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Anisotropy degree of elastic materials

J. RYCHLEWSKI (WARSZAWA) and ZHANG JIN MIN (SHANGHAI)

THE DEFINITION of the anisotropy degree of tensors, functions and functionals with respect to some given operation group is presented. The anisotropy degree of fourth-order tensors is investigated in details. Numerical examples are given for cubic, transversely-isotropic and orthotropic linear elastic materials.

Podano definicję stopnia anizotropii tensorów, funkcji lub funkcjonałów, względem odpowiedniej grupy. Szczegółowo rozważono stopień anizotropii tensorów euklidesowych czwartego rzędu. Podano przykłady numeryczne dla tensorów sztywności materiałów liniowo sprężystych o symetrii kubicznej, transwersalnie izotropowych i ortotropowych.

Дано определение степени анизотропности тензоров, функций и функционалов, относительно соответствующей группы. Детально рассмотрен случай евклидовых тензоров четвертого ранта. Даны численные примеры описания степени анизотропии тензоров жесткости линейно упругого тела для случая кубической симметрии, трансверсальной изотропии и ортотропии.

Introduction

MOST NATURAL materials, like rocks, bones, wood, are anisotropic. In highly advanced technology, isotropic materials have come into use rather by way of exception.

It would be good to have the possibility to say that one elastic material is more or less anisotropic than another one. The possibility deserves some measures or coefficients of anisotropy. There have been some attempts. For example, in rock mechanics the parameter $a = (C_{11}-C_{12})/C_{44}$ is in use. Here $C_{11} = C_{1111}$, $C_{12} = C_{1122}$, $C_{44} = C_{1212}$ are elastic coefficients in some special orthogonal base, see [2, 7]. In [1], it is mentioned that such a measure is suited only for cubic crystals. For general elastic materials, six independent parameters of such a kind should be used. All six parameters are equal to 1 for isotropic materials and degenerate to one parameter for cubic crystals. For transversely-isotropic materials, the parameters are equal in pairs and one of the pairs is equal to 1. In [5] it is mentioned that for any anisotropic elastic material, its deviation from some isotropic material can be measured. In [6] a measure of this kind is presented.

Let us stress that the quantitative evaluation of the anisotropy of a material and its symmetry group are quite different matters. A matrial can be very close to an isotropic material but have no symmetry at all.

We present here a concept of the anisotropy degree of elastic materials. The concept is based on the investigation of orbits of the orthogonal group in tensor space [17, 18].

1. Anisotropy degree

1.1. Let Γ be a group and $\mathcal{L}$ a set. The group Γ is said to *operate* on the set $\mathcal{L}$ if there exists such a mapping

$$\Gamma \times \mathcal{L} \to \mathcal{L}, \quad (\alpha, x) \to \alpha * x \tag{1.1}$$

that for each $x \in \mathcal{L}$ and any $\alpha, \beta \in \Gamma$

$$(\alpha\beta) * x = \alpha * (\beta * x), \quad i * x = x, \tag{1.2}$$

where $(\alpha, \beta) \to \alpha\beta$ is the group operation and i the identity of the group Γ.

The *orbit* of the element x is

$$\Gamma * x = \{\alpha * x \mid \alpha \in \Gamma\}. \tag{1.3}$$

The *symmetry group of the element* x is

$$\Gamma(x) = \{\alpha \in \Gamma \mid \alpha * x = x\}. \tag{1.4}$$

If $\Gamma(x) = \Gamma$, i.e., the orbit of x is a singleton, $\Gamma * x = \{x\}$, we say that x is an *isotropic* element. All other elements are called *anisotropic*. For details see, for example, [18].

1.2. We are interested in the case when the set $\mathcal{L}$ is a Banach space, and the group Γ preserves the structure, i.e.,

$$\alpha * (ax + by) = a(\alpha * x) + b(\alpha * y), \qquad ||\alpha * x|| = ||x|| \tag{1.5}$$

for all arguments written. Here $||\ldots||$ is the norm in $\mathcal{L}$.

The size of the orbit $\Gamma * x$ is described by its *diameter*

$$d(\Gamma * x\} = \max_{y \in \Gamma * x} ||y - x|| = \max_{\alpha \in \Gamma} ||\alpha * x - x|| \tag{1.6}$$

and its *radius*

$$r(\Gamma * x) = ||x||. \tag{1.7}$$

1.3. Now we may introduce the main parameter that is supposed to measure anisotropy.

DEFINITION. *The parameter*

$$\Delta(\Gamma * x) = \frac{d(\Gamma * x)}{2r(\Gamma * x)} \tag{1.8}$$

is said to be the anisotropy degree of the orbit $\Gamma * x$. We introduce the anisotropy degree $\delta(x)$ of the element x as anisotropy degree of its orbit

$$\delta(x) \equiv \Delta(\Gamma * x) = \max_{\alpha \in \Gamma} \frac{||\alpha * x - x||}{2||x||}. \tag{1.9}$$

The geometric meaning of $\delta(x)$ is obvious

$$\delta(x) = \operatorname{tg} \frac{\varphi}{2}, \tag{1.10}$$

where φ is the apex angle of the cone $\{a\alpha * x | a \in R, \alpha \in \Gamma\}$. Therefore

(1.11) $$0 \leqslant \delta(x) \leqslant 1.$$

The element x is isotropic if and only if its anisotropy degree is equal to zero, $\delta(x) = 0$. The set of all isotropic elements in $\mathscr{L}$ is a linear subspace

(1.12) $$J = \{x \in \mathscr{L} | \delta(x) = 0\}.$$

The element x with $\delta(x)$ equal to 1 is said to be *extremely anisotropic*.

1.4. The above definition is good for functions (mappings, functionals) as well. Let f be a function $f: \mathscr{X} \to \mathscr{Y}$ where $\mathscr{X}, \mathscr{Y}$ are two Banach spaces. Let Γ be a group operating both on $\mathscr{X}$ and $\mathscr{Y}$:

(1.13) $$x \to \alpha \cdot x, \quad y \to \alpha \times y.$$

It leads to the operation of Γ on the set of functions $\mathscr{X}^{\mathscr{Y}}$.

(1.14) $$f \to \alpha * f \equiv (\alpha \times) \circ f \circ (\alpha^{-1} \cdot).$$

A function is called an *isotropic function* if $\Gamma(f) = \Gamma$, i.e.,

(1.15) $$\alpha \times f(\alpha^{-1} \cdot x) = f(x) \quad \text{for all } x \in \mathrm{Dom}(f) \quad \text{and} \quad \alpha \in \Gamma.$$

Otherwise we say that the function f is an *anisotropic function*. For details, see [18].

Introducing in $\mathscr{X}^{\mathscr{Y}}$ the linear structure and norm such that Eq. (1.5) is valid, we can define the anisotropy degree of the function f as

(1.16) $$\delta(f) = \max_{\alpha \in \Gamma} \frac{||(\alpha \times) \circ f \circ (\alpha^{-1} \cdot) - f||}{2||f||}.$$

2. Anisotropy degree of tensors, tensor functions and functionals

2.1. The properties of materials in continuum physics are described by Euclidean tensors, tensor functions and functionals of various kinds. We have to apply the formal scheme of Sect. 1 to these situations.

Let us take

(2.1) $$\mathscr{L} = \mathscr{T}_p = \otimes \mathfrak{Z}^p, \quad \Gamma = \mathcal{O} = S0(3).$$

We assume that the norm in $\mathscr{T}_p$ is given by

(2.2) $$||\mathbf{A}|| = (\mathbf{A} \cdot \mathbf{A})^{1/2}.$$

Tensor operations are described in the Notations.

The orthogonal group $\mathcal{O}$ operates on $\mathscr{T}_p$ according to

(2.3) $$\mathbf{A} \to \mathbf{R} * \mathbf{A}, \quad \mathbf{R} \in \mathcal{O}.$$

It is obvious that the conditions (1.2) and (1.5) are satisfied.

