Arch. Mech., 41, 6, pp. 857–865, Warszawa 1989

Spectral problems for semidiscrete and discrete models of the Boltzmann equation Part II

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We study spectral problems for the semidiscrete and the regular $2r$ velocities discrete models of the Boltzmann equation which result in the course of analyzing forced sound waves. It is shown that eigenvalues of the operator corresponding to the continuous distribution of velocities can be approximated by eigenvalues of the operator generated by the discrete case. The eigenfunctions of the continuous velocity distribution operator can be approximated by a sequence of step functions built of suitable eigenvectors of the discrete velocity distribution operator. On the other hand, the essential spectrum of the continuous velocity distribution operator, being unbounded, cannot be approximated by the spectrum of any discrete velocity distribution operator, however large r can be.

Badane są zagadnienia spektralne dla półdyskretnych i regularnych dyskretnych o $2r$ prędkościach modeli równania Boltzmanna, wynikające z analizy propagacji wymuszonych fal dźwiękowych. Pokazuje się, że wartości własne operatora odpowiadającego ciągłemu rozkładowi prędkości mogą być przybliżane przez wartości własne operatora generowanego przez przypadek dyskretny. Funkcje własne operatora o ciągłym rozkładzie prędkości mogą być przybliżane przez ciąg funkcji schodkowych zbudowanych z odpowiednich wektorów własnych operatora o dyskretnym rozkładzie prędkości. Z drugiej strony, widmo istotne operatora o ciągłym rozkładzie prędkości, będąc nieograniczonym, nie może być przybliżane przez widmo żadnego operatora o dyskretnym rozkładzie prędkości, bez względu na to jak duże będzie r.

Исследуем спектральные задачи для полудискретных и регулярных дискретных, с $2r$ скоростями, моделей уравнения Больцмана, вытекающие из анализа распространения вынужденных звуковых волн. Показывается, что собственные значения оператора, отвечающего непрерывному распределению скорости, могут приближаться собственными значениями оператора, генерированного дискретным случаем. Собственные функции оператора, с непрерывным распределением скорости, могут приближаться последовательностью ступенчатых функций, построенных из соответствующих собственных векторов оператора, с дискретным распределением скорости. С другой стороны, существенный спектр оператора с непрерывным распределением скорости, будучи неограниченным, не может приближаться спектром никакого оператора с дискретным распределени м скорости, несмотря на то, как большим будет.

1. Introduction

In 1980 Cabannes [1] proposed the so-called semidiscrete model of the Boltzmann equation. It is the following integro-differential equation for the distribution function $N(t, \mathbf{x}, \theta)$:

$$(1.1)\qquad \frac{\partial}{\partial t} N(t, \mathbf{x}, \theta) + \mathbf{c} \cdot \frac{\partial}{\partial \mathbf{x}} N(t, \mathbf{x}, \theta) = \frac{cS}{\pi} \int_0^{2\pi} N(t, \mathbf{x}, \varphi) N(t, \mathbf{x}, \varphi + \pi) d\varphi - 2cSN(t, \mathbf{x}, \theta) N(t, \mathbf{x}, \theta + \pi),$$

where $t \geqslant 0$ is the time, $\mathbf{x} = (x, y) \in \mathscr{R}^2$ is the position, $\theta \in \langle 0, 2\pi \rangle$ and $\mathbf{c} = c(\cos\theta, \sin\theta)$ is the velocity vector with constant modulus c, $\partial/\partial\mathbf{x}$ stands for the gradient operator with respect to $\mathbf{x}$ and the dot in $\mathbf{c}\cdot(\partial/\partial\mathbf{x})$ denotes the standard scalar products in $\mathscr{R}^2$. Finally, the positive constant S is proportional to the collisional cross-section. The distribution function $N(t, \mathbf{x}, \theta)$ is assumed to be 2π-periodic with respect to θ. The above model was introduced as a formal limit to the discrete regular $2r$ velocity model as $r \to \infty$. The latter model of the Boltzmann equation proposed by GATIGNOL [2] is a system od $2r$ semilinear partical differential equations of the hyperbolic type:

$$\frac{\partial}{\partial t} N_m + \mathbf{c}_m \cdot \frac{\partial}{\partial \mathbf{x}} N_m = \frac{cS}{r} \sum_{j=1}^{2r} N_j N_{j+r} - 2cSN_m N_{m+r}, \qquad m = 1, 2, \ldots, 2r, \tag{1.2}$$

where $N_m = N_m(t, \mathbf{x})$ ($m = 1, 2, \ldots, 2r$) is a density of particles moving at the velocity $\mathbf{c}_m$

$$\mathbf{c}_m = c\left(\cos\frac{(m-1)\pi}{r}, \sin\frac{(m-1)\pi}{r}\right), \qquad m = 1, 2, \ldots, 2r. \tag{1.3}$$

