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Growth of voids in a ductile matrix: a review

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THE MAIN models concerning the growth of voids or the macroscopic behavior of porous materials are reviewed. The bases and the main results of these are given. The influence of triaxiality, strain rate sensitivity, and porosity are detailed. It appears that the growth of isolated voids is correctly modelled. By contrast, interaction effects between cavities is not well accounted for, as shown by experimental evidence, as well as strain and damage anisotropy.

Omówiono podstawowe modele opisujące wzrost porów lub zjawiska makroskopowe zachodzące w materiałach porowatych. Podano podstawy ogólne tych modeli i podstawowe wyniki. Wyróżniono wpływ takich czynników jak trójwymiarowość, wrażliwość na prędkość odkształcenia i porowatość. Okazuje się, że wzrost pojedynczych porów jest modelowany właściwie, w odróżnieniu od wzajemnego oddziaływania pustek, wzmocnienia i anizotropii zniszczenia, na co wskazują wyniki doświadczeń.

Обсуждены основные модели, описывающие рост поров или макроскопические явления, происходящие в пористых материалах. Приведены общие основы этих моделей и основные результаты. Отмечено влияние таких факторов как трехмерность, чувствительность на скорость деформирования и пористость. Оказывается, что рост единичных поров правильно моделируется, в отличие от взаимодействия пустот, упрочнения и анизотропии разрушения, на что указывают результаты экспериментов.

Notations

μ	material parameter in the viscous law (1.4),
s	rate sensitivity parameter (1.4),
$\boldsymbol{\sigma}, \sigma_{ij}$	microscopic stress tensor,
$\mathbf{s}, s_{ij}$	microscopic deviatoric stress tensor,
σ_{eq}	microscopic equivalent stress,
σ_0	microscopic equivalent yield stress,
$\dot{\boldsymbol{\epsilon}}, \dot{\varepsilon}_{ij}$	microscopic strain rate tensor,
Y	representative volume element,
.v.e.	abbreviated form for the representative volume element,
Y^*	part of the r.v.e. occupied by the matrix,
V	part of the r.v.e. occupied by the cavities,
$\mathbf{n}$	outer normal vector to ∂Y^*,
f	void volume fraction (porosity) $= V/Y$,
a, b	radii of the cavity,
R	mean radius of the cavity,
β	excentricity of the cavity,
$\langle . \rangle$	average symbol on the r.v.e.,
$\boldsymbol{\Sigma}, \Sigma_{ij}$	macroscopic stress tensor,
$\dot{\mathbf{E}}, \dot{E}_{ij}$	macroscopic strain-rate tensor,
φ	microscopic dissipation potential (A.5),
Φ	macroscopic dissipation potential (4.4),

ψ microscopic force potential (A.5),
Ψ macroscopic force potential (4.5),
$\dot{\mathscr{E}}$ measure of the macroscopic rate of deformation.

1. Introduction

1.1. Damage and voids

"DAMAGE" is a general word used to express the decay of mechanical properties of a given material during its history. Damage is measured with a scalar or tensorial variable the changes of which are modelled through macroscopic considerations (sample stiffness measures, use of thermodynamics) or microscopic investigations (based on the local meaning of damage). From the latter point of view, which is adopted in this paper, the damage variable is explicitly related to the density of local defects in the material, and these variable changes are governed by the alteration of the geometry, distribution, etc. of the defects. More precisely, the development of a macroscopic crack in a ductile material can be divided into three main stages which have a microscopic interpretation:

(i) void initiation: microcrack opening at the surface of local inhomogeneities,

(ii) void growth: the plasticity of the surrounding material allows the growth of microvoids,

(iii) void coalescence: plastic instability between microvoids leads to their coalescence, and a macrocrack appears.

The above three phenomena are closely interrelated and can occur simultaneously. For instance, it has been observed that a distribution of very small voids (i.e. which are at the beginning of the growth stage).

1.2. Damage and micromechanics

The study of porosity change is carried out on a representative volume element (r.v.e.) of the porous matrix. In what follows, the r.v.e. is denoted by Y, and is split into Y^* and V which are the domains of material and void in Y, respectively (Fig. 1a).

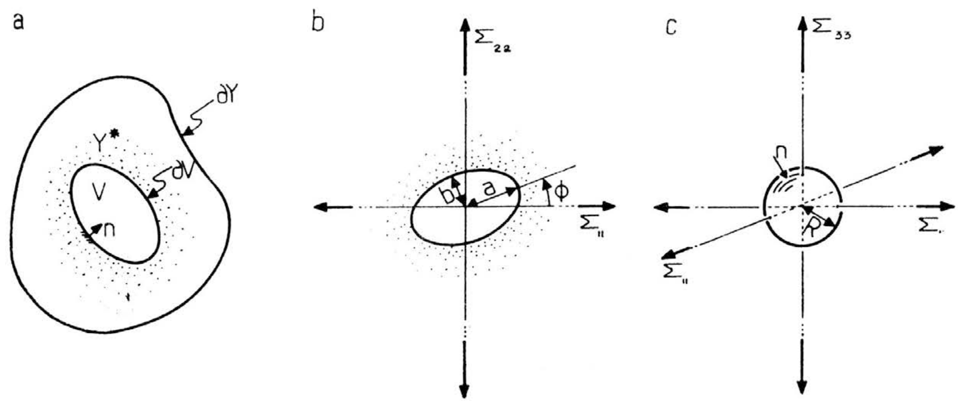

FIG. 1. General representative volume element (a), and geometry of the problem for a cylindrical (b) or spherical (c) void.

The microscopic variables concern the local state in the r.v.e., they are denoted with lower-case letters ($\boldsymbol{\sigma}$, $\dot{\boldsymbol{\varepsilon}}$, etc.). Upper-case letters ($\boldsymbol{\Sigma}$, $\dot{\mathbf{E}}$, etc.) are used for the macroscopic conditions applied to the r.v.e. The classical relations (HILL (1967)) between these variables are recalled:

$$(1.1)\qquad \begin{aligned} \Sigma_{ij} &= \frac{1}{Y}\int_{Y} \sigma_{ij}dx = \frac{1}{Y}\int_{\partial Y} \sigma_{ip}n_p x_j ds \ (^1),\\ \dot{E}_{ij} &= \frac{1}{Y}\int_{\partial Y} \frac{1}{2}(\dot{u}_i n_j + \dot{u}_j n_i)ds = \frac{1}{Y}\int_{Y} \dot{\varepsilon}_{ij}dx - \frac{1}{Y}\int_{\partial V} \frac{1}{2}(\dot{u}_i n_j + \dot{u}_j n_i)ds, \end{aligned}$$

where the unit normal $\mathbf{n}$ to the void surface points into V.

The rate of porosity change $\dot{f}$ (with $f = V/Y$) is easily deduced from the macroscopic condition for mass balance:

$$\dot{\varrho} + \varrho \dot{E}_{ii} = 0 \quad \text{and} \quad \varrho = \varrho_0(1-f)$$

and leads to

$$(1.2)\qquad \dot{f} = (1-f)\dot{E}_{ii}$$

can play an important role in the coalescence of bigger voids. Similarly, damage increase is due to the growth of previously created voids and initiation of new voids simultaneously. This coupling is neglected in the present paper which is devoted to the longest stage in damage development, i.e. *void growth*. This paper is also restricted to *isotropic damage*, which can be described by using porosity (f) as a single scalar variable.

Isotropic damage through void growth in a ductile matrix can be modelled by writing two equations:

$$\dot{f} = \dot{f}(\dot{\mathbf{E}}, \boldsymbol{\Sigma}, f) \quad \text{and} \quad \dot{\mathbf{E}} = \dot{\mathbf{E}}(\boldsymbol{\Sigma}, f).$$

The first equation gives the porosity change associated with external loading (defined by the macroscopic strain rate $\dot{\mathbf{E}}$ and stress $\boldsymbol{\Sigma}$) applied to a porous material. The second equation defines the macroscopic mechanical behavior of the porous material, as a function of porosity. These two equations can be studied separately; the porosity change can be studied apart from the macroscopic behavior, for instance, and this is done in Sects. 2 and 3 below. But the true nature of the above equations is to be coupled by the mass balance equation Eq. (1.2), and this coupling would even be stronger and affect all the components of $\dot{E}$ if anisotropic damage was taken into account, which would be due to anisotropic growth of voids (DRAGON (1985)). Consequently, Sect. 4 is devoted to macroscopic behavior which is a way of getting the void growth law.

In the case of an incompressible matrix ($\dot{\varepsilon}_{ii} = 0$), Eqs. (1.1) lead to the following relations between the rate of porosity change $\dot{f}$, the voids' logarithmic growth rate $\dot{V}/V$, and the rate of volume expansion $\dot{E}_{ii}$:

$$(1.3)\qquad \frac{\dot{V}}{V} = -\frac{1}{V}\int_{\partial V} \dot{u}_i n_j ds, \quad \dot{E}_{ii} = f\frac{\dot{V}}{V}, \quad \dot{f} = f(1-f)\frac{\dot{V}}{V}.$$

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(1) For the sake of simplicity V and Y also denote the volume of V and Y. We shall use in the sequel the average symbol $\langle F \rangle = \frac{1}{Y}\int_{Y^*} F(x)dx$.

Thus $\dot{f}$ can be deduced either from $\dot{V}/V$ or from $\dot{E}_{ii}$. The first relation in Eqs. (1.3) shows that $\dot{V}/V$ is calculated from the microscopic velocity field $\dot{\mathbf{u}}$ (at least from its values at the cavity boundary), which is obtained from the solution of a *micromechanics problem in the r.v.e.* submitted to $\mathbf{\Sigma}$ and $\dot{\mathbf{E}}$ on the average.

One of the equations in this micromechanics problem expresses the mechanical behavior of the matrix. The matrix is viscoplastic in most of the cases considered in this paper:

$$(1.4)\qquad \begin{gathered} \operatorname{div}\dot{\mathbf{u}} = 0 \quad \text{(matrix incompressibility)},\\ \sigma_{ij} = s_{ij} + \sigma_m \delta_{ij} \quad (\sigma_m = \sigma_{ii}/3,\ \mathbf{s} \text{ is the deviatoric part of } \sigma),\\ \frac{3}{2}\frac{s_{ij}}{\sigma_{eq}} = \frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}_{eq}} \quad \text{with} \quad \sigma_{eq} = \mu \dot{\varepsilon}_{eq}^{s-1}. \end{gathered}$$

The classical definitions for equivalent stress, strain rate, and strain are used in this paper:

$$\sigma_{eq} = \left(\frac{3}{2} s_{ij} s_{ij}\right)^{1/2}, \quad \dot{\varepsilon}_{eq} = \left(\frac{2}{3}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}\right)^{1/2}, \quad \varepsilon_{eq}(t) = \int_0^t \dot{\varepsilon}_{eq}(\tau)\,d\tau.$$

In Eq. (1.4), s denotes the strain rate sensitivity: $s = 1.2$ for a hot-worked metal, for instance, and $s = 1.5$ for a superplastic material. The special cases $s = 2$ and 1 correspond to the linearly viscous Newtonian behavior and to rigid-perfect plasticity, respectively. Various equivalent forms of the flow rule (1.4) are given in Appendix 1, in the notations used by other authors.

The macroscopic stress or strain rate is usually prescribed on the r.v.e. outer surface with the following boundary conditions:

$$(1.5)\qquad \sigma_{ij} n_j = \Sigma_{ij} n_j \quad \text{on } \partial Y, \quad \text{or} \quad \dot{u}_i = \dot{E}_{ij} x_j \quad \text{on } \partial Y.$$

HILL (1967) has derived the following relation between the microscopic and macroscopic works:

$$(1.6)\qquad \frac{1}{Y}\int_{Y^*} \tilde{\sigma}_{ij}\,\varepsilon_{ij}(\dot{\tilde{u}}) = \tilde{\Sigma}_{ij}\dot{\tilde{E}}_{ij}$$

for a stress field $\tilde{\boldsymbol{\sigma}}$ in equilibrium [2] and a velocity field $\dot{\tilde{u}}$, provided that $\tilde{\boldsymbol{\sigma}}$ or $\dot{\tilde{\mathbf{u}}}$ fulfills one of the conditions in the set (1.5). Equation (1.6) is easily obtained when $\tilde{\boldsymbol{\sigma}}$ and $\boldsymbol{\epsilon}(\dot{\tilde{\mathbf{u}}})$ are split into their volume average over Y and a fluctuating term the average of which is zero:

$$(1.7)\qquad \begin{gathered} \tilde{\boldsymbol{\sigma}}(x) = \tilde{\mathbf{\Sigma}} + \tilde{\boldsymbol{\sigma}}^*(x),\\ \boldsymbol{\epsilon}(\dot{\tilde{\mathbf{u}}}(x)) = \dot{\tilde{\mathbf{E}}} + \dot{\boldsymbol{\epsilon}}(\dot{\tilde{\mathbf{u}}}^*(x)) \quad \text{with} \quad \dot{\tilde{u}}_i^* = \dot{\tilde{u}}_i - \dot{\tilde{E}}_{ij} x_j. \end{gathered}$$

Equation (1.7) means that the microscopic quantities are the sum of the average of the corresponding quantities in the material and perturbation terms due to the void. HILL (1967) has extended the validity of Eq. (1.6) to fields $\tilde{\boldsymbol{\sigma}}$ and $\dot{\tilde{\mathbf{u}}}$ which do not satisfy one of the conditions (1.5), using a *macrohomogeneity assumption*. In the context of elastic media this assumption is clearly related to St Venant's principle, but its use for nonlinear material behaviors, and especially for plasticity, seems more difficult. Equation (1.6)

[2] in equilibrium means $\operatorname{div}\tilde{\boldsymbol{\sigma}} = 0$ in Y^*, $\tilde{\boldsymbol{\sigma}}\cdot \mathrm{n} = 0$ on ∂V.

is rigorously generalized in Sect. 3 to microscopic fields with boundary conditions different from the relations (1.5) which are more suitable for periodic media.

The problem of micromechanics to be solved in the r.v.e. can be summarized as follows:

$$\begin{gathered}\operatorname{div}\boldsymbol{\sigma}=0, \quad \sigma_{ij}n_j=0 \quad \text{on } \partial V,\\ \operatorname{div}\dot{\mathbf{u}}=0, \quad s_{ij}=\frac{2}{3}\mu\dot{\varepsilon}_{eq}^{s-2}\dot{\varepsilon}_{ij},\\ \sigma_{ij}n_j=\Sigma_{ij}n_j \quad \text{or} \quad \dot{u}_i=\dot{E}_{ij}x_j \quad \text{on } \partial Y.\end{gathered} \tag{1.8}$$

The macroscopic mechanical behavior, which relates $\mathbf{\Sigma}$, $\dot{\mathbf{E}}$, and the other geometrical and mechanical parameters of the problem, is deduced by solving the problem (1.8) and applying the average conditions (1.1). For a given $\dot{\mathbf{E}}$, $\boldsymbol{\sigma}$ is deduced from the problem (1.8) and its average $\mathbf{\Sigma}$ is the macroscopic stress associated to $\dot{\mathbf{E}}$ through the macroscopic behavior.