The isotropic tensor $\mathbf{A}$ is defined by $\mathcal{O}(\mathbf{A}) = \mathcal{O}$ (see Eq. (1.4)), i.e.,

(2.4) $$\mathbf{R} * \mathbf{A} = \mathbf{A} \quad \text{for any } \mathbf{R} \in \mathcal{O}.$$

The subspace $J \subset \mathscr{T}_p$ for any p is described in many papers.

The *anisotropy degree of an anisotropic tensor* **A** is equal, according to the definition (1.9), to

$$\delta(\mathbf{A}) = \max_{R\in\mathcal{O}} \frac{||\mathbf{R}^*\mathbf{A}-\mathbf{A}||}{2||\mathbf{A}||}. \tag{2.5}$$

For extremely anisotropic tensors we have $\delta(\mathbf{A}) = 1$, i.e.,

$$||\mathbf{R}*\mathbf{A}-\mathbf{A}|| = 2||\mathbf{A}|| \tag{2.6}$$

for some $\mathbf{R} \in \mathcal{O}$. It is possible only for $\mathbf{A}\cdot(\mathbf{R}*\mathbf{A}) = -\mathbf{A}\cdot\mathbf{A}$, i.e., the tensor **A** is extremely anisotropic if

$$\mathbf{R}*\mathbf{A} = -\mathbf{A}. \tag{2.7}$$

2.2. Let us take several examples

EXAMPLE 1. Vectors, $\mathscr{L} = э\otimes э$, $\Gamma = \mathcal{O}$. The only isotropic vector is **0**. For any $\mathbf{a} \neq \mathbf{0}$ we have a rotation **R** that, according to (2.7), $\mathbf{R}*\mathbf{a} = -\mathbf{a}$. Therefore

$$\delta(\mathbf{a}) = 1 \quad \text{for all} \quad \mathbf{a} \neq \mathbf{0}. \tag{2.8}$$

EXAMPLE 2. Vectors, $\mathscr{L} = э\otimes э$, $\Gamma = \mathcal{O}(\mathbf{k})$, where $\mathbf{k} \neq \mathbf{0}$ is given. For any $\mathbf{a} \neq \mathcal{O}$, we have

$$\delta(\mathbf{a}) = \sin(\mathbf{a}, \mathbf{k}). \tag{2.9}$$

Vectors $\lambda\mathbf{k}$ are isotropic with respect to group $\mathcal{O}(\mathbf{k})$. The extremely anisotropic vectors are $\lambda\mathbf{n}$, $\mathbf{nk} = 0$.

EXAMPLE 3. Symmetric second order tensors, $\mathscr{L} = \mathrm{Sym}э\otimes э$, $\Gamma = \mathcal{O}$. This case has been studied in [8]. For any $\boldsymbol{\alpha} = \alpha_1\mathbf{n}_1\otimes\mathbf{n}_1+\alpha_2\mathbf{n}_2\otimes\mathbf{n}_2+\alpha_3\mathbf{n}_3\otimes\mathbf{n}_3$, $\alpha_3 \geqslant \alpha_2 \geqslant \alpha_1$, we have

$$\delta(\boldsymbol{\alpha}) = \frac{\sqrt{2}\,|\alpha_3-\alpha_1|}{2(\alpha_1^2+\alpha_2^2+\alpha_3^2)^{1/2}}. \tag{2.10}$$

Hydrostatic pressures $\alpha_1 = \alpha_2 = \alpha_3$ are isotropic, pure shears $\alpha_2 = 0$, $\alpha_3 = -\alpha_1$ are extremely anisotrpic.

Let $\mathscr{N}$ be the subset of tensors with nonnegative eigenvalues: $\lambda_1 \geqslant 0$, $\alpha_2 \geqslant 0$, $\alpha_3 \geqslant 0$. We have

$$\max_{\alpha\in\mathscr{N}} \delta(\boldsymbol{\alpha}) = \frac{\sqrt{2}}{2}. \tag{2.11}$$

EXAMPLE 4. Let us take into consideration a linear tensor function $l:\mathscr{T}_p \to \mathscr{T}_q$, which is uniquely determined by a tensor **L**

$$l(\mathbf{X}) = \mathbf{L}\cdot\mathbf{X} \quad \text{for any} \quad \mathbf{X} \in \mathscr{T}_p. \tag{2.12}$$

There are two interesting possibilities, among others, to define the anisotropy degree of the relation (2.12) with respect to the orthogonal group $\mathcal{O}$. One is to consider the function l as an element in the Banach space of all linear tensor functions from $\mathscr{T}_p$ into $\mathscr{T}_q$ with the induced norm defined by

$$||f|| \equiv \sup_{\mathbf{X}} \frac{||f(\mathbf{X})||}{||\mathbf{X}||}. \tag{2.13}$$

An other problem is to consider l or its equivalence $\mathbf{L}$ as an element in the Banach space $\mathscr{T}_{p+q}$ with the norm defined by Eq. (2.2). The following discussions are restricted to the last consideration. The first case deserves another paper.

EXAMPLE 5. The same scheme as the last paragraph can be applied to tensor functionals.

3. Solving equation

3.1. In the investigation of the anisotropy degree of the p-th order tensor which is considered as an element in the Banach space with the norm defined by Eq. (2.2), it is fundamental to calculate the diameter

$$d(\mathbf{A}) = \max_{\mathbf{X},\mathbf{Y}\in\mathcal{O}*\mathbf{A}} ||X-Y|| = \max_{\mathbf{R}\in\mathcal{O}} ||R*A-A||. \tag{3.1}$$

If one discusses the maximum values directly from Eq. (3.1), the calculation will be very complicated because of the root. But we have

$$d^2(\mathbf{A}) = (\max_{\mathbf{R}\in\mathcal{O}} ||\mathbf{R}*\mathbf{A}-\mathbf{A}||)^2 = \max_{\mathbf{R}\in\mathcal{O}} ||\mathbf{R}*\mathbf{A}-\mathbf{A}||^2. \tag{3.2}$$

So one can first calculate $d^2(\mathbf{A})$ which in fact, is equal to

$$d^2(\mathbf{A}) = \max 2(\mathbf{A}\cdot\mathbf{A}-(\mathbf{R}*\mathbf{A})\cdot\mathbf{A}). \tag{3.3}$$

Introducing the Lagrange multiplier $\boldsymbol{\alpha}$, one has the Lagrangian function as follows:

$$\Phi^{\mathbf{A}}(\mathbf{R}) = 2(\mathbf{A}\cdot\mathbf{A}-(\mathbf{R}*\mathbf{A})\cdot\mathbf{A})-\boldsymbol{\alpha}\cdot(\mathbf{R}^T\mathbf{R}-\mathbf{I}). \tag{3.4}$$

After calculating the partial derivative of the obtained Lagrangian function with respect to $\mathbf{R}$, we get the condition for the stationary values

$$\frac{\partial\Phi^{\mathbf{A}}(\mathbf{R})}{\partial\mathbf{R}} = -2\sum_{i=1}^{p}\sigma_i\mathbf{A}\odot[(\mathbf{1}\otimes\mathbf{R}\otimes\ldots\otimes\mathbf{R})\sigma_i\mathbf{A}]-2\mathbf{R}\boldsymbol{\alpha} \tag{3.5}$$

$$= 2\sum_{i=1}^{p}(\mathbf{1}\otimes\mathbf{R}^T\otimes\ldots\otimes\mathbf{R}^T)(\sigma_i\mathbf{A})\odot\sigma_i\mathbf{A}-2\mathbf{R}\boldsymbol{\alpha} = \mathbf{0}.$$

