Arch. Mech., 41, 5, pp. 659–678, Warszawa 1989

Controlling agents in dynamics of rigid bodies

J. J. SŁAWIANOWSKI (WARSZAWA)

WE DISCUSS certain problems concerning control and programme motion of rigid bodies and their systems. The special stress is laid on phenomenological time-dependent moments, rotors, manipulated inertial tensors as controlling agents. For such models we formulate certain statements concerning controllability. Differential-geometric techniques (Pfaff problem, Lie algebras of vector fields) are used as mathematical tools.

Praca dotyczy zagadnień sterowania i ruchu programowego w mechanice bryły sztywnej i układu brył sztywnych. Szczególny nacisk położono na takie czynniki sterowania jak fenomenologiczne momenty skręcające zależne od czasu, rotory wmontowane w obiekt i regulowany moment bezwładności. Przedstawiono pewne wyniki dotyczące sterowalności takich modeli. Wyniki te otrzymano przy użyciu metod geometrii różniczkowej (zagadnienie Pfaffa, algebry Liego pól wektorowych itp).

Работа касается задач управления и программного движения в механике жесткого тела и системы жестких тел. Особенное внимание уделено таким фактором управления, как феноменологические скручивающие моменты, зависящие от времени, роторы вмонтированные в объект и регулированный момент инерции. Представлены некоторые результаты, касающиеся управляемости таких моделей. Эти результаты получены при использовании методов дифференциальной геометрии (задача Пфаффа, алгебры Ли векторных полей и т.п.).

e differential-geometric aspects of controllability

LET US BEGIN by recalling a few elementary concepts concerning control processes. We shall use standard symbols, i.e., state and control parameters will be denoted, respectively, by $x^1 \ldots x^n$ and $u^1 \ldots u^m$. The manifold of states will be denoted by P and that of controls by C. Control processes are ruled by differential equations of the form

$$\frac{dx^i}{dt} = f^i(x^1 \ldots x^n;\ u^1 \ldots u^m), \tag{1.1}$$

or, using obvious abbreviations,

$$\frac{dx}{dt} = f(x, u), \tag{1.2}$$

where the control signals $t \mapsto u(t)$ are assumed to obey certain mathematical conditions (admissible controls); usually they are piecewise continuous-differentiable functions. One can also consider processes for which f depends explicitly on the time variable

$$\frac{dx}{dt} = f(t, x, u). \tag{1.3}$$

It is said that the system (1.1) is *controllable* if for any pair of states x_1, x_2 there exist functions $t \mapsto x(t)$, $t \mapsto u(t)$ (u being within the class of admissible controls) such that:

(i) (1.1) is satisfied,

(ii) $x(t_i) = x_1, x(t_f) = x_2$;

t_i denotes the initial time instant of our controlling operation, and t_f is some later, in general non-specified a priori, final instant of time.

Roughly speaking, any state is approachable from any state by a suitable choice of control.

With any state x we can associate the set $A(x)$ of all states attainable from x under the influence of a proper controlling signal $u(t)$. If the system is controllable, then $A(x) = P$ for any state x.

Simple and effective criteria of controllability exist only for exceptional problems, first of all for stationary linear systems described by the equations

$$\frac{dx}{dt} = Ax + Bu, \tag{1.4}$$

A and B being, respectively, $n \times n$ and $n \times m$ constant matrices. Namely, Kalman's theorem states that the system (1.4) is controllable if and only if

$$\mathrm{Rank}[B, AB, A^2B, \ldots, A^{n-1}B] = n. \tag{1.5}$$

There is no general theory for nonlinear systems. There are methods based on the analysis of linearized models. Another, relatively modern, approach consists in using Lie-algebraic and Lie-group techniques. These methods are applicable to systems for which certain vector-fields on P, characterizing the dynamical quantity f, generate a finite-dimensional Lie algebra. The problem (1.1) can then be reduced in some sense to the control problem on a Lie group; this enables one to use powerful and effective geometric techniques elaborated in the Lie group theory [3, 6].

In general, the global controllability problem is very difficult and as no effective solution. However, quite often we are interested in certain weaker aspects of controllability which can be relatively easily determined by using non-complicated algebraic and differential-geometric techniques [3, 6].

