Arch. Mech., **41**, 5, pp. 725–733, Warszawa 1989

Dusty hydrodynamics oscillation between two perturbed parallel plates

E. F. ELSHEHAWEY (CAIRO)

THE PROPAGATION of coupled modes in a viscous incompressible dusty fluid confined between two perturbed parallel plates is considered. The treatment is confined to the first-order perturbation theory in order to simplify the algebra. The multiple scales approach yields information on the transition curve separating passbands from stopbands as well as the interaction equations which govern the amplitudes and phases of the coupled modes.

Rozważono propagację sprzężonych postaci drgań cieczy nieściśliwej zawierającej zawiesinę pyłową pod wpływem zakłóceń wprowadzonych przez równoległe ścianki przewodu. Dla uproszczenia obliczeń analizę ograniczono do uwzględnienia teorii perturbacji pierwszego rzędu. Podejście wieloskalowe pozwala uzyskać informacje dotyczące krzywej przejścia oddzielającej pasma przepustowe od pasm tłumieniowych, jak również określające współzależność amplitud i faz sprzężonych postaci drgań.

Рассмотрено распространение сопряженных типов колебений несжимаемой жидкости, содержавшей пылевую взвесь, под влиянием возмущений введенных параллельными стенками провода. Дла упрощения расчетов анализ ограничивается учетом теории пертурбаций первого порядка. Многомасштабный подход позволяет получить информации, касающиеся кривой перехода, отделяющей пропускные полосы от полос задерживания, как тоже определяющие взаимо зависимость амплитуд и фаз сопряженных типов колебаний.

1. Introduction

RECENTLY, there has been considerable interest in periodic boundary perturbation problems, and a review of much of the literature is presented in ASFAR and NAYFEH [1].

The study of solid particles-fluid flow systems has been the object of scientific and engineering research for a long time. The problem has appeared in various forms such as sediment transport by water and by air, the centrifugal separation of particulate matter from fluids, fluid-droplet sprays, fluidized beds and other two—phase phenomena of interest in chemical processing. A number of studies of fluid embedded with particles have appeared in the literature [2–8].

The problem of stability of a fluid layer has recently attracted investigators in various fields as a result of its important physical applications in chemical engineering, medicine, etc. Much work has been done on the hydrodynamic stability of a fluid layer without periodic boundaries [9–13].

The aim of the present study is to examine the application of the method of multiple scales [14] to the problem of propagation of coupled modes in a dusty viscous incompressible fluid confined between two periodic plates.

2. The system in equilibrium state

The equations of motion of a dusty, unsteady, viscous incompressible fluid based on SAFFMAN's [3] model of a dusty gas are

$$\bar{\varrho}\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot \operatorname{grad}\mathbf{v}\right) = \bar{\varrho}\mathbf{g}-\operatorname{grad}p+\mu\nabla^2\mathbf{v}+\bar{K}N(\mathbf{u}-\mathbf{v}), \tag{2.1}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{2.2}$$

$$\bar{M}\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot \operatorname{grad}\mathbf{u}\right) = \bar{K}(\mathbf{v}-\mathbf{u}), \tag{2.3}$$

$$\frac{\partial N}{\partial t}+\operatorname{div}(N\mathbf{u}) = 0, \tag{2.4}$$

where $\mathbf{v}$ is the velocity of the fluid, $\mathbf{u}$ is the dusty velocity, p, $\bar{M}$, $\bar{K}$, $\bar{N}$, and $\bar{\varrho}$ are the fluid pressure, the mass concentration of the dust particles, the Stokes resistance coefficient which, for spherical particles, is $6\pi\mu d$ (μ being the coefficient of viscosity and d the radius of dust particles; assumed to be spherical), the number density of the dust particles and the fluid density, respectively. $\mathbf{g} = (0, 0, -g)$ is the gravitational accelaration.