The Lagrange multiplier can be derived out from Eq. (3.5)

$$\boldsymbol{\alpha} = -\sum_{i=1}^{p}\sigma_i\mathbf{A}\odot[(\mathbf{R}\otimes\mathbf{R}\otimes\ldots\otimes\mathbf{R})\sigma_i\mathbf{A}] \tag{3.6}$$

$$= -\sum_{i=1}^{p}[(\mathbf{R}^T\otimes\mathbf{R}^T\otimes\ldots\otimes\mathbf{R}^T)\sigma_i\mathbf{A}]\odot\sigma_i\mathbf{A} = -\sum_{i=1}^{p}(\mathbf{R}^T*\sigma_i\mathbf{A})\odot\sigma_i\mathbf{A}.$$

As the restriction condition $\mathbf{R}^T\mathbf{R}-\mathbf{I} = \mathbf{0}$ is symmetric, it is enough to take a symmetric multiplier, i.e.,

$$\boldsymbol{\alpha}^a \equiv \boldsymbol{\alpha}^T-\boldsymbol{\alpha} = \mathbf{0} \tag{3.7}$$

which can be further expressed in terms of Eq. (3.6) as

$$\sigma_i\mathbf{A}\odot(\mathbf{R}^T*\sigma_i\mathbf{A})-(\mathbf{R}^T*\sigma_i\mathbf{A})\odot\sigma_i\mathbf{A} = \mathbf{0}. \tag{3.8}$$

Because of the asymmetry of the tensor $\boldsymbol{\alpha}^a$, Eq. (3.8), in fact, is equivalent to three scalar equations which, in addition to the three orthogonal conditions and three normal conditions, can be used to determine all rotations $\mathbf{R}$ which make $\|\mathbf{R} * \mathbf{A}-\mathbf{A}\|$ have stationary values.

3.2. When $\mathbf{A} = \boldsymbol{\alpha}$ is a tensor of the second order, Eq. (3.8) degenerates into

$$\mathbf{R}^T\boldsymbol{\alpha}\mathbf{R}\boldsymbol{\alpha}^T+\mathbf{R}^T\boldsymbol{\alpha}^T\mathbf{R}\boldsymbol{\alpha}-\boldsymbol{\alpha}\mathbf{R}^T\boldsymbol{\alpha}^T\mathbf{R}-\boldsymbol{\alpha}^T\mathbf{R}^T\boldsymbol{\alpha}\mathbf{R} = 0. \tag{3.9}$$

When $\boldsymbol{\alpha}$ further is a symmetric tensors, Eq. (3.9) is simplified into

$$\mathbf{R}^T\boldsymbol{\alpha}\mathbf{R}\boldsymbol{\alpha} = \boldsymbol{\alpha}\mathbf{R}^T\boldsymbol{\alpha}\mathbf{R}. \tag{3.10}$$

It is well known that two symmetric tensors of the second order are commutative if and only if for some orthogonal base $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$

$$\boldsymbol{\beta} \equiv \mathbf{R}^T\alpha\mathbf{R} = \beta_1\mathbf{n}_1\otimes\mathbf{n}_1+\beta_2\mathbf{n}_2\otimes\mathbf{n}_2+\beta_3\mathbf{n}_3\otimes\mathbf{n}_3, \tag{3.11}$$

$$\boldsymbol{\alpha} = \alpha_1\mathbf{n}_1\otimes\mathbf{n}_1+\alpha_2\mathbf{n}_2\otimes\mathbf{n}_2+\alpha_3\mathbf{n}_3\otimes\mathbf{n}_3. \tag{3.12}$$

On the other hand, $\boldsymbol{\beta}$ has the same eigenvalues as $\boldsymbol{\alpha}$. The eigenvalues of $\boldsymbol{\alpha}$ must be some permutations of those of $\boldsymbol{\alpha}$. Therefore, Eq. (3.10) holds if and only if $\mathbf{R}$ is such that the eigenvectors of $\boldsymbol{\alpha}$ permute. This result is the same as [8] and leads to equation (2.10).

3.3. When $\mathbf{A}$ is a tensor $\mathbf{C}$ of the fourth order with the symmetry properties

$$C_{ijkl} = C_{ijlk} = C_{klij}, \tag{3.13}$$

then Eq. (3.8) becomes

$$4\big((\mathbf{R}^T*\mathbf{C})\odot\mathbf{C}-\mathbf{C}\odot(\mathbf{R}^T*\mathbf{C})\big) = 0. \tag{3.14}$$

This case is especially interesting to us as the mechanical behaviour of a linear elastic solid can be described completely by such tensor. In the two following paragraphs, the attention will be paid to a detailed investigation of various equivalent conditions of Eq. (3.14) and the calculation formulae of the anisotropy degree $\delta(\mathbf{C})$ for transversely-isotropic, cubic and orthotropic linear elastic materials.

4. Elasticity, eigenstates and true rigid moduli

4.1. The constitutive equation of elastic materials has the form

$$\boldsymbol{\sigma} = f(\mathbf{F}) = \mathbf{R}*g(\mathbf{U}), \tag{4.1}$$

where $\mathbf{F} = \mathbf{RU}$ is the deformation gradient with respect to a fixed undistorted configuration, $\mathbf{R}, \mathbf{U}$ are, respectively, the rotation tensor and the right stretch tensor, $\boldsymbol{\sigma}$ is the Cauchy stress tensor.

Let us rotate the reference configuration by a rotation $\mathbf{Q}$ and then apply the same deformation $\mathbf{F}$. We have

$$f(\mathbf{FQ}) = \mathbf{R}*[\mathbf{Q}*g(\mathbf{Q}^T*\mathbf{U})]. \tag{4.2}$$

After taking

$$\Delta\boldsymbol{\sigma} \equiv \mathbf{R}*[f(\mathbf{FQ}^T)-f(\mathbf{F})], \tag{4.3}$$

we obtain for $\Delta\tilde{\sigma} \equiv R*\Delta\sigma$ that

$$\Delta\tilde{\boldsymbol{\sigma}} = (\mathbf{Q}*g-g)(\mathbf{U}). \tag{4.4}$$

We take $\delta(g)$ as the anisotropy degree of the elastic material.

4.2. For small rotations and deformations, the constitutive equation (4.1) takes the classical form of Hooke's law

$$\boldsymbol{\sigma} = \mathbf{C}\cdot\boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} = \mathbf{S}\cdot\boldsymbol{\sigma}, \tag{4.5}$$

where $\boldsymbol{\epsilon}$ is the tensor of small deformation and $\mathbf{C}$ the rigidity tensor, $\mathbf{S}$ is the inverse of $\mathbf{C}$ satisfying

$$\mathbf{C}\circ\mathbf{S} = \mathbf{S}\circ\mathbf{C} = \mathbf{I}. \tag{4.6}$$

According to Eq. (1.8), we take $\delta(\mathbf{C})$ *as the anisotropy degree of the linear elastic material.*

4.3. We are going to present closed formulae for the anisotropy degree of several kinds of linear elastic matrials. The method is based on the approach to linear elasticity given in [9]–[16]. Let us present a brief outline of the approach.