The densities N_m are assumed to satisfy

$$N_{m+2r} = N_m, \qquad m = 1, 2, \ldots, 2r. \tag{1.4}$$

The aim of the paper is to analyse the forced sound wave propagation described by the two types of models (1.1) and (1.2). LONGO, MONACO and PŁATKOWSKI [3] were the first who investigated sound waves for the semidiscrete model.

This paper is a continuation of the previous one [4], and its objective is forced sound waves. Here we do not give full proofs and omit some calculations since they are similar to those given in [4].

2. Forced sound waves by the semidiscrete model

We linearize Eq. (1.1) around a uniform state of rest by setting

$$N(t, \mathbf{x}, \theta) = N_0\big(1 + P(t, \mathbf{x}, \theta)\big),$$

where P is a small perturbation. The function $P(t, \mathbf{x}, \theta)$ is 2π-periodic in θ. We introduce

$$A(t, \mathbf{x}, \theta) = \frac{1}{2}\,[P(t, \mathbf{x}, \theta) + P(t, \mathbf{x}, \theta+\pi)],$$

$$B(t, \mathbf{x}, \theta) = \frac{1}{2}\,[P(t, \mathbf{x}, \theta) - P(t, \mathbf{x}, \theta+\pi)].$$

The linearized Cabannes equation can be reduced to an equation for A only (see [4]):

$$\frac{\partial^2}{\partial t^2} A - \left(\mathbf{c}\cdot\frac{\partial}{\partial \mathbf{x}}\right)^2 A + 4cSn_0 \frac{\partial}{\partial t} A = \frac{4cSN_0}{\pi} \int_0^{\pi} \frac{\partial}{\partial t} A\, d\varphi, \qquad (0 \leqslant \theta \leqslant \pi). \tag{2.1}$$

In what follows, a one-dimensional flow in the x-direction, periodic in time, is considered. Therefore we look for solutions of Eq. (2.1) in the form

$$A(t, \mathbf{x}, \theta) = a(\theta)\exp\big(i(kx - \omega t)\big), \tag{2.2}$$

where the complex wave number k is treated as an unknown function of the real and positive frequency ω.

From Eqs. (2.1) and (2.2) we obtain the following equation for the amplitude $a(\theta)$:

$$(1+iR-2\lambda^2\cos^2\theta)a(\theta)-iR\frac{1}{\pi}\int\limits_0^\pi a(\varphi)d\varphi = 0, \tag{2.3}$$

where

$$\lambda = \frac{kc}{\sqrt{2}\omega}, \tag{2.4}$$

$$R = \frac{4cSN_0}{\omega}. \tag{2.5}$$

The parameter R can be interpreted as the inverse of the Knudsen number. We treat it as fixed but arbitrary and usually it is not indicated as an argument.

Calling $T(\lambda)$ the operator generated by the left-hand side of Eq. (2.3), we write this equation as follows:

$$T(\lambda)[a] = 0. \tag{2.6}$$

Similarly to [4] we take the set $C(\langle 0, \pi\rangle)$ of continuous functions defined on $\langle 0, \pi)$ as the domain of $T(\lambda)$. The eigenproblem (2.6) is not of the classical form $(\lambda I - A)$, for some A and I being the identity operator, hence, as usual, we call it generalized, and the spectrum of $T(\lambda)$ is called generalized.

As it is seen from Eq. (2.3), the essential generalized spectrum of $T(\lambda)$ consists of such λ that the coefficient of $a(\theta)$ can vanish for some $\theta \in \langle 0, \pi)$. Thus

$$\sigma_e = \left\{\lambda \in \mathrm{C}: 0 \leqslant \mathrm{Re}\frac{2(1+iR)}{\lambda^2} \leqslant 1, \quad \mathrm{Im}\frac{2(1+iR)}{\lambda^2} = 0\right\},$$

where Re and Im stand for the real and imaginary parts of a complex number.