We say that the control system is *dimensionally-controllable* if for any x the accessibility set $A(x)$ is an n-dimensional submanifold of P. Thus m control parameters $(u^1 \ldots u^m)$ are sufficient for shifting the system from any state into n-dimensional open domains of P. In many problems, the first step of controllability analysis consists in deciding whether the system is or is not dimensionally-controllable. In this step one uses only local differential concepts. In the case of an affirmative answer, the problem of global controllability is studied with the use of other, e.g., topological or qualitative techniques elaborated in the dynamical systems theory.

Obviously, if $m > n$, then dimensionally noncontrollable systems are rather pathological and nonrealistic; it is clear that some of their parameters must be artificial. However, the point is that in realistic models we often have $m < n$ and the problem becomes non-academic. Moreover, from the, so to speak, "naive" point of view it seems rather natural to expect that if $m < n$, our control possibilities are too poor for the effective manipulation with n state parameters. A more careful analysis shows, however, that with a suitable structure of f in (1.1), dimensional control is possible even if m is drastically smaller than n. To see this, let us formulate the control problem (1.1) in the following way:

We construct the $(n+m+1)$-dimensional evolution manifold $E := P \times C \times \mathsf{R}$ of variables x, u, t. The dynamical law (1.1) is equivalent to endowing E with some kind of geometry based on the system of differential one-forms:

$$\Omega^i = dx^i - f^i(x, u, t)dt. \tag{1.6}$$

More rigorously, this geometry is represented by the $(m+1)$-dimensional distribution [field of $(m+1)$-dimensional tangent subspaces] solving the Pfaff problem

$$\Omega^i = 0. \tag{1.7}$$

Solutions of Eq. (1.7) are represented in E by one-dimensional integral manifolds (curves) $t \mapsto (x(t), u(t), t)$ of this Pfaff system. It is obvious that controllability has something to do with the degree of non-integrability of this distribution and with the location of integral surfaces of maximal dimension with respect to the components of the Cartesian splitting $E = P \times C \times \mathsf{R}$, [6].

If f in Eq. (1.1) is not subject to additional restrictions, it is rather hard to formulate effective criteria in a closed, explicit form. However, this is possible for a mathematically narrow, but technically very important class of problems with *multiplicative controls*. For such systems f is a linear-nonhomogeneous function of u,

$$\frac{dx}{dt} = Y_0(x) + \sum_{a=1}^{m} u_a Y_a(x), \tag{1.8}$$

i.e., our control problem is fully described by the $(m+1)$-tuple of vector fields Y_μ, $\mu = 0, 1, \ldots, m$ on the state manifold P. Y_0 describes the natural, non-influenced, dynamics of the system, and Y_a are *basic control modes*.

Let us perform the *Lie-bracket extension* of the system of fields Y_μ, $\mu = 0, 1, \ldots, m$. Recall (cf. [3, 4, 6]) that the Lie bracket $[A, B]$ of the vector fields A, B with the components $A^i(x)$, $B^i(x)$, $i = 1 \ldots m$, is defined as a new vector field with the components

$$[A, B]^i = A^j(x)\frac{\partial B^i}{\partial x^j} - B^j(x)\frac{\partial A^i}{\partial x^j}; \tag{1.9}$$

this definition is independent of the choice of coordinates $(x^1 \ldots x^n)$. The construction of the extended system of $(Y_0, Y_1 \ldots Y_m)$ proceeds in the following way:

(i) We calculate Lie brackets $Y_{\mu\nu} := [Y_\mu, Y_\nu]$.

(ii) We form the *first-step extended system* consisting of Y and their Lie brackets

$$(\ldots Y_\mu \ldots ; \ldots [Y_\mu, Y_\nu] \ldots). \tag{1.10}$$

Let us introduce the new index $\mu(1)$ with a suitable range and the new kernel symbol $Y(1)$; the system (1.10) will be written as

$$(\ldots Y(1)_{\mu(1)} \ldots). \tag{1.11}$$

(iii) We calculate the rank of the components matrix of the Equation (1.11),

$$r(1) := \operatorname{rank} \| \ldots Y(1)^i_{\mu(1)} \ldots \|; \quad i = 1 \ldots n. \tag{1.12}$$

