem

We consider a uniaxial tensile test of an initially cylindrical bar (see Fig. 4). For the formulation of the thermo-mechanical boundary conditions we use space-fixed coordinates x^α and body-fixed (co-deforming) coordinates ξ^i coinciding in the initial state. We assume

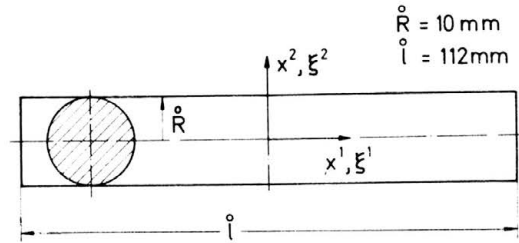


FIG. 4. Initial configuration of the bar.

the ends of the specimen are free of shear stress and kept at a constant temperature \dot{T} by cooled grips. The heat transfer across the lateral surface to the surrounding air is neglected. The prescribed elongation rate is v (see Fig. 5).

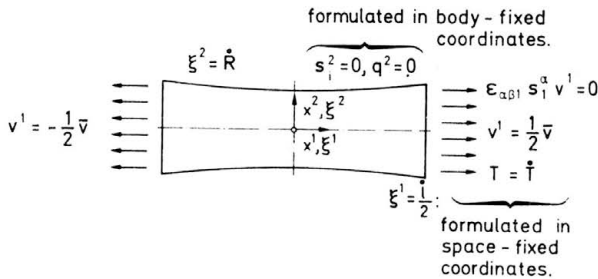
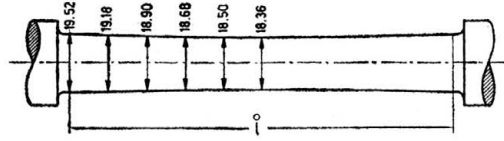


FIG. 5. Thermo-mechanical boundary conditions.

The material used in experiments was Al 99. For theoretical calculations a simplified constitutive law was introduced taking into account only isotropic hardening (i.e. $\alpha_k^i \equiv 0$ in Eq. (4.5)₁ etc.) and unifying local and global processes ($d_k^i = 0$ in (4.6)₃ etc.). For more details of the constitutive law see [19 to 21].

The mechanical boundary conditions admit a deformation process which does not depend on ξ^1 . The thermal boundary conditions, however, contradict such a process. Therefore the corresponding symmetry conditions break down with the onset of inelastic deformations. The specimen loses its cylindrical shape already before the proper necking process starts (see Fig. 6).



$$\dot{l} = 112 \text{ mm} , \quad \Delta l = 20 \text{ mm} , \quad \bar{v} = 5 \text{ mm/s}$$

FIG. 6. Deformation before necking.

It is an open question whether or not still a real bifurcation problem exists under these conditions. With respect to the isothermal process it is known that a bifurcation problem does not exist any more when the boundary conditions (geometrical imperfections of the cylindrical shape or bar with ends cemented to rigid grips) contradicts the corresponding

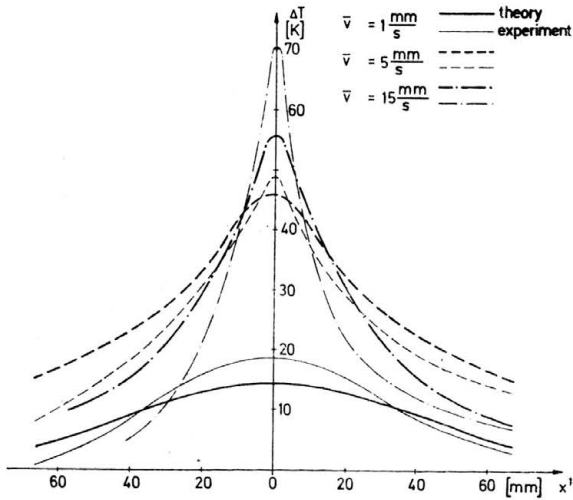


FIG. 7. Surface temperature ($\Delta T \approx 35 \text{ mm}$).