In this paper, a special emphasis is put on axisymmetric loadings (the only nonzero remote stresses are $\Sigma_{11} = \Sigma_{22}$ and Σ_{33}), where the void is also axisymmetric with respect to the same axis. This type of loading is of interest for the study of necking during uniaxial tension of a cylindrical sample, where the stress state is triaxial when necking develops, due to a non-uniform change in the cross-section of the specimen. An estimate of the principal stresses in the neck has been proposed by Bridgman, and the stress state is characterized by the *triaxiality ratio* Σ_m/Σ_{eq} which is given by

$$\frac{\Sigma_m}{\Sigma_{eq}}=\frac{\Sigma_{33}+2\Sigma_{11}}{3|\Sigma_{33}-\Sigma_{11}|}$$

in the axisymmetric case.

The studies which are detailed in the following Section show a power-law or even an exponential influence of triaxiality on the growth rate of voids. The importance of the mean stress and, consequently, of triaxiality, on void expansion is easy to imagine, but the exponential influence was found only two decades ago, and the power law influence related to strain rate sensitivity was found even more recently. Most of the studies reported in the present paper were carried out in the United States, mainly at Brown and Harvard Universities, with J. Rice, B. Budiansky, J. W. Hutchinson and A. Needleman.

2. Isolated voids

The first type of void growth models which have been proposed in the last two decades have concerned an isolated void in an unbounded matrix uniformly loaded at infinity. Various void geometries (cylindrical, spherical, spheroidal), matrix behaviors (rigid-plastic, linearly or power-law viscous), and types of approach (analytical solution, variational. semi-analytical methods) have been used.

2.1. Cylindrical voids

2.1.1. Linearly viscous matrix (BERG (1962)). Berg has considered the case of a cylindrical void with an elliptical cross-section in a linearly viscous material. This corresponds to $s = 2$ in Eq. (1.4). The following discussion is restricted to plane strain, although Berg calculations also include plane stress. The linearity of the matrix behavior allows an exact analytical solution of the problem, and Berg uses methods which were originally developed for linear elasticity. The relations applying to the case of a cylindrical void embedded in an infinite linearly viscous material in plane strain are the following:

$$(2.1)\qquad \begin{aligned} \frac{1}{\dot{E}_{\mathrm{eq}}}\frac{\dot{R}}{R} &= \frac{3}{2}\frac{\Sigma_m}{\Sigma_{\mathrm{eq}}},\\ \frac{\dot{\beta}}{\dot{E}_{\mathrm{eq}}} &= \sqrt{3}\left(\chi-\sqrt{3}\beta\frac{\Sigma_m}{\Sigma_{\mathrm{eq}}}\right),\\ \frac{\dot{V}}{V} &= 2\left(\frac{\dot{R}}{R}-\frac{|\beta||\dot{\beta}|}{1-|\beta|^2}\right). \end{aligned}$$

In the above equations, $\dot{R}$ is the mean radius of the elliptical cross-section of the void, and β is a complex number whose modulus and argument define, respectively, the shape and the orientation of the cavity with respect to the principal axes of loading (Fig. 1b):

$$R=\frac{a+b}{2}\quad\text{and}\quad \beta=\frac{a-b}{a+b}e^{2i\phi}\qquad(a>b).$$

The loading at infinity is defined by the principal stresses Σ_{11} and Σ_{22}, and can be characterized by the triaxiality coefficient $\Sigma_m/\Sigma_{\mathrm{eq}}$. The following well-known relations apply in the case of plane strain:

$$\Sigma_{33}=\frac{\Sigma_{11}+\Sigma_{22}}{2},\qquad \Sigma_{\mathrm{eq}}=\frac{\sqrt{3}}{2}|\Sigma_{11}-\Sigma_{22}|,\qquad \Sigma_m=\frac{\Sigma_{ii}}{3}=\frac{\Sigma_{11}+\Sigma_{22}}{2}=\frac{\Sigma_{\alpha\alpha}}{2},$$

where the Greek (respectively Roman) indices vary from 1 to 2 (respectively 1 to 3). The χ value defines an extension of the matrix at infinity in the ($\Sigma_{11}>\Sigma_{22}$, $\chi=1$) or ($\Sigma_{22}>\Sigma_{11}$, $\chi=-1$) direction. It can be observed that $\dot{R}$ depends *linearly* on the triaxiality but is unaffected by the void aspect ratio(3). By contrast, the changes in shape and orientation of the void are influenced by both quantities.

2.1.2. Rigid-plastic matrix (MCCLINTOCK (1968)). The linearly viscous behavior is not satisfactory for metals, and McClintock has proposed an approximate solution for the case of a cylindrical void in a rigid-plastic strain-hardening medium. This author first gives the exact analytical solution for an axisymmetric loading and a rigid-perfectly plastic material (Appendix 2) containing a cylindrical cavity with a circular cross-section. In these conditions the void growth rate is given by

$$(2.2)\qquad \frac{1}{\dot{E}_{\mathrm{eq}}}\frac{\dot{R}}{R}=-\frac{1}{2}+\frac{\sqrt{3}}{2}\sinh\left(\frac{\sqrt{3}}{2}\frac{\Sigma_{\alpha\alpha}}{\sigma_0}\right).$$

The first term in the right-hand side of Eq. (2.2) is due to incompressibility and leads to a shortening of the void radius when a zero radial stress is applied. The exponential term grows very fast when a tensile radial stress is applied, leading to a significant increase of the void radius. In these conditions the void growth rate is

(3) It should be noted that the rate of void *volume* change depends on both the triaxiality and the aspect ratio if the void is not circular, since $\dot{V}/V$ is a function of R, $\dot{R}$, $|\beta|$ and $|\dot{\beta}|$, as shown in Eq. (2.1).

$$\frac{1}{\dot{E}_{eq}}\frac{\dot{V}}{V} = \sqrt{3}\sinh\left(\frac{\sqrt{3}}{2}\frac{\Sigma_{\alpha\alpha}}{\sigma_0}\right). \tag{2.3}$$

The second step of McClintock calculation concerns a power-law strain-hardening material

$$\frac{3}{2}\frac{s_{ij}}{\sigma_{eq}} = \frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}_{eq}} \quad \text{with} \quad \sigma_{eq} = \mu\varepsilon_{eq}^n,$$

and uses the following analogic method:

(i) The exact solutions are known for the simple problems of a circular cylindrical void in a linearly viscous or rigid-perfectly plastic medium with an axisymmetric loading.

(ii) A set of formal substitutions can be deduced from these solutions, which can be used to go from one to the other.

(iii) The same substitutions are applied to Berg's solution and lead to an approximate solution for a cylindrical void with an elliptical cross-section in a rigid-perfectly plastic matrix in a generalized plane strain (Σ_{11}, Σ_{22} and $\Sigma_{33} \neq 0$, $\dot{\varepsilon}_{33} = \dot{E}_{33} \neq 0$). Finally, the strain-hardening coefficient n is introduced as an interpolator between the two basic solutions.

The geometry of the problem is a special case of Fig. 1b with $\phi = 0$, and, consequently, β is a real number. The results can be written as follows:

$$\begin{aligned} \frac{1}{\dot{E}_{eq}}\frac{\dot{R}}{R} &= \frac{\sqrt{3}}{2(1-n)}\sinh\left[\frac{\sqrt{3}}{2}(1-n)\frac{\Sigma_{\alpha\alpha}}{\sigma_0}\right] + \frac{1}{2}\frac{\dot{E}_{\alpha\alpha}}{\dot{E}_{eq}}, \\ \frac{\dot{\beta}}{\dot{E}_{eq}} &= \frac{-\sqrt{3}}{1-n}\left(\beta - \frac{\Sigma_{11}-\Sigma_{22}}{\Sigma_{11}+\Sigma_{22}}\right)\sinh\left[\frac{\sqrt{3}}{2}(1-n)\frac{\Sigma_{\alpha\alpha}}{\sigma_0}\right]. \end{aligned} \tag{2.4}$$

The plane strain case is deduced easily from Eqs. (2.4) with

$$\dot{E}_{11} + \dot{E}_{22} = -\dot{E}_{33} = 0 \quad \text{(incompressibility and plane strain)},$$

$$\Sigma_{11} - \Sigma_{22} = \frac{2}{\sqrt{3}}\chi\Sigma_{eq}, \qquad \Sigma_{\alpha\alpha} = \Sigma_{11} + \Sigma_{22} = 2\Sigma_m.$$

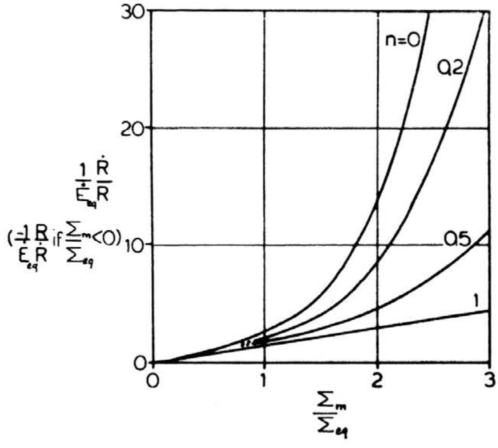

FIG. 2. Influence of triaxiality and strain-hardening on the growth rate of a cylindrical void in plane strain (McClintock model).

The rate of change of the void mean radius does not depend on the aspect ratio (similarly to what was found in the linearly viscous case) but increases *exponentially* with triaxiality.

Figure 2 compares the influence of triaxiality on the rate of change of the mean radius of a given void (which coincides with the void volume growth rate in the case of a circular cavity only) given by Berg and McClintock models in plane strain. The latter model predicts a decrease of the void growth when strain-hardening is increased. The rate of shape change

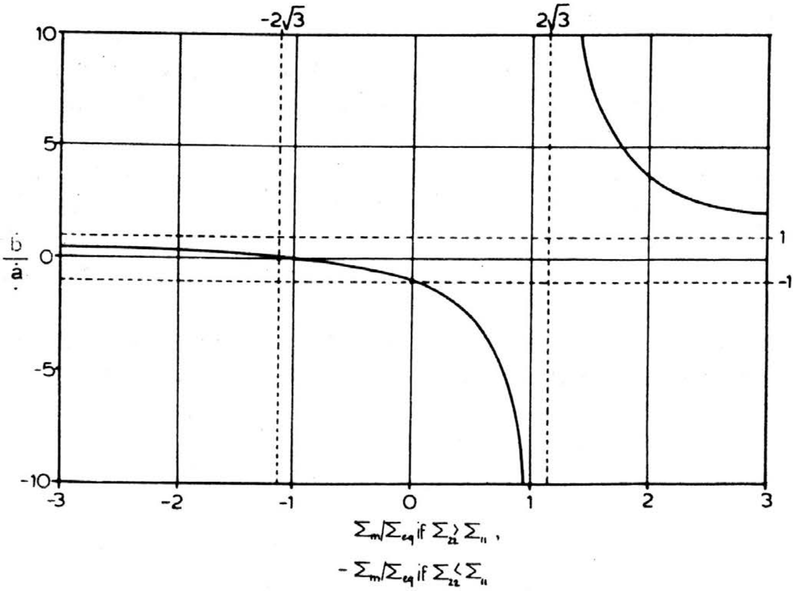

FIG. 3. Influence of triaxiality on the ratio of the radial velocities at a circular cylindrical void boundary. Identical results in Berg and McClintock models.

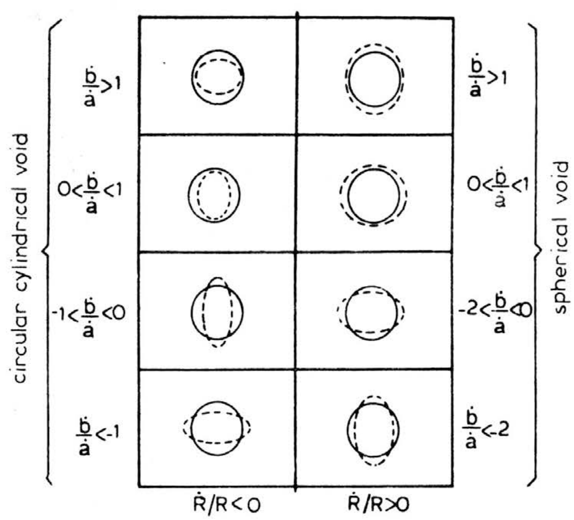

FIG. 4. Qualitative relation between the $\dot{b}/\dot{a}$ value and the initial shape change of cylindrical or spherical void.

of a circular cavity can be related to $\dot{b}/\dot{a}$, where $\dot{a}$ and $\dot{b}$ are the radial velocities of the points located on the void boundary in the 1 and 2 directions, respectively. This ratio is obtained easily from Eqs. (2.4) in the case of plane strain; it does not depend on n, and the result is the same as the one given by Eqs. (2.1). This does not mean that the shape change of the void does not depend on n, since $\dot{b}/\dot{a}$ is only the *ratio* of two velocities. In agreement with what can be deduced intuitively, a cylindrical cavity with a circular cross-section tends to elongate in the direction of the maximum principal applied stress. This corresponds to a location of the curve in Fig. 3 between $\dot{b}/\dot{a} = 1$ and -1 for $\Sigma_m/\Sigma_{eq} < 0$ and above or under these lines for $\Sigma_m/\Sigma_{eq} > 0$ when $\Sigma_{22} > \Sigma_{11}$, and the opposite when $\Sigma_{22} < \Sigma_{11}$ (Fig. 4). The vertical asymptote of Fig. 3 for $\Sigma_m/\Sigma_{eq} = 2/\sqrt{3}$ (resp. $-2/\sqrt{3}$) when $\Sigma_{22} > \Sigma_{11}$ (resp. $\Sigma_{22} < \Sigma_{11}$) corresponds to $\dot{a} = 0$, i.e. the diameter of the cavity in the 1 direction does not change, and $\dot{b}$ is zero for the opposite value of Σ_m/Σ_{eq}. The horizontal asymptote $\dot{b}/\dot{a} = 1$ is associated with a cavity which keeps a circular cross-section. This would correspond to an equibiaxial tension or compression which is excluded in plane strain, since $\Sigma_{11} = \Sigma_{22}$ leads to $\Sigma_{eq} = 0$.

2.1.3. TRACEY'S **Contribution** (1971). The nonhomogeneity of strain-hardening in the matrix is not taken into account in McClintock's model. Equations (2.4) show that the growth rate is lower in the strain-hardening case ($n > 0$) than in the perfectly plastic one ($n = 0$), this difference being constant when time is increased. The nonhomogeneity of strain-hardening in the matrix has been taken into account by Tracey who proposed upper and lower bounds of the growth rate by using a variational principle. This study shows that the growth rate is all along the time lower in a strain-hardening matrix than in a perfectly

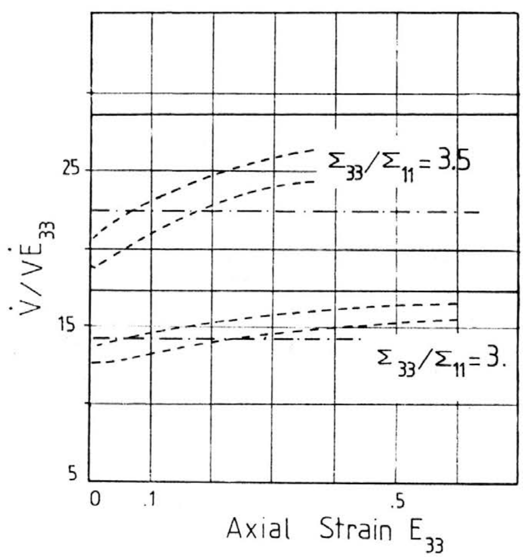

FIG. 5. Growth rate of a circular cylindrical cavity
—— rigid perfectly plastic material (McClintock's model),
—.— hardening material (McClintock's model),
- - - bounds for the growth rate of the cavity in a strain hardening material (Tracey's model).
[After TRACEY (1971)].

plastic one. At the beginning of the deformation, the growth rate is lower than the one given by McClintock, but it quickly becomes greater and tends to the value associated with a perfectly plastic material, as shown in Fig. 5.