(iv) We repeat everything for the extended system $(\ldots Y(1)_{\mu(1)} \ldots)$, and so on. After k steps, we obtain the k-th order extended system

$$(\ldots\, Y(k)_{\mu(k)} \ldots) \tag{1.13}$$

with the rank

$$r(k) = \operatorname{rank} \| Y(k)^i_{\mu(k)} \| . \tag{1.14}$$

(v) We continue until, at a certain final stage $k = f$, the rank stops increasing:

$$r(f+1) = r(f) = p. \tag{1.15}$$

Obviously, f exists because $r(k) \leqslant r(k+1) \leqslant n$.
The final system of fields

$$(\ldots\, Y(f)_{\mu(f)} \ldots) \tag{1.16}$$

is just the Lie-bracket extension of $(Y_0, Y_1 \ldots Y_m)$.

The final value $p = r(f) = r(f+1)$ determines dimensional controllability. Namely, if $p < n$, then certainly the system is incontrollable. If $p = n$, it is at least *dimensionally-controllable*.

The reason is that, for $p < n$, the extended system $(\ldots Y(f)_{\mu(f)} \ldots)$ locally determines a p-dimensional involutive distribution. The state manifold P is foliated by the family of p-dimensional surfaces $F_a = \text{const}$, $a = 1, \ldots (n-p)$, where F_a satisfy differential equations:

$$\hat{Y}(f)_{\mu(f)} \cdot F_a = \sum_{i=1}^{n} Y(f)^i_{\mu(f)} \frac{\partial F_a}{\partial x^i} = 0, \tag{1.17}$$

and all vectorfields $Y(f)_{\mu(f)}$ [thus also $Y_0, Y_1 \ldots Y_m$] are tangent to those surfaces. Therefore, independently of the shape of control signals $u(t)$, all trajectories $x(t)$ are placed on p-dimensional value surfaces $F_a = \text{const}$, $a = 1 \ldots (n-p)$, and there is no possibility of transition between different surfaces. If $p = n$, then starting from any state x we can reach n-dimensional manifolds of final states; [3, 6].

The answer to other controllability questions, e.g., global controllability, or local controllability in open neighbourhoods of initial states, depends on the detailed shape of local *cones*

$$\left\{ Y_0(x) + \sum_{i=1}^{m} u^i Y_i(x), \quad x\text{—fixed}, \quad u\text{—variable within its range} \right\}$$

of admissible "forces" and of the character of their dependence of $x \in P$. Let us illustrate this with the help of the following trivial example: $P = \mathrm{R}^2$, $C = \mathrm{R}$,

$$\frac{dx^1}{dt} = 1, \ \frac{dx^2}{dt} = u,$$

thus $\hat{Y}_0 = \dfrac{\partial}{\partial x^1}$, $\hat{Y}_1 = \dfrac{\partial}{\partial x^2}$. Obviously this system is dimensionally-controllable; however, from any initial state (a, b) only points of the half-plane $x^1 > a$ are reachable. Thus there is neither global nor local controllability because in any neighbourhood of the point (a, b) there are states which cannot be reached from (a, b). This kind of non-attainability does not occur when f is homogeneous-linear in controls and $m = n$ or $p = n$, and the range of admissible u contains an open neighbourhood of zero.

Let us quote a simple example of determining controllability through the above criterion based on the Lie-bracket extension of $(Y_0, Y_1 \ldots Y_m)$. As a manifold of states we take $GL^+(n)$—the group of all positive-determinant real $n \times n$ matrices. Control manifolds C will be submanifolds of $L(n)$ — the space of all real $n \times n$ matrices. We consider processes described by equations of the form

$$\frac{dX}{dt} = Y(X) + UX, \tag{1.18}$$

or,

$$\frac{dX}{dt} = Y(X) + XU, \tag{1.19}$$

where U runs over $C \subset P$. Both equations are multiplicative in controls.

(i) If $C = \mathrm{Sym}\,L(n) \subset L(n)$ — the space of all symmetric matrices, then the problems (1.18), and (1.19) are dimensionally-controllable (and if $Y = 0$, they are controllable without adjectives).

