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# BRIEF NOTES

## Two-phase flow with condensation in cylindrical tube (*)

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A TYPICAL industrial process is analyzed: slow, laminar flow of a fluid in a tube of circular cross-section, account being taken of phase changes. An approximate method of solution is proposed and illustrated by the example of flow with steam condensation and freezing at the wall.

### 1. Introduction

IN MANY THERMAL plants chemical reactions or changes of phase occur in a fluid flowing in a cylindrical tube, for instance during cracking operations in petroleum chemistry or the fouling of heat exchangers.

Generally, the rate of flow is low and the lengths are great so that the flow is laminar and established. In these conditions, the equations of the problem are reduced to the stationary convective equations of the temperature $T$:

$$V \cdot \operatorname{grad} T - a \Delta T = 0,$$

where $a$ is the thermal diffusivity and $V$ the flow speed reduced to the axial component $u$ with parabolic distribution against the radial coordinate $r$

$$u = (2Q/\pi R^2)\,[1-(r/R)^2],$$

$Q$ is the rate of flow, $R$ the radius of the tube.

This equation is known as Graetz equation but the presence of chemical reactions or changes of phase introduces a right-hand member to the equation: the present paper is limited to the case of change of phase.

### 2. Two-phase equations

It is assumed that the fluid is a mixture of an uncondensable gas and a vapor which is capable of getting solid at the wall temperature. In these conditions three phases are present in the flow: gas (index $g$), vapor (index $v$), liquid (index $l$); it is supposed that the solid phase is immediately attached to the wall.

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In order to simplify the problem, several hypotheses will be made [1]:
the axial components (along the $x$ axis) $u_g, u_v, u_l$ have the same value;
the radial components (along the $y$ axis) $v_g, v_v, v_l$ are null;
diffusion affects only gas and vapor;
it introduces only a radial component $\Delta$ (along the $y$ axis) of velocity

$$\Delta_g = -D\frac{\varrho_g+\varrho_v}{\varrho_g}\frac{\partial}{\partial r}\frac{\varrho_g}{\varrho_g+\varrho_v},$$

$$\Delta_v = -D\frac{\varrho_g+\varrho_v}{\varrho_v}\frac{\partial}{\partial r}\frac{\varrho_v}{\varrho_g+\varrho_v},$$

$D$ is a constant diffusion coefficient, it appears that $\varrho_g\Delta_g+\varrho_v\Delta_v = 0$; thus, for the mixture, $\varrho_v = 0$.

But when a change of phase occurs, it is convenient to use the molecular fractions $X_i = n_i/n$ ($n_i$ is the number density of molecules, $n = \Sigma n_i$) instead of the densities $\varrho_i$

$$\Delta_g = -D\frac{X_g M_g+X_v M_v}{X_g M_g}\frac{\partial}{\partial r}\frac{X_g M_g}{X_g M_g+X_v M_v},$$

where $M_g$ and $M_v$ are the molecular mass of gas and vapor.

### 2.1. Mass-transfer equations

According to the above hypothesis, the mass-transfer equations of each component are: for the gas

$$u\frac{\partial}{\partial x}(nX_g)+\frac{1}{r}\frac{\partial}{\partial r}(rnX_g\Delta_g) = 0,$$

for the vapor

$$u\frac{\partial}{\partial x}(nX_v)+\frac{1}{r}\frac{\partial}{\partial r}(rnX_v\Delta_v) = -\sigma,$$

for the liquid

$$u\frac{\partial}{\partial x}(nX_l) = \sigma;$$

by summing these equations multipled, respectively, by $M_g$, $M_v$, $M_l = M_v$, the equation $\frac{\partial}{\partial x}[n(X_gM_g+X_vM_v+X_lM_l)] = 0$ is obtained.