Moreover, Tracey emphasized the interest of Eqs. (2.4) in the case of necking. McClintock assumed that the void axis is parallel to the tensile direction, and Tracey considered that the axis is *perpendicular* to this direction. This leads to a drastic decrease of the fracture strain given by the model, as shown in Fig. 6 (curve III).

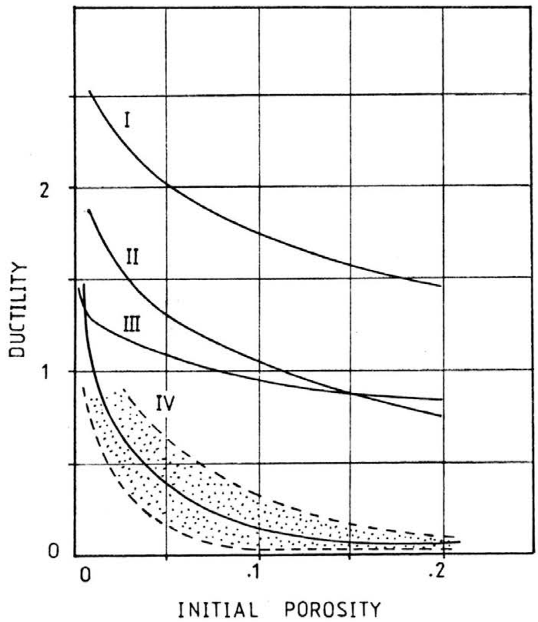

FIG. 6. Fracture strains for different models of growth
I: rigid plastic infinite volume (McClintock's model)
II: rigid plastic finite volume (Tracey's model) axis of the cylinder in the tensile direction.
III: rigid plastic infinite volume (McClintock's model) axis of the cylinder transverse to the tensile direction.
IV: envelope of experimental results.
[After TRACEY (1971)].

In the same study, Tracey showed the influence of a non-zero porosity (i.e. of a *finite* r.v.e.) on the growth rate, and noted a significant decrease of fracture strain when a finite cylindrical r.v.e. was considered (curve II in Fig. 6). This result is discussed in Sect. 3 below.

It should be noted that BUDIANSKY, HUTCHINSON and SLUTSKY (1982) have solved the problem of a circular cylindrical void in a viscoplastic infinite medium with an axisymmetric loading (Appendix 2), and did not propose to follow McClintock method to extend their solution to generalized plane strain. This may be due to the uncertainty which surrounds McClintock analysis.

The above models concern cylindrical cavities which are not very much satisfactory, but can be considered as a good approximation of cigar-shaped voids aligned with the plane strain direction. McClintock model is obtained by an analogic process and, consequently, remains questionable. These limitations have stimulated the development of the models which follow.

2.2. Spherical voids: the importance of variational principles

2.2.1. Rigid-plastic matrix (RICE and TRACEY (1969)). In 1969, RICE and TRACEY proposed a new model of void growth for the case of a spherical cavity embedded in an unbounded *rigid-perfectly plastic* medium ($s = 1$ in Eq. (1.4)) with an axisymmetric loading at infinity (Fig. 1c). A *variational principle* was used, which is derived from the material flow rule. This principle states that any solution of the micromechanics problem defined by Eqs. (1.8) corresponds to a stationary value of the following functional:

$$Q(\dot{\mathbf{u}}^*) = \int_{Y^*} [\varphi(\dot{\mathbf{E}}+\boldsymbol{\epsilon}(\dot{\mathbf{u}}^*)) - \varphi(\dot{\mathbf{E}}) - \Sigma_{ij}\dot{\varepsilon}_{ij}(\dot{\mathbf{u}}^*)]\,dx + \int_{\partial V} \Sigma_{ij} n_j \dot{u}_i^*\,ds, \tag{2.5}$$

where $\dot{\mathbf{u}}$ is a kinematically admissible velocity field: $\dot{u}_i = \dot{E}_{ij}x_j + \dot{u}_i^*$

$$\operatorname{div}\dot{\mathbf{u}} = 0 \quad \text{in } Y^*, \qquad \lim_{|x|\to\infty} \dot{u}_i^*(x) = \dot{u}_i - \dot{E}_{ij}x_j = 0.$$

In Eq. (2.5) above, $\dot{\mathbf{u}}^*$ is the perturbation velocity field associated with $\dot{\mathbf{u}}$, $\mathbf{n}$ is the unit normal to the void surface ∂V pointing into the void, $\boldsymbol{\Sigma}$ is the stress state prescribed at infinity, $\dot{\mathbf{E}}$ is associated to $\boldsymbol{\Sigma}$ through the flow rule of the matrix which is deduced from the dissipation potential of a rigid-plastic material, i.e. $\varphi(\dot{\boldsymbol{\epsilon}}) = \sigma_0 \dot{\varepsilon}_{eq}$. The above minimum principle is a special form (with an infinite medium) of a classical principle for a finite matrix, and this is detailed below in Sect. 3.

As mentioned in the Introduction, and as was outlined by Rice and Tracey, the velocity field $\dot{u}$ can be split into a macroscopic reference part $\dot{E}_{ij}x_j$ and a perturbation term $\dot{\mathbf{u}}^*$ due to the void. The latter contribution can in turn be split into a spherically symmetric term $\dot{\mathbf{u}}^D$ giving a change in void volume without shape change, and a part $\dot{\mathbf{u}}^F$ corresponding to a change in void shape without volume change. This can be written as follows:

$$\dot{u}_i = \dot{E}_{ij}x_j + \dot{u}_i^* = \dot{E}_{ij}x_j + D\dot{u}_i^D + \dot{u}_i^F$$

with

$$\dot{u}_i^D = \dot{E}_{eq} R^3 \frac{x_i}{(x_R x_R)^{3/2}}.$$

The $\dot{\mathbf{u}}^F$ term defines the class of velocity fields which is studied. Rice and Tracey considered six such classes, including two which were quickly eliminated because of paradoxical results (this point has been detailed recently by Budiansky, Hutchinson and Slutsky). Each of these $\dot{\mathbf{u}}^F$ velocity fields is characterized by only one parameter which defines the rate of change of void shape $(\dot{a}-\dot{b})/R$ [4]. Instead of minimizing Q with respect to the two par-

[4] R is the void radius, $\dot{a}$ and $\dot{b}$ are the radial velocities of the points at the void equator and poles, respectively. The void growth rate is given by:

$$\frac{\dot{R}}{R} = \frac{1}{3}\frac{\dot{V}}{V} = \frac{2\dot{a}+\dot{b}}{R} = D\dot{E}_{eq}.$$

ameters of $\dot{\mathbf{u}}^*$, Rice and Tracey looked for the triaxiality value (which defines the external loading) leading to zero partial derivatives of Q with respect to the parameters, for each pair of the latter. The Q value at the minimum is not calculated in this method and, consequently, the solutions which are obtained with various velocity fields cannot be compared. Nevertheless Rice and Tracey could show (i) that the void growth rate for a given value of triaxiality is almost independent of the class of velocity fields $\dot{u}^F$, and (ii) that the value of Q is mostly defined by the volume change term. The result can be written as follows for $|\Sigma_m/\Sigma_{eq}| > 1$ (in axisymmetric deformation $\Sigma_m = (2\Sigma_{11}+\Sigma_{33})/3$ and $\Sigma_{eq} = |\Sigma_{33}-\Sigma_{11}|$):

$$\frac{1}{\dot{E}_{eq}}\frac{\dot{R}}{R} = 0.283\chi\exp\left(\frac{3}{2}\chi\frac{\Sigma_m}{\Sigma_{eq}}\right) - 0.275\chi\exp\left(-\frac{3}{2}\chi\frac{\Sigma_m}{\Sigma_{eq}}\right), \tag{2.6}$$

where $\chi = 1$ for $\Sigma_{33} > \Sigma_{11}$ (extension along the symmetry axis of loading), and $\chi = -1$ for $\Sigma_{33} < \Sigma_{11}$ (radial extension). Equation (2.6) gives reasonable, but much less precise values for $-1 < \Sigma_m/\Sigma_{eq} < 1$. Only the first term in the right-hand side of Eq. (2.6) is kept usually, since high positive triaxialities and axial extension are considered in most cases. Similarly to what was found by McClintock for cylindrical voids, Rice and Tracey have shown the *exponential* influence of triaxiality on the growth rate of spherical voids in a rigid-perfectly plastic matrix.

It should be noted that the $\dot{R}/R$ value given by Eq. (2.6) is not exactly the same for opposite values of triaxiality. As a consequence, Eq. (2.6) gives a small (about 0.01) but non-zero growth rate for $\Sigma_m/\Sigma_{eq} = 0$. This leads to the following paradoxical effect: for low negative (respectively positive) triaxiality, a spherical void grows (respectively shrinks). This void growth is low, and so is the critical triaxiality for which this paradox occurs ($\sim \pm 0.01$). This phenomenon is not contradictory to the basic symmetry of the problem, i.e. the sign of the void growth is changed when the stresses applied at infinity are reversed, since both Σ and χ change sign in these conditions.

Rice and Tracey have proposed to extend their results to the more general case of a spherical void in a rigid-perfectly plastic matrix whose loading at infinity is *not axisymmetric*. They based their analysis on the fact that the void growth term is predominant in Q for the axisymmetric case. By considering only spherically symmetric velocity fields, these authors have generalized Eq. (2.6) as follows:

$$\frac{1}{\dot{E}_{eq}}\frac{\dot{R}}{R} = 0.558\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_{eq}}\right) + 0.008\,\chi\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_{eq}}\right), \tag{2.7}$$

where $\chi = -3\dot{E}_{22}/(\dot{E}_{11}-\dot{E}_{33})$ defines the applied deformation mode through the principal values of the applied strain rate ($\dot{E}_{11} \geqslant \dot{E}_{22} \geqslant \dot{E}_{33}$). It can be checked that Eq. (2.7) leads to Eq. (2.6) in the axi-symmetric case since $\chi = \pm 1$ in these conditions.

2.2.2. Power-law viscous matrix (BUDIANSKY, HUTCHINSON and SLUTSKY (1982)). The results obtained by Rice and Tracey have been extended recently by Budiansky, Hutchinson and Slutsky to the case of a spherical void in a power-law viscous matrix ($s \neq 1$ in Eq. (1.4)) with a uniform axisymmetric loading at infinity. The above variational principle (Eq. (2.5)) is still valid, with $\varphi(\dot{\boldsymbol{\epsilon}}) = \frac{\mu}{s}\dot{\varepsilon}^s_{eq}$.

The minimum principle was applied to a large class of velocity fields depending on seven parameters, including six parameters for the change of void shape. It was found that the change of void volume was predominant in Q for $|\Sigma_m|/\Sigma_{eq} > 1$, similarly to what was observed by Rice and Tracey. Equation (2.6) can be generalized as follows for this triaxiality range:

(2.8)

$$\frac{1}{\dot{E}_{eq}}\frac{\dot{R}}{R} = \frac{1}{2}\frac{|\Sigma_m|}{\Sigma_m}\left[\frac{3}{2}(s-1)\frac{|\Sigma_m|}{\Sigma_{eq}} + (2-s)\left[1+(s-1)\left(0.4175+0.0144\chi\frac{|\Sigma_m|}{\Sigma_m}\right)\right]\right]^{\frac{1}{s-1}}.$$

It should be noted that Eq. (2.8) reduces to Eq. (2.6) for high triaxiality when s tends to 1, and this provides continuity with Rice and Tracey results for a rigid-perfectly plastic matrix. For the other limiting case, i.e. linearly viscous matrix ($s = 2$), Eq. (2.8) gives the exact analytical result:

$$\frac{1}{\dot{E}_{eq}}\frac{\dot{R}}{R} = \frac{3}{4}\frac{\Sigma_m}{\Sigma_{eq}} = \frac{1}{4}\frac{\Sigma_{33}+2\Sigma_{11}}{\Sigma_{33}-\Sigma_{11}}. \tag{2.9}$$

The influence of triaxiality and strain-rate sensitivity on void growth rate is illustrated in Fig. 7. Strain-rate sensitivity decreases void growth rate, and the latter depends on

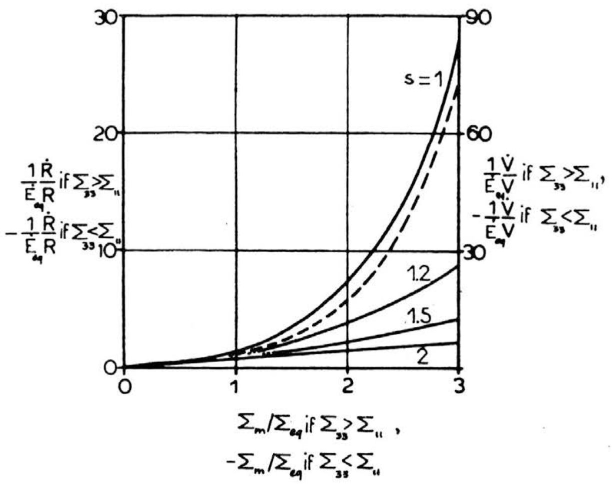

FIG. 7. Influence of triaxiality and strain rate sensitivity on the growth rate of a spherical void in axisymmetric deformation.

(——): Budiansky, Hutchinson and Slutsky; (– –): Rice and Tracey.

triaxiality through a *power law* for $1 < s < 2$ according to Eq. (2.8). There is a difference for $s = 1$ between Rice and Tracey results on the one hand, and those obtained by Budiansky, Hutchinson and Slutsky on the other hand as shown in Fig. 7. This difference is reduced for high triaxiality and is probably due to the higher number of parameters in the class of velocity fields used by the latter authors, which leads to a better minimum for Q.

Equation (2.8) gives slightly different results for $\dot{R}/R$ when the sign of triaxiality is reversed, similarly to what was noted for Eq. (2.6), and the results given by Budiansky, Hutchinson and Slutsky show clearly that $\dot{R}/R$ is zero for $\Sigma_m/\Sigma_{eq} = 0$ in the case of a linearly viscous material only, in agreement with Eq. (2.9). It is nevertheless possible to extend the curves in Fig. 7 to the negative (respectively positive) values of Σ_m/Σ_{eq} when $\Sigma_{33} > \Sigma_{11}$ (respectively $\Sigma_{33} < \Sigma_{11}$) by symmetry with respect to the origin since the variation in $\dot{R}/R$ is very small when Σ_m/Σ_{eq} is changed into $-\Sigma_m/\Sigma_{eq}$. Equation (2.8) fits approximately the curves in Fig. 7 and the precision is increased for high s values.