It is clear that there exist the following steady state solutions, when the number density N of the dust particles is taken as a constant N_0

$$\bar{v} = 0, \quad \bar{u} = 0 \quad \text{and} \quad \bar{\varrho}\mathbf{g} = \operatorname{grad}\bar{p}. \tag{2.5}$$

3. Perturbation equations

Let the initial steady state be slightly perturbed. Following the classical lines of the linear stability theory as presented by CHANDRASEKHAR [9], the equations governing two-dimensional small perturbations may be written as

$$\bar{\varrho}\frac{\partial v_x'}{\partial t} = -\frac{\partial p'}{\partial x}+\mu\nabla^2 v_x'+\bar{K}N_0(u_x'-v_x'), \tag{3.1}$$

$$\bar{\varrho}\frac{\partial v_z'}{\partial t} = -\frac{\partial p'}{\partial z}+\mu\nabla^2 v_z'+\bar{K}N_0(u_z'-v_z'), \tag{3.2}$$

$$\frac{\partial v_x'}{\partial x}+\frac{\partial v_z'}{\partial z} = 0, \tag{3.3}$$

$$\bar{M}\frac{\partial u_x'}{\partial t} = \bar{K}(v_x'-u_x'), \tag{3.4}$$

$$\bar{M}\frac{\partial u_z'}{\partial t} = \bar{K}(v_z'-u_z'), \tag{3.5}$$

$$\frac{\partial u_x'}{\partial x}+\frac{\partial u_z'}{\partial z} = 0, \tag{3.6}$$

where the bar and the prime indicate steady-state and perturbed quantities, respectively, and

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

Equations (3.1)–(3.6) are simplified in the usual manner by decomposing the solution in terms of normal modes, so that the space and the time dependence of the perturbed quantities are of the type

$$F(x, z; t) = \hat{f}(x, z)e^{\omega t}, \tag{3.7}$$

where ω is the time constant which is complex in general. Thus we arrive at

$$\nabla^2 \hat{v}_x - \left[\frac{\bar{\varrho}\omega}{\mu} + \frac{\bar{M}\omega\bar{K}N_0}{\mu(\bar{M}\omega+\bar{K})}\right]\hat{v}_x(x, z) = \frac{1}{\mu}\frac{\partial\hat{p}(x, z)}{\partial x}, \tag{3.8}$$

$$\nabla \hat{v}_z - \left[\frac{\bar{\varrho}\omega}{\mu} + \frac{\bar{M}\omega\bar{K}N_0}{\mu(\bar{M}\omega+\bar{k})}\right]\hat{v}_z(x, z) = \frac{1}{\mu}\frac{\partial\hat{p}(x, z)}{\partial z}, \tag{3.9}$$

$$\frac{\partial\hat{v}_x}{\partial x} + \frac{\partial\hat{v}_z}{\partial z} = 0, \tag{3.10}$$

$$\hat{u}_x = \frac{\bar{K}}{\bar{M}\omega+\bar{K}}\hat{v}_x, \tag{3.11}$$

$$\hat{u}_z = \frac{\bar{K}}{\bar{M}\omega+\bar{K}}\hat{v}_z. \tag{3.12}$$

4. A periodic wall distortion function

We consider an infinite layer of dusty viscous fluid of depth 1. The lower surface at $z = 0$ and the upper surface at $z = 1$.

The boundary condition to be satisfied at a hard wall is the vanishing of the normal component of fluid and dust velocity. Let the two infinite plates be perturbed according to the wall distortion functions

$$z = \varepsilon \sin K_\omega x \qquad \text{lower plate}, \tag{4.1}$$

$$z = 1+\varepsilon\sin(K_\omega x+\tau) \quad \text{upper plate}, \tag{4.2}$$

where ε is the dimensionless amplitude of the wall undulations and is assumed to be much smaller than unity, K_ω is the spatial wavenumber of the perturbed plate and τ is phase shift.

