Arch. Mech., 41, 5, pp. 717–724, Warszawa 1989

Elastic eigenstates of a medium with transverse isotropy

P. S. THEOCARIS and T. P. PHILIPPIDIS (ATHENS)

THE SPECTRAL decomposition of the fourth-rank compliance tensor of an elastic solid with an axis of symmetry of infinite order was given, and the characteristic values (eigenvalues) of the same tensor were calculated in terms of its Cartesian components. Energy-orthogonal stress states for the transversely isotropic solid were explicitly calculated and the elastic energy associated with these stress tensors was given a special identification with dilatational or distortional strain energy density. Bounds on the values of Poisson's ratios of the transversely isotropic solid were established by imposing the eigenvalues of the compliance tensor to be strictly positive.

Podano rozkład widmowy tensora podatności czwartego rzędu dla ciała sprężystego z osią symetrii nieskończonego rzędu, a wartości własne tego tensora przedstawiono we współrzędnych kartezjańskich. Energetycznie ortogonalne stany naprężeń ciała sprężystego o izotropii poprzecznej wyznaczono w sposób jawny i podano wyrażenie na energię sprężystą stowarzyszoną z tymi tensorami naprężeń identyfikując człony energii odpowiadające odkształceniom dylatacyjnym i dystorsyjnym. Ustalono ograniczenia dla stałej Poissona ciała poprzecznie izotropowego zakładając, że wartości własne tensora podatności są dodatnio określone.

Приведено спектральное разложение тензора податливости четвертого порядка для упругого тела с осью симметрии бесконечного порядка, а собственные значения этого тензора представлены в декартовых координатах. Энергетически ортогональные состояния напряжений упругого тела с поперечной изотропией определены явным образом и приведено выражение для упругой энергии ассоциированной с этими тензорами напряжений, идентифицируя члены энергии отвечающие дилатационным и дисторсионным деформациям. Установлены ограничения для постоянной Пуассона поперечно изотропного тела, предполагая, что собственные значения тензора податливости положительно определены.

1. Introduction

THE SPECTRAL decomposition of a fourth-rank symmetric tensor in elementary ones (idempotent tensors) was used for the energy orthogonal decomposition of the second order symmetric tensor space [1–2]. For the compliance tensor **S** of a generic elastic solid, when spectrally decomposed, the following relation holds:

$$\mathbf{S} = \lambda_1 \mathbf{E}_1 + \ldots + \lambda_m \mathbf{E}_m, \qquad m \leqslant 6, \tag{1.1}$$

where the roots of the minimum polynomial of **S** were denoted by λ_m [3]. The tensors $\mathbf{E}_m$ decompose the unit element, **I**, of the fourth rank symmetric tensor space and the following is valid

$$\begin{gathered} \mathbf{I} = \mathbf{E}_1 + \ldots + \mathbf{E}_m, \\ \mathbf{E}_K \cdot \mathbf{E}_N = 0 \quad \text{for} \quad K \neq N, \\ \mathbf{E}_K \cdot \mathbf{E}_K = \mathbf{E}_K. \end{gathered} \tag{1.2}$$

If $\boldsymbol{\sigma}$ is a symmetric second order tensor, by means of the relation (1.2) one has

$$\mathbf{I}\cdot\boldsymbol{\sigma} = \mathbf{E}_1\cdot\boldsymbol{\sigma} + \ldots + \mathbf{E}_m\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \ldots + \boldsymbol{\sigma}_m = \boldsymbol{\sigma}, \tag{1.3}$$

and the following properties of $\boldsymbol{\sigma}_K$ are derived [1–2]:

$$\boldsymbol{\sigma}_K\cdot\boldsymbol{\sigma}_N = 0, \quad K \neq N, \qquad \mathbf{S}\cdot\boldsymbol{\sigma}_K = \lambda_K\boldsymbol{\sigma}_K. \tag{1.4}$$

Moreover, if $\boldsymbol{\sigma}_K$ represents a stress tensor, the corresponding strain tensor (elastic eigenstrain) is given by the simple expression

$$\boldsymbol{\epsilon}_K = \lambda_K\boldsymbol{\sigma}_K. \tag{1.5}$$

The tensors $\boldsymbol{\sigma}_K$ which are eigenstates of the compliance tensor $\mathbf{S}$ were called by RYCHLEWSKI [1] *energy orthogonal stress states* and they were shown to possess a remarkable property. Namely, they decompose the elastic potential function, T, into discrete components which correspond to the respective eigendeformation tensors, thus having a specified meaning depending on the material symmetry properties. Then the following relation is satisfied by the stress eigenstates $\boldsymbol{\sigma}_K$:

$$T(\boldsymbol{\sigma}_1 + \ldots + \boldsymbol{\sigma}_m) = T(\boldsymbol{\sigma}_1) + \ldots + T(\boldsymbol{\sigma}_m). \tag{1.6}$$