2.2.3. Change of void shape. Although this paper is devoted mainly to isotropic damage described only by porosity, it is interesting to discuss the validity of this assumption by considering the rate of change of void shape. Figure 8 shows the influence of triaxiality

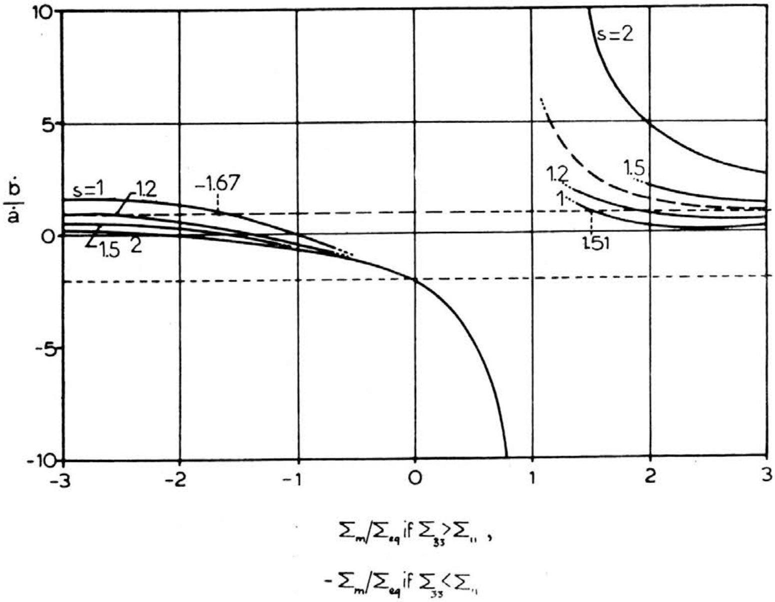

FIG. 8. Influence of triaxiality and strain rate sensitivity on the ratio of the radial velocities at the poles and equator of a spherical void.
(——): Budiansky, Hutchinson and Slutsky; (– –): Rice and Tracey.

and strain rate sensitivity on $\dot{b}/\dot{a}$ which characterizes the initial shape change of a spherical void. The curves in Fig. 8 were deduced from partial results given by Rice and Tracey on the one hand, and Budiansky, Hutchinson and Slutsky on the other. The only complete result is for the closed form solution in the case of a linearly viscous matrix:

$$\frac{\dot{b}}{\dot{a}} = \frac{\dfrac{9}{10}\dfrac{\Sigma_m}{\Sigma_{eq}}+2}{\dfrac{9}{10}\dfrac{\Sigma_m}{\Sigma_{eq}}-1}$$

which was found by Budiansky, Hutchinson and Slutsky.

Although the curves are not complete, a *second paradox* of spherical void growth can be noted, which was outlined by Budiansky, Hutchinson and Slutsky. Intuition would suggest that a void elongates in the direction of the maximum applied principal stress, according to what happens to the surrounding matrix far from the cavity. In these conditions, the curves in Fig. 8 should be located between $\dot{b}/\dot{a} = 1$ and -2 for $\Sigma_m/\Sigma_{eq} < 0$, and above or under these lines for $\Sigma_m/\Sigma_{eq} < 0$, when $\Sigma_{33} < \Sigma_{11}$ (and the opposite when $\Sigma_{33} > \Sigma_{11}$, see Fig. 4). It can be seen that this is not true for a part of the curves given by Budiansky, Hutchinson and Slutsky when $s < 1.5$. In the case of a rigid-perfectly plastic matrix ($s = 1$); for instance, a spherical cavity grows towards an oblate shape although the surrounding matrix elongates along the axis ($\Sigma_{33} > \Sigma_{11}$ case), when triaxiality is greater than 1.51 or lower than -1.67. Similarly, the cavity grows to a prolate shape although the matrix elongates radially ($\Sigma_{33} < \Sigma_{11}$ case) if triaxiality is greater than 1.67 or lower than -1.51. It should be noted that in the first (respectively second) case the void does grow (respectively shrink), but the radial velocity is greater (respectively lower) at the equator than at the poles. This paradox occurs when the curves cross the $\dot{b}/\dot{a} = 1$ asymptote which is associated with a spherically symmetric loading ($\Sigma_{33} = \Sigma_{11}$, $\Sigma_{eq} = 0$) and corresponds to a void which remains spherical. Budiansky, Hutchinson and Slutsky mentioned that Rice and Tracey rejected two classes of velocity fields because of this paradox, although a better minimum would have been obtained for Q. This explains why the Rice and Tracey results in Fig. 8 are in agreement with the intuitive trend.

The above models consider velocity fields which are built around spherical voids only, and it is of interest to know if the results are still valid for spheroidal voids since Fig. 8 shows that a spherical cavity will generally not remain spherical. The stable limit shape a void will tend to has been determined by Budiansky and Hutchinson (1980) who used velocity fields around spheroidal cavities. Figure 9 shows that the limit shape is nearly a sphere for very ligh triaxiality only. The limit aspect ratio b/a (where a and b are the void radii at the equator and poles, respectively) is closer to unity when s is low. The above

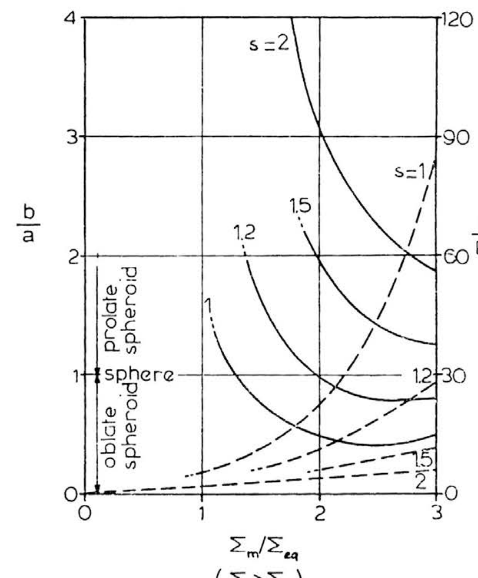

FIG. 9. Limit aspect ratio (——) and growth rate (– –) for an initially spherical void in a power law viscous matrix (adapted from Budiansky and Hutchinson).

mentioned paradox leads to oblate shapes for high triaxiality and s close to unity, and induces critical values of triaxiality for which the spherical shape is maintained (for $s = 1$ this value is about 1.33 in Budiansky and Hutchinson results, although it was about 1.51 in their previous results mentioned in Fig. 7). Figure 9 also shows that the void growth rate for the limit shape is very close to the one obtained for a sphere (Fig. 7). This is even exactly true for a linearly viscous matrix ($s = 2$), and the exact closed form solution shows that the void growth rate is only slightly lower for the intermediate shapes in this special case. As a consequence, Eq. (2.8) gives a good estimate of the limit growth rate, the correct value being more underestimated when s is closer to unity. Finally, Eqs. (2.6) and (2.8) give a reasonable result for the *size change* of an initially spherical void, with a better precision when both triaxiality and strain-rate sensitivity are high.

3. Growth of non-isolated cavities

3.1. Effect of porosity on growth law

In the preceding section the cavity was assumed to be isolated in an infinite matrix, and the evolution of the damage parameter, related to the growth law by the relations (1.3), depends only on the applied stresses and on the rheology of the matrix. No account

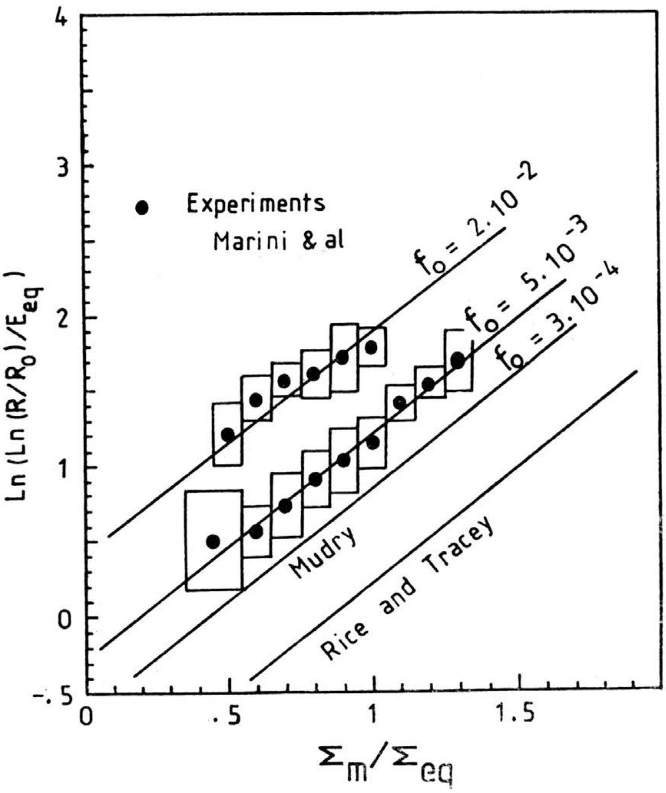

FIG. 10. Influence of initial void volume fraction (f_0) and of triaxiality on the growth rate of voids. Experimental results obtained by MARINI *et al.* (1985) and MUDRY (1985). [After MARINI *et al.* (1985)].

for the existent porosity is given. This view point has the merit of a relative simplicity and is likely valid at the very beginning of the growth stage, when interactions between cavities are negligible. However, a more thorough analysis must include the case of non-isolated cavities, and at least must account for the effect of porosity.

The first evidence of the necessity for such a deeper study is of experimental nature. MARINI, MUDRY and PINEAU (1985) have observed the cavity growth in a sintered-forged steel containing spherical Al_2O_3 particles, from which cavities are nucleated. They conclude that the growth law can be well approximated by an exponential dependence on the triaxiality ratio:

$$\frac{\dot{R}}{\dot{E}_{eq}R} = \alpha \exp\left(\beta \frac{\Sigma_m}{\Sigma_{eq}}\right). \tag{3.1}$$

The value $\beta = 3/2$ derived by Rice and Tracey is in good agreement with the experimental results of various authors, in the range of porosity $3 \cdot 10^{-4}$ to $2 \cdot 10^{-2}$ (Fig. 10). However, the pre-exponential factor α depends strongly on the initial porosity and largely exceeds the value $\alpha = 0.28$ derived by Rice and Tracey. An extrapolation in the low porosity range of Marini *et al.* results, leads to $\alpha = 0.5$ (cf. Fig. 11). The discrepancy between the theoretical

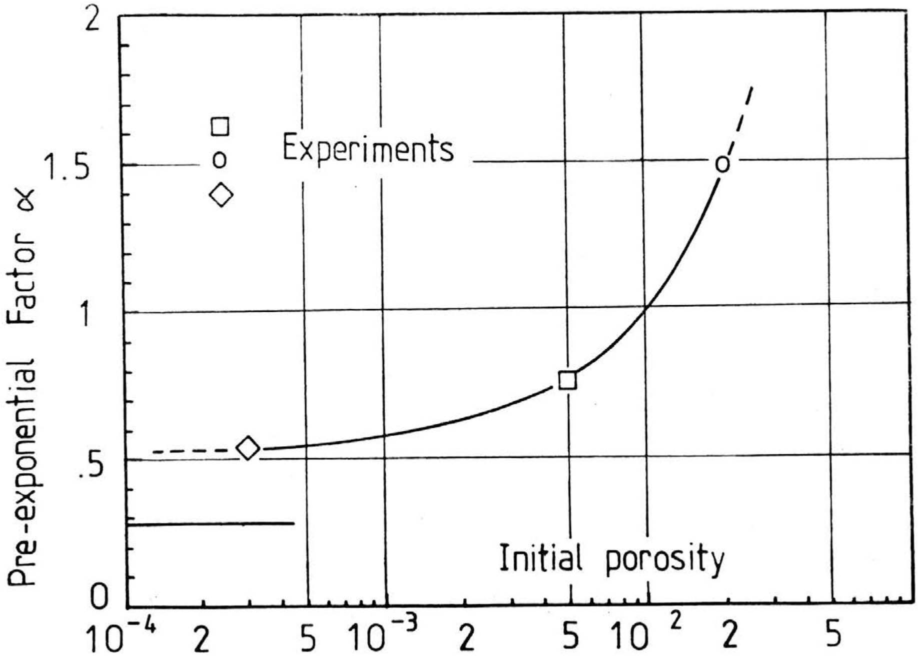

FIG. 11. Influence of initial void volume fraction on the pre-exponential factor α in Eq. (3.1). [After MARINI *et al.* (1985)].

model and the experimental results is "associated with the interactions between neighbouring cavities and (or) the formation of a second population of microcavities" (from Marini *et al.*).

A second striking argument, of theoretical nature, showing the role of porosity in the growth law, is given by BUDIANSKY *et al.* (1982): the growth rate of a spherical cavity in a spherical shell of finite outer radius, under hydrostatic loading ($\Sigma_{11} = \Sigma_{33} = \Sigma$) is given by

$$\frac{\dot{V}}{V} = \frac{3}{2}\left(\frac{s-1}{\mu}\right)^{\frac{1}{s-1}} \operatorname{sgn}(\Sigma)\left|\frac{3}{2}\Sigma\right|^{\frac{1}{s-1}} (1-f^{s-1})^{-\frac{1}{s-1}}.$$

For small f it can be developed into

$$\frac{\dot{V}}{V}(f) = \frac{\dot{V}}{V}(0)\left(1 + \frac{1}{s-1} f^{s-1} + \ldots\right). \tag{3.2}$$

For $s = 1.33$, a porosity of only $f = 10^{-3}$ increases the growth rate by 30% over the prediction for the infinite region ($f = 0$).

Two aspects in the discussion about non-isolated cavities can be discerned. The first aspect is associated with the effect of porosity as illustrated by the above theoretical argument by Budiansky *et al.* The relevant micromechanics theory is to be developed on a *finite representative volume element* submitted to classical boundary conditions (1.5). In the second stage it can be desirable to evaluate *interactions between cavities* by considering that several cavities are included in the micromechanics problem. In these conditions, it is not possible to assume that stress or strain fields become homogeneous on the boundary of a volume surrounding each cavity. A typical illustration of the difference between these two effects in the study of non-isolated cavities, is given by a pair of neighbouring cavities in an infinite matrix: the interaction is certainly great but porosity is zero.

The effect of interactions is often assimilated to the effect of porosity in the literature (see TRACEY (1971), STIGH (1986)). The next section is primarily devoted to the effect of porosity, and the difficult problem of interactions is briefly discussed in Sect. 4.