The boundary condition can be written as

$$\begin{aligned} &\hat{v}_z(x, z) = \varepsilon K_\omega \hat{v}_x(x, z)\operatorname{Cos} K_\omega x, \\ &\hat{u}_z(x, z) = \frac{\varepsilon\bar{K}K_\omega}{\bar{M}\omega+\bar{K}}\hat{v}_x(x, z)\operatorname{Cos} K_\omega x \quad \text{at} \quad z = \varepsilon\sin K_\omega x, \end{aligned} \tag{4.3}$$

and

$$(4.4)\qquad \begin{aligned} \hat{v}_z(x, z) &= \varepsilon K_\omega \hat{v}_x(x, z)\operatorname{Cos}(K_\omega x + \tau), \\ \hat{u}_z(x, z) &= \frac{\varepsilon \bar{K} K_\omega}{\bar{M}\omega + \bar{K}} \hat{v}_x(x, z)\operatorname{Cos}(K_\omega x + \tau) \quad \text{at} \quad z = 1 + \varepsilon \sin(K_\omega x + \tau). \end{aligned}$$

We now introduce the stream function Ψ such that

$$(4.5)\qquad \hat{v}_x = \frac{\partial \Psi}{\partial z}, \quad \hat{v}_z = -\frac{\partial \Psi}{\partial x},$$

$$(4.6)\qquad \hat{u}_x = \frac{\bar{K}}{\bar{M}\omega + \bar{K}} \frac{\partial \Psi}{\partial z}, \quad \hat{u}_z = -\frac{\bar{K}}{\bar{M}\omega + \bar{K}} \frac{\partial \Psi}{\partial x},$$

and the vorticity $\hat{\Omega}$ such that

$$(4.7)\qquad \hat{\Omega} = \frac{\partial \hat{v}_x}{\partial z} - \frac{\partial \hat{v}_z}{\partial x} = \frac{\bar{M}\omega + \bar{K}}{\bar{K}} \left(\frac{\partial \hat{u}_x}{\partial z} - \frac{\partial \hat{u}_z}{\partial x} \right) = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial z^2}.$$

From Eqs. (3.8)–(3.12) and (4.5)–(4.7) we get the equation of conservation of vorticity

$$(4.8)\qquad \nabla^2 \hat{\Omega} + \lambda_p^2 \hat{\Omega} = 0,$$

where

$$(4.9)\qquad \lambda_p^2 = -\frac{\bar{\varrho}\omega}{\mu} - \frac{\bar{M}\omega \bar{K} N_0}{\mu(\bar{M}\omega + \bar{K})}.$$

The corresponding boundary conditions will be

$$(4.10)\qquad -\frac{\partial \Psi(x, z)}{\partial x} = \varepsilon K_\omega \frac{\partial \Psi(x, z)}{\partial z} \operatorname{Cos} K_\omega x, \quad \text{at} \quad z = \varepsilon \sin K_\omega x,$$

$$(4.11)\qquad -\frac{\partial \Psi(x, z)}{\partial x} = \varepsilon K_\omega \frac{\partial \Psi(x, z)}{\partial z} \operatorname{Cos}(K_\omega x + \tau), \quad \text{at} \quad z = 1 + \varepsilon \sin(K_\omega x + \tau).$$

5. The method of multiple scales

We seek a first-order uniform asymptotic expansion of $\hat{\Omega}$ and Ψ from

$$(5.1)\qquad \hat{\Omega}(x, z) = \hat{\Omega}_0(x_0, x_1;\, z) + \varepsilon \hat{\Omega}_1(x_0, x_1;\, z) + \ldots,$$

$$(5.2)\qquad \Psi(x, z) = \Psi_0(x_0, x_1;\, z) + \varepsilon \Psi_1(x_0, x_1;\, z) + \ldots,$$