The generalized essential spectrum of $T(\lambda)$ is shown in Fig. 1. In this figure the dashed line represents the boundary of the union over $R \geqslant 0$ of the generalized essential spectra of $T(\lambda, R)$, hence it is the boundary of the set

$$\bigcup_{R \leqslant 0} \sigma_e(R).$$

This boundary is a part of the following hyperbola:

$$2(\mathrm{Re}\lambda)^2 - 2(\mathrm{Im}\lambda)^2 = 1.$$

For every fixed R, the essential spectrum of $T(\lambda; R)$ consists of two infinite rays belonging to the same straight line: one ray is contained in the first quadrant and the second in the third one. In Fig. 1 the rays for $R = 0$ to 3 every 0.5 are shown.

The dots in Fig. 1 represent the point spectra of $T(\lambda, R)$ for $R = 0$ to 3 with step 0.5. If $\lambda \notin \sigma_e$, then the solution of Eq. (2.3) must be of the form

$$a(\theta) = \frac{C}{1+iR-2\lambda^2\cos^2\theta}, \tag{2.7}$$

where C is a constant.

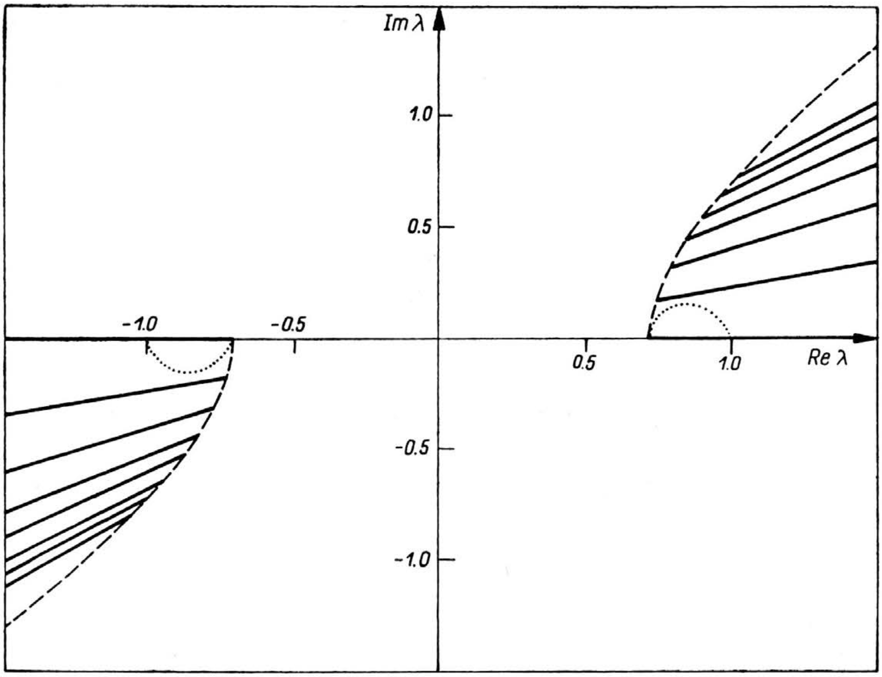

FIG. 1. The generalized spectrum of $T(\lambda)$.

Substituting Eq. (2.7) into Eq. (2.3), we check easily that $C \neq 0$ if and only if

$$\lambda^2 = \frac{1+2iR}{2(1+iR)}. \tag{2.8}$$

Thus the generalized point spectrum of $T(\lambda)$ consists of two eigenvalues. The corresponding eigenfunctions are given by Eq. (2.7). The eigenvalues of $T(\lambda)$ have a direct physical interpretation. Namely the real part of the eigenvalue λ is the sound dispersion, i.e., it is the inverse of the dimensionless sound speed, and the imaginary part of λ is the sound attenuation. They are shown in Figs. 2 and 3 versus the rarefaction parameter R.