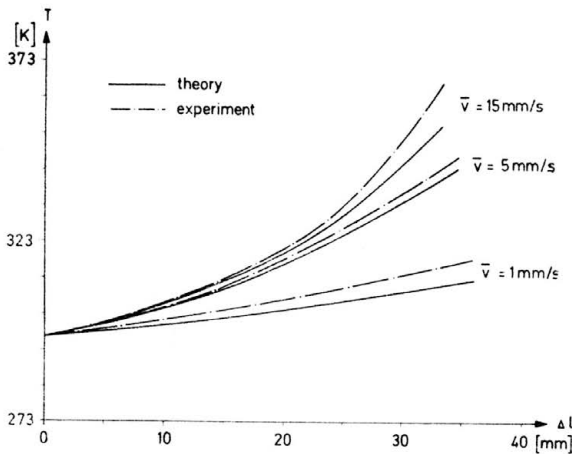


FIG. 8. Temperature in the neck.

symmetry conditions [22]. We suppose that the same happens when the thermal boundary conditions break the corresponding symmetry. However, there exists no mathematical proof.

The necking process itself becomes rate dependent as the temperature distribution and evolution (Figs. 7 and 8) as well as the evolution of the mean axial stress $\bar{\sigma}$ (Fig. 9) show. With respect to more details we refer to [19 to 21].

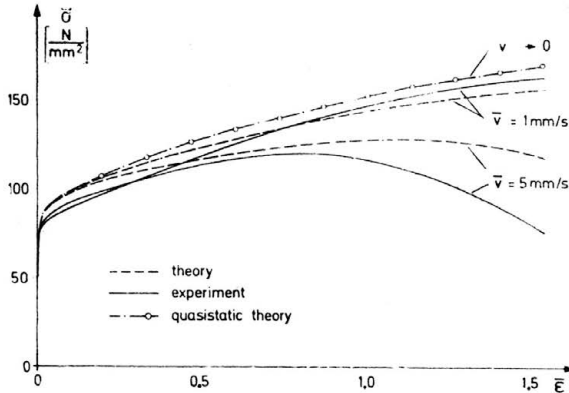


FIG. 9. Mean axial stress vs mean axial strain in the neck.

7. Closing remarks

As pointed out in Chapters 5 and 6, the coupling between mechanical and thermal processes may change the feature of a process essentially in comparison to the corresponding isothermal process. This is true for elastic-plastic materials (see also [23]). Exact investigations of the mathematical structure of the coupled problem are still missing. Particularly, variational formulations within the frame of convex functional analysis are unknown. It seems also to be questionable whether such general formulations are possible. Therefore general methods for the proof of existence and uniqueness of solutions from which further restrictions for the formulation of constitutive laws may arise does not exist as yet. This does not exclude that for particular constitutive laws a variational formulation can be found on which numerical methods can be based [24, 25].

References

1. TH. LEHMANN, *General frame for the definition of constitutive laws for large nonisothermic elastic-plastic and elastic-viscoplastic deformations*, in: *The Constitutive Law in Thermo-plasticity* (ed. TH. LEHMANN), CISM Courses and Lectures No. 281, Springer-Verlag, Wien-New York, 379-463, 1984.
2. TH. LEHMANN, *Thermo-mechanical coupling in large deformations particularly in bifurcation problems*, Proc. IUTAM Symposium "Thermomechanical Coupling in Solids" (ed. BUI NGUYEN), Elsevier Sci. Publ., 277-290, 1987.
3. D. B. MACVEAN, *Die Elementararbeit in einem Kontinuum und die Zuordnung von Spannungs- und Verzerungstensorsoren*, ZAMP, 19, 157-185, 1969.