where $x_0 = x$ is a length scale characterizing distances that are the order of a wawelength and $x_1 = \varepsilon x$ is a long length scale characterizing the spatial amplitude and phase modulations. Substituting Eqs. (5.1) and (5.2) into Eqs. (4.8)–(4.11), we transfer the boundary conditions (4.10) and (4.11) to the uniform boundaries $z = 0$ and $z = 1$ by developing Ψ and its derivatives in Taylor series around $z = 0$ and $z = 1$. Then we equate the coefficients of equal powers of ε on both sides of every equation and obtain

$$(5.3)\qquad \frac{\partial^2 \hat{\Omega}_0}{\partial x_0^2} + \frac{\partial^2 \hat{\Omega}_0}{\partial z^2} + \lambda_p^2 \hat{\Omega}_0 = 0,$$

$$(5.4)\qquad \hat{\Omega}_0 = \frac{\partial \Psi_0}{\partial x_0^2} + \frac{\partial^2 \Psi_0}{\partial z^2},$$

$$\frac{\partial \Psi_0}{\partial x_0} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1, \tag{5.5}$$

$$\frac{\partial^2 \hat{\Omega}_1}{\partial x_0^2} + \frac{\partial^2 \hat{\Omega}_1}{\partial z^2} + \lambda_p^2 \hat{\Omega}_1 = -2 \frac{\partial^2 \hat{\Omega}_0}{\partial x_0 \partial x_1}, \tag{5.6}$$

$$\hat{\Omega}_1 = \frac{\partial^2 \Psi_1}{\partial z^2} + 2 \frac{\partial^2 \Psi_0}{\partial x_0 \partial x_1} + \frac{\partial^2 \Psi_1}{\partial x_0^2}, \tag{5.7}$$

$$-\frac{\partial \Psi_1}{\partial x_0} = \sin(K_\omega x_0) \frac{\partial^2 \Psi_0}{\partial x_0 \partial z} + \frac{\partial \Psi_0}{\partial x_1} + K_\omega \frac{\partial \Psi_0}{\partial z} \operatorname{Cos} K_\omega x_0 \quad \text{at} \quad z = 0, \tag{5.8}$$

$$-\frac{\partial \Psi_1}{\partial x_0} = \sin(K_\omega x_0 + \tau) \frac{\partial^2 \Psi_0}{\partial x_0 \partial z} + \frac{\partial \Psi_0}{\partial x_1} + K_\omega \frac{\partial \Psi_0}{\partial z} \operatorname{Cos}(K_\omega x_0 + \tau) \quad \text{at} \quad z = 1. \tag{5.9}$$

The solution of the reduced problem (5.3)–(5.5), for the case of resonance, with the two dominant resonant modes with wave numbers K_m and K_n can be expressed as

$$\hat{\Omega}_0 = A_n(x_1) \sin(n\pi z) e^{iK_n x_0} + A_m(x_1) \sin(m\pi z) e^{iK_m x_0}, \tag{5.10}$$

$$\Psi_0 = -\frac{A_n(x_1)}{n^2\pi^2 + K_n^2} \sin(n\pi z) e^{iK_n x_0} - \frac{A_m(x_1)}{m^2\pi^2 + K_m^2} \sin(m\pi z) e^{iK_m x_0}, \tag{5.11}$$

where

$$\begin{aligned} K_n^2 &= \lambda_p^2 - n^2\pi^2 > 0, \\ K_m^2 &= \lambda_p^2 - m^2\pi^2 > 0 \quad \text{for a propagating mode,} \end{aligned} \tag{5.12}$$

n and m are integers, and $A_n(x_1)$ and $A_m(x_1)$ are to be determined at the next level of approximation.