Besides the spectral decomposition of the compliance tensor $\mathbf{S}$, there is the possibility of others which also yield invariant scalars in terms of S_{ijkl} and elementary fourth-rank tensors [3–5]. The decompositions of $\mathbf{S}$ obtained in these papers were mainly used in order to simplify calculations in the formulation of elastic inclusion and related problems [6].

In this paper the authors succeeded to decompose spectrally the compliance tensor of a medium with transverse isotropy, which describes satisfactorily the behaviour of unidirectional fiber-reinforced composites, and to evaluate its characteristic values. Based on the properties of this decomposition, energy orthogonal stress states were established and both the eigenvalues as well as the stress eigenstates were expressed in terms of the Cartesian components of the compliance tensor. It was further shown that positiveness of the characteristic values of $\mathbf{S}$ establishes bounds for the values of Poisson's ratios of the transversely isotropic solid, which are necessary in the qualification of their experimentally measured values.

2. Spectral decomposition of the transversely isotropic compliance tensor

Let us consider a medium with transverse isotropy, its elastic properties characterized by the components of the compliance tensor $\mathbf{S}$. We suppose a Cartesian frame oriented along the principal material directions, with axis-33 normal to the isotropic (transverse) plane defined by axes-11 and –22. Using engineering constants in which the subscript (T) denotes elastic properties on the isotropic plane and the subscript (L) the corresponding ones on the normal (longitudinal) plane, the components of the compliance tensor, associated with the adopted Cartesian frame, are given in terms of elastic moduli and Poisson's ratios by

$$\begin{aligned}
&S_{1111} = S_{2222} = 1/E_T, \quad S_{3333} = 1/E_L,\\
&S_{1122} = S_{2211} = -\nu_T/E_T,\\
&S_{1133} = S_{3311} = S_{2233} = S_{3322} = -\nu_L/E_L,\\
&S_{2323} = S_{2332} = S_{3223} = S_{3232} = 1/4G_L,\\
&S_{1313} = S_{1331} = S_{3113} = S_{3131} = 1/4G_L,\\
&S_{1212} = S_{1221} = S_{2112} = S_{2121} = 1/4G_T.
\end{aligned} \tag{2.1}$$

All the remaining S_{ijkl} are zero. Between the engineering constants of the transverse plane the well-known isotropic relation holds:

$$1/2G_T = (1+\nu_T)/E_T.$$

The characteristic values of the associated square matrix of rank six to tensor **S** were found to be given in terms of the engineering constants by the following relations:

$$\begin{aligned}
\lambda_1 &= (1+\nu_T)/E_T = 1/2G_T,\\
\lambda_2 &= 1/2G_L,\\
\lambda_3 &= (1-\nu_T)/2E_T+1/2E_L+\{[(1-\nu_T)/2E_T-1/2E_L]^2+2\nu_L^2/E_L^2\}^{1/2},\\
\lambda_4 &= (1-\nu_T)/2E_T+1/2E_L-\{[(1-\nu_T)/2E_T-1/2E_L]^2+2\nu_L^2/E_L^2\}^{1/2}.
\end{aligned} \tag{2.2}$$

That is, two of its characteristic values, namely λ_1 amd λ_2, are of multiplicity two. Then the minimum polynomial of the tensor **S** is a quartic and has as roots the eigenvalues λ_1, λ_2, λ_3 and λ_4. The associated four idempotent tensors of the sepctral decomposition of **S** were also found as follows

$$\begin{aligned}
\mathbf{E}_1 &= E^1_{ijkl} = \frac{1}{2}(b_{ik}b_{jl}+b_{jk}b_{il}-b_{ij}b_{kl}),\\
\mathbf{E}_2 &= E^2_{ijkl} = \frac{1}{2}(b_{ik}a_{jl}+b_{il}a_{jk}+b_{jl}a_{ik}+b_{jk}a_{il}),\\
\mathbf{E}_3 &= E^3_{ijkl} = \mathbf{f}\otimes\mathbf{f} = f_{ij}f_{kl},\\
\mathbf{E}_4 &= E^4_{ijkl} = \mathbf{g}\otimes\mathbf{g} = g_{ij}g_{kl}.
\end{aligned} \tag{2.3}$$