(ii) If $C = so(n) \subset L(n)$ — the space of skew-symmetric matrices, and $Y = 0$, then both problems are incontrollable. There exist $\frac{1}{2}n(n-1)$ controllable parameters (orthogonal group $SO(n) \subset GL^+(n)$ and its cosets as attainability surfaces $F = \mathrm{const}$) and $\frac{1}{2}n(n+1)$ noncontrollable parameters (deformations–symmetric matrices).

(iii) If $C = sl(n) \subset L(n)$ — the space of trace-less matrices, then (if $Y = 0$), both problems (1.18) and (1.19) are incontrollable. There exist (n^2-1) controllable parameters (unimodular group $SL(n) \subset GL^+(n)$ and its cosets) and one completely incontrollable (dilatational) parameter.

Control problems faced with in mechanics have certain specific features. Namely, the dynamical systems $dx/dt = f(x)$ used in mechanics are reducible to second-order differential equations, usually written as Lagrange equations of the second kind,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \phi_i, \qquad i = 1 \ldots n; \tag{1.20}$$

L denotes the Lagrangian, and ϕ — additional forces nonderivable from L (e.g., dissipative ones). Using the Newtonian concept of mechanical states as pairs position-velocity, $x = (q, v)$, we can write (1.20) in the form

$$\begin{gathered} \frac{dq^i}{dt} = v^i, \\ \frac{\partial^2 L}{\partial v^i \partial v^j}\frac{dv^j}{dt} + \frac{\partial^2 L}{\partial v^i \partial q^j}v^j - \frac{\partial L}{\partial q^i} = \phi_i(q, v). \end{gathered} \tag{1.21}$$

This is almost the form $dx/dt = f(x)$ because in realistic problems the matrix $\partial^2 L/\partial v^i \partial v^j$ is nonsingular and usually well-known (kinematical metric tensor underlying the kinetic energy form).

The mathematical asymmetry of the variables (q, v) in Eq. (1.21) has also some practical aspects. Namely, quite often it happens that one is interested in *partial controllability problems* (controllability with *moving ends*), first of all — controllability in the configuration space of variables q^i, or in the space of velocities v^i.

Control parameters u may be introduced through generalized forces ϕ or Lagrangian L. However, it happens quite often that before turning Eq. (1.21) into a control system we introduce additional degrees of freedom $(Q^1 \dots Q^k)$ (configurations of steering instruments) and consider richer systems described by $L(q, \dot{q}; Q, \dot{Q})$, $\phi\ (q, \dot{q}; Q, \dot{Q})$. The discussion of controllability is based on equations of motion written in the form $dx/dt = f(x, u)$. However, the proper, or convenient, choice of (x, u)-variables is quite often nonautomatic (cf. examples in the next sections). Usually, u are some functions of $(Q, \dot{Q}, \phi)$, and state variables x are functions of $(q, \dot{q}; Q, \dot{Q})$; their proper choice depends on our inventiveness.

2. Rigid body conventions

If we fix some standard system of Cartesian coordinates in our three-dimensional space, then the configuration space of a rigid body can be identified with $SO(3, \mathsf{R}) \times \mathsf{R}^3$, i.e., with the semi-direct product of the proper orthogonal group in 3 dimensions and the three-dimensional numerical space. Thus configurations are represented as pairs $(\mathbf{R}, \mathbf{x})$, where $\mathbf{R}$ is a real 3×3 matrix satisfying the conditions $\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathrm{Id}$, $\det \mathbf{R} = 1$, and $\mathbf{x}$ is a triple of real numbers. $\mathbf{R}$ describes the degrees of freedom of the relative motion, i.e., the orientations of the body, and $\mathbf{x}$ represents the radius vector of some fixed point of the body (e.g., the centre of mass) with respect to the origin of coordinate systems; thus it refers to translational motion of the body.

When the body moves in space, $(\mathbf{R}, \mathbf{x})$ are functions of time. Let us introduce the matrices

$$\mathbf{w} = \frac{d\mathbf{R}}{dt}\mathbf{R}^T, \quad \mathbf{W} = \mathbf{R}^T\frac{d\mathbf{R}}{dt}. \tag{2.1}$$

They are skew-symmetric, thus we identify them with numerical axial vectors $\mathbf{w} = (w_1, w_2, w_3)^T$, $\mathbf{W} = (W_1, W_2, W_3)^T$, where

$$w_{ij} = -\varepsilon_{ijk}w_k, \quad W_{AB} = -\varepsilon_{ABC}W_C, \tag{2.2}$$

ε being totally antisymmetric symbol, $\varepsilon_{123} = 1$, and we apply the summation convention (summation over repeated indices, unless otherwise stated).