Substituting Eqs. (5.10) and (5.11) into Eqs. (5.6)–(5.9) yields

$$\frac{\partial^2 \hat{\Omega}_1}{\partial x_0^2} + \frac{\partial^2 \hat{\Omega}_1}{\partial z^2} + \lambda_p^2 \hat{\Omega}_1 = -2iK_n \frac{\partial A_n}{\partial x_1} \sin(n\pi z) e^{iK_0 x_0} - 2iK_m \frac{\partial A_m}{\partial x_1} \sin(m\pi z) e^{iK_m x_0}, \tag{5.13}$$

$$\hat{\Omega}_1 = \frac{\partial^2 \Psi_1}{\partial x_0^2} + \frac{\partial^2 \Psi_1}{\partial z^2} - \frac{2iK_n}{n^2\pi^2 + K_n^2} \frac{\partial A_n}{\partial x_1} \sin(n\pi z) e^{iK_n x_0} - \frac{2iK_m}{m^2\pi^2 + K_m^2} \frac{\partial A_m}{\partial x_1} \sin(m\pi z) e^{iK_m x_0}, \tag{5.14}$$

$$\begin{aligned} -\frac{\partial \Psi_1}{\partial x_0} = &-\frac{n\pi(K_n + K_\omega)}{2(n^2\pi^2 + K_n^2)} A_n(x_1) e^{i(K_\omega + K_n)x_0} - \frac{m\pi(K_m + K_\omega)}{2(m^2\pi^2 + K_m^2)} A_m e^{i(K_\omega + K_m)x_0} \\ &+ \frac{n\pi(K_n + K_\omega)}{2(n^2\pi^2 + K_n^2)} A_n(x_1) e^{i(K_n - K_\omega)x_0} + \frac{m\pi(K_m - K_\omega)}{2(m^2\pi^2 + K_m^2)} A_m(x_1) e^{i(K_m - K_\omega)x_0} \quad \text{at} \quad z = 0, \end{aligned} \tag{5.15}$$

$$\begin{aligned} -\frac{\partial \Psi_1}{\partial x_0} = &\frac{-n\pi(-1)^n (K_n + K_\omega)}{2(n^2\pi^2 + K_m^2)} A_n e^{i[(K_\omega + K_n)x_0 + \tau]} \\ &- \frac{m\pi(-1)^m (K_m + K_\omega)}{2(m^2\pi^2 + K_m^2)} A_m e^{i[(K_\omega + K_m)x_0 + \tau]} + \frac{n\pi(-1)^n (K_n - K_\omega)}{2(n^2\pi^2 + K_m^2)} A_n e^{i[(K_n - K_\omega)x_0 - \tau]} \\ &+ \frac{m\pi(-1)^m (K_m - K_\omega)}{2(m^2\pi^2 + K_n^2)} A_m e^{i[(K_m - K_\omega)x_0 - \tau]} \quad \text{at} \quad z = 1, \end{aligned} \tag{5.16}$$

The appearance of resonant terms in the right-hand sides of Eqs. (5.13) is undesirable because they make $\hat{\Omega}_1$ unbounded. However, these secular producing terms can be eliminated by a proper choice of the arbitrary functions $A_n(x_1)$ and $A_m(x_1)$.

This particular choice is furnished by the solvability (integrability) condition of the system (5.13)–(5.16). Since the homogeneous parts of Eqs. (5.3) and (5.13) are identical, the right-hand sides (or inhomogeneous parts) of Eqs. (5.13) must be orthogonal to the solutions of the homogeneous problem. This is the solvability condition we are seeking. To achieve this, we seek a particular solution of Eqs. (5.13)–(5.16) in the form

$$\hat{\Omega}_1 = i\Phi_n(x_1, z)e^{iK_nx_0} + i\Phi_m(x_1, z)e^{iK_mx_0}, \tag{5.17}$$

$$\Psi_1 = i\Phi_n^*(x_1, z)e^{iK_nx_0} + i\Phi_m^*(x_{,1} z)e^{iK_mx_0}. \tag{5.18}$$