3.2. Effect of porosity

For *cylindrical cavities* growing in a viscous matrix, a closed form solution is available (LICHT and SUQUET (1986)): the cavity rate of growth is related to the ratio $\Sigma_{11}/(\Sigma_{33} - \Sigma_{11})$(5) by (Appendix 2)

$$\frac{\Sigma_{11}}{\Sigma_{33}-\Sigma_{11}} = \frac{1}{\sqrt{3}\,\omega f} \frac{\int_{\omega f}^{\omega} (1+x^2)^{\frac{s-2}{2}} dx}{\int_{\omega f}^{\omega} (1+x^2)^{\frac{s-2}{2}} \frac{dx}{x^2}}, \tag{3.3}$$

where

$$\omega = \frac{1}{\sqrt{3}\dot{E}_{33}} \frac{\dot{V}}{V}.$$

The ratio $\dot{V}/V\dot{\mathscr{E}}$, as a function of porosity, is plotted in Figs. 11 and 12. Several choices of the strain rate $\dot{\mathscr{E}}$ can be used in the dimensionless growth rate $\dot{V}/V\dot{\mathscr{E}}$. The most relevant ones are

(5) related to the triaxiality ratio by $\Sigma_m/\Sigma_{eq} = \chi(\Sigma_{11}/(\Sigma_{33}-\Sigma_{11})+1/3)$ where $\chi = 1$ if $\Sigma_{33} > \Sigma_{11}$, $\chi = -1$ if $\Sigma_{33} < \Sigma_{11}$.

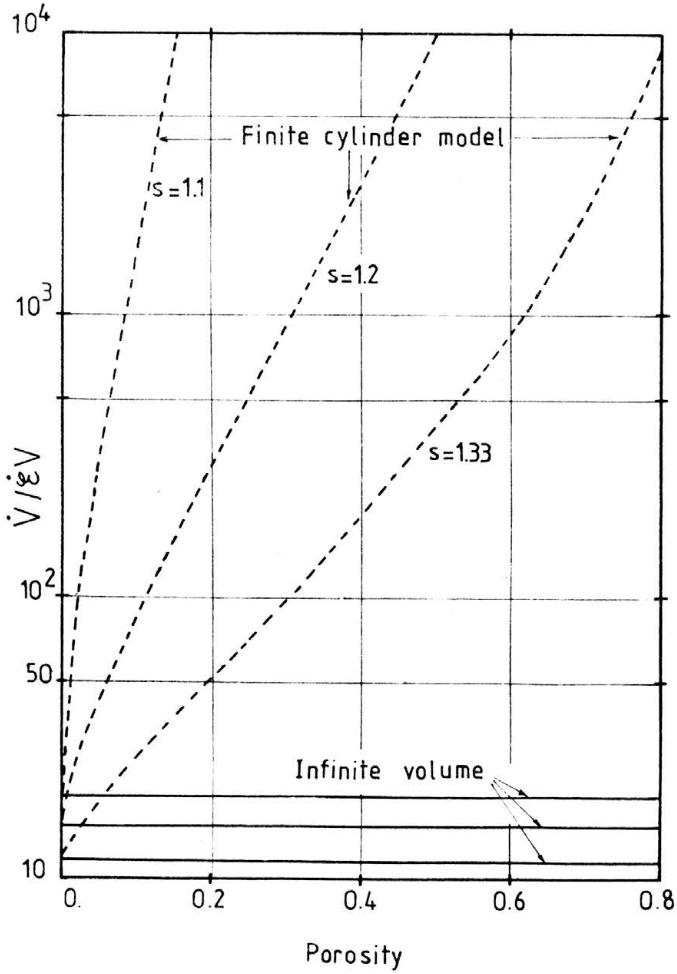

FIG. 12. Circular cylindrical void under axisymmetric loading: influence of porosity and of strain rate sensitivity on the growth rate. $\Sigma_{11}/(\Sigma_{33}-\Sigma_{11}) = 2$.
—— infinite volume approximation,
- - - - finite cylindrical shell,
choice (3.4) (a) of the strain rate $\dot{\mathscr{E}}$.

$$\text{(3.4)}\quad \begin{aligned} &\text{a)}\quad \dot{\mathscr{E}} = \mu^{1-s'}|\Sigma_{33}-\Sigma_{11}|^{s'-2}(\Sigma_{33}-\Sigma_{11}), \quad \text{where} \quad s' = s(s-1),\\ &\text{b)}\quad \dot{\mathscr{E}} = \dot{E}_{33}. \end{aligned}$$

Choice (a) corresponds to the axial strain which would take place in the original undamaged matrix submitted to the same loading, while choice (b) is the actual axial strain. Both quantities coincide in the infinitely dilute case, but differ in a finite volume. Porosity affects the second choice which is relevant in simple tension, but not the first one which reflects the true variations of $\dot{V}/V$ with respect to f. Figure 12 shows a significant increase in the growth rate due to porosity. The deviation between the dilute case and the finite case increases when the ratio $\Sigma_{11}/(\Sigma_{33}-\Sigma_{11})$ increases, and when s approaches 1.

The results obtained with the choice (b) $\dot{\mathscr{E}} = \dot{E}_{33}$ are plotted in Fig. 13. The decrease in $\dot{V}/V\dot{E}_{33}$ which is to be noted, should not be attributed to a decrease in $\dot{V}/V$, which is an increasing quantity as can be deduced from Fig. 12, but to an increase in $\dot{E}_{33}$. In a Newtonian matrix ($s = 2$) the dimensionless growth rate does not depend on porosity since Eq. (3.3) reduces to

$$\frac{\dot{V}}{\dot{E}_{33}V} = \frac{3\Sigma_{11}}{\Sigma_{33}-\Sigma_{11}}. \tag{3.5}$$

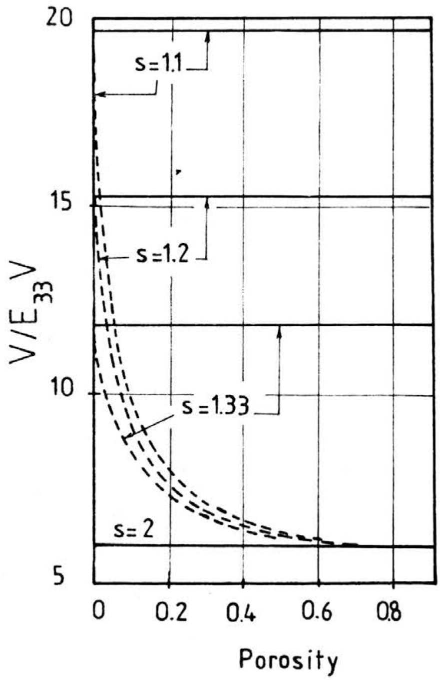

FIG. 13. Circular cylindrical void under axisymmetric loading: influence of porosity and of strain rate sensitivity on the growth rate. $\Sigma_{11}/(\Sigma_{33}-\Sigma_{11}) = 2$.
—— infinite volume approximation,
- - - - finite cylindrical shell,
choice (3.4) (b) of the strain rate $\dot{\mathscr{E}}$.

TRACEY (1971) has derived upper and lower bounds for the growth rate of cylindrical cavities in a finite volume of a strain-hardening matrix. Figure 6, taken from Tracey's work, shows a significant decrease in the fracture strain due to porosity during a tensile test, and illustrates the role of the existent damage on its own evolution.

DUVA and HUTCHINSON (1984) have applied the above variational principle (2.5) to a *spherical cavity* growing in a finite spherical shell of a viscous material and have derived through numerical calculations the growth rate of the cavity. Figure 14 illustrates their results (choice (a) of the strain rate $\dot{\mathscr{E}}$ was made). The general conclusions drawn from the cylindrical model are still valid, and the effect of porosity increases with the triaxiality ratio, and when s approaches 1. In a slightly different direction, STIGH (1986) has simulated

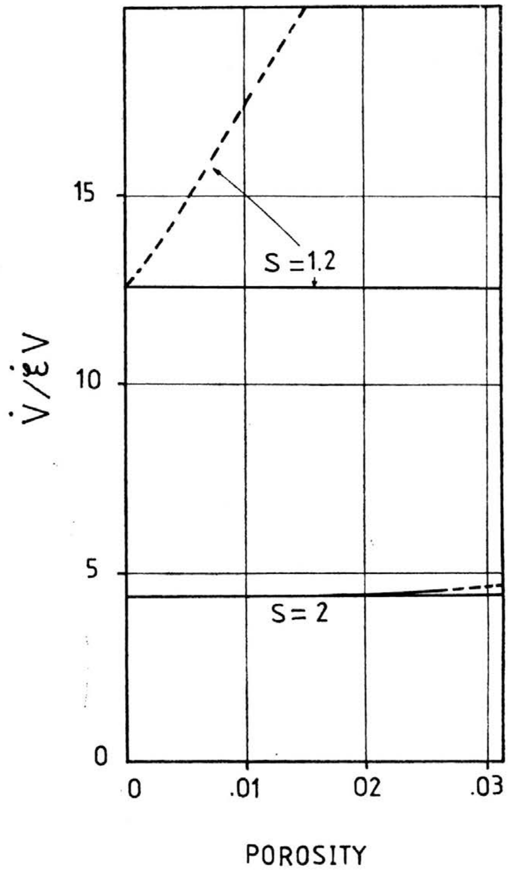

FIG. 14. Spherical void under axisymmetric loading: influence of porosity and of strain rate sensitivity on the growth rate. $\Sigma_m/\Sigma_{eq} = 2$.
—— infinite volume approximation,
- - - - finite spherical shell,
choice (3.4) (a) of strain rate $\dot{\mathscr{E}}$.
[Adapted from DUVA and HUTCHINSON (1984)].

by the finite element method the growth of a spherical cavity in a cylindrical volume of viscoplastic material, submitted to a uniaxial tension ($\Sigma_{33} \neq 0$, $\Sigma_{11} = 0$). The relevant strain rate $\dot{\mathscr{E}}$ for this problem is $\dot{E}_{33}$, and the dimensionless growth rates computed by this author are reported in Fig. 15 (triaxiality ratio = 1/3). At low porosity, and for a rigid plastic material ($s = 1$), Rice and Tracey's result $\dot{V}/V\dot{E}_{33} = 0.85 \exp(1/2)$ is found, but a small oscillation appears for viscous materials ($s \neq 1$). As above, the decrease in the growth rate in the range of porosity 0.01, 0.4 should be attributed to an increase in $\dot{E}_{33}$, rather than to a decrease in $\dot{V}/V$. For high porosities the sudden increase in the growth rate, which is not observed for cylindrical cavities (cf. Fig. 13), will result in a sudden increase of damage at the macroscopic level, leading to final fracture.

It is impossible to report here the abundant literature devoted to the numerical study of cavity growth (see NEEDLEMAN (1972), ANDERSSON (1977), NEMAT-NASSER and TAYA

(1980), TVERGAARD (1981)). A common feature of these works is to underline the effect of porosity which, however, has not been incorporated in a closed form expression of the growth rate such as Eqs. (2.6) or (2.8). Approximations on microscopic tensorial fields are unavoidable, and of a difficult use (see the discussion by DUVA and HUTCHINSON (1984)

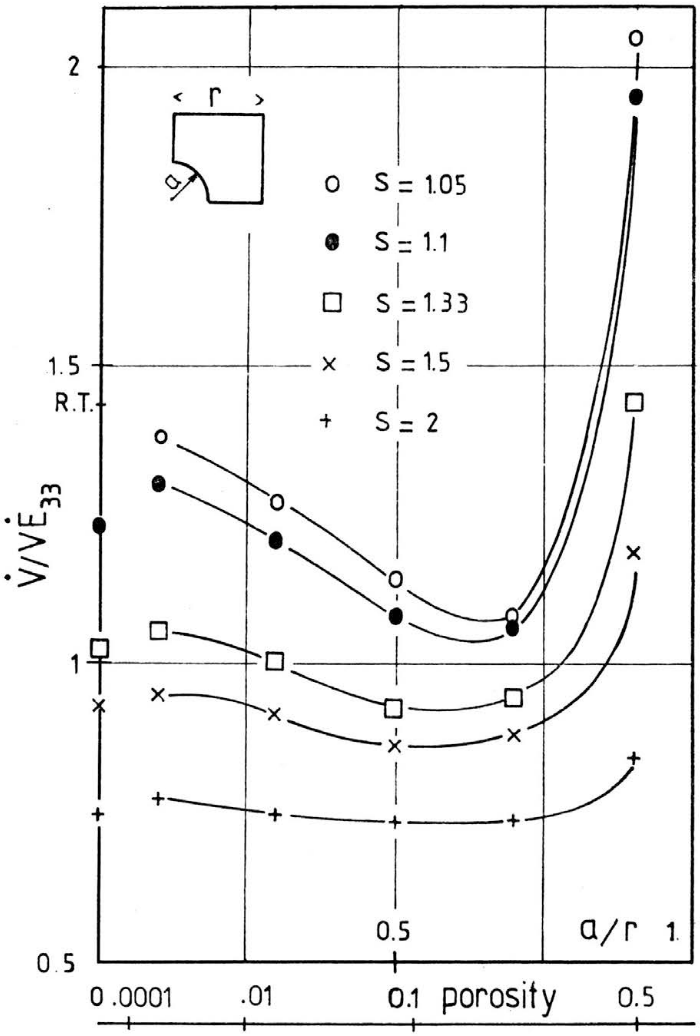

FIG. 15. Growth of a spherical void in a cylindrical shell during a tensile test ($\Sigma_m/\Sigma_{eq} = 1/3$). Influence of porosity and of strain rate sensitivity on the growth rate.
[After STIGH (1986)].

of MCMEEKING'S work (1984)). For this reason the next section is devoted to the macroscopic mechanical behavior of a porous material which yields, through the mass balance equation (1.2), the growth law of the cavities.

4. Macroscopic behavior of a porous ductile material

As already mentioned in the introduction, the porosity change can be derived from the *macroscopic behavior* of the porous material, by means of the mass balance equation

$$\dot{f} = (1-f)\dot{E}_{ii}.$$

The derivation of the macroscopic behavior is a more general problem than the cavity growth problem, and is treated in this section from the micromechanical point of view.

4.1. Macroscopic potentials

The macroscopic constitutive law is derived from the micromechanics problem (1.8) which includes the rheological behavior of the matrix, the equilibrium equations and the boundary conditions related to the macroscopic loading. For an infinite r.v.e. the macroscopic rheology of the material is the matrix one, and the only case of interest here is a *finite r.v.e.* For a given macroscopic stress $\mathbf{\Sigma}$ the micromechanics problem is solved for the microscopic tensorial fields $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$, and the macroscopic strain rate $\dot{\mathbf{E}}$ is derived from the average relations (1.1): the macroscopic constitutive law relates $\dot{E}$ and Σ. Alternatively, a macroscopic strain rate $\dot{\mathbf{E}}$ can be imposed through the uniform boundary conditions (1.5), and the macroscopic stress $\mathbf{\Sigma}$ obtained as the average of the microscopic stress $\boldsymbol{\sigma}$, is related to $\dot{\mathbf{E}}$ through the macroscopic constitutive law. Although these two procedures for deriving the mechanical behavior of a porous material are not equivalent, it is often implicitly claimed that the results are sufficiently closed to be considered as identical.