The rigidity tensor $\mathbf{C}$ is considered as a linear operator mapping 6-dimensional space $\mathscr{S}$ of the symmetrical second order tensor into itself. We have in $\mathscr{S}$ the scalar product $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \to \boldsymbol{\alpha}\cdot\boldsymbol{\beta}$. There exists the following spectrum theorem [12] for $\mathbf{C}$:

For every elastic material $\mathbf{C}$, there exists such an orthogonal expansion

$$\begin{gathered}\mathscr{S} = \mathscr{P}_1\otimes \dots \otimes\mathscr{P}_\varrho, \quad \varrho \leqslant 6,\\ \mathscr{P}_i \perp \mathscr{P}_j \quad \text{for} \quad i \neq j,\end{gathered} \tag{4.7}$$

and such a sequence

$$\lambda_1, \dots, \lambda_\varrho, \quad \lambda_\alpha \neq \lambda_\beta \quad \text{for} \quad \alpha \neq \beta \tag{4.8}$$

that

$$\mathbf{C} = \lambda_1 \mathbf{P}_1 + \dots + \lambda_\varrho \mathbf{P}_\varrho, \tag{4.9}$$

where $\mathbf{P}_i$ is the orthogonal projector from $\mathscr{S}$ on P_i, and

$$\begin{gathered}\mathbf{I} = \mathbf{P}_1 + \dots + \mathbf{P}_\varrho,\\ \mathbf{P}_i\circ\mathbf{P}_i = \mathbf{P}_i, \quad \mathbf{P}_i\circ\mathbf{P}_j = \mathbf{P}_j\circ\mathbf{P}_i = \mathbf{0} \quad \text{for} \quad i \neq j.\end{gathered} \tag{4.10}$$

For every $\omega \in \mathscr{P}_i$, we have

$$\mathbf{C}\cdot\boldsymbol{\omega} = \lambda_i\boldsymbol{\omega}. \tag{4.11}$$

so λ_i are eigenvalues and $\boldsymbol{\omega}$ eigenelements of the operator $\mathbf{C}$. The parameters λ_i are called the *true rigidity moduli* and $\boldsymbol{\omega}$ are called the *eigenstates* of the elastic material $\mathbf{C}$.

For elastic energy, we have the expression

$$2E(\boldsymbol{\epsilon}) = \boldsymbol{\epsilon}\cdot\mathbf{C}\cdot\boldsymbol{\epsilon} = \lambda_1 e_1^2 + \dots + \lambda_\varrho e_\varrho^2 \tag{4.12}$$

where $e_i^2 = \boldsymbol{\epsilon}\cdot\mathbf{P}_i\cdot\boldsymbol{\epsilon}$. Therefore we must have

$$\lambda_1 \geqslant 0, \dots, \lambda_\varrho \geqslant 0. \tag{4.13}$$

If the orthogonal base $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_6$ in $\mathscr{S}$, $\boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_k = \delta_{ik}$, is taken in such a way that every $\boldsymbol{\omega}_i$ belongs to some $\mathscr{P}_k$, then

$$\mathbf{C} = \lambda_1 \omega_1 \otimes \omega_1 + \ldots + \lambda_6 \boldsymbol{\omega}_6 \otimes \boldsymbol{\omega}_6. \tag{4.14}$$

For *isotropic material* the expansion is as follows:

$$\mathbf{C} = \lambda_{\mathscr{P}} \mathbf{P}_{\mathscr{P}} + \lambda_{\mathscr{D}} \mathbf{P}_{\mathscr{D}}, \tag{4.15}$$

where

$$\mathbf{P}_{\mathscr{P}} \equiv \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \tag{4.16}$$

is the orthogonal projector from $\mathscr{S}$ onto the 1-dimensional space of isotropic tensors,

$$\mathbf{P}_{\mathscr{D}} \equiv \mathbf{I} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \tag{4.17}$$

is the orthogonal projector from $\mathscr{S}$ onto the 5-dimensional space of deviators, and

$$\lambda_{\mathscr{P}} = \lambda + 2\mu, \quad \lambda_{\mathscr{D}} = 2\mu, \tag{4.18}$$

where λ, μ are the Lamé's moduli. It corresponds to the case

$$\boldsymbol{\omega}_1 = \frac{1}{3} \mathbf{1}, \quad \lambda_1 = \lambda_{\mathscr{P}}, \quad \lambda_2 = \lambda_3 = \ldots = \lambda_6 = 2\mu = \lambda_{\mathscr{D}} \tag{4.19}$$

in the formula (4.14).

For *cubic crystals*, we have

$$\mathbf{C} = \lambda \mathbf{P}_{\mathscr{P}} + \mu \mathbf{P}_Q + \nu \mathbf{P}_{\mathscr{R}}, \tag{4.20}$$

where $P_{\mathscr{P}}$ is given by Eq. (4.16) and

$$\mathbf{P}_Q \equiv \mathbf{K} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1}, \quad \mathbf{P}_{\mathscr{R}} \equiv \mathbf{I} - \mathbf{K} \tag{4.21}$$

where

$$\mathbf{K} \equiv \mathbf{n}_1 \otimes \mathbf{n}_1 \otimes \mathbf{n}_1 \otimes \mathbf{n}_1 + \mathbf{n}_2 \otimes \mathbf{n}_2 \otimes \mathbf{n}_2 \otimes \mathbf{n}_2 + \mathbf{n}_3 \otimes \mathbf{n}_3 \otimes \mathbf{n}_3 \otimes \mathbf{n}_3. \tag{4.22}$$

For *transversely-isotropic materials*, the expansion is

$$\mathbf{C} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \lambda_3 \mathbf{P}_3 + \lambda_4 \mathbf{P}_4, \tag{4.23}$$

where

$$\begin{aligned}
\mathbf{P}_1 &= \frac{1}{2} \sin^2 \varkappa \mathbf{1} \otimes \mathbf{1} + \frac{\sqrt{3}}{2} \sin \varkappa \sin(\varkappa_0 - \varkappa)(\mathbf{1} \otimes \boldsymbol{\pi} + \boldsymbol{\pi} \otimes \mathbf{1}) + \frac{3}{2} \sin^2(\varkappa_0 - \varkappa) \boldsymbol{\pi} \otimes \boldsymbol{\pi}, \\
\mathbf{P}_2 &= \frac{1}{2} \cos^2 \varkappa \mathbf{1} \otimes \mathbf{1} - \frac{\sqrt{3}}{2} \cos \varkappa \cos(\varkappa_0 - \varkappa)(\mathbf{1} \otimes \boldsymbol{\pi} + \boldsymbol{\pi} \otimes \mathbf{1}) + \frac{3}{2} \cos^2(\varkappa_0 - \varkappa) \boldsymbol{\pi} \otimes \boldsymbol{\pi}, \\
\mathbf{P}_3 &\equiv \frac{1}{2} (\sigma_1 + \sigma_2 - \varepsilon) \times [(\mathbf{1} - \boldsymbol{\pi}) \otimes (\mathbf{1} - \boldsymbol{\pi})], \\
\mathbf{P}_4 &= \frac{1}{2} (\sigma_1 + \sigma_2) \times [\boldsymbol{\pi} \otimes (\mathbf{1} - \boldsymbol{\pi}) + (\mathbf{1} - \boldsymbol{\pi}) \otimes \boldsymbol{\pi}], \quad \boldsymbol{\pi} \equiv \mathbf{k} \otimes \mathbf{k},
\end{aligned} \tag{4.24}$$

where $0 \leqslant \varkappa < \frac{\pi}{2}$ is called the distributor, σ_1, σ_2 are two permutation operators $\sigma_1 = [1324]$, $\sigma_2 = [1432]$ and ε the unit permutation operator, $\varepsilon = [1234]$, $\operatorname{tg} \varkappa_0 = \sqrt{2}$.