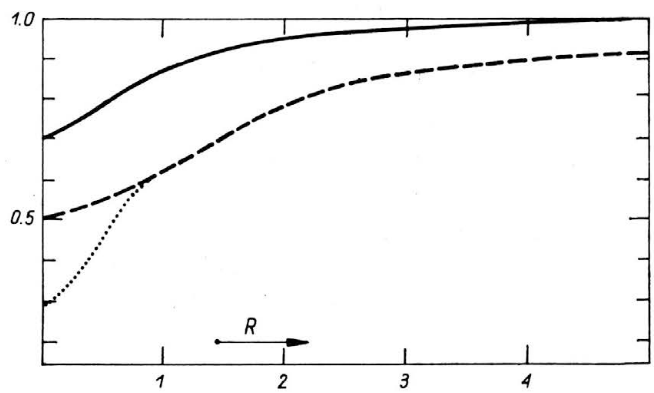

FIG. 2. Sound dispersion. ----- PEKERIS *et al.* [5] ——— present theory, ... SIROVICH, THURBER [8].

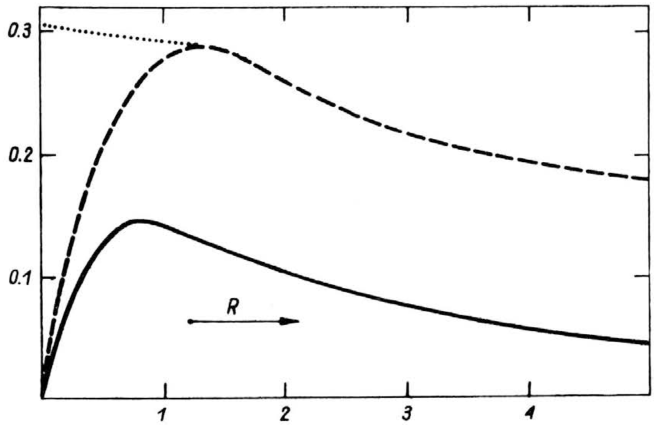

FIG. 3. Sound attenuation. ----- PEKERIS *et al.* [5] —— present theory, ... SIROVICH, THURBER [8].

The present results show some qualitative similarity to those obtained by PEKERIS, ALTERMAN, FINKELSTEIN and FRANKOWSKI [5] from the true linearized Boltzmann equation. In particular, we see from Fig. 3 that the attenuation coefficient vanishes as the sound frequency ω tends to infinity. This is, however, an unrealistic result since both the experiments [6, 7] at the theoretical results by SIROVICH and THURBER [8], predict finite positive attenuation for large values of ω. To explain the agreement between the present results and those of PEKERIS *et al.* [5], we remind first the essence of their method. The authors looked for solutions of the linearized Boltzmann equation by expanding the perturbation to the distribution function in a series of Sonine polynomials, and next they truncated the series at a finite number of terms. Since such a truncation gives an asymptotic expansion which is not uniform as the molecular velocity tends to infinity, they ignored, as a matter of fact, high molecular velocities. Similarly, in the Cabannes model the high molecular velocities are ignored since they all have the same absolute value. Thus in both theories the high molecular velocities are missing, and, in our opinion, it is this feature which makes the results similar.

3. Forced sound waves by discrete models

We linearize Eqs. (1.2) around a uniform state of equilibrium by setting

$$N_m(t, \mathbf{x}) = N_0\big(1+P_m(t, \mathbf{x})\big), \quad m = 1, 2, \ldots, 2r,$$

where P_m is a small perturbation.

We define

$$A_m = \frac{1}{2}(P_m+P_{m+r}),$$

$$B_m = \frac{1}{2}(P_m-P_{m+r}), \quad m = 1, 2, \ldots, 2r.$$

In [4] the quantities A_m are shown to satisfy the following system of equations:

(3.1) $$\frac{\partial^2}{\partial t^2}A_m-\left(\mathbf{c}_m\cdot\frac{\partial}{\partial \bar{x}}\right)^2 A_m+4cSN_0\frac{\partial}{\partial t}A_m=\frac{4cSN_0}{r}\sum_{k=1}^{r}\frac{\partial}{\partial t}A_k,\qquad m=1,2,\ldots,r$$

and

(3.2) $$A_{m+r}=A_m.$$

Knowing A_m, we can find B_m and, consequently, P_m (see [4]). We look for solutions of Eqs. (3.1) in the form

(3.3) $$A_m(t,\bar{x})=a_m\exp\big(i(kx-\omega t)\big),$$

where k, x, t and ω have the same meaning as in the previous chapter.