The above described procedure for the derivation of macroscopic equations involves computations and approximations of tensorial fields, and is therefore of difficult use as already mentioned. DUVA and HUTCHINSON (1984) noticed that the derivation of the *scalar macroscopic potentials*, which define the macroscopic rheology as well, involves only scalar approximations. The stress-strain relations at the microscopic scale may be derived from two companion potentials $\varphi(\dot{\boldsymbol{\epsilon}})$, $\psi(\mathbf{s})$

$$s_{ij} = \frac{\partial \varphi}{\partial \dot{\varepsilon}_{ij}}(\dot{\boldsymbol{\epsilon}}) \quad \text{or} \quad \dot{\varepsilon}_{ij} = \frac{\partial \psi}{\partial s_{ij}}(\mathbf{s}). \tag{4.1}$$

The macroscopic stress-strain relation can also be derived from two macroscopic potentials Φ and Ψ

$$\Sigma_{ij} = \frac{\partial \Phi}{\partial \dot{E}_{ij}}(\dot{\mathbf{E}}) \quad \text{or} \quad \dot{E}_{ij} = \frac{\partial \Psi}{\partial \Sigma_{ij}}(\mathbf{\Sigma}), \tag{4.2}$$

and we claim that the macroscopic potentials are the averages of the microscopic ones, calculated from the local values $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\epsilon}}$ of the stress and strain rate

$$\Phi(\dot{\mathbf{E}}) = \langle \varphi(\dot{\boldsymbol{\epsilon}}) \rangle \quad \text{and} \quad \Psi(\mathbf{\Sigma}) = \langle \psi(s) \rangle. \tag{4.3}$$

In order to prove that the relations (4.2) are a consequence of the relations (4.3) and of the average relations (1.1), we note that, on the one hand,

$$\frac{\partial}{\partial \dot{E}_{ij}} \langle \varphi(\dot{\boldsymbol{\epsilon}}) \rangle = \left\langle \frac{\partial}{\partial \dot{\varepsilon}_{kh}} \varphi(\dot{\boldsymbol{\epsilon}}) \frac{\partial \dot{\varepsilon}_{kh}}{\partial \dot{E}_{ij}} \right\rangle = \left\langle s_{kh} \frac{\partial \dot{\varepsilon}_{kh}}{\partial \dot{E}_{ij}} \right\rangle.$$

On the other hand, Hill's relation (1.6) yields

$$\tilde{\Sigma}_{kh}\dot{E}_{kh} = \langle \tilde{\sigma}_{kh}\dot{\varepsilon}_{kh}\rangle = \langle \tilde{s}_{kh}\dot{\varepsilon}_{kh}\rangle$$

for every stress field $\tilde{\boldsymbol{\sigma}}$ in equilibrium. Derivation with respect to $\dot{E}_{ij}$ yields

$$\tilde{\Sigma}_{ij} = \left\langle \tilde{s}_{kh}\frac{\partial \dot{\varepsilon}_{kh}}{\partial \dot{E}_{ij}}\right\rangle.$$

The special choice $\tilde{\sigma} = \sigma$ in the above relation gives

$$\Sigma_{ij} = \left\langle s_{kh}\frac{\partial \dot{\varepsilon}_{kh}}{\partial \dot{E}_{ij}}\right\rangle = \frac{\partial}{\partial \dot{E}_{ij}}\langle \varphi(\dot{\boldsymbol{\epsilon}})\rangle$$

and justifies the expression (4.3) of the macroscopic potentials.

The variational properties of $\boldsymbol{\sigma}$ and $\dot{\mathbf{u}}$ can be used to give a variational characterization of Φ and Ψ. For instance,

(4.4) $$\Phi(\dot{E}) = \operatorname*{Inf}_{\dot{\tilde{\mathbf{u}}}}\langle \varphi(\boldsymbol{\epsilon}(\dot{\tilde{u}}))\rangle,$$

(4.5) $$\Psi(\boldsymbol{\Sigma}) = \operatorname*{Inf}_{\tilde{\boldsymbol{\sigma}}}\langle \psi(\tilde{\mathbf{s}})\rangle,$$

where $\dot{\tilde{\mathbf{u}}}$ and $\tilde{\boldsymbol{\sigma}}$ satisfy [6]

(4.6) $$\dot{\tilde{u}}_i = \dot{E}_{ij}x_j \quad \text{on } \partial Y,$$

(4.7) $$\tilde{\sigma}_{ij}n_j = 0 \quad \text{on } \partial V, \quad \operatorname{div}\tilde{\boldsymbol{\sigma}} = 0, \quad \langle \tilde{\boldsymbol{\sigma}}\rangle = \boldsymbol{\Sigma}.$$

The two following points are worth noting:

i) approximate expressions, of the stress or velocity fields, depending on a finite number of scalar parameters, can be inserted in the variational expressions (4.4) or (4.5), following Rice and Tracey, or Budiansky, Hutchinson and Slutsky. By the variational property, such an approximation always leads to an *overestimate* of the potentials

(4.8) $$\Phi(\dot{\mathbf{E}}) \leqslant \Phi^a(\dot{\mathbf{E}}), \quad \Psi(\boldsymbol{\Sigma}) \leqslant \Psi^a(\boldsymbol{\Sigma}),$$

where the superscript a denotes an approximation of the potential obtained by an approximate minimization;

ii) the potentials φ and ψ are homogeneous functions of degree s and $s' = s/(s-1)$, respectively. The same property holds for the macroscopic potentials of a voided viscous material

(4.9) $$\Phi(\lambda\dot{\mathbf{E}}) = |\lambda|^s\Phi(\dot{\mathbf{E}}), \quad \Psi(\eta\boldsymbol{\Sigma}) = |\eta|^{s'}\Psi(\boldsymbol{\Sigma}).$$

4.2. First order expansion of the potentials: Duva and Hutchinson's model for dilutely voided materials

For low values of porosity the expansion of the macroscopic potentials Φ and Ψ to the first order in f, is a meaningful approximation:

$$\Phi(\dot{\mathbf{E}}, f) = \Phi(\dot{\mathbf{E}}, O) + f\frac{\partial \Phi}{\partial f}(\dot{\mathbf{E}}, f) + O(f^2).$$

---

[6] For the sake of clarity, a specific choice of boundary conditions, namely uniform strains has been made. The theory also holds true for the other set of boundary conditions, uniform stresses, although the expressions of the potentials derived by the two approaches are different (SUQUET (1985)).

The term of order zero in this expansion is the potential $\varphi(\dot{E})$ of the original matrix. The value of the derivative $\partial\Phi/\partial f|_{f=0}$ is closely related to the minimum value of the functional Q in the variational property (2.5). Let a *finite* r.v.e. Y_ϱ be considered with the outer radius ϱ (ϱ is intended to tend to ∞), containing the fixed void V. The variational property (4.4) of Φ

$$\Phi(\dot{\mathbf{E}},f) = \operatorname*{Inf}_{\dot{\mathbf{u}}^*=0 \text{ on } \partial Y} \frac{1}{Y_\varrho} \int\limits_{Y_\varrho^*} \varphi\big(\dot{\mathbf{E}}+\boldsymbol{\epsilon}(\dot{\mathbf{u}}^*)\big)\,dx$$

leads to

$$\frac{\Phi(\dot{\mathbf{E}},f)-\varphi(\dot{\mathbf{E}})}{f} = \operatorname*{Inf}_{\dot{\mathbf{u}}^* \text{ on } \partial Y} \left[\frac{1}{V}\int\limits_{Y_\varrho^*} \big(\varphi(\dot{\mathbf{E}}+\boldsymbol{\epsilon}(\dot{\mathbf{u}}^*))-\varphi(\dot{\mathbf{E}})\big)\,dx\right]-\varphi(\dot{\mathbf{E}}).$$

The constant tensor $\dfrac{\partial\varphi}{\partial\dot{\mathbf{E}}}(\dot{\mathbf{E}})$ is divergence free and $\dot{\mathbf{u}}^*$ vanishes on the outer boundary of Y, thus

$$\int\limits_{Y_\varrho^*} \frac{\partial\varphi}{\partial\dot{E}_{ij}}(\dot{\mathbf{E}})\varepsilon_{ij}(\dot{\mathbf{u}}^*)\,dx = \int\limits_{\partial V} \frac{\partial\varphi}{\partial\dot{E}_{ij}}(\dot{\mathbf{E}})n_j\dot{u}_i^*\,ds.$$

Finally, the following result is obtained:

$$\frac{\Phi(\dot{\mathbf{E}},f)-\varphi(\dot{\mathbf{E}})}{f} = \operatorname*{Inf}_{\dot{\mathbf{u}}^*=0 \text{ on } \partial Y} \left[\frac{1}{V} Q_\varrho(\dot{\mathbf{u}}^*)\right]-\varphi(\dot{\mathbf{E}}),$$

where

$$Q_\varrho(\dot{\mathbf{u}}^*) = \int\limits_{Y_\varrho^*} \left(\varphi(\dot{\mathbf{E}}+\boldsymbol{\epsilon}(\dot{\mathbf{u}}^*))-\varphi(\dot{\mathbf{E}})-\frac{\partial\varphi}{\partial\dot{E}_{ij}}(\dot{\mathbf{E}})\varepsilon_{ij}({}^*\dot{\mathbf{u}}^*)\right)dx+\int\limits_{\partial V}\frac{\partial\varphi}{\partial\dot{E}_{ij}}(\dot{\mathbf{E}})n_j\dot{u}_i^*\,ds.$$

The limit of the above relation, as ϱ tends to ∞, is

$$\frac{\partial\Phi}{\partial f}(\dot{\mathbf{E}},0) = \operatorname*{Inf}_{\substack{\lim \dot{\mathbf{u}}^*=0\\ |x|\to\infty}} \left(\frac{1}{V}Q(\dot{\mathbf{u}}^*)\right)-\varphi(\dot{\mathbf{E}}), \tag{4.10}$$

where Q has the form (2.5). The above relation (4.10) is of a double interest:

i) it establishes the validity of the variational property (2.5), which was used above without proof;

ii) it allows a direct computation of the first order term in the expansion of Φ by means of a minimization already performed by Budiansky *et. al.*, in an infinite r.v.e. The first order term of the expansion of Ψ can also be derived from Eq. (4.10) since

$$\frac{\partial\Psi}{\partial f}(\boldsymbol{\Sigma},0) = \frac{\partial\Phi}{\partial f}(\dot{\mathbf{E}},0) \tag{4.11}$$

where

$$\dot{\mathbf{E}} = \frac{\partial\psi}{\partial\Sigma}(\boldsymbol{\Sigma})$$

Equation (4.11) is a direct consequence of Young's equality for the convex conjugate functions Ψ and Φ

$$\Psi(\mathbf{\Sigma},f)+\Phi(\dot{\mathbf{E}}(f),f) = \Sigma_{ij}\dot{E}_{ij}(f),$$

where

$$\dot{E}(f) = \frac{\partial\psi}{\partial\mathbf{\Sigma}}(\mathbf{\Sigma},f).$$

Equations (4.10) and (4.11) lead to

$$\frac{\partial\Psi}{\partial f}(\mathbf{\Sigma},0) = \varphi(\dot{\mathbf{E}})-\frac{1}{V}\operatorname*{Inf}_{\substack{\lim \dot{\tilde{\mathbf{u}}}^*=0\\ |x|\to\infty}} Q(\dot{\mathbf{u}}^*). \tag{4.12}$$

After these preliminaries of general validity, Duva and Hutchinson's approach can be detailed, which concerns spherical cavities in a viscous matrix obeying the relations (1.4) and corresponding to the following potential ψ:

$$\psi(\boldsymbol{\sigma}) = \frac{\mu^{1-s'}}{s'}\sigma_{\mathrm{eq}}^{s'}, \qquad s' = s(s-1).$$

The porous material is isotropic at the macroscopic scale, and its force potential Ψ depends only on the three invariants of $\mathbf{\Sigma}$:

$$\Psi(\Sigma_m, \Sigma_{\mathrm{eq}}, \Sigma_{\mathrm{III}}, f) = \mu^{1-s'}\Sigma_{\mathrm{eq}}^{s'}h(X, Y, f).$$

The homogeneity of Ψ has been used together with the following notations:

$$X = \Sigma_m/\Sigma_{\mathrm{eq}}, \qquad Y = \Sigma_{\mathrm{III}}/\Sigma_{\mathrm{eq}} \qquad \Sigma_{\mathrm{III}} = (S_{ij}S_{jk}S_{ki})^{1/3}.$$

The dependence of Ψ on the third invariant Σ_{III} is shown to be negligible, at least in the small porosity range, by the numerical study of axisymmetric loadings. Therefore the first order term in the expansion of Ψ reads as

$$\frac{\partial\Psi}{\partial f}(\mathbf{\Sigma},0) = \mu^{1-s'}\Sigma_{\mathrm{eq}}^{s'}F(X).$$

F is determined separately for high and low values of triaxiality.

For high triaxiality ratios the void growth rate deduced from Eq. (1.3) should reduce to the expression (2.8) derived by Budiansky *et al.* The following equation, valid to the second order in f, relates F and the void growth rate:

$$\frac{\dot{V}}{V} = \frac{\operatorname{tr}\dot{\mathbf{E}}}{f} = \frac{1}{f}\frac{\partial\Psi}{\partial\Sigma_m}(\mathbf{\Sigma},f) \simeq \frac{\partial}{\partial\Sigma_m}\left(\frac{\partial\Psi}{\partial f}(\mathbf{\Sigma},0)\right) = \dot{E}_{\mathrm{eq}}\frac{\partial F}{\partial X}(X).$$

Comparison with the expression (2.8) yields the following expression of F in the high triaxiality range:

$$F_H(X) = \frac{1}{s}\left[\frac{3}{2}(s-1)|X|+G(s,1)\right]^{\frac{s}{s-1}},$$

$$G(s,m) = (2-s)(1+(s-1)(0.4175+0.0144\chi m)), \qquad m = \operatorname{sgn}X.$$

For low triaxiality ratios F is determined by means of the expression (4.12) of $\partial\Psi/\partial f$ and depends on the computation of the infimum of the functional Q. An accurate approximation is given by

$$F(X) = F^* + \frac{1}{2} KX^2,$$

where F^* and K, depending on s, have been tabulated by Duva and Hutchinson. For arbitrary triaxiality F is interpolated between its values F_L and F_H in the low and high triaxiality ranges:

$$(4.13)\qquad \begin{gathered} F(X) = c_1 + c_2|X| + c_3X^2 + \frac{1}{s}\left[\frac{3}{2}(s-1)|X| + G(s,1)\right]^{\frac{s}{s-1}}, \\ c_1 = F^* - \frac{1}{s} G^{\frac{s}{s-1}}, \quad c_2 = -\frac{3}{2}^{\frac{s}{s-1}}, \quad c_3 = \frac{1}{2}K - \frac{9}{8}^{\frac{2-s}{s-1}}. \end{gathered}$$

Therefore the first order expansion of the force potential Ψ is

$$\Psi(\mathbf{\Sigma}, f) \simeq \frac{\mu^{1-s'}}{s'} \Sigma_{\text{eq}}^{s'}\big(1 + s' f F(X)\big).$$

The macroscopic stress-strain rate relation derived from Ψ is

$$(4.14)\qquad \begin{aligned} \dot{E}_{ij} &= \frac{3}{2}\mu^{1-s'}\Sigma_{\text{eq}}^{s'-2} S_{ij} + f\dot{E}_{ij}^*, \\ \dot{E}_{ij}^* &= \mu^{1-s'}\Sigma_{\text{eq}}^{s'-1}\left[\frac{3}{2}\frac{\delta_{ij}}{\Sigma_{\text{eq}}}\left(s'F(X) - X\frac{\partial F}{\partial X}(X)\right) + \frac{1}{3}\frac{\partial F}{\partial X}(X)\delta_{ij}\right]. \end{aligned}$$

Porosity is present at the first order in the constitutive law (4.14) via the additional strain $\dot{E}_{ij}^*$. We emphasize, as Duva and Hutchinson did, that the complete constitutive law (with both deviatoric and dilatational strain rates) has been derived only through a scalar approximation on Ψ.