For *orthotropic materials*, the eigenstates are

$$\boldsymbol{\omega}_L = \begin{vmatrix} \omega_{L1} & 0 & 0 \\ 0 & \omega_{L2} & 0 \\ 0 & 0 & \omega_{L3} \end{vmatrix}, \quad L = 1, 2, 3, \tag{4.25}$$

where

$$\boldsymbol{\omega}_L \cdot \boldsymbol{\omega}_N = \sum_{i=1}^{3} \omega_{Li}\omega_{Ni} = \delta_{LN}, \tag{4.26}$$

$$\begin{gathered} \boldsymbol{\omega}_4 = \frac{\sqrt{2}}{2} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \\ \boldsymbol{\omega}_5 = \frac{\sqrt{2}}{2} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \\ \boldsymbol{\omega}_6 = \frac{\sqrt{2}}{2} \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}. \end{gathered} \tag{4.27}$$

5. Anisotropy degree of linear elastic materials

5.1. For linear elastic materials, after substituting Eq. (4.9) into (3.14) and taking into consideration Eq. (4.10), we have

$$\sum_{\alpha,\beta=1}^{\varrho-1} (\lambda_\alpha - \lambda_\varrho)(\lambda_\beta - \lambda_\varrho)[(\mathbf{R}^T * \mathbf{P}_\alpha) \odot \mathbf{P}_\beta - \mathbf{P}_\beta \odot (\mathbf{R}^T * \mathbf{P}_\alpha)] = \mathbf{0} \tag{5.1}$$

or, written in terms of eigenstates,

$$\sum_{i,j=1}^{5} (\lambda_i - \lambda_6)(\lambda_j - \lambda_6)\big((\mathbf{R}^T * \boldsymbol{\omega}_i) \cdot \boldsymbol{\omega}_j\big)[(\mathbf{R}^T * \boldsymbol{\omega}_i)\boldsymbol{\omega}_j - \boldsymbol{\omega}_j(\mathbf{R}^T * \boldsymbol{\omega}_i)] = \mathbf{0}. \tag{5.2}$$

In a similar way, the calculation formula of the norm $||\mathbf{R} * \mathbf{C} - \mathbf{C}||$ can be simplified to

$$||\mathbf{R} * \mathbf{C} - \mathbf{C}||^2 = 2 \sum_{i=1}^{5} (\lambda_i - \lambda_6)^2 - 2 \sum_{i,j=1}^{5} (\lambda_i - \lambda_6)(\lambda_j - \lambda_6)[(\mathbf{R}^T * \boldsymbol{\omega}_i) \cdot \boldsymbol{\omega}_j]^2 \tag{5.3}$$

or

$$||\mathbf{R} * \mathbf{C} - \mathbf{C}||^2 = 2 \sum_{\alpha=1}^{\varrho-1} d_\alpha (\lambda_\alpha - \lambda_\varrho)^2 - 2 \sum_{\alpha,\beta=1}^{\varrho-1} [(\lambda_\alpha - \lambda_\varrho)(\lambda_\beta - \lambda_\varrho)[(\mathbf{R}^T * \mathbf{P}_\alpha) \cdot \mathbf{P}_\beta], \tag{5.4}$$

$$d_\alpha \equiv \mathrm{Tr}\mathbf{P}_\alpha .$$

5.2. For the fourth order tensors satisfying the symmetry property (3.13), the condition (2.7) can be expanded into

$$\lambda_1 \mathbf{R}*\mathbf{P}_1 + \ldots + \lambda_\varrho \mathbf{R}*\mathbf{P}_\varrho = -\lambda_1 \mathbf{P}_1 - \ldots - \lambda_\varrho \mathbf{P}_\varrho . \tag{5.5}$$

For each tensor, there exists exactly one system of pairwise mutually orthogonal projectors. As its result, Eq. (5.5) holds if and only if there exists a permutation σ of $[1, 2, \ldots, \varrho]$ such that

$$\mathbf{R}*\mathbf{P}_\alpha = \mathbf{P}_{\sigma(\alpha)}, \tag{5.6}$$

$$\lambda_\alpha = -\lambda_{\sigma(\alpha)}, \tag{5.7}$$

$$\mathrm{Tr}\mathbf{P}_\alpha = \mathrm{Tr}\mathbf{P}_{\sigma(\alpha)}. \tag{5.8}$$

For elasticity tensors of real materials, Eq. (5.7) can not be satisfied because all rigidity moduli are nonnegative and there exists at least one nonnegative rigidity modulus. Therefore there is no extremely anisotropic linear elastic material at all.

5.3. Cubic crystals. Taking into account Eq. (4.20), the solving equation for cubic crystals can be simplified to

$$(\mu-\nu)^2\big((\mathbf{R}^T*\mathbf{K})\odot\mathbf{K} - \mathbf{K}\odot(\mathbf{R}^T*\mathbf{K})\big) = \mathbf{0} \tag{5.9}$$

which can be led to

$$(\mu-\nu)^2 \sum_{i,j=1}^{3} (\mathbf{n}_i \mathbf{R}\mathbf{n}_j)^3 (\mathbf{R}^T\mathbf{n}_i \otimes \mathbf{n}_j - \mathbf{n}_j \otimes \mathbf{R}^T\mathbf{n}_i) = \mathbf{0}. \tag{5.10}$$

When $\mu = \nu$, the material degenerates to a trivial case: the isotropic material for which the parameter constantly equals zero for all rotations $\mathbf{R}$. If $\mu \neq \nu$, let $\mathbf{R} = R_{ij}\mathbf{n}_i \otimes \mathbf{n}_j$, Eq. (5.10) is led to

$$\sum_{i=1}^{3} (R_{ij}^3 R_{ik} - R_{ik}^3 R_{ij}) = 0. \tag{5.11}$$

In order to find the solutions of the above equation, we will use the following genera representation formula of the rotation tensor

$$R_{ij} = \cos\varphi\,\delta_{ij} + (1-\cos\varphi)k_i k_j + \sin\varphi\,\varepsilon_{isj} k_s, \tag{5.12}$$

where $\mathbf{k}$ is the rotation axis and φ the rotation angle of $\mathbf{R}$.

If one tries to find all solutions of $\mathbf{R}$ directly from Eq. (5.11), the calculation will be very complicated. The solution for maximum can be guessed in terms of the characteristics of the problem. The solutions are believed to be

$$k_1^2 = k_2^2 = k_3^2 = \frac{1}{3}, \qquad \varphi = \frac{\pi}{3}. \tag{5.13}$$

In this case,

$$(5.14)\qquad \mathbf{R} = \begin{vmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & & \pm 1 \end{vmatrix} \begin{vmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{vmatrix} \begin{vmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{vmatrix}.$$

It is obvious that such $\mathbf{R}$ satisfy Eq. (5.11).

The diameter of the $\mathcal{O}$-orbit is

$$(5.15)\qquad d^2(\mathbf{C}) = \frac{128}{27}(\mu-\nu)^2$$

because of

$$(5.16)\qquad (\mathbf{R}^T*\mathbf{K})\cdot\mathbf{K} = \frac{17}{27}.$$

The anisotropy degree $\delta(\mathbf{C})$ is

$$(5.17)\qquad \delta(\mathbf{C}) = \frac{d(\mathbf{C})}{2\|\mathbf{C}\|} = \left(\frac{32(\mu-\nu)^2}{27(\lambda^2+2\mu^2+3\nu^2)}\right)^{1/2}.$$

The maximum anisotropy degree for materials with cubic symmetry is

$$(5.18)\qquad \operatorname{Sup}\delta(\mathbf{C}) = \lim_{\lambda/\mu,\,\nu/\mu\to 0}\delta(\mathbf{C}) = 0.7698.$$

From Eq. (5.15) the diameter $d(\mathbf{C})$ is independent of the rigidity modulus λ whose projector represents an isotropic volume deformation. When $\mu = \nu$, $d(\mathbf{C}) = 0$, the material degenerates to an isotropic one.

The parameter $a = (C_{11}-C_{12})/C_{44}$ for cubic crystals, expressed by eigenvalues, is μ/ν, which, like $d(\mathbf{C})$, is independent of the rigidity modulus λ. The parameter a represents the ratio of two rigidity moduli μ and ν which are equal to each other for isotropic materials.