Second-rank symmetric tensors **a** and **b** figuring in the above cited relations for $\mathbf{E}_1$ and $\mathbf{E}_2$ are defined by

$$\begin{aligned}
\mathbf{a} &= \mathbf{k}\otimes\mathbf{k},\\
\mathbf{a}+\mathbf{b} &= \mathbf{1},
\end{aligned} \tag{2.4}$$

with **k** the unit vector of $\mathbf{R}^3$, associated with the 33-direction of the Cartesian coordinate system. The tensors **f** and **g** are axisymmetric too, and depend on the components of the tensor **S**. They are given by

$$\begin{aligned}
\mathbf{f} &= \frac{1}{\sqrt{2}}\cos\omega\mathbf{b}+\sin\omega\mathbf{a},\\
\mathbf{g} &= \frac{1}{\sqrt{2}}\sin\omega\mathbf{b}-\cos\omega\mathbf{a}
\end{aligned} \tag{2.5}$$

with

$$\cos 2\omega = [(1-\nu_T)/2E_T - 1/2E_L]/[\{(1-\nu_T)/2E_T - 1/2E_L\}^2 + 2\nu_L^2/E_L^2]^{1/2}. \tag{2.6}$$

It can be readily checked that in case $E_L = E_T$, $G_L = G_T$ and $\nu_L = \nu_T$, all the above cited relations degenerate in the respective ones of the isotropic solid, yielding the parameters of its spectral decomposition. In particular, the relations (2.2) yield only two distinct eigenvalues and thus two idempotent tensors $\mathbf{E}_m$ which decompose the unit element, $\mathbf{I}$, of the fourth-rank tensor space and the following well-known relations are valid [1, 2]:

$$\begin{aligned} \mathbf{S} &= \frac{1}{3K}\mathbf{E}_p + \frac{1}{2G}\mathbf{E}_D, \\ \mathbf{E}_p &= \frac{1}{3}\mathbf{1}\otimes\mathbf{1}. \end{aligned} \tag{2.7}$$

Let us now define the orthogonal subspaces in terms of which the space of second-rank symmetric tensors is expressed as their direct sum, and which also constitute characteristic states of the tensor $\mathbf{S}$, that is, they satisfy the following relations:

$$\mathbf{S}\cdot\boldsymbol{\sigma}_m = \lambda_m \boldsymbol{\sigma}_m \tag{2.8}$$

with λ_m given by the relations (2.2). These stress states are simply defined by equations of the form

$$\boldsymbol{\sigma}_m = \mathbf{E}_m\cdot\boldsymbol{\sigma} \tag{2.9}$$

with $\mathbf{E}_m$ given by the relations (2.3). Denoting by $\boldsymbol{\sigma}$ the contracted stress tensor, which in the form of a 6-D vector is written as follows:

$$\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6]^T,$$

and carrying out the calculations implied by the relations (2.9), one finally has

$$\begin{aligned} \boldsymbol{\sigma}_1 &= \left[\frac{1}{2}(\sigma_1-\sigma_2), \frac{1}{2}(\sigma_2-\sigma_1), 0, 0, 0, \sigma_6\right]^T, \\ \boldsymbol{\sigma}_2 &= [0, 0, 0, \sigma_4, \sigma_5, 0]^T, \\ \boldsymbol{\sigma}_3 &= \left(\frac{1}{\sqrt{2}}\cos\omega(\sigma_1+\sigma_2)+\sin\omega\sigma_3\right)\left[\frac{1}{\sqrt{2}}\cos\omega, \frac{1}{\sqrt{2}}\cos\omega, \sin\omega, 0, 0, 0\right]^T, \\ \boldsymbol{\sigma}_4 &= \left(\frac{1}{\sqrt{2}}\sin\omega(\sigma_1+\sigma_2)-\cos\omega\sigma_3\right)\left[\frac{1}{\sqrt{2}}\sin\omega, \frac{1}{\sqrt{2}}\sin\omega, -\cos\omega, 0, 0, 0\right]^T. \end{aligned} \tag{2.10}$$

As it may be derived from the relations (2.10), the characteristic states of stress, which correspond to the spectral decomposition of the compliance tensor $\mathbf{S}$ of a transtropic material, decompose any stress tensor in a prescribed manner. That is, the states $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are *shears*, with $\boldsymbol{\sigma}_2$ simple shear and $\boldsymbol{\sigma}_1$ a superposition of pure and simple shear. The sum of $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$ is the orthogonal supplement to the shear subspace of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. These two states, i.e., $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$, constitute equilateral stressing in the plane of isotropy and prescribed tension or compression along the material axis of symmetry.