The quantities w_i are components of the angular velocity with respect to the space-fixed reference frame, whereas W_A are projections of this vector onto orthonormal axes co-moving with the body. Similarly the quantities characterizing translational motion, e.g., $d\mathbf{x}/dt$, may be represented through laboratory and co-moving components, respectively v^i and V^A. Obviously,

$$\mathbf{w} = \mathbf{R}\mathbf{W}, \quad \mathbf{v} = \mathbf{R}\mathbf{V}. \tag{2.3}$$

The axial vector of the moment of forces with respect to the centre of mass will be denoted by $\mathbf{M}$ of $\mathbf{m}$, depending on whether we use the co-moving or laboratory representation; $\mathbf{m} = \mathbf{RM}$.

Almost all physical quantities will be represented through their co-moving components.

Inertial properties of the body are described by the total mass M and the constant matrix $\mathbf{I}$ expressing the ininertial tensor of the body with respect to the centre of mass through co-moving components. In laboratory representation, the inertial tensor is described by the matrix $\mathbf{RIR}^T$; it is nonconstant unless $\mathbf{I}$ is completely degenerate, i.e., the top is spherical, $\mathbf{I} = I\mathbf{Id}$. Without any loss of generality, we can choose as co-moving axes the principal axes of inertia, thus $\mathbf{I} = \mathrm{diag}(I_1, I_2, I_3)$.

Kinetic energies of rotational and translational motions are given by

$$T_{\mathrm{rot}} = \frac{1}{2}\mathbf{W}^T\mathbf{I}\mathbf{W}, \quad T_{\mathrm{tr}} = \frac{M}{2}\mathbf{v}^T\mathbf{v} = \frac{M}{2}\mathbf{V}^T\mathbf{V}; \tag{2.4}$$

the total kinetic energy is $T_{\mathrm{tot}} = T_{\mathrm{rot}} + T_{\mathrm{tr}}$.

Internal angular momentum, i.e., angular momentum with respect to the centre of mass, is given by

$$\mathbf{S} = \mathbf{IW} \tag{2.5}$$

in the co-moving representation, and

$$\mathbf{s} = \mathbf{RIR}^T\mathbf{w} \tag{2.6}$$

in laboratory representation.

In this paper we shall deal almost exclusively with the relative, i.e., rotational motion.

As it is well-known, equations of rotational motion can be written in the following form:

$$\begin{gathered} \mathbf{I}\frac{d\mathbf{W}}{dt} = (\mathbf{IW}) \times \mathbf{W} + \mathbf{M}, \\ \frac{d\mathbf{R}}{dt} = \mathbf{RW}, \end{gathered} \tag{2.7}$$

$\mathbf{a} \times \mathbf{b}$ denoting the vector product of vectors $\mathbf{a}, \mathbf{b} \in \mathrm{R}^3$.

Equations (2.7) represent a system of 6 ordinary differential equations of first order for 6 gyroscopic parameters of state, i.e., for 3 variables W_A and 3 generalized coordinates parametrizing the manifold of $\mathbf{R}$, e.g., Euler angles φ, ψ, ϑ or the rotation vector $\mathbf{k} = (k_1, k_2, k_3)^T$. Let us recall that $\mathbf{k}$ is defined by the formulas

$$\mathbf{R}(\mathbf{k})\mathbf{k} = \mathbf{k}, \quad \mathrm{Tr}\,\mathbf{R}(\mathbf{k}) = 1 + 2\cos k, \tag{2.8}$$

which mean that the versor $\mathbf{n} = \frac{1}{k}\mathbf{k}$ defines the rotation axis (in the right-handed screw convention) and $k = |\mathbf{k}|$ is the rotation angle. Explicitly, $\mathbf{R}(\mathbf{k})$ is given by

$$\mathbf{R}(\mathbf{k})\mathbf{x} = \cos k\mathbf{x} + (1 - \cos k)\left(\mathbf{x}^T\frac{\mathbf{k}}{k}\right)\frac{\mathbf{k}}{k} + \sin k\frac{\mathbf{k}}{k} \times \mathbf{x}, \tag{2.9}$$

thus, for small rotations, $\mathbf{k} \approx \mathbf{0}$,

$$\mathbf{R}(\mathbf{k})\mathbf{x} \approx \mathbf{x}+\mathbf{k}\times\mathbf{x}.$$