5.4. Transversely-isotropic materials. Taking into account Eqs. (4.23) and (4.24), after some tedious but simple calculations, one has

$$(5.19)\qquad (\mathbf{R}^T*\mathbf{P}_1)\odot\mathbf{P}_2-\mathbf{P}_2\odot(\mathbf{R}^T*\mathbf{P}_1)$$
$$= \gamma\frac{27}{16}\sin 2(\varkappa_0-\varkappa)[\gamma^2\sin 2(\varkappa_0-\varkappa)-\sin 2(\varkappa_0+\varkappa)](\mathbf{t}\otimes\mathbf{k}-\mathbf{k}\otimes\mathbf{t}),$$

where

$$(5.20)\qquad \gamma = \mathbf{tk},\quad \mathbf{t} = \mathbf{R}^T\mathbf{k}.$$

Similarly, for any α and β, it is easy to prove that

$$(5.21)\qquad (\mathbf{R}^T*\mathbf{P}_\alpha)\odot\mathbf{P}_\beta-\mathbf{P}_\beta\odot(\mathbf{R}^T*\mathbf{P}_\alpha) = \gamma f_{\alpha\beta}(\gamma^2)(\mathbf{t}\otimes\mathbf{k}-\mathbf{k}\otimes\mathbf{t}),\ \alpha\neq\beta,$$

where $f_{\alpha\beta}$ are functions of γ^2.

Now the solving equation is led to

$$\gamma\Big[\sum_{\alpha,\beta=1}^{4} f_{\alpha\beta}(\gamma^2)\,\lambda_\alpha\lambda_\beta)\Big](\mathbf{t}\otimes\mathbf{k}-\mathbf{k}\otimes\mathbf{t}) = \mathbf{0}. \tag{5.22}$$

It can be proved by direct calculation that if $\lambda_i \neq \lambda_j$, $i \neq j$, then,

$$\sum_{\alpha,\beta=1}^{4} \lambda_\alpha\lambda_\beta f_{\alpha\beta}(\gamma^2) > 0. \tag{5.23}$$

There exist two possibilities for the solving equation to be satisfied:

$$\mathbf{t}\otimes\mathbf{k}-\mathbf{k}\otimes\mathbf{t} = \mathbf{0} \tag{5.24}$$

or

$$\gamma = 0. \tag{5.25}$$

Equation (5.24) holds when and only when $\mathbf{t} = \mathbf{R}^T\mathbf{k} = \mathbf{k}$, i.e., $\mathbf{R}$ is a symmetric rotation of the matrial. At this case, $\|\mathbf{R} * \mathbf{C}-\mathbf{C}\|$ equals zero.

Equation (5.25) means, in fact, that

$$\mathbf{tk} = 0, \quad \text{or equivalently, } \mathbf{kRk} = 0 \tag{5.26}$$

which implies that the rotation $\mathbf{R}$ rotates $\mathbf{k}$ to the plane perpendicular to $\mathbf{k}$ itself. The stationary value in this case can be calculated in the way presented in Tabl. 1.

Table 1. The coefficients $(\mathbf{R}^T * \mathbf{P}_\alpha)\cdot\mathbf{P}_\beta$.

	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$
$\mathbf{R}^T * \mathbf{P}_1$	$\frac{9}{4}\sin^2\varkappa\sin^2(2\varkappa_0-\varkappa)$	$\frac{9}{16}\sin^2 2(\varkappa_0-\varkappa)$	0	$\frac{3}{4}\sin^2(\varkappa_0-\varkappa)$
$\mathbf{R}^T * \mathbf{P}_2$	$\frac{9}{16}\sin^2 2(\varkappa_0-\varkappa)$	$\frac{9}{4}\cos^2\varkappa\cos^2(2\varkappa_0-\varkappa)$	0	$\frac{3}{4}\cos^2(\varkappa_0-\varkappa)$
$\mathbf{R}^T * \mathbf{P}_3$	0	0	1	1
$\mathbf{R}^T * \mathbf{P}_4$	$\frac{3}{4}\sin^2(\varkappa_0-\varkappa)$	$\frac{3}{4}\cos^2(\varkappa_0-\varkappa)$	1	$\frac{1}{4}$

According to Eq. (5.4), the stationary value is

$$d^2(\mathbf{C}) = (\lambda_1-\lambda_2)^2+2(\lambda_3-\lambda_4)^2+(\lambda_1-\lambda_4)^2+(\lambda_2-\lambda_4)^2 - \frac{9}{2}\,[(\lambda_1-\lambda_4)\sin\varkappa\sin(2\varkappa_0-\varkappa)-(\lambda_2-\lambda_4)\cos\varkappa\cos(2\varkappa_0-\varkappa)]^2, \tag{5.27}$$

where we have used the equalities

$$\sin 2\varkappa_0 = \frac{2\sqrt{2}}{3}, \quad \cos 2\varkappa_0 = -\frac{1}{3}, \quad \cos 4\varkappa_0 = -\frac{7}{9}. \tag{5.28}$$

For transversely-isotropic materials,

$$\text{(5.29)} \qquad \operatorname{Sup}\delta(\mathbf{C}) = \lim_{\frac{\lambda_2}{\lambda_4} = \frac{\lambda_3}{\lambda_4} \to 0,\ \frac{\lambda_1}{\lambda_4} \to 2} \frac{d(\mathbf{C})}{2\|\mathbf{C}\|} = \frac{\sqrt{7}}{4} = 0.66144,$$

where the supremum is taken with respect to the elasticity tensors of all transversely isotropic materials.

The following special cases seem to be interesting to us:

i) When $\lambda_1 = \lambda_2$,

$$\text{(5.30)} \qquad d^2(\mathbf{C}) = 2(\lambda_3-\lambda_4)^2 + \frac{3}{2}(\lambda_1-\lambda_4)^2.$$

ii) When $\varkappa = 0$

$$\text{(5.31)} \qquad d^2(\mathbf{C}) = (\lambda_1-\lambda_2)^2 + 2(\lambda_3-\lambda_4)^2 + (\lambda_1-\lambda_4)^2 + \frac{1}{2}(\lambda_2-\lambda_4)^2.$$

iii) When $\varkappa = \frac{\pi}{2}$

$$\text{(5.32)} \qquad d^2(\mathbf{C}) = (\lambda_1-\lambda_2)^2 + 2(\lambda_3-\lambda_4)^2 + (\lambda_2-\lambda_4)^2 + \frac{1}{2}(\lambda_1-\lambda_4)^2.$$

iv) When $\varkappa = \varkappa_0$

$$\text{(5.33)} \qquad d^2(\mathbf{C}) = 2(\lambda_3-\lambda_4)^2 + \frac{3}{2}(\lambda_2-\lambda_4)^2.$$

v) When

$$\text{(5.34)} \qquad \varkappa = \varkappa_0 \quad \text{and} \quad \lambda_2 = \lambda_3 = \lambda_4.$$

the material becomes isotropic and, therefore, $d^2(\mathbf{C}) = 0$

In engineering applications, the material constants of transversely isotropic materials usually are given through elastic moduli $E(\mathbf{k})$, $E(\mathbf{n})$, Poisson's ratios $\nu(\mathbf{n})$, $\nu(\mathbf{k})$ and shear moduli $G(\mathbf{n}, \mathbf{k})$, where $\mathbf{n}$ is an arbitrary direction normal to the symmetry axis $\mathbf{k}$. The relations between the engineering constants, the rigidity modululi and the distributor are

$$\text{(5.35)} \qquad \frac{1}{\lambda_1} = \frac{1}{2}\left(\frac{1}{E(\mathbf{n})} + \frac{1}{E(\mathbf{k})} - \frac{\nu(\mathbf{n})}{E(\mathbf{n})} - \frac{2\nu(\mathbf{k})}{E(\mathbf{k})\sin 2\varkappa}\right),$$

$$\text{(5.36)} \qquad \frac{1}{\lambda_2} = \frac{1}{2}\left(\frac{1}{E(\mathbf{n})} + \frac{1}{E(\mathbf{k})} - \frac{\nu(\mathbf{n})}{E(\mathbf{n})} + \frac{2\nu(\mathbf{k})}{E(\mathbf{k})\sin 2\varkappa}\right),$$

$$\text{(5.37)} \qquad \lambda_3 = 2G(\mathbf{n}, \mathbf{k}),$$

$$\text{(5.38)} \qquad \lambda_4 = \frac{E(\mathbf{n})}{1+\nu(\mathbf{n})}.$$

The distributor is determined by means of Eq. (5.36) and the equation

$$\text{(5.39)} \qquad \operatorname{tg}\varkappa = \frac{\lambda_2\nu(\mathbf{k})}{E(\mathbf{k})-\lambda_2}.$$