Substituting Eq. (3.3) into Eqs. (3.1), we obtain for a_m the following set of linear algebraic equation:

(3.4) $$\left(1+iR-2\lambda^2\cos^2\frac{(m-1)\pi}{r}\right)a_m-\frac{iR}{r}\sum_{k=1}^{r}a_k=0,$$

where λ and R are the same as in Eqs. (2.4) and (2.5). Let $T_r(\lambda)$ stand for an operator given by the left hand side of Eqs. (3.4). It is an operator mapping the complex r-dimensional space $\mathbf{C}^r$ into itself.

Let us notice that the last term on the left hand side of Eqs. (3.4) is constant, i.e., it does not depend on m. Therefore the general solution of these equations is of the form

(3.5) $$a_m=\frac{C}{1+iR-2\lambda^2\cos^2\dfrac{(m-1)\pi}{r}},\qquad m=1,2,\ldots,r,$$

where C is a constant, unless the denominator vanishes. This happens if

(3.6) $$\lambda_m^{\pm}=\pm\sqrt{\frac{1+iR}{2\cos^2\dfrac{(m-1)\pi}{r}}},\qquad m=1,2,\ldots,r.$$

All complex numbers of the form (3.6) are denoted by $\sigma_r^{(1)}$. We have obviously

(3.7) $$\sigma_r^{(1)}\subset\sigma_r\subset\sigma_e,$$

where σ_r is the generalized spectrum of $T_r(\lambda)$, σ_e is the generalized essential spectrum of $T(\lambda)$. We add the word "generalized" to mark that Eq. (3.4) is not of the classical eigenproblem of the form

$$\det(\lambda I-A)=0,$$

where I is a unit matrix, and A is a quadratic matrix. Substituting Eq. (3.5) into Eq. (3.4), we obtain an equation for C, which has a nonzero solution if and only if λ is such that

(3.8) $$F_r(\lambda)=0,$$

where

(3.9) $$F_r(\lambda)=1-\frac{iR}{r}\sum_{i=1}^{r}\frac{1}{1+iR-2\lambda^2\cos^2\dfrac{(m-1)\pi}{r}}.$$

With the symbol $\sigma_r^{(2)}$ we denote the set of the eigenvalues of $T_r(\lambda)$ which are solutions of Eq. (3.8). We have obviously

$$\sigma_r = \sigma_r^{(1)} \cup \sigma_r^{(2)}, \qquad \sigma_r^{(1)} \cap \sigma_r^{(2)} = \emptyset. \tag{3.10}$$

LEMMA 1. Let $R > 0$, and let $\lambda^{\pm}$ be the eigenvalues of $T(\lambda)$. For every real ε such that $0 < \varepsilon < \operatorname{dist}(\lambda^{\pm}, \sigma_e)$ where $\operatorname{dist}(z, A)$ is the distance between a point $z \in \mathbf{C}$ and a set $A \subset \mathbf{C}$, there exists an integer r_ε such that for every $r \geqslant r_\varepsilon$ the operator $T_r(\lambda)$ has exactly one eigenvalue in an ε-neighbourhood of λ^+ and exactly one eigenvalue in an ε-neighbourhood of λ^-.

SKETCH OF THE PROOF. We define

$$\tilde{F}_r(\mu) = F_r\left(\frac{1}{\mu}\right), \tag{3.11}$$

$$\tilde{F}(\mu) = F\left(\frac{1}{\mu}\right), \tag{3.12}$$

where

$$F(\lambda) = 1 - \frac{iR}{\pi}\int_0^{\pi} \frac{1}{1+iR-2\lambda^2\cos^2\varphi}\, d\varphi. \tag{3.13}$$

The zeros of $\tilde{F}(\mu)$ are

$$\mu^{\pm} = \frac{1}{\lambda^{\pm}}.$$

Now, proceeding as in the proof of Lemma 2 of [4], we obtain easily the assertion. We proceed to determine the eigenvectors of $T_r(\lambda)$.

We introduce the function

$$a(\theta, \lambda) = \frac{1}{1+iR-2\lambda^2\cos^2\theta}.$$

This is well defined if $\lambda \notin \sigma_e$ where σ_e is the generalized essential spectrum of $T(\lambda)$.