If we use the **k**-parametrization, then the explicit representation of (2.7) as a dynamical system, i.e., a system of differential equations solved with respect to derivatives, has the form

$$\begin{aligned}
&\frac{d\mathbf{W}}{dt} = \mathbf{I}^{-1}\left((\mathbf{I}\mathbf{W})\times\mathbf{W}\right)+\mathbf{I}^{-1}\mathbf{M},\\
(2.10)\qquad &\frac{d\mathbf{k}}{dt} = \mathbf{C}(\mathbf{k})\mathbf{W} = \frac{k}{2}\operatorname{ctg}\frac{k}{2}\mathbf{W}+\left(1-\frac{k}{2}\operatorname{ctg}\frac{k}{2}\right)\left(\mathbf{W}^T\frac{\mathbf{k}}{k}\right)\frac{\mathbf{k}}{k}-\frac{1}{2}\mathbf{W}\times\mathbf{k}.
\end{aligned}$$

The advantage of Eqs. (2.7) and (2.10) is that their first subsystems, i.e., *Euler equations*, can be independently solved in numerous practically relevant problems when the moment $\mathbf{M}$ depends only on $\mathbf{W}$ and t, but not on $\mathbf{R}$ (i.e., on $\mathbf{k}$). Then, substituting the solution for $\mathbf{W}(t)$ into second subsystems of Eqs. (2.7) and (2.10) we can, in principle, find the time dependence of the orientation variable $\mathbf{R}(t)$, i.e., $\mathbf{k}(t)$.

The dynamical structure of the problem is fully described by Euler equations. On the contrary, there is nothing but geometry and group theory in Eqs. (1.7.b) and (2.10.b). Incidentally, let us notice that the relationship between $d\mathbf{k}/dt$ and the laboratory representation $\mathbf{w}$ of the angular velocity differs from $(2.10)_2$ only in sign of the vector product term,

$$(2.11)\qquad \frac{d\mathbf{k}}{dt} = \mathbf{C}(\mathbf{k})^T\mathbf{w} = \frac{k}{2}\operatorname{ctg}\frac{k}{2}\mathbf{w}+\left(1-\frac{k}{2}\operatorname{ctg}\frac{k}{2}\right)\left(\mathbf{w}^T\frac{\mathbf{k}}{k}\right)\frac{\mathbf{k}}{k}+\frac{1}{2}\mathbf{w}\times\mathbf{k}.$$

If co-moving axes coincide with the principal directions of $\mathbf{I}$, i.e., $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$, then Eq. $(2.7)_1$ takes on the standard form

$$\begin{aligned}
&I_1\frac{dW_1}{dt} = (I_2-I_3)W_2W_3+M_1,\\
(2.12)\qquad &I_2\frac{dW_2}{dt} = (I_3-I_1)W_3W_1+M_2,\\
&I_3\frac{dW_3}{dt} = (I_1-I_2)W_1W_2+M_3.
\end{aligned}$$

However, there are problems when it is more convenient to use the formulas (2.7), for example, when we decide to use the inertial tensor as an input of controlling influences.

3. Time-dependent moments as control parameters

The simplest possible formulation of the control problem in rigid body mechanics consists in introducing formally three control parameters identified with Cartesian components of certain additional moments of forces. This kind of control is achieved without introducing additional (steering) degrees of freedom. There are two classes of problems of this type:

(i) Inner steering problems

These are problems where the *co-moving* components of the controlling momentum are used as directly manipulated quantities subject to our will. Equations of controlled rotational motion then have the form

$$\text{(3.1)}\quad \mathbf{I}\frac{d\mathbf{W}}{dt} = (\mathbf{IW})\times\mathbf{W}+\mathbf{M}(\mathbf{R},\mathbf{W},t)+\mathbf{U}(t),\qquad \frac{d\mathbf{R}}{dt} = \mathbf{RW},$$

or, when we use the representation (2.10),

$$\text{(3.2)}\quad \frac{d\mathbf{W}}{dt} = \mathbf{I}^{-1}\big((\mathbf{IW})\times\mathbf{W}\big)+\mathbf{I}^{-1}\mathbf{M}(\mathbf{k},\mathbf{W},t)+\mathbf{I}^{-1}\mathbf{U}(t),\qquad \frac{d\mathbf{k}}{dt} = \mathbf{C}(\mathbf{k})\mathbf{W}.$$