EXAMPLE. Most of fine-grained and laminated rocks can be regarded as transversely isotropic materials. LEKHNITSKII [4] has given the elastic constants of a coarse dark-grey aleurolith containing 60–70 percent of fragments of quartz and feldspar and 30–40 percent of argil. The constants are as follows:

(5.40)
$$E(\mathbf{n}) = 6.09\times 10^9 \text{ Pa}, \quad E(\mathbf{k}) = 5.57\times 10^9 \text{ Pa},$$
$$G(\mathbf{n},\mathbf{k}) = 2.24\times 10^9 \text{ Pa}, \quad \nu(\mathbf{n}) = 0.22,$$
$$\nu(\mathbf{k}) = 0.24.$$

The rigidity moduli and the distributor for such material are

(5.41)
$$\lambda_1 = 9.74\times 10^9 \text{ Pa}, \quad \lambda_2 = 4.90\times 10^9 \text{ Pa},$$
$$\lambda_3 = 4.49\times 10^9 \text{ Pa}, \quad \lambda_4 = 4.99\times 10^9 \text{ Pa},$$
$$\varkappa = 0.955.$$

The $\mathcal{O}$-orbit diameter $d(\mathbf{C})$ and the anisotropy parameter $\delta(\mathbf{C})$ of this matrial are

(5.42)
$$d(\mathbf{C}) = 1.84\times 10^9 \text{ Pa}, \quad \delta(\mathbf{C}) = 0.072.$$

From Eq. (5.41), it is obvious that the material is very close to isotropic material whose elastic constants satisfy Eq. (5.34).

It seems that the following material [1] deviates a little more from the isotropic material. The elastic constants are

(5.43)
$$E(\mathbf{n}) = 11.17\times 10^9 \text{ Pa}, \quad E(\mathbf{k}) = 5.19\times 10^9 \text{ Pa},$$
$$G(\mathbf{n},\mathbf{k}) = 1.94\times 10^9 \text{ Pa}, \quad \nu(\mathbf{n}) = 0.067,$$
$$\nu(\mathbf{k}) = 0.328.$$

The rigidity moduli and the distributor for such material are

(5.44)
$$\lambda_1 = 18.32\times 10^9 \text{ Pa}, \quad \lambda_2 = 4.52\times 10^9 \text{ Pa},$$
$$\lambda_3 = 3.88\times 10^9 \text{ Pa}, \quad \lambda_4 = 10.47\times 10^9 \text{ Pa},$$
$$\varkappa = 1.141.$$

The $\mathcal{O}$-orbit diameter $d(\mathbf{C})$ and the anisotropy parameter $\delta(\mathbf{C})$ of this matrial, respectively, are

(5.45)
$$d(\mathbf{C}) = 11.23\times 10^9 \text{ Pa}, \quad \delta(\mathbf{C}) = 0.256.$$

5.5. Orthotropic materials. If one substitutes Eqs. (4.25) and (4.27) into Eq. (5.2) directly the equation will be very complicated. It is easy to see that if the rotation $\mathbf{R}$ is such that $\boldsymbol{\omega}_4, \boldsymbol{\omega}_5, \boldsymbol{\omega}_6$ permute with each other, then all rotated tensors of $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_6$ remain perpendicular to each other. Equation (5.2) is satisfied because

(5.46)
$$\text{i)}\quad i,j \geqslant 4, \quad (\mathbf{R}^T * \boldsymbol{\omega}_i)\cdot\boldsymbol{\omega}_j = 0 \quad \text{or} \quad \mathbf{R}^T * \boldsymbol{\omega}_i = \boldsymbol{\omega}_j,$$
$$\text{ii)}\quad i \geqslant 4,\ j \leqslant 3, \quad (\mathbf{R}^T * \boldsymbol{\omega}_i)\cdot\boldsymbol{\omega}_j = 0,$$
$$i \leqslant 3,\ j \geqslant 4,$$
$$\text{iii)}\quad i,j \leqslant 3, \quad (\mathbf{R} * \boldsymbol{\omega}_i)\boldsymbol{\omega}_j = \boldsymbol{\omega}_j(\mathbf{R}^T * \boldsymbol{\omega}_i).$$

The correspondence relation between the exchanges of [456] and the exchanges of [123] is as follows:

$$(45) \to (23),\quad (56) \to (12),\quad (46) \to (13). \tag{5.47}$$

Because each permutation can be decomposed into multiplications of finite exchanges, the relation (5.47) induces naturally a correspondence relation between the permutations of [456] and [123].

The stationary values in these cases are

$$d^2(\mathbf{C}) = 2\Big[\sum_{i=1}^{6} \lambda_i^2 - \sum_{i,j=4}^{6} \lambda_i \lambda_j \delta_{\sigma(i)j} - \sum_{k,i,j=1}^{3} \lambda_i \lambda_j (\omega_{i\sigma(k)}\omega_{jk})^2\Big], \tag{5.48}$$

where σ is a permutation of [123456] such that the first half part and the last half part satisfy the correspondence relation (5.47). For different permutation σ, Eq. (5.48) will lead to different stationary values, the largest of which is what we need: maximum $d(\mathbf{C})$.

For orthotropic matrials, the relations between the moduli of rigidity λ_1, λ_2, λ_3, λ_4 λ_5, λ_6, the distributors ω_{ij} and the engineering constants are

$$\frac{1}{E(X)} = \frac{\omega_{11}^2}{\lambda_1} + \frac{\omega_{21}^2}{\lambda_2} + \frac{\omega_{31}^2}{\lambda_3},\quad \frac{1}{E(Y)} = \frac{\omega_{12}^2}{\lambda_1} + \frac{\omega_{22}^2}{\lambda_2} + \frac{\omega_{32}^2}{\lambda_3},\quad \frac{1}{E(Z)} = \frac{\omega_{13}^2}{\lambda_1} + \frac{\omega_{23}^2}{\lambda_2} + \frac{\omega_{33}^2}{\lambda_3}, \tag{5.49}$$

$$-\frac{\nu(X,Y)}{E(Y)} = \frac{\omega_{11}\omega_{12}}{\lambda_1} + \frac{\omega_{21}\omega_{22}}{\lambda_2} + \frac{\omega_{31}\omega_{32}}{\lambda_3},\quad -\frac{\nu(Y,Z)}{E(Z)} = \frac{\omega_{12}\omega_{13}}{\lambda_1} + \frac{\omega_{22}\omega_{23}}{\lambda_2} + \frac{\omega_{32}\omega_{33}}{\lambda_3},\quad -\frac{\nu(X,Z)}{E(Z)} = \frac{\omega_{11}\omega_{13}}{\lambda_1} + \frac{\omega_{21}\omega_{23}}{\lambda_2} + \frac{\omega_{31}\omega_{33}}{\lambda_3}, \tag{5.50}$$

$$\frac{1}{4G(X,Y)} = \frac{1}{2\lambda_4},\quad \frac{1}{4G(Z,X)} = \frac{1}{2\lambda_5},\quad \frac{1}{4G(Z,Y)} = \frac{1}{2\lambda_6}. \tag{5.51}$$

Equations (5.49) and (5.50) can be combined into a matrix equation

$$W^T\Big[\frac{1}{\lambda}\Big]W = U, \tag{5.52}$$

where U, $\left[\frac{1}{\lambda}\right]$ and W are defined as follows:

$$(5.53)\qquad U = \begin{vmatrix} \frac{1}{E(X)} & -\frac{\nu(X,Y)}{E(Y)} & -\frac{\nu(X,Z)}{E(Z)} \\ & \frac{1}{E(Y)} & -\frac{\nu(Y,Z)}{E(Z)} \\ \text{Symmetry} & & \frac{1}{E(Z)} \end{vmatrix},$$

$$(5.54)\qquad \left[\frac{1}{\lambda}\right] = \begin{vmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{vmatrix}, \qquad W = \begin{vmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{vmatrix}.$$

In other words, U can be diagonalized into $\left[\frac{1}{\lambda}\right]$ by the orthogonal matrix W. In order to find the rigidity moduli λ_1, λ_2, λ_3 and the distributors W, one needs only to consider the diagonalizing problem of U. The following are four calculation examples for orthotropic rocks. The material coefficients are taken from LUMA and VUTUKURU's "Handbook on Mechanical Properties of Rocks" [3]

Table 2. Engineering elastic constants of the four rocks.