### 2.2. Equation of equilibrium

In equilibrium state the partial pressure of vapor is given by various formulae, the simplest of which is (for perfect gases):

$$P_v/P = (P_c/P)\exp[-A(T_c-T)/T] = f(T) = X_v/(X_v+X_g).$$

Thus $n$ and the molecular fractions $X_i$ are determined by four equations

$$X_g+X_v+X_l = 1, \quad X_v/(X_v+X_g) = f(T),$$

$$u\frac{\partial}{\partial x}(nX_g)-\frac{D}{r}\frac{\partial}{\partial r}\left[rn(X_gM_g+X_vM_v)\frac{\partial}{\partial r}\frac{X_g}{X_gM_g+X_vM_v}\right]=0,$$

$$\frac{\partial}{\partial x}[n(X_gM_g+X_vM_v+X_lM_l)]=0,$$

the solution of the last equation is obvious:

$$n(X_gM_g+X_vM_v+X_lM_l)=\tilde{\varrho}=\tilde{n}(\tilde{X}_gM_g+\tilde{X}_vM_v)=\tilde{n}\tilde{M}$$

($\sim$ indicates inlet values, $\tilde{X}_g+\tilde{X}_v=1$).

### 2.3. Energy-transfer equation

Let $\theta=(T-T_a)/(T_e-T_a)$ where $T_a$ is the wall temperature and $T_e$ the inlet temperature. The equation of energy is

$$\frac{\partial}{\partial x}\left[(X_gM_gC_g+X_vM_vC_v+X_lM_lC_l)nu\left(\theta+\frac{T_a}{\Delta T}\right)-\frac{Lnu}{\Delta T}X_lM_l\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial r}\left[r(X_gM_gC_g\Delta_g+X_vM_vC_v\Delta_v)n\left(\theta+\frac{T_a}{\Delta T}\right)\right]-\lambda\Delta\theta=0$$

with the boundary conditions: $\theta(0,r)=1$; $\theta(x,R)=0$ and the symmetry condition $\theta'(x,0)=0$.

## 3. Proposed method of resolution

The mass-transfer system is obviously coupled with the energy-transfer equation by the temperature $T$ in the function $f(T)$. The proposed method introduces alternatively in the $X$-system a previous approximation of $T$ and in the $T$-equation a previous approximation of $\{X\}$. The starting approximation $\theta^0$ for $\theta$ is, of course, the solution of the $\theta$-equation without diffusion or condensation ($X_l=0, X_g$ and $X_v$ constant, $\Delta_g=0$, $\Delta_v=0$)

$$u\tilde{C}\tilde{\varrho}\frac{\partial\theta^0}{\partial x}-\lambda\Delta\theta^0=0$$

with

$$\tilde{C}\tilde{\varrho}=\tilde{n}(C_g\tilde{X}_gM_g+C_v\tilde{X}_vM_v).$$

The starting approximation of $\{X\}$ without condensation is not obvious because of the boundary conditions: instead of writing $X_l=0$ on the wall, $X_v=0$ will be written, what is a rather crude approximation; thus the starting value $n^0$ of $n$ is given by the equation

$$u\frac{\partial n^0}{\partial x}-\frac{D}{r}\frac{\partial}{\partial r}\left[r\frac{\partial n^0}{\partial r}\right]=0.$$

Introducing dimensionless variables and the parameters $\xi=x/R$, $\eta=r/R$; $\mathrm{Gz}=\pi R\lambda/2Q\tilde{C}\tilde{\varrho}$ (Graetz Number); $\mathrm{Di}=\pi RD/2Q$ (Diffusion Number) leads to the equations below:

$$(1-\eta^2)\frac{\partial\theta^0}{\partial\xi}-\mathrm{Gz}\left(\frac{\partial^2\theta^0}{\partial\xi^2}+\frac{\partial^2\theta^0}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\theta^0}{\partial\eta}\right)=0,$$

$$(1-\eta^2)\frac{\partial n^0}{\partial\xi}-\mathrm{Di}\left(\frac{\partial^2 n^0}{\partial\xi^2}+\frac{\partial^2 n^0}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial n^0}{\partial\eta}\right)=0,$$

$$[n^0(\xi,1)=\tilde{\varrho}/M_g;\, n^0(0,\eta)=\tilde{n}=\tilde{\varrho}/(\tilde{X}_g M_g+\tilde{X}_v M_v)];$$

they are identical (by changing Gz into Di) and can be solved by the exponential and hypergeometric functions [2], $(b_j=(\beta_j-2)/4)$

$$\theta^0=\Sigma_j A_{j1}F_1(b_j,1;\beta_j\eta^2)\exp(-\beta_j\eta^2/2)\exp(-\beta_j^2\mathrm{Gz}\xi)$$

when

$$\mathrm{Di}=\mathrm{Gz},\qquad n^0=\tilde{n}\theta^0+\tilde{\varrho}(1-\theta^0)/M_g=\tilde{n}(1-\tilde{M}/M_g)\theta^0+\tilde{n}\tilde{M}/M_g.$$