We also consider the case of near resonance; that is, the two modes are coupled in a frequency range close to the perfectly tuned case. To describe quantitatively the nearness of K_m to $K_n \pm K_\omega$, we introduce a detuning parameter defined by

$$K_m = K_n \pm K_\omega + \varepsilon\sigma, \qquad \sigma = O(1). \tag{5.19}$$

This equation could be used with either the plus or minus sign and in each case the condition could imply either one of two types of interaction: the interaction of two codirectional modes or the interaction of two contradirectional modes. However, if we choose coupled modes such that $|K_n| > |K_m|$, then the plus sign corresponds to two opposite modes whereas the minus sign corresponds to two codirectional modes. We use Eq. (5.19) with the minus sign and express $e^{i(K_n-K_\omega)x_0}$ and $e^{i(K_m+K_\omega)x_0}$ as

$$e^{i(K_n-K_\omega)x_0} = e^{iK_mx_0-i\varepsilon\sigma x_0} = e^{iK_mx_0-i\sigma x_1}, \tag{5.20}$$

$$e^{i(K_m+K_\omega)x_0} = e^{iK_nx_0+i\sigma x_1}. \tag{5.21}$$

Substituting Eqs. (5.17), (5.18), (5.20) and (5.21) into Eqs. (5.13)–(5.16) and equating the coefficients of $e^{iK_mx_0}$ and $e^{iK_nx_0}$ on both sides, we obtain

$$\frac{\partial^2\Phi_n}{\partial z^2} + n^2\pi^2\Phi_n = -2K_n\frac{\partial A_n}{\partial x_1}\sin(n\pi z), \tag{5.22}$$

$$\frac{\partial^2\Phi_n^*}{\partial z^2} - K_n^2\Phi_n^* = \frac{K_n}{n\pi}\frac{\partial A_n}{\partial x_1}z\,\mathrm{Cos}(n\pi z) + \frac{2K_n}{n^2\pi^2+K_n^2}\frac{\partial A_n}{\partial x_1}\sin(n\pi z), \tag{5.23}$$

$$\frac{\partial^2\Phi_m}{\partial z^2} + m^2\pi^2\Phi_m = -2K_m\frac{\partial A_m}{\partial x_1}\sin(m\pi z), \tag{5.24}$$

$$\frac{\partial^2\Phi_m^*}{\partial z^2} - K_m^2\Phi_m^* = \frac{K_m}{m\pi}\frac{\partial A_m}{\partial x_1}z\,\mathrm{Cos}(m\pi z) + \frac{2K_m}{m^2\pi^2+K_m^2}\frac{\partial A_m}{\partial x_1}\sin(m\pi z), \tag{5.25}$$

$$\Phi_n^*(x_1, 0) = \frac{-m\pi A_m(x_1)}{2(m^2\pi^2+K_m^2)}\frac{K_n+\varepsilon\sigma}{K_n^2}e^{i\sigma x_1}, \tag{5.26}$$

$$\Phi_m^*(x_1, 0) = \frac{n\pi A_n(x_1)}{2(n^2\pi^2+K_n^2)}\frac{K_m-\varepsilon\sigma}{K_m^2}e^{-i\sigma x_1}, \tag{5.27}$$

(5.28) $$\Phi_n^*(x_{1,1}) = \frac{-m\pi A_m(x_1)(-1)^m}{2(m^2\pi^2+K_m^2)} \frac{K_n+\varepsilon\sigma}{K_m^2} e^{i(\sigma x_1+\tau)},$$

(5.29) $$\Phi_m^*(x_{1,1}) = \frac{n\pi A_n(x_1)(-1)^n}{2(n^2\pi^2+K_n^2)} \frac{K_m-\varepsilon\sigma}{K_m^2} e^{-i(\sigma x_1+\tau)}.$$

Hence the particular solution of Eqs. (5.13)–(5.16) is given by

(5.30) $$\hat{\Omega}_1 = \frac{iK_n}{n\pi}\frac{\partial A_n}{\partial x_1} z\,\mathrm{Cos}(n\pi z)\, e^{iK_n x_0} + \frac{iK_m}{m\pi}\frac{\partial A_m}{\partial x_1} z\,\mathrm{Cos}(m\pi z)\, e^{iK_m x_0},$$