The stress eigenstates (2.10) decompose the elastic potential of the transversely isotropic solid in four distinct components and by means of the relation (1.6) the decomposition is expressed by

$$T(\boldsymbol{\sigma}) = T(\boldsymbol{\sigma}_1)+T(\boldsymbol{\sigma}_2)+T(\boldsymbol{\sigma}_3)+T(\boldsymbol{\sigma}_4). \tag{2.11}$$

As it is known from isotropic elasticity, the strain energy density can be separated into two parts, *voluminal and distortional*, accounting for recoverable elastic energy stored by dilation and distortion of the solid respectively.

By considering the relations (2.10) and (2.11) it can be deduced that for a medium with transverse isotropy, such a decomposition is not in general conceivable. Although the components of the elastic potential associated with the eigentensors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are of *pure distortional* type, the remaining parts of the decomposition are not associated solely with distortional or dilatational elastic energy. However, for some loading configurations or material properties, the work produced by the stresses $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$ could be identified with dilatational strain energy or a distortional one.

Consider as an example transversely isotropic materials which satisfy the following equation:

$$(1-\nu_T)/E_T = (1-\nu_L)/E_L, \tag{2.12}$$

where ν_L, ν_T, E_L and E_T can take any value, but of course the moduli E_L, E_T must be positive and ν_L, ν_T assume values for which all λ_K are positive in order to maintain the positive definite character of the elastic potential function T.

Then it can be readily proved that the work of stresses $\boldsymbol{\sigma}_4$

$$\lambda_4 \boldsymbol{\sigma}_4 \cdot \boldsymbol{\sigma}_4$$

is dilatational strain energy, whereas the work of $\boldsymbol{\sigma}_3$ is a distortional one.

An interesting geometric interpretation arises for the energy-orthogonal stress states if we consider the "projections" of $\boldsymbol{\sigma}_K$ in the principal 3-D stress-space. Then the characteristic state $\boldsymbol{\sigma}_2$ vanishes, whereas the stress states $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$ are represented by three mutually orthogonal vectors oriented along the directions with the following associated unit vectors:

$$\begin{aligned}
&\mathbf{e}_1:\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right),\\
&\mathbf{e}_3:\left(\frac{1}{\sqrt{2}}\cos\omega, \frac{1}{\sqrt{2}}\cos\omega, \sin\omega\right),\\
&\mathbf{e}_4:\left(\frac{1}{\sqrt{2}}\sin\omega, \frac{1}{\sqrt{2}}\sin\omega, -\cos\omega\right).
\end{aligned} \tag{2.13}$$

The argument ω figuring in the relations (2.13) was defined in terms of the engineering constants in the relation (2.6). If we denote by $(0-\sigma_1'\sigma_2'\sigma_3')$ the Cartesian frame of principal stresses (see Fig. 1), then, as it can be seen by the relations (2.13), the vectors $\mathbf{e}_3$ and $\mathbf{e}_4$ are equally inclined with respect to the axes $0\sigma_1'$ and $0\sigma_2'$ and thus they lie on the main diagonal plane $\sigma_1' = \sigma_2'$. The vector $\mathbf{e}_3$ subtends with axis $0\sigma_3'$ an angle equal to $(\omega-\pi/2)$, whereas the vector $\mathbf{e}_4$ subtends with the same axis an angle $(\pi-\omega)$. The vector $\mathbf{e}_1$ is perpendicular to the axis $0\sigma_3'$ and to the plane $\sigma_1' = \sigma_2'$ lying on the deviatoric π-plane.