4.3. Rigid plastic matrix: GURSON'S criterion (1977)

GURSON (1977) has applied the variational property (4.4) of the dissipation potential to a porous rigid plastic matrix ($s = 1$), and used local deformation modes depending on a finite number of parameters to derive an approximate value of Φ. The macroscopic yield locus of the porous material can be deduced from the expression of Φ, since it defines the macroscopic stress Σ which can be derived from Φ, for any value of $\dot{\mathbf{E}}$

$$(4.15)\qquad \mathbf{\Sigma} = \frac{\partial \Phi}{\partial \dot{\mathbf{E}}}(\dot{\mathbf{E}}, f).$$

At this point we can notice, as Gurson did after BERG (1970) and BISHOP and HILL (1951), that the normality rule is valid at the macroscopic scale, provided that it is valid at the microscopic scale. This is a straightforward consequence of the constitutive laws (4.2).

Gurson considered two different deformation modes of the r.v.e.. In the first type of deformation, suggested by the solution for a Newtonian material ($s = 2$), the r.v.e.

is *fully plastified.* The problem is reduced to a scalar minimization which is performed numerically. From this first choice of the deformation mode, Gurson derived the following yield conditions:

i) Cylindrical voids

$$C\frac{\Sigma_{eq}^2}{\sigma_0^2}+2f\cosh\left(\frac{\sqrt{3}}{2}\frac{\Sigma_{\alpha\alpha}}{\sigma_0}\right)-1-f^2=0, \tag{4.16}$$

where $C = 1$ for axisymmetric loadings (exact result, see Appendix 2), $C = 1+3f+24f^6$ for a plane strain problem (C was determined numerically). Surprisingly, it seems that Gurson considers that under the plane strain hypothesis the classical identities $\Sigma_{33} = \frac{\Sigma_{11}+\Sigma_{22}}{2}$ and $\Sigma_{eq} = |\Sigma_{22}-\Sigma_{11}|$ still hold, which is incorrect for a porous material (pressure dependent criterion).

ii) Spherical cavities under axisymmetric loading

$$\frac{\Sigma_{eq}^2}{\sigma_0^2}+2f\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\sigma_0}\right)-1-f^2=0 \tag{4.17}$$

(the coefficient 1 of Σ_{eq}^2/σ_0^2 was obtained numerically).

In both cases, the yield condition reduces to the one of the original matrix when $f = 0$.

The assumption of a fully plastified r.v.e. is likely to be valid in the low porosity range and for high triaxiality ratios. For high triaxiality, the effect of the hydrostatic part of the macroscopic stress is predominant and enhances the deformation modes with spherical symmetry and deformation of the whole r.v.e.. On the opposite, in the high porosity range, localization of deformation between cavities seems to be the rule, and the induced deformation modes affect only a part of the r.v.e..

Probably motivated by this last argument Gurson considered a second type of deformation modes in which entire blocks of the r.v.e. remain rigid. In the deformed zones, which are sections of cones, the velocity fields are expanded into trigonometric series. The minimization problem involved in the definition of Φ is reduced to a scalar minimization. From the numerical study of this last problem, Gurson proposed the following form of the yield criterion for a porous material with spherical voids:

$$\frac{\Sigma_{eq}^2}{\sigma_0^2}-\left(B_0(f)+B_1(f)\frac{\Sigma_{ii}}{\sigma_0}+B_2(f)\left(\frac{\Sigma_{ii}}{\sigma_0}\right)^2\right)=0 \tag{4.18}$$

(for a cylindrical void Σ_{ii} is to be changed into $\Sigma_{\alpha\alpha}$). The porosity-depending coefficients B_0, B_1, B_2 have been tabulated by Gurson.

The yield locus derived from the rigid blocks velocity fields are clearly located inside the previous yield surface derived on the basis of a full plastification of the r.v.e., except in the high triaxiality range ($\Sigma_m/\Sigma_{eq} \simeq 5$, see Fig. 16). Surprisingly, the second form (4.18) of the yield condition proposed by Gurson has not retained the attention of subsequent authors, though its superiority is clear in Fig. 16. We emphasize that both approaches lead to an *overestimate of the actual macroscopic yield locus,* and it is not surprising that more refined numerical studies of the problem show lower yield stresses.

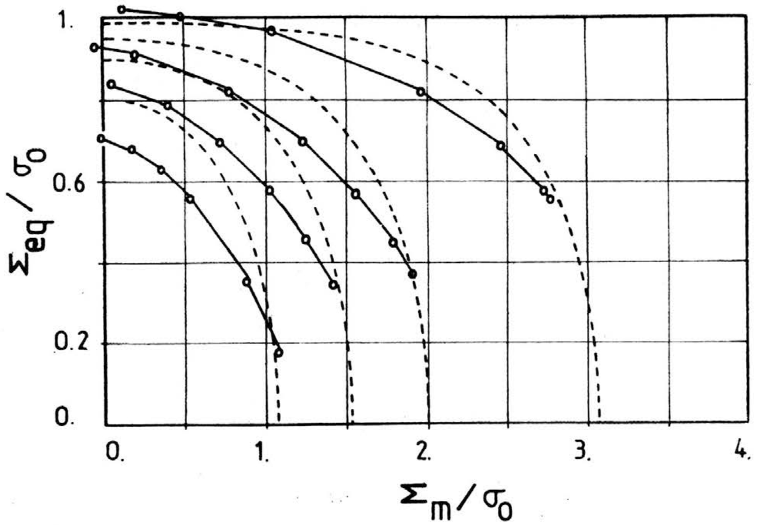

FIG. 16. Spherical void under axisymmetric loading; Gurson's model
---- fully plastified r.v.e. criterion (4.17),
— ∘ — deformation of the r.v.e. with rigid wedges, linear interpolation of the computed results.
[From GURSON (1977)].

A possible reason why the yield condition (4.17) is so popular may be the good agreement between the void growth rate it leads to by using the normality law, and the previous expressions derived by McClintock, and Rice and Tracey. Applying the normality law to Eq. (4.16) and Eq. (4.17) yields

i) for cylindrical cavities

$$\frac{\dot{V}}{\dot{\mathscr{E}}V} = \frac{\mathrm{Tr}\dot{\mathbf{E}}}{\dot{\mathscr{E}}f} = \sqrt{3}\,\frac{\sigma_0}{\Sigma_{eq}}\sinh\left(\frac{\sqrt{3}}{2}\,\frac{\Sigma_{\alpha\alpha}}{\sigma_0}\right), \tag{4.19}$$

where

$$\dot{\mathscr{E}} = \dot{\tilde{E}}_{eq}, \quad \tilde{E}_{\alpha\beta} = \dot{E}_{\alpha\beta} - \frac{\mathrm{Tr}\,\dot{E}}{2}\,\delta_{\alpha\beta} \quad \text{if} \quad 1 \leqslant \alpha, \quad \beta \leqslant 2, \quad \dot{\tilde{E}}_{ij} = \dot{E}_{ij}$$

otherwise,

ii) for spherical cavities

$$\frac{\dot{V}}{\dot{\mathscr{E}}V} = \frac{3}{2}\,\frac{\sigma_0}{\Sigma_{eq}}\sinh\left(\frac{3}{2}\,\frac{\Sigma_m}{\sigma_0}\right), \tag{4.20}$$

where $\dot{\mathscr{E}} = \dot{\tilde{E}}_{eq}$ and $\dot{\tilde{\mathbf{E}}}$ is the deviatoric part of $\dot{\mathbf{E}}$.

For very low porosity ($f \simeq 0$) $\dot{\mathscr{E}}$ and Σ_{eq} reduce to $\dot{E}_{eq}$ and σ_0, respectively. Equation (4.19) is exactly McClintock's result, and Eq. (4.20) which reduces to 0.75 exp $\left(\frac{3}{2}\,\frac{\Sigma_m}{\sigma_0}\right)$ in the high triaxiality range, differs from Rice and Tracey's result by 10% in the pre-exponential factor (0.75 instead of 0.85 for Rice and Tracey).

For arbitrary porosity f, the pre-exponential factor $\alpha = \sigma_0/4\Sigma_{eq}$ depends on f and on the triaxiality ratio Σ_m/Σ_{eq} through $Z = \sigma_0/\Sigma_{eq}$ which can be derived from the yield condition. In the case of spherical voids, Z is given by

$$\left(\frac{1}{Z}\right)^2 + 2f\cosh\left(\frac{3}{2} Z \frac{\Sigma_m}{\Sigma_{eq}}\right) - 1 - f^2 = 0.$$

It can easily be shown that $Z > 1/(1-f)$. This porosity dependence of the pre-exponential factor is in good qualitative agreement with Marini *et al.* conclusions, but is still insufficient to reflect quantitatively the results of these authors in the range of porosity under consideration, and more specific analyses remain to be developed.

The *strain hardening of the matrix* can be inserted into Gurson's model via a power law dependence of the yield stress $\sigma_0 = \sigma_{eq}$ on equivalent strain:

$$\sigma_{eq} = \mu \varepsilon_{eq}^n. \tag{4.21}$$

The effect of strain hardening is a lessening of the void growth rate.

Several improvements of Gurson's criterion have been proposed. From the general form of this yield condition, TVERGAARD (1981) proposed to introduce additional parameters:

$$\frac{\Sigma_{eq}^2}{\sigma_0^2} + 2fq_1 \cosh\left(\frac{3q_2}{2} \frac{m}{\sigma_0}\right) - (1 + q_3 f^2) = 0 \tag{4.22}$$

(Gurson's model corresponds to $q_1 = q_2 = q_3 = 1$). The parameters q_i are determined from comparison between two numerical studies of instability in plane strain tension: a first computation is performed on the original matrix weakened by a periodic array of holes, and the second computation is performed on the homogenized porous material obeying Eq. (4.22). The values of q_i leading to a satisfactory agreement between both calculations depend on the hardening exponent (n in Eq. (4.21)), but for the most common values of this exponent the following choice of q_i seems to be adequate:

$$q_1 = 1.5, \quad q_2 = 1, \quad q_3 = q_1^2.$$

The change of q_1 from 1 to 1.5 into Eq. (4.22) results in an increase by 50% of the void growth rate, but this is still insufficient to reflect the observed values of the pre-exponential factor.

Another improvement, due to MEAR and HUTCHINSON (1985), concerns the hardening of the matrix which is usually assumed to be isotropic. Mear and Hutchinson proposed the following criterion, where isotropic and kinematic hardening are coupled:

$$\frac{(\mathbf{\Sigma} - \mathbf{A})_{eq}^2}{\sigma_F^2} + 2f\cosh\left(\frac{3}{2} \frac{(\mathbf{\Sigma} - \mathbf{A})_m}{\sigma_F}\right) - (1 + f^2) = 0, \tag{4.23}$$

where

$$\sigma_F = b\sigma_Y + (1-b)\sigma_{eq}.$$

σ_Y and σ_{eq} are, respectively, the stress at first yield, and the current yield stress of the matrix. b is an interpolator between the isotropic model ($b = 0$) and the kinematic model ($b = 1$). The evolution law for the internal stress $\mathbf{A}$ has a complex expression and is not given here.

The main feature of Eq. (4.23), as compared to the usual isotropic model, is to predict a stronger curvature of the yield surface, and this point turns out to be of great importance for the study of instability in simple tension.

4.4. Interacting cavities

When the cavities are located at sufficiently short distances they interact, and it is no more possible to consider each of them as isolated in a finite volume submitted to uniform boundary conditions, such as the conditions (1.5). The latter type of model, which was adopted in the preceding section, would assume that there exist regions in the matrix, surrounding cavities, where the fluctuations of stress and strain are negligible. This is clearly incorrect in the high porosity range. Two other possible types of approach can be proposed.

On the one hand the r.v.e. can be assumed to contain a large number of cavities of small size in comparison with its overall dimensions, and to be submitted to uniform boundary conditions. In such a situation obtaining a closed form solution is hopeless, and numerical approaches will require long computational times and be difficult to interpret.

On the other hand it is possible to consider a finite r.v.e., containing only one cavity, but submitted to relevant boundary conditions, expressing the *in situ state* of the r.v.e. inside the original porous material, i.e. the effect of other cavities. The derivation of these proper boundary conditions is intractable, except in the simple case of periodic arrays of cavities.

This second point of view is rapidly illustrated hereafter. In a periodic array the typical representative volume element is the unit cell, i.e. a polyhedric volume of R^3 which generates by periodicity a regular covering of the space. The average relations (1.1) are still valid for periodic arrays, the spatial average being performed on the unit cell. The boundary conditions, relevant for this choice of the r.v.e., reflect the periodic character of stress and strain at the microscopic scale (SUQUET (1985)):

$$(4.24)\qquad \begin{aligned} &\sigma_{ij}n_j \quad \text{opposite on opposite sides of } \partial Y\ (^7),\\ &\dot{u}_i^* = \dot{u}_i - \dot{E}_{ij}x_j \quad \text{equal on opposite sides of } \partial Y. \end{aligned}$$

These boundary conditions may seem complex by comparison with the conditions (1.5) where the uniformity of the stress or the strain on Y was assumed. However, they are more satisfactory since they ensure that both the force and the velocity are continuous from one cell to the neighbouring ones.

It is worth noting that Hill's equality (1.6) remains valid

$$\langle \tilde{\sigma}_{ij}\,\varepsilon_{ij}(\dot{\tilde{u}})\rangle = \tilde{\Sigma}_{ij}\dot{\tilde{E}}_{ij}$$

provided that $\tilde{\boldsymbol{\sigma}}$ is in equilibrium and that $\tilde{\boldsymbol{\sigma}}$ and $\dot{\tilde{\mathbf{u}}}$ satisfy the conditions (4.24). This remark allows the extension for periodic arrays of the above developments about the macroscopic constitutive law and the macroscopic potentials Φ and Ψ. Indeed, Hill's equality is the main

(7) Since the outward normal vector **n** are opposite on opposite sides of ∂Y, the periodicity of $\boldsymbol{\sigma}$ implies that the forces $\boldsymbol{\sigma}\cdot\mathbf{n}$ are opposite.

ingredient used in the derivation of the macroscopic constitutive law (4.2), and the expressions (4.4) and (4.5) of the macroscopic potentials are still valid provided that the periodic boundary conditions (4.24) are substituted to the classical ones in the conditions (4.6) (4.7).