(5.31)
$$\Psi_1 = \frac{-iK_n}{n\pi(K_n^2+n^2\pi^2)}\frac{\partial A_n}{\partial x_1} z\,\mathrm{Cos}(n\pi z)\, e^{iK_n x_0} - \frac{iK_m}{m\pi(K_m^2+m^2\pi^2)}\frac{\partial A_m}{\partial x_1} z\,\mathrm{Cos}(m\pi z)\, e^{iK_m x_0},$$

and the solvability condition gives

(5.32) $$\frac{\partial A_n}{\partial x_1} = \frac{mn\pi^2}{2K_n^3}(K_n+\varepsilon\sigma)A_m(x_1)[-1+(-1)^{m+n}e^{i\tau}]e^{i\sigma x_1},$$

(5.33) $$\frac{\partial A_m}{\partial x_1} = \frac{nm\pi^2}{2K_m^3}(K_m-\varepsilon\sigma)A_n(x_1)[1-(-1)^{m+n}e^{-i\tau}]e^{-i\sigma x_1}.$$

Equations (5.32) and (5.33) govern the amplitudes and phases of the coupled modes.

We seek the solutions of Eqs. (5.32) and (5.33) in the form

(5.34) $$A_m = O_m e^{Sx_1},$$

(5.35) $$A_n = O_n e^{(S+i\sigma)x_1},$$

where O_m and O_n are constants. Substituting Eqs. (5.34) and (5.35) into Eqs. (5.32) and (5.33), and eliminating the O, we obtain

(5.36) $$S^2+i\sigma S-\hat{\omega} = 0,$$

where

(5.37) $$\hat{\omega} = \left(\frac{nm\pi^2}{\sqrt{2}K_nK_m}\right)^2\left(\frac{K_\omega}{K_m}+1\right)\left(\frac{K_\omega}{K_n}-1\right)[1-(-1)^{m+n}\mathrm{Cos}\,\tau].$$

The characteristic exponent S is thus given by

(5.38) $$S_{1,2} = \frac{i}{2}[-\sigma\pm(\sigma^2-4\hat{\omega})^{1/2}].$$

6. The transition curve

The relevant first order approximation for the stream function Ψ and vorticity $\hat{\Omega}$ for the given system are

(6.1) $$\Psi = \frac{-O_n}{n^2\pi^2+K_n^2}\left[\mathrm{Sin}(n\pi z)+\frac{\varepsilon K_n(\sigma-iS)}{n\pi} z\,\mathrm{Cos}\,n\pi z\right]e^{i(K_n+\varepsilon\sigma)x+\varepsilon Sx}$$
$$+\frac{-O_m}{m^2\pi^2+K_m^2}\left\{\mathrm{Sin}(m\pi z)+\frac{i\varepsilon SK_m}{m\pi} z\,\mathrm{Cos}\,m\pi z\right\}e^{i(K_m x)+\varepsilon Sx}+O(\varepsilon^2),$$

$$(6.2)\qquad \hat{\Omega} = O_n\left[\operatorname{Sin}(n\pi z)+\frac{\varepsilon K_n(\sigma-iS)}{n\pi}z\operatorname{Cos}n\pi z\right]e^{i(K_n+\varepsilon\sigma)x+\varepsilon Sx}$$

$$+O_m\left[\operatorname{Sin}(m\pi z)+\frac{i\varepsilon SK_m}{m\pi}z\operatorname{Cos}m\pi z\right]e^{iK_m x+\varepsilon Sx}+O(\varepsilon^2),$$

where

$$(6.3)\qquad K_n^2 = -\frac{\varrho\omega}{\mu}-\frac{\bar{M}\omega\bar{K}N_0}{\mu(\bar{M}\omega+\bar{K})}-n^2\pi^2,$$

$$(6.4)\qquad K_m^2 = -\frac{\varrho\omega}{\mu}-\frac{\bar{M}\omega\bar{K}N_0}{\mu(\bar{M}\omega+\bar{K})}-m^2\pi^2.$$