### 3.1. First approximation for concentrations

For $X_l\neq 0$, the three algebraic equations of mass-transfer give

$$nX_g=(\tilde{\varrho}-M_v n)/(M_g-M_v);\; M_gX_g+M_vX_v=[M_g+M_vf/(1-f)]X_g=FX_g,$$

whereas the differential equation can be written as

$$(1-\eta^2)\frac{\partial n}{\partial\xi}-\frac{\mathrm{Di}}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial n}{\partial\eta}\right)=H(\xi,\eta)=\frac{M_g-M_v}{M_v}\frac{\mathrm{Di}}{\eta}\frac{\partial}{\partial\eta}\left[\eta nX_g\frac{\partial}{\partial\eta}\log\left(\frac{n}{\tilde{n}}X_gF\right)\right]$$

the left-hand side is the same as for $n^0$ (save the term $\partial^2n^0/\partial\xi^2$ generally omitted), in the right-hand side the starting solution $n^0X_g^0$ and $\theta^0$ will be introduced. The method of resolution by matching finite pieces for such an equation with the right-hand side will be explained in a later section. In this approximation, the boundary condition on the wall is $X_l^1(\xi,1)=0$; this means $X_g^1(\xi,1)=1-f^a$, $X_v^1(\xi,1)=f^a$, $n^1(\xi,1)=\tilde{\varrho}/[M_g(1-f^a)+M_vf^a]$, with $f^a=f(T_a)$. From the approximation $n^1$ the corresponding molecular fractions are easily deduced:

$$X_g^1=(\tilde{\varrho}-n^1M_v)/[n^1(M_g-M_v)],$$

$$X_v^1=[f/(1-f)]X_g^1,\qquad X_l^1=1-X_g^1/(1-f).$$

The following approximation needs also the approximation $\theta^1$ deduced from the first approximation of the energy-transfer equation.

### 3.2. First approximation for energy-transfer equation

In the energy-transfer equation, the enthalpy coefficient will be written as

$$n(M_gC_gX_g+M_vC_vX_v+M_lC_lX_l)/\tilde{C}\tilde{p}=$$

$$=1+[M_gC_g(nX_g-\tilde{n}\tilde{X}_g)+M_vC_v(nX_v-\tilde{n}\tilde{X}_v)+M_lC_lnX_l]/\tilde{C}\tilde{p}$$

so that an approximation for the energy-transfer equation is

$$(1-\eta^2)\frac{\partial\theta^1}{\partial\xi}-\mathrm{Gz}\Delta\theta^1=\Theta(\xi,\eta)$$

$$=-(1-\eta^2)\frac{\partial}{\partial\xi}\left[\frac{M_gC_g(nX_g-\tilde{n}\tilde{X}_g)+M_vC_v(nX_v-\tilde{n}\tilde{X}_v)+M_lC_lnX_l}{C\tilde{p}}\left(\theta+\frac{T_a}{\Delta T}\right)\right.$$

$$\left.-\frac{LM_l}{C\tilde{\varrho}\Delta T}nX_l\right]+\frac{\mathrm{Di}}{\eta}\frac{\partial}{\partial\eta}\left[\eta\frac{C_g-C_v}{C\tilde{\varrho}}n(X_gM_g+X_vM_v)\frac{\partial}{\partial\eta}\frac{X_gM_g}{X_gM_g+X_vM_v}\left(\theta+\frac{T_a}{\Delta T}\right)\right].$$

As for mass equation, the left-hand side is identical to the starting equation.