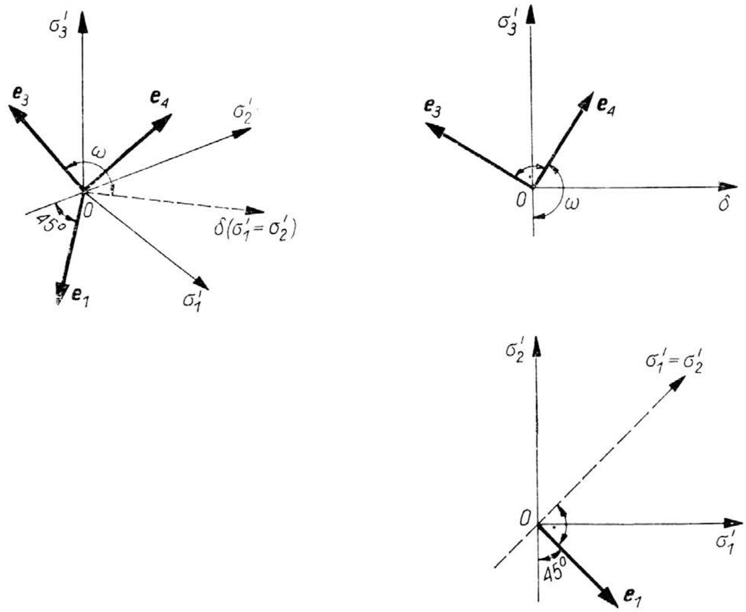

FIG. 1. Projections of stress eigenstates in principal stress coordinate frame.

For any transversely isotropic material, the direction of the vector $\mathbf{e}_1$, and thus the eigentensor $\boldsymbol{\sigma}_1$, remains constant, whereas the vectors $\mathbf{e}_3$ and $\mathbf{e}_4$ rotate with respect to the origin of the coordinate frame lying always on the plane $\sigma_1' = \sigma_2'$.

It can be readily checked from the relation (2.6) that for the isotropic material the argument ω takes the value $\omega = 125.26°$. Then, the vector $\mathbf{e}_4$ has the direction of the hydrostatic axis ($\sigma_1' = \sigma_2' = \sigma_3'$), i.e., the eigenstate $\boldsymbol{\sigma}_4$ becomes the spherical tensor, whereas the vector $\mathbf{e}_3$ lies on the deviatoric plane.

In general the angle ω takes values in the interval $(0, \pi)$, and it is valid that the materials characterized by high anisotropy yield values of ω near to π.

Then, since the isotropic solid has its argument ω equal to 125.26°, it is reasonable to accept finally that the angle ω varies between the values of 125.26° and 180°. Indeed, it can be checked that the engineering materials with transverse isotropy, including also unidirectional fiber reinforced composites, behave in this manner, yielding values of ω belonging to the above cited interval.

Let now the principal stress coordinate system $(0-\sigma_1'\sigma_2'\sigma_3')$ transform and have the directions of $\mathbf{e}_1$, $\mathbf{e}_3$ and $\mathbf{e}_4$, with the axis σ_3' having the direction of $\mathbf{e}_3$ and the axis σ_1', this of $\mathbf{e}_1$. If by $(0-\bar{\sigma}_1'\bar{\sigma}_2'\bar{\sigma}_3')$ we denote the new coordinate system, then it is obvious that the expression for the elastic energy function becomes

$$2T = \lambda_1 \bar{\sigma}_1^2 + \lambda_4 \bar{\sigma}_2^2 + \lambda_3 \bar{\sigma}_3^2. \tag{2.14}$$

By giving the value $2T = 1$, Eq. (2.14) represents an *ellipsoid* centered at the origin 0 of the coordinate system and having axes of symmetry along the directions, $\mathbf{e}_1$, $\mathbf{e}_3$ and $\mathbf{e}_4$.

The lengths of the semi-axes of the ellipsoid along the axes of the coordinate system are, respectively, $1/\sqrt{\lambda_1}$, $1/\sqrt{\lambda_4}$ and $1/\sqrt{\lambda_3}$.

Moreover, if the fourth-rank tensor **S** describes the isotropic solid, then the relation (2.14) represents an *ellipsoid of revolution* with its major semi-axis along $\mathbf{e}_4$ having the direction of the hydrostatic axis, i.e., $\sigma'_1 = \sigma'_2 = \sigma'_3$, and the equal two semi-axes lying on the deviatoric π-plane. In this case, $\lambda_1 = \lambda_3 = 1/2G$ and $\lambda_4 = 1/3K$. The representation of the elastic energy for the isotropic solid by the ellipsoid of revolution is due to BELTRAMI [7].