Therefore, the theoretical investigation of periodic arrays of cavities is not significantly different from that of more general distributions. However, the modification of the boundary conditions can lead to significant quantitative modifications of the resulting potentials. A typical illustration of this fact is provided by the study of yield stresses in an anisotropic perforated plate, following the experimental work of LITEWKA *et al.* (1984). The elongated perforations are located at the nodes of a triangular or square pattern, and the influence of the angle between the tensile direction and the pattern direction is investigated by Litewka *et al.* (cf. Fig. 17). A numerical simulation of these experiments is reported in

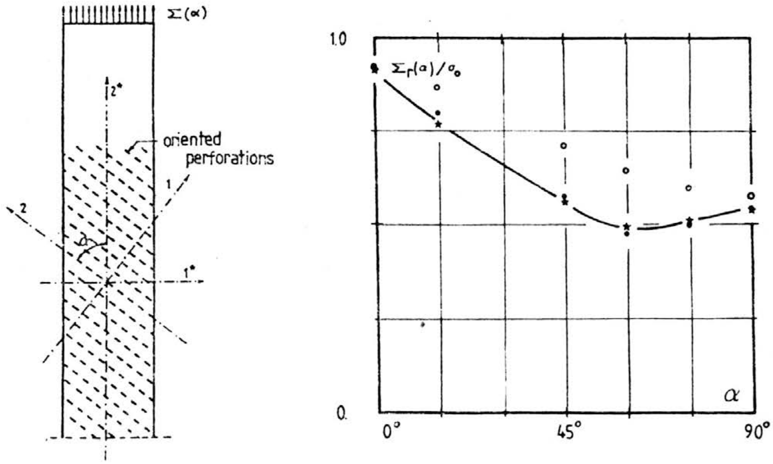

FIG. 17. Yield stresses of a perforated plate in a tensile test. (a) Specimen geometry; (b) comparison experiment/computations.
○ uniform strain on the r.v.e. boundary,
• periodicity conditions (MARIGO *et al.* (1987)),
* experiment (LITEWKA *et al.* (1984)).

MARIGO *et al.* (1987) with two choices of boundary conditions: uniform strain and periodicity conditions. A significant discrepancy (30%) between the results can be observed in Fig. 17. The results obtained with the periodicity conditions are in good agreement with experiments, and the assumption of uniform strains leads to an overestimate of the yield stresses and to a bad prediction of the direction of minimal strength (90° with the assumption of uniform strain, 60° with the assumption of periodic forces and displacements, 60° observed experimentally). The superiority of the periodic model is to be attributed to a better description of the in situ mechanical state of the r.v.e. inside the specimen. The various perturbations due to other cavities are globally taken into account by the boundary conditions. It is worth noting that, in the elastic range, the difference between the two theories is significantly reduced. The interaction effect increases with the nonlinearity of the material behavior.

Apart from the specific case of periodic arrays of cavities, for which the choice of the r.v.e. and of the boundary conditions are clear as illustrated above, the problem of interacting cavities is still completely open to discussion: the choice of the r.v.e., the statistical description of a population of cavities, and the boundary conditions are some difficult problems which remain to be solved in the future.

5. Conclusions

(i) The growth of an isolated void in an infinite matrix has been extensively studied by several authors for cylindrical or spheroidal voids and rigid-perfectly plastic or power-law viscous matrices. The main trends which are observed experimentally are correctly predicted: the void growth rate is enhanced by a high applied triaxiality (this influence is linear, power-law or exponential for a linearly viscous, power-law viscous, or rigid-perfectly plastic matrix, respectively) and by a low strain rate sensitivity of the matrix. The influence of matrix strain hardening has been taken into account approximately and leads to a decrease in the void growth rate.

(ii) Experimental investigation has shown that void interaction increases void growth rate, even for low porosity. This effect is enhanced for high triaxiality and low strain rate sensitivity. The exponential influence of triaxiality is maintained for strain rate insensitive materials, but the pre-exponential term is higher than in the isolated void case.

(iii) The mass balance equation allows to derive the voids growth law from the macroscopic constitutive equations of porous materials. The last point presently receives an increasing attention, and several models of macroscopic behavior have been proposed for power law viscous or rigid plastic matrices, but there still remains uncertainties in the high porosity range.

(iv) The field of damage, and more specifically the application of micromechanics methods to the derivation of damage evolution laws, has received a noticeable development in the last two decades. But the need for further research is still important and interactions between cavities, influence of matrix strain hardening, and anisotropic development of damage are some examples of widely open subjects for further work.

Appendix 1. Viscous materials

For the class of materials under consideration in this paper, the strain-rate and the stress are related in a tensile test through the Norton law (power law)

$$\sigma = \mu \dot{\varepsilon}^{s-1}. \tag{A.1}$$

This equation can be written in a dimensionless form

$$\frac{\sigma}{\sigma_0} = \left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)^{s-1} \tag{A.2}$$

with

$$\mu = \sigma_0 (\dot{\varepsilon}_0)^{1-s}.$$

For multiaxial states of stress the extension of the law (A.1) reads as follows:

(A.3) $$\operatorname{div}\dot{\mathbf{u}} = 0 \quad \text{and} \quad \frac{s_{ij}}{\sigma_{eq}} = \frac{2}{3}\frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}_{eq}},$$

with

$$\sigma_{eq} = \mu\dot{\varepsilon}_{eq}^{s-1},$$

The expressions (A.3) can be written alternatively:

(A.4) $$s_{ij} = \frac{2}{3}\mu\dot{\varepsilon}_{eq}^{s-2}\dot{\varepsilon}_{ij} \quad \text{or} \quad \dot{\varepsilon}_{ij} = \frac{3}{2}\mu^{1-s'}\sigma_{eq}^{s'-2}s_{ij},$$

where $s' = s/(s-1)$. The correspondence with BUDIANSKY *et al.* (1982) notations

$$\dot{\varepsilon}_{ij} = \frac{1}{2\eta}\sigma_{eq}^{n-1}s_{ij}$$

is obtained by the following change of material parameters:

$$n = 1/(s-1), \qquad \eta = \mu^n/3 = \sigma_0^n/3\dot{\varepsilon}_0.$$

The viscous laws under consideration in this paper can be derived from two dual potentials φ and ψ:

(A.5) $$s_{ij} = \frac{\partial\varphi}{\partial\dot{\varepsilon}_{ij}}(\dot{\boldsymbol{\varepsilon}}), \qquad \dot{\varepsilon}_{ij} = \frac{\partial\psi}{\partial s_{ij}}(s),$$
$$\varphi(\dot{\varepsilon}) = \frac{\mu}{s}\dot{\varepsilon}_{eq}^{s}, \qquad \psi(s) = \frac{\mu^{1-s'}}{s'}\sigma_{eq}^{s'},$$

φ and ψ are respectively called the dissipation potential and the force potential of the material. The correspondence with Duva and Hutchinson's notations

$$\psi(\sigma) = \frac{\alpha\sigma_0}{n+1}\left(\frac{\sigma_{eq}}{\sigma_0}\right)^{n+1}$$

is obtained by the following change of material parameters:

$$n = 1/(s-1), \qquad \alpha = (\mu/\sigma_0)^{1-s'} = \dot{\varepsilon}_0.$$

Appendix 2. Cylindrical cavities under axisymmetric loading

A cylindrical shell of inner radius a, outer radius ϱ, is submitted to an axisymmetric loading. The radial and axial components of the velocity field read as

$$\dot{u}_r = \dot{u}_r(r), \qquad \dot{u}_z(z) = \dot{E}_{33}z,$$

and the non-vanishing components of the strain rate tensor are

$$\dot{\varepsilon}_{rr} = \frac{\partial\dot{u}_r}{\partial r}, \qquad \dot{\varepsilon}_{\theta\theta} = \frac{\dot{u}_r}{r}, \qquad \dot{\varepsilon}_{zz} = \dot{E}_{33}.$$

Matrix incompressibility yields

(A.6) $$\dot{u}_r = -\frac{\dot{E}_{33}}{2}r + \frac{A}{r}.$$

where A has to be determined from the mechanical behavior of the matrix, the equilibrium equations, and the boundary conditions. The logarithmic growth rate is

$$\frac{\dot V}{V} = \frac{2\dot a}{a} + \frac{\dot H}{H} = \frac{2A}{a^2},$$

where H is the height of the cylinder.

The equivalent strain rate is

$$\dot\varepsilon_{eq} = \left(\frac{2}{3}\dot\varepsilon_{ij}\dot\varepsilon_{ij}\right)^{1/2} = \left(\dot E_{33} + \frac{4A^2}{3r^4}\right)^{1/2}.$$

The mechanical behavior

$$s_{ij} = \frac{2}{3}\mu\dot\varepsilon_{eq}^{s-2}\dot\varepsilon_{ij}$$

yields

$$\sigma_{rr} - \sigma_{\theta\theta} = s_{rr} - s_{\theta\theta} = -\frac{4}{3}\mu\left(\dot E_{33}^2 + \frac{4A^2}{3r^4}\right)^{\frac{s-2}{2}}\frac{1}{r^2},$$

$$\sigma_{zz} - \frac{\sigma_{rr}+\sigma_{\theta\theta}}{2} = s_{zz} - \frac{s_{rr}+s_{\theta\theta}}{2} = \mu\dot\varepsilon_{eq}^{s-2}\dot E_{33}.$$

Equilibrium equations and boundary conditions allow to relate A, $\dot E_{33}$, Σ_{11} and Σ_{33}

$$\Sigma_{11} = \sigma_{rr}(\varrho) - \sigma_{rr}(a) = \int_a^\varrho \frac{\partial\sigma_{rr}}{\partial r}dr = \int_a^\varrho \frac{\sigma_{\theta\theta}-\sigma_{rr}}{r}dr,$$

$$\Sigma_{11} = \frac{4}{3}\mu\int_a^\varrho\left(\dot E_{33}^2 + \frac{4A^2}{3r^4}\right)^{\frac{s-2}{2}}\frac{A}{r^3}dr,$$

$$\Sigma_{33} - \Sigma_{11} = \frac{1}{Y}\int_{Y*}\left(\sigma_{zz} - \frac{\sigma_{rr}+\sigma_{\theta\theta}}{2}\right)dr = \frac{2\mu}{\sigma^2}\int_a^\varrho\left(\dot E_{33}^2 + \frac{4A^2}{3r^4}\right)^{\frac{s-2}{2}}\dot E_{33}r\,dr.$$

With the following change of variables:

$$\omega = \frac{2A}{\sqrt{3}\dot E_{33}a^2} = \frac{1}{\sqrt{3}}\frac{\dot V}{\dot E_{33}V}, \quad x = \omega\left(\frac{a}{r}\right)^2, \tag{A.7}$$

we obtain

$$\begin{aligned}\Sigma_{11} &= \frac{\mu}{\sqrt{3}}|\dot E_{33}|^{s-2}\dot E_{33}\int_{\omega f}^{\omega}(1+x^2)^{\frac{s-2}{2}}dx \quad (f = a^2/\varrho^2),\\ \Sigma_{33} - \Sigma_{11} &= \mu\omega f|\dot E_{33}|^{s-2}E_{33}\int_{\omega f}^{\omega}(1+x^2)^{\frac{s-2}{2}}\frac{dx}{x^2}.\end{aligned} \tag{A.8}$$

Consequences

a) Finite volume. Viscous material

Equation (A.8) leads to

(A.9)
$$\frac{\Sigma_{33}-\Sigma_{11}}{\Sigma_{11}} = \sqrt{3}\,\omega f \frac{\int\limits_{\omega f} (1+x^2)^{\frac{s-2}{2}} \frac{dx}{x^2}}{\int\limits_{\omega f}^{\omega} (1+x^2)^{\frac{s-2}{2}} dx},$$

$$\omega = \frac{1}{\sqrt{3}} \frac{\dot{V}}{V\dot{E}_{33}}.$$

b) Infinite volume. Viscous matrix

When f tends to 0 in Eq. (A.9), a result derived by BUDIANSKY *et al.* (1982) is obtained:

(A.10)
$$\frac{\Sigma_{33}-\Sigma_{11}}{\Sigma_{11}} = \frac{\sqrt{3}}{\int\limits_{0}^{\omega} (1+x^2)^{\frac{s-2}{2}} dx},$$

$$\omega = \frac{1}{\sqrt{3}} \frac{\dot{V}}{\dot{E}_{33} V}.$$

c) Infinite volume. Rigid-plastic matrix

When $s = 1$, the material parameter μ reduces to the yield stress σ_0 of the matrix. For an infinite matrix ($f = 0$) $\Sigma_{33}-\Sigma_{11}$ reduces to $\sigma_0 \dot{E}_{33}/|\dot{E}_{33}|$ Moreover,

$$\Sigma_{11} = \frac{\sigma_0}{\sqrt{3}} \int\limits_{0}^{\omega} (1+x^2)^{-1/2} dx = \frac{\sigma_0}{\sqrt{3}} \operatorname{Argsinh}\omega,$$

i.e.

(A.11)
$$\frac{\dot{V}}{V\dot{E}_{33}} = \sqrt{3}\sinh\left(\sqrt{3}\frac{\Sigma_{11}}{\Sigma_{33}-\Sigma_{11}}\right).$$

This last result was found by MCCLINTOCK (1968).

d) Finite volume. Rigid-plastic matrix

From Eq. (A.8):

$$\frac{\sqrt{3}}{2}\frac{|\Sigma_{\alpha\alpha}|}{\sigma_0} = \sqrt{3}\frac{|\Sigma_{11}|}{\sigma_0} = \int\limits_{\omega f}^{\omega} (1+x^2)^{-1/2} dx = \operatorname{sgn}(\dot{E}_{33}) \operatorname{Log}\left[\frac{(1+\omega^2)^{1/2}+\omega}{(1+\omega^2 f^2)^{1/2}+\omega f}\right].$$

Moreover,

$$\frac{\Sigma_{eq}}{\sigma_0} = \frac{|\Sigma_{33}-\Sigma_{11}|}{\sigma_0} = \omega f\left[-\frac{(1+x^2)^{1/2}}{x}\right]_{\omega f}^{\omega} = (1+\omega^2 f^2)^{1/2} - f(1+\omega^2)^{1/2}.$$

Let the following intermediate quantities be introduced:

$$A_1 = \frac{\sqrt{3}}{2}\frac{|\Sigma_{\alpha\alpha}|}{\sigma_0}, \qquad A_2 = \frac{\Sigma_{eq}}{\sigma_0},$$
$$A_3 = (1+\omega^2)^{1/2}+\omega, \qquad A_4 = (1+\omega^2 f^2)^{1/2}+\omega f.$$

The following relations apply:

$$A_2 = (A_4 - fA_3), \quad \exp(A_1) = A_3/A_4,$$

(A.12)
$$A_4 = \frac{A_2}{1-f\exp(A_1)}, \quad A_3 = \frac{A_2\exp(A_1)}{1-f\exp(A_1)},$$

(A.13)
$$2\omega = \frac{A_3^2-1}{A_3} = \frac{A_4^2-1}{fA_4}.$$

A_4 and A_3 can be eliminated in Eq. (A.13), and this leads to

$$A_2^2 = (1-fe^{A_1})(1-fe^{-A_1}) = 1-f^2-2f\cosh(A_1)$$

i.e.

(A.14)
$$\frac{\Sigma_{eq}^2}{\sigma_0^2} + 2f\cosh\left(\frac{\sqrt{3}}{2}\frac{\Sigma_{\alpha\alpha}}{\sigma_0}\right) - 1 - f^2 = 0.$$

This expression of the macroscopic yield criterion has been derived by GURSON (1977).

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