LEMMA 2. Let $R > 0$

i) The components $a_m (m = 1, 2, \ldots, r)$ of the eigenvector $a(\lambda)$ corresponding to $\lambda \in \sigma_r^{(2)}$ are given by

$$a_m = a\left(\frac{(m-1)\pi}{r}, \lambda\right), \qquad m = 1, 2, \ldots, r, \qquad \lambda \in \sigma_r^{(2)}. \tag{3.14}$$

ii) If $\lambda \in \sigma_r^{(1)}$, then the components a_m of the eigenvector $a(\lambda)$ are given by

$$a_m(\lambda_j^{\pm}) = \delta_{m,j+1} - \delta_{m,2s-j+1}, \qquad m = 1, 2, \ldots, r, \qquad j = 1, 2, \ldots, s-1$$

in the case of $r = 2s$ where δ_{kl} is the Kronecker's delta, and $\lambda_j^{\pm}$ are given by Eq. (3.11); any by

$$a_m(\lambda_j^{\pm}) = \delta_{m,j+1} - \delta_{m,2s-j+2}, \qquad m = 1, 2, \ldots, r, \qquad j = 1, 2, \ldots, s$$

where $\lambda_j^{\pm}$ are given by Eq. (3.6).

The proof of the above Lemma is similar to that of Lemma 6 of [4].

4. Approximation of the spectrum problem of $T(\lambda)$ with that of $T_r(\lambda)$

We put

$$\sigma_\infty = \bigcup_{r=2}^{\infty} \sigma_r$$

and

$$\sigma_T = \sigma_e \cup \{\lambda^+, \lambda^-\}.$$

We define

$$\tilde{\sigma}_\infty = \left\{\mu \in \mathsf{C}: \frac{1}{\mu} \in \sigma_\infty\right\},$$

$$\tilde{\sigma}_T = \left\{\mu \in \mathsf{C}: \frac{1}{\mu} \in \sigma_T\right\} \cup \{0\}.$$

Following [4] it can be shown that for every fixed $R \geqslant 0$, the set $\tilde{\sigma}_\infty$ is bounded. Using that we can prove

THEOREM 1. $\tilde{\sigma}^d_\infty = \tilde{\sigma}_T$ *where* $\tilde{\sigma}^\infty_d$ *is the derivative of* σ_∞ *i.e.,* $\tilde{\sigma}^d_\infty$ *is the set of points of accumulation of* σ_∞.

From this theorem it follows immediately

COROLLARY

Given a $\lambda \in \sigma_T$ and a positive ε, then there is $\tilde{\lambda} \in \sigma_\infty$ such that $|\lambda - \tilde{\lambda}| < \varepsilon$.

Of course, there is r such that $W_r(\tilde{\lambda}) = 0$. The number r depends generally on ε, what is natural, but also it depends on λ what is undesirable. Unfortunately, this last dependence cannot be removed. This is due to the fact that for every fixed r (and R) the generalized spectrum of $T_r(\lambda)$ is contained in a bounded subset of the complex plane C, whereas the generalized essential spectrum of $T(\lambda)$ is unbounded. In other words, with the generalized spectrum of one fixed $T_r(\lambda)$, however large r can be, it is impossible to approximate the generalized spectrum of $T(\lambda)$. Consequently, in this aspect the Cabannes semidiscrete model can be hardly treated as a limit of the regular $2r$ velocity model.

Let $\lambda_r^{(\pm)}$ be one of these eigenvalues of $T_r(\lambda)$ Lemma 1 says about, and let $a\left(\frac{(m-1)\pi}{r}, \lambda_r^{(\pm)}\right)$ $(m = 1, 2, \ldots, r)$ be the components of the corresponding eigenvector. We form the following step function:

$$a_r^{(\pm)}(\theta) = \sum_{m=1}^{r} a\left(\frac{(m-1)\pi}{r}, \lambda_r^{(\pm)}\right)\chi_m^{(r)}(\theta),$$

where $\chi_m^{(r)}(\theta)$ is the characteristic function (indicator) of the interval

$$\left\langle \frac{(m-1)\pi}{r}, \frac{m\pi}{r}\right).$$

Following [4] it can be proved:

THEOREM 2. *Let* $R > 0$, *then*

$$\lim_{r\to\infty} a_r^{\pm}(\theta) = a^{\pm}(\theta),$$

uniformly in $\theta \in \langle 0, \pi)$.

Acknowledgment

This paper was supported by the Polish Government Scientific Program C.P.B.P.—01.02.

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Received September 14, 1988.