3. Restrictions upon the values of the compliance tensor components

As it is implied by thermodynamics, the elastic potential must be positive. This is guaranteed by the positive definite nature of the compliance or stiffness tensor which in turn implies that

$$\lambda_m > 0, \quad m = 1, \ldots, 4. \tag{3.1}$$

In the initial Cartesian coordinate frame with respect to which the compliance tensor components were defined in the relations (2.1), all elastic moduli appear as diagonal elements of the associated square matrix of the contracted tensor and thus they must be positive. Then, by imposing the eigenvalues of the transversely isotropic medium expressed by the relations (2.2) to the conditions (3.1), it is seen that the values of Poisson's ratios ν_L and ν_T are bounded by the validity of these inequalities. The set of the following two inequalities must be satisfied by the values of Poisson's ratios:

$$\begin{aligned} |\nu_T| &\leqslant 1, \\ |\nu_L| &\leqslant ((1-\nu_T)E_L/2E_T)^{1/2}. \end{aligned} \tag{3.2}$$

In these relations, it is interesting to notice the bounds for the "isotropic" ν_T which differ from the bounds of Poisson's ratio for the isotropic solid, i.e.,

$$-1 \leqslant \nu \leqslant \frac{1}{2}.$$

Similar expressions with the relations (3.2) were found by other authors also [8, 9], by following different, but mathematically equivalent, procedures satisfying positiveness of the elastic potential.

The relations (3.2) can be used as well in the qualification of experimental results concerning the values of Poisson's ratios, especially for unidirectional fiber-composites which usually possess the symmetry of the transversely isotropic configuration. It has to be pointed out, however, that *both inequalities* (3.2) *should be satisfied* in order to have positive elastic potential and not only one of them.

CHRISTENSEN [10], following a procedure based on physical considerations, deduced for the "longitudinal" Poisson's ratio, ν_T, the following bounding inequality

$$|\nu_L| < \left(\frac{E_L}{E_T}\right)^{1/2}. \tag{3.3}$$

Comparing with the second inequality of the set (3.2), it can be seen that the relation (3.3) *overestimates* the bounding interval of ν_L and is only exact for the limiting value of $\nu_T = -1$.

A similar procedure to that of Christensen was also followed in [11] where bounds only for the transverse Poisson's ratio were given and, erroneously, the interval [0, 1] was indicated as appropriate for ν_T.

4. Conclusions

The energy-orthogonal decomposition of the second-rank symmetric tensor space consisting of subspaces of eigentensors of the compliance tensor, **S**, of a transversely isotropic medium was obtained by means of the spectral decomposition of **S**.

The decomposition of the stress tensor $\boldsymbol{\sigma}$ gave four energy-orthogonal eigenstates which decompose appropriately the elastic energy function. Two of them were shown to be solely associated with distortional elastic energy, whereas the remaining two denote, in general, both voluminal and distortional elastic energies.

Imposing the calculated eigenvalues of the compliance tensor **S** to be strictly positive, bounds were deduced for the values of Poisson's ratios, ν_L and ν_T.

References

1. J. RYCHLEWSKI, *Elastic energy decompositions and limit criteria*, Adv. in Mech., **7**, 3, 51–80, 1984.
2. J. RYCHLEWSKI, *On Hooke's law*, PMM, **48**, 3, 303–314, 1984.
3. L. J. WALPOLE, *Fourth-rank tensors of the thirty-two crystal classes: multiplication tables*, Proc. Royal Soc., London, **A391**, 149–179, 1984.
4. T. P. SRINIVASAN and S. D. NIGAM, *Invariant elastic constants for crystals*, J. Math. Mech., **19**, 5, 411–420, 1969.
5. R. HILL, *Continuum micro-mechanics of elastoplastic polycrystals*, J. Mech. Phys. Sol., **13**, 89–101, 1965. See also, R. HILL, *A self-consistent mechanics of composite materials*, Ibid **13**, 213–222, 1965.
6. J. R. WILLIS, *A polarization approach to the scattering of elastic waves. I. Scattering by a single inclusion*, J. Mech. Phys. Sol., **28**, 287–305, 1980.
7. F. STASSI, *Teoria della plasticita e sue applicazioni*, G. Denaro ed., Palermo 1958.
8. B. M. LEMPRIERE, *Poisson's ratios in orthotropic materials*, AIAA J., **6**, 11, 2226–2227, 1968.
9. R. A. EUBANKS and E. STERNBERG, *On the axisymmetric problem of elasticity theory with tansverse isotropy*, J. Rat. Mech. Anal., **3**, 89–101, 1964.
10. R. M. CHRISTENSEN, *Mechanics of composite materials*, J. Wiley and Sons, New York 1979.
11. M. KNIGHT, *Three-dimensional elastic moduli of graphite/epoxy composites*, J. Comp. Mat., **16**, 3, 153–159, 1982.

ATHENS NATIONAL TECHNICAL UNIVERSITY, ATHENS, GREECE.

Received August 25, 1988.