For codirectional modes $\hat{\omega} < 0$ and S is purely imaginary. Consequently, the amplitude functions $A_m(x_1)$ and $A_n(x_1)$ are bounded. This is so-called passband interaction, i.e.,

$$(6.5)\qquad K_\omega < \left\{-\frac{\varrho\omega}{\mu}-\frac{\bar{M}\omega\bar{K}N_0}{\mu(\bar{M}\omega+\bar{K})}-n^2\pi^2\right\}^{1/2}.$$

If one of the modes is reversed, then $\hat{\omega} > 0$ and S is complex ($\sigma^2 < 4\hat{\omega}$), indicating that the modes are evanescent. This occurs for a frequency range different from that of the passband interaction. Since the modes are attenuated as they propagate down the guide, this range of frequencies is known as a stopband or attenuation band, i.e., $\sigma^2 < 4\hat{\omega}$ and

$$(6.6)\qquad \omega^2+\frac{\omega}{\varrho\bar{M}}[\bar{K}(\varrho+\bar{M}N_0)+\mu\bar{M}(n^2\pi^2+K_\omega^2)]+\frac{\bar{K}\mu}{\varrho\bar{M}}(n^2\pi^2+K_\omega^2) > 0.$$

The transition curve which defines the frequency at which the behaviour of the solutions changes from one type of interaction to the other (transition frequency) is a solution of

$$(6.7)\qquad \sigma^2 = 4\left(\frac{nm\pi^2}{\sqrt{2}K_nK_m}\right)^2\left(\frac{K_\omega}{K_m}+1\right)\left(\frac{K_\omega}{K_n}-1\right)[1-(-1)^{m=n}\operatorname{Cos}\tau].$$

It is clear from the given problem that if the masses of the dust particles are small, their influence on fluid flow is reduced and in the limit as $\bar{M}\to 0$, the fluid becomes ordinary viscous.

References

1. O. R. ASFAR and A. H. NAYFEH, SIAM Rev., **25**, 4, 1983.
2. TH. VON KÁRMÁN, *Sand ripples in the desert*, Technion Year Book, 1947.
3. P. G. SAFFMANN, J. Fluid Mech., **13**, 120, 1962.
4. D. H. MICHAEL, Proc. Camb. Phil. Soc., **61**, 659, 1965.
5. D. H. MICHAEL and D. A. MILLER, Mathematika, **13**, 97, 1966.
6. J. R. HOWARD, *Fluidized beds*, Appl. Sci. Publ. LTD, London and New York 1983.
7. A. A. HAJJ and E. F. ELSHEHAWEY, *A variational formalism for steady flow of dusty fluid problems. I*, to be published in the Quart. of Appl. Math.
8. E. F. ESHEHAWEY, *On dusty hydrodynamic instability conditions. I*, (to be published).
9. S. CHANDRASEKHAR, *Hydrodynamic and hydromagnetic stability*, Cambr. Univers., Cambridge 1961.

10. E. F. ELSHEHAWEY, Y. O. EL-DIB and A. A. MOHAMED, *Field*, Il Nuovo Cim. Serie 1, 6D, 291, 1985.
11. A. A. MOHAMED, E. F. ELSHEHAWEY and Y. O. EL-DIB, Il Nuovo Cim., Serie 1, 8D, 177, 1986.
12. N. T. ELDABE, E. F. ELHEHAWEY, G. M. MOATIMID and A. A. MOHAMED, J. Math. Phys., **26**, 207, 1985.
13. A. A. MOHAMED, E. F. ELSHEHAWEY and Y. O. EL-DIB, J. Chem. Phys., **85**, 445, 1986.
14. A. N. NAYFEH, *Perturbation methods*, Wiley, Interscience, New York 1973.

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION
AIN SHAMS UNIVERSITY, HELIOPOLIS, CAIRO, EGYPT.

Received September 5, 1988.