In the right-hand side the previous approximations of $n$ and $\theta$ will be introduced; the last line can be reduced to

$$-\frac{C_g-C_v}{C\tilde{\varrho}}\frac{\mathrm{Di}}{\eta}\frac{\partial}{\partial\eta}\left[\eta n X_g M_g^!\left[\left(\theta+\frac{T_a}{\Delta T}\right)\frac{\partial}{\partial\eta}\log F-\frac{\partial\theta}{\partial\eta}\right]\right].$$

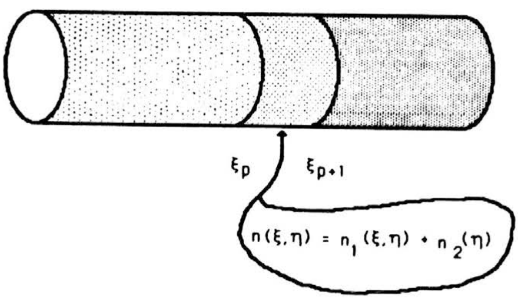
## 4. A method of matched finite pieces

It has been observed that the evolution of temperatures and molecular fractions along the $x$-axis is very slow. Thus it is convenient to divide the tube into finite pieces where the radial distributions of $\theta$ and $n$ are taken independent of $\xi$. In each finite piece the solutions $Z$ (this means $\theta$ or $n$) will be the sum of two terms [3]:

one term is a two-dimensional solution $Z_1(\xi, \eta)$ of the equation without the right-hand side (without RHS);

the other term is a one-dimensional solution $Z_2(\eta)$ of the equation with the right-hand side (with RHS);

the boundary condition on the wall is satisfied by the second one but the total solutions are matched at each junction of two consecutive finite pieces (Fig. 1).




$$\left[\begin{array}{l} (1-\eta^2)\dfrac{\partial n_1}{\partial\xi}-\dfrac{\mathrm{Di}}{\eta}\dfrac{\partial}{\partial\eta}\left(\eta\dfrac{\partial n_1}{\partial\eta}\right)=0, \quad n_1(\xi,1)=0 \quad \Rightarrow \quad \beta_j \\ n_1(\xi,\eta)=\Sigma_j B_j\,{}_1F_1\left(\dfrac{\beta_j-2}{4}, 1, \beta_j\eta^2\right)\exp(-\beta_j\eta^2)\exp(-\beta_j G\mathcal{Z}\xi) \end{array}\right.$$

$$\left[\begin{array}{l} \mathrm{Di}\left[\dfrac{d^2n_2}{d\eta^2}+\dfrac{1}{\eta}\dfrac{dn_2}{d\eta}\right]=H(\xi_p,\eta), \quad n_2(\xi,1)=\tilde{n}\dfrac{\tilde{M}}{M_g(1-f^\theta)+M_v f^\theta} \\ \mathrm{Di}\, n_2(\eta)=\log\eta\displaystyle\int_0^\eta H\eta\,d\eta+\int_\eta^1 H\log\eta\,\eta\,d\eta+\mathrm{Di}\dfrac{\tilde{n}\tilde{M}}{[M_g(1-f^\theta)+M_v f^\theta]} \end{array}\right.$$

$$\left[\begin{array}{l} \displaystyle\int_0^1 [n_1^p(\xi_p,\eta)+n_2^p(\eta)]\,{}_1F_1(\beta_k\eta^2)\exp\left(-\beta_k\frac{\eta^2}{2}\right)d\eta = \\ \displaystyle\int_0^1 [n_1^{p+1}(\xi_p,\eta)+n_2^{p+1}(\eta)]\,{}_1F_1(\beta_k\eta^2)\exp\left(-\beta_k\frac{\eta^2}{2}\right)d\eta \end{array}\right] \Rightarrow B_j$$

Fig. 1.

### 4.1. Solutions of the differential equations without RHS

The solution $Z_1(\xi, \eta) = \theta_1^1$ of the equation without RSH is identical to $\theta^0$

$$\theta^0 = \Sigma_j A_{j1} F_1(b_j, 1; \beta_j\eta^2)\exp(-\beta_j\eta^2/2)\exp(-\beta_j^2 \mathrm{Gz}\xi)$$

the eigenvalues $\beta_j$ are determined by the boundary condition on the wall

$$_1F_1(b_j, 1; \beta_j) = 0 \quad \text{with} \quad b_j = (\beta_j - 2)/4.$$

The coefficients $A_j$ will be determined either by the inlet condition or by matching two consecutive finite pieces. The mass transfer solution $Z_1(\xi, \eta) = n_1^1$ has practically the same form.