4. TH. LEHMANN, *Some remarks on kinematics and constitutive equations of inelastic solids*, in: *Mechanics of Inelastic Media and Structures* (ed. O. MAHRENHOLTZ and A. SAWCZUK), Pol. Sci. Publ., Warszawa, 161–178, 1982.
5. A. PHILLIPS and LEE KHONG-WAI, *Yield surfaces and loading surfaces, Experiments and recommendations*, *Int. J. Sol. Struct.*, **15**, 715–729, 1979.
6. K. IKEGAMI, *Experimental plasticity in the anisotropy of metals*, in: *Proc. Euromech-Coll. 115, Villard-de-Lans, 1979* (ed. J.-P. BOEHLER), Ed. CNRS Paris, 201–242, 1982.
7. TH. LEHMANN, *On a generalized constitutive law in thermoplasticity taking into account different yield mechanisms*, *Acta Mech.*, **57**, 1–23, 1985.
8. TH. LEHMANN, *On a generalized constitutive law in thermoplasticity*, in: *Plasticity Today* (ed. A. SAWCZUK and G. BIANCHI), Elsevier Appl. Sci. Publ., London–New York, 115–134, 1985.
9. A. L. GURSON, *Continuum theory of ductile rupture by void nucleation and growth*, *J. Eng. Mater. Technol.*, 2–15, 1977.
10. V. TVERGAARD and A. NEEDLEMAN, *Analysis of the cup-cone fracture in a round tensile bar*, *Techn. Univ. of Denmark, Rep. No. 264*, 1983.
11. P. PERZYNA, *On constitutive modelling of dissipative solids for plastic flow, instability and fracture*, in: *Plasticity Today* (ed. A. SAWCZUK and G. BIANCHI), Elsevier Appl. Sci. Publ., London New–York, 657–679, 1985.
12. E. H. LEE, *Some remarks on elastic-plastic analysis*, *Int. J. Sol. Struct.*, **17**, 859–872, 1981.
13. U. ZDEBEL, *Theoretische und experimentelle Untersuchungen zu einem thermoplastischen Stoffgesetz*, *Diss. Ruhr-Univ. Bochum, Mitt. Inst. Mech.*, **43**, 1984.
14. U. ZDEBEL and TH. LEHMANN, *Some theoretical considerations and experimental investigations on a constitutive law in thermoplasticity* *Int. J. Plasticity*, **3**, 369–389, 1987.
15. TH. LEHMANN, *Some theoretical considerations and experimental results concerning elastic-plastic stress-strain relations*, *Ing. Arch.*, **52**, 391–403, 1982.
16. P. J. ARMSTRONG and C. D. FREDERIK, *A mathematical representation of the multiaxial Bauschinger effect*, *G.E.G.B. Report RD/B/N 731*, 1966.
17. C. CALLETAUD and J. L. CHABOCHE, *Macroscopic description of the microstructural changes induced by varying temperature: Example of in 100 cyclic behaviour*, *Onera-Rep. No. 1979–112*.
18. TH. LEHMANN, *Some remarks on the evolution of anisotropic hardening in plasticity*, *ZAMM*, **66**, 163–165, 1986.
19. U. BLIX, *Zur Berechnung der Einschnürung von Zugstäben unter Berücksichtigung thermischer Einflüsse mit Hilfe der Finite-Element-Method₂*, *Diss. Ruhr-Univ. Bochum, Mitt. Inst. Mechanik RUB*, **35**, 1983.
20. TH. LEHMANN and U. BLIX, *On the coupled thermo-mechanical process in the necking problem*, *Int. J. Plasticity*, **1**, 175–188, 1985.
21. TH. LEHMANN and U. BLIX, *On the influence of the coupling of thermal and mechanical processes in the necking problem in uniaxial tension*, *J. Thermal Stresses*, **8**, 153–165, 1985.
22. A. NEEDLEMAN, *A numerical study of necking in circular cylindrical bars*, *J. Mech. Phys. Solids*, **20**, 111–127, 1972.
23. TH. LEHMANN, *On the coupling of thermal and mechanical processes in large inelastic nonisothermic deformations of solid bodies*, *Proc. Congress of Appl. Math. I, Bratislava 1984* (in press).
24. J. F. BESSELING, *Finite element methods*, in: *Trends in Solid Mechanics, 1979* (ed. J. F. BESSELING and A. M. A. VAN DER HEIJDEN), Univ. Press, Delft, 53–78, 1979.
25. J. F. BESSELING, *Constitutive law in inelastic structural mechanics*, in: *The Constitutive Law in Thermo-plasticity* (ed. TH. LEHMANN), *CISM Courses and Lectures No. 281*, Springer-Verlag, Wien, 541–601, 1984.

RUHR-UNIVERSITÄT BOCHUM
 INSTITUT FÜR MECHANIK, BOCHUM, FRG.

Received June 26, 1986.