	$E(X)$ ($\times 10^9$ Pa)	$E(Y)$ ($\times 10^9$Pa)	$E(Z)$ ($\times 10^9$ Pa)	$G(X,Y)$ ($\times 10^9$ Pa)	$G(X,Z)$ ($\times 10^9$ Pa)	$G(Y,Z)$ ($\times 10^9$ Pa)	$V(X,Y)$	$V(X,Z)$	$V(Y,Z)$
Rock I	1.70	1.48	0.72	0.61	0.63	0.75	0.005	0.058	0.087
Rock II	8.30	7.85	7.33	2.91	3.01	3.19	0.314	0.353	0.306
Rock III	11.27	10.39	9.60	3.58	3.72	4.17	0.278	0.276	0.317
Rock IV	9.90	8 30	6.08	2.36	2.33	3.55	0.185	0.258	0.329

Table 3. The rigidity moduli of the four rocks (unit = 10^9 Pa).

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
Rock I	1.72	1.48	0.72	1.22	1.25	1.49
Rock II	24.29	5.59	6.10	5.82	6.02	6.39
Rock III	8.49	26.87	7.46	7.13	7.45	9.31
Rock IV	7.76	21.09	4.85	4.72	4.66	7.10

Table 4. The distributors for the four rocks.

	ω_{11}	ω_{12}	ω_{13}	ω_{21}	ω_{22}	ω_{23}	ω_{31}	ω_{32}	ω_{33}
Rock I	0.994	0.048	0.099	−0.050	1.000	0.011	−0.100	−0.016	0.995
Rock II	0.618	0.556	0.556	−0.474	−0.301	0.828	−0.628	0.775	−0.077
Rock III	0.779	−0.571	−0.258	0.605	0.579	0.546	−0.162	−0.582	0.797
Rock IV	0.732	−0.667	−0.142	0.634	0.589	0.502	−0.251	−0.457	0.853

Table 5. Orbit diameters and anisotropy degrees.

	$d(\mathbf{C})$, Pa	$\delta(\mathbf{C})$
Rock I	1.449×10^9	0.219
Rock II	2.498×10^9	0.045
Rock III	4.079×10^9	0.063
Rock IV	6.548×10^9	0.131

Now we have established a *total order* in the set consisting of all linearly elastic materials according to their anisotropy degrees. For two arbitrary elastic material $\mathbf{C}_1$ and $\mathbf{C}_2$, we write $\mathbf{C}_1 > \mathbf{C}_2$ if $\delta(\mathbf{C}_1) > \delta(\mathbf{C}_2)$. Therefore, according to Table 5, we have the following relation:

(5.55) $$ROCK\ I > ROCK\ IV > ROCK\ III > ROCK\ II.$$

Notations

A three-dimensional Euclidean vector space is denoted by э, its elements by $\mathbf{x}, \mathbf{n}, \ldots,$ scalar product by $\mathbf{x}y$.

The Euclidean tensors of the p-th order are elements of the p-th tensor power $\otimes^p$э.

Second order symmetric tensors are denoted by $\boldsymbol{\alpha}, \boldsymbol{\beta}\ldots$, orthogonal tensors of the second order are denoted by $\mathbf{R}, \mathbf{Q}$. They are elements in the orthogonal group $\mathcal{O} = SO(3) = $ = Aut э $= \{\mathbf{Q} \in$ э$\otimes$э$|\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{1}\}$.

Group $\mathcal{O}$ operates on $\otimes^p$э according to the rule $\mathbf{A} \to Q * \mathbf{A}$, $Q*$ is a linear operation defined on decomposable tensors by

$$Q*(\mathbf{a}\otimes \ldots \otimes\mathbf{b}) = Q\mathbf{a}\otimes \ldots \otimes Q\mathbf{b}.$$

Permutation of the sequence $(1, \ldots, p)$ is denoted by $\sigma = (\sigma(1) \ldots \sigma(p))$. Permutations form a group which operates on $\otimes^p$э according to $\mathbf{A} \to \sigma\times\mathbf{A}$, where $\sigma\times$ is a linear operator prescribed on decomposable tensors by

$$\sigma\times(\mathbf{a}_1\otimes \ldots \otimes\mathbf{a}_p) = \mathbf{a}_{\sigma(1)}\otimes \ldots \otimes\mathbf{a}_{\sigma(p)}.$$

The permutations σ_i, $i = 1, \ldots, p$ are given as $\sigma_i = (i\,12\ldots(i-1)\,1\,(i+1)\ldots p)$.

To translate all formulae to well-known Cartesian index language, you may use the table

$\mathbf{x}, \mathbf{n}$	x_i, n_i
$\mathbf{x}\mathbf{y}$	$x_i y_i$
$\mathbf{x}\otimes\mathbf{y}$	$x_i y_j$
$\boldsymbol{\alpha}, \mathbf{1}$	α_{ij}, δ_{ij}
$\boldsymbol{\alpha}\boldsymbol{\beta}$	$\alpha_{ip}\beta_{pj}$
$\boldsymbol{\alpha}\otimes\boldsymbol{\beta}$	$\alpha_{ij}\beta_{kl}$
$\boldsymbol{\alpha}\cdot\boldsymbol{\beta}$	$\alpha_{pq}\beta_{pq}$
$\mathbf{A}$	$A_{i\dots j}$
$\mathbf{A}\odot\mathbf{B}$	$A_{ip\dots q}B_{jp\dots q}$
$\mathbf{R}*\mathbf{n}$	$R_{ip}n_p$
$\mathbf{R}*\boldsymbol{\alpha} = \mathbf{R}\boldsymbol{\alpha}\mathbf{R}^T$	$R_{ip}R_{jq}\alpha_{pq}$
$\mathbf{R}*\mathbf{A}$	$R_{ip}\dots R_{jq}A_{p\dots q}$
$(\mathbf{1}\otimes\mathbf{R}\otimes\dots\otimes\mathbf{R})\mathbf{A}$	$\delta_{ip}R_{jp}\dots R_{kr}A_{pq\dots r}$
$\mathbf{C}$	C_{ijkl}
$\mathbf{C}\cdot\mathbf{D}$	$C_{pqrs}\ D_{pqrs}$
$\mathbf{C}\circ\mathbf{D}$	$C_{ijpq}\ D_{pqkl}$

$$\mathbf{I}_{ijkl} = \frac{1}{2}(\delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl}).$$

We have

$$\mathbf{1x} = \mathbf{x} \qquad \mathbf{I}\cdot\boldsymbol{\alpha} = \boldsymbol{\alpha}$$

for all $\mathbf{x} \in э$, $\boldsymbol{\alpha} \in \mathrm{sym}э\otimesэ$.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
and
SHANGHAI INSTITUTE OF APPLIED MATHEMATICS AND MECHANICS, CHINA.

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