### 4.2. Solutions of differential equations with RHS

Concerning the one-dimensional solutions $Z_2$, the equations are reduced to ordinary differential equations:

$$\mathrm{Gz}\left[\frac{d^2\theta_2}{d\eta^2} + \frac{1}{\eta}\frac{d\theta_2}{d\eta}\right] = \Theta(\xi_p, \eta),$$

$$\mathrm{Di}\left[\frac{d^2 n_2}{d\eta^2} + \frac{1}{\eta}\frac{dn_2}{d\eta}\right] = H(\xi_p, \eta),$$

where $\xi_p$ is a reference point of the piece numbered $p$ (the index $p$ is omitted here for $\theta$ and $n$).

The solution of such equations is classical:

$$\mathrm{Gz}\,\theta_2(\eta) = \log\eta \int_0^\eta \Theta\eta d\eta + \int_0^l \Theta \log\eta\,\eta d\eta,$$

$$\mathrm{Di}\, n_2(\eta) = \log\eta \int_0^\eta H\eta d\eta + \int_\eta^1 H\log\eta\,\eta d\eta + \mathrm{Di}\,\tilde{\varrho}/[M_g(1-f^a) + M_v f^a],$$

the last additive constant is introduced in order to satisfy the specific boundary condition of $n_2$ on the wall.

In the consideration of the derivative terms with respect to $\eta$ in $\Theta$ and $H$, the above integrals are reduced to very simple forms:

$$+\mathrm{Di}\frac{C_g - C_v}{C\tilde{\varrho}} \int_\eta^1 nX_g M_g \left[\left(\theta + \frac{T_a}{\Delta T}\right)\frac{\partial}{\partial\eta}\log F - \frac{\partial\theta}{\partial\eta}\right] d\eta,$$

$$-\mathrm{Di}\frac{M_g - M_v}{M_v} \int_\eta^1 nX_g \frac{\partial}{\partial\eta}\log\left(\frac{n}{\tilde{n}} X_g F\right) d\eta;$$

no simplification can be introduced in the derivative term with respect to $\xi$.

### 4.3. Computation of coefficients $A_j$

The inlet conditions

$$\theta_1(0, \eta) + \theta_2(\eta) = 1$$

and

$$n_1(0, \eta)+n_2(\eta) = \tilde{n}$$

make it possible to determine the coefficients $A_j$ (an $B_j$ for $n_1$) for the first piece of the tube by means of the linear algebrical system

$$\int_0^1 \theta_1(0, \eta)_1F_1(b_k, 1; \beta_k\eta^2)\exp(-\beta_k\eta^2/2)d\eta = J_k$$

with

$$J_k = \int_0^1 {}_1F_1(b_k, 1; \beta_k\eta^2)\exp(-\beta_k\eta^2/2)d\eta.$$

For the other pieces, the coefficients $A_j$ and $B_j$ are determined by the system

$$\int_0^1 [\theta_1^p(\xi_p, \eta)+\theta_2^p(\eta)]_1F_1(b_k, 1; \beta_k\eta^2)\exp(-\beta_k\eta^2/2)\,d\eta$$

$$= \int_0^1 [\theta_1^{p+1}(\xi_p, \eta)+\theta_2p^{+1}(\eta)]_1F_1(b_k, 1; \beta_k\eta^2)\exp(-\beta_k\eta^2/2)\,d\eta,$$

$$\int_0^1 [n_1^p(\xi_p, \eta)+n_2^p(\eta)]_1F_1(b_k, 1; \beta_k\eta^2)\exp(-\beta_k\eta^2/2)\,d\eta$$

$$= \int_0^1 [n_1^{p+1}(\xi_p, \eta)+n_2^{p+1}(\eta)]_1(b_k, 1; \beta_k\eta^2)\exp(-\beta_k\eta^2/2)\,d\eta,$$

that is the matching condition of two consecutive pieces at the junction point.

## 5. Conclusion

An example of this computation process is given (Fig. 2) for a flow with steam condensation and freezing on the wall. The proposed method should be useful to solve any prac-

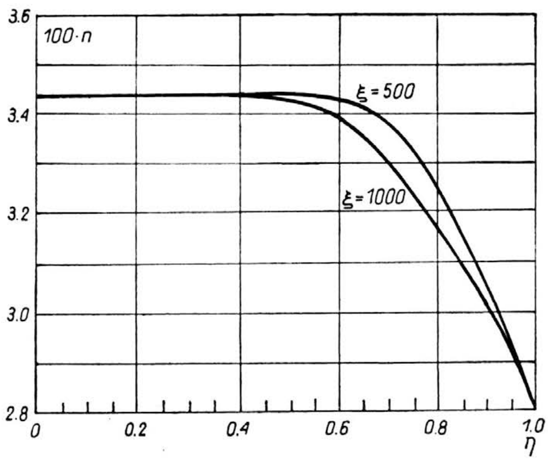



FIG. 2.

tical problem of flow with change of phase or chemical reaction [4] in cylindrical tubes because it can be performed with a small personal computer.

## Appendix. A simplification for $n_2$ and $\theta_2$ when $\mathrm{Di} = \mathrm{Gz}$

An alternative form of $n_2$ is

$$n_2 = \frac{\tilde{\varrho}}{M_g(1-f^a)+M_v f^a} - \frac{M_g - M_v}{M_v}\left[ |n^0 X_g^0|_\eta^1 + \int_\eta^1 n^0 X_g^0 \frac{\partial}{\partial \eta} \log F^0 d\eta \right]$$

with

$$\frac{\partial}{\partial \eta}\log F = \frac{AM_v}{M_g(1-f)+M_v f}\,\frac{f}{(1-f)}\,\frac{T_c(T_e - T_a)}{T^2}\,\frac{\partial \theta}{\partial \eta}$$

but for

$$\mathrm{Di} = \mathrm{Gz}, \quad n^0 X_g^0 = \tilde{n}\,\frac{\tilde{M}(1-f^a) - M_v(\tilde{X}_v - f^a)\theta^0}{M_g(1-f^a)+M_v f^a}$$

so that [1]

$$n_2 = \frac{\tilde{\varrho}\left(1-(M_g - M_v)(\tilde{X}_v - f^a)\theta^0/\tilde{M}\right)}{M_g(1-f^a)+M_v f^a}$$
$$+ \frac{\tilde{n}A(M_g - M_v)}{M_g(1-f^a)+M_v f^a}\int_0^{\theta^0} \frac{\tilde{M}(1-f^a) - M_v(\tilde{X}_v - f^a)\theta^0}{M_g(1-f)+M_v f}\,\frac{f}{1-f}\,\frac{T_c(T_e - T_a)}{T^2}\,d\theta^0$$

since the function to be integrated is then an explicit function of $\theta$. In the same manner, concerning the derivative term with respect to $\eta$ in $\theta_2$:

$$\theta_2 = -\frac{C_g - C_v}{\tilde{Cp}}\,\tilde{n}M_g \int_0^{\theta^0} \frac{\tilde{M}(1-f^a) - M_v(\tilde{X}_v - f^a)\theta^0}{Mg(1-f^a)+M_v f^a}$$
$$\times \left[\left(\theta^0 + \frac{T_a}{\Delta T}\right)\frac{AM_v}{M_g(1-f)+M_v f}\,\frac{f}{1-f}\,\frac{T_c(T_e - T_a)}{T^2} - 1\right] d\theta^0.$$

As far as the derivative term with respect to $\xi$ is concerned, it is consistent with the proposed method to replace the derivation by a finite difference such as

$$[G(\xi_{p+1}, \eta) - G(\xi_p, \eta)]/(\xi_{p+1} - \xi_p)\mathrm{Gz}$$

with

$$G = \log\eta \int_0^\eta (1-\eta^2)\Xi\eta d\eta + \int_\eta^1 (1-\eta^2)\,\Xi \log\eta\eta d\eta,$$

where $\Xi$ denotes the part in brackets of $\Theta$ (p. 6).

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(1) The starting solution (p. 4) should lead to $n^0 X_g^0 = \tilde{n}(\tilde{M} - M_v\theta^0)M_g$ but a better convergence is obtained by assigning the boundary condition on the wall corresponding to $X_l = 0$ instead of $X_v = 0$.

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