The control vector $\mathbf{U} = (U_1, U_2, U_3)^T$ depends only on time, but not on the state variables **k, W.**

Such models describe situations where the controlling instruments, e.g., reaction motors or thrust-based propeller motors are immovably fastened to the "deck" of the object (cf. Fig. 1.a).

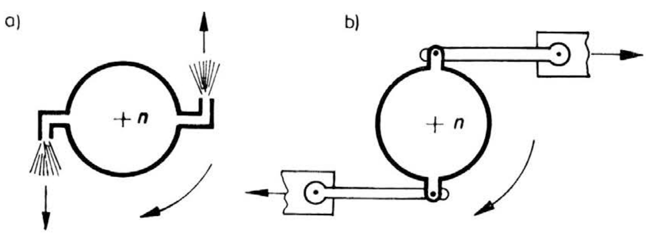

FIG. 1.

(ii) Outer-steering problems

In this class of problems *laboratory* components of the controlling moment are directly manipulated. Control equations have the form

$$\text{(3.3)}\quad \mathbf{I}\frac{d\mathbf{W}}{dt} = (\mathbf{IW})\times\mathbf{W}+\mathbf{M}(\mathbf{R},\mathbf{W},t)+\mathbf{R}^{-1}\mathbf{u}(t),\qquad \frac{d\mathbf{R}}{dt} = \mathbf{RW},$$

or, equivalently,

$$\text{(3.4)}\quad \frac{d\mathbf{W}}{dt} = \mathbf{I}^{-1}\big((\mathbf{IW})\times\mathbf{W}\big)+\mathbf{I}^{-1}\mathbf{M}(\mathbf{k},\mathbf{W},t)+\mathbf{I}^{-1}\mathbf{R}(\mathbf{k})^{-1}\mathbf{u}(t),\qquad \frac{d\mathbf{k}}{dt} = \mathbf{C}(\mathbf{k})\mathbf{W}.$$

Such models describe situations where the control forces are produced by external instruments like servomotors, pull rods, etc. or by external physical fields (cf. Fig. 1.b).

The mathematical difference between Eqs. (3.1) and (3.3) is that the controlling term on the right-hand side of Eq. $(3.3)_1$ depends explicitly on the state variables. Moreover, it depends on the configuration $\mathbf{k}$, thus, even if the background (non-controlled) moment $\mathbf{M}$ is independent of $\mathbf{k}$, the system (3.3) is non-splitting. Euler equations cannot be separately solved. Thus outer problems are mathematically more complicated. At the same time, the inner problem is more interesting for modern technical applications (autonomous, remotely operated apparata), thus, from now on, we concentrate on it.

For the completely general background dynamics $\mathbf{M}(\mathbf{k}, \mathbf{W}, t)$, it is rather difficult to formulate any statements concerning controllability. Let us notice that the problem is nonacademic, at least from the dimensional point of view, because we have 6 state parameters $(\mathbf{k}, \mathbf{W})$ and only 3 controlling variables $\mathbf{U}$.

For simplicity, let us consider the problem (3.1) with interaction–free background, $\mathbf{M} = 0$. Assuming, in addition, that the co-moving reference axes coincide with the inertial ones, we have

$$(3.5)\qquad \begin{aligned} \frac{dW_1}{dt} &= \frac{I_2 - I_3}{I_1} W_2 W_3 + \frac{U_1}{I_1}, \\ \frac{dW_2}{dt} &= \frac{I_3 - I_1}{I_2} W_3 W_1 + \frac{U_2}{I_2}, \\ \frac{dW_3}{dt} &= \frac{I_1 - I_2}{I_3} W_1 W_2 + \frac{U_3}{I_3}, \\ \frac{dk^i}{dt} &= \sum_j C_{ij}(\mathbf{k}) W_j, \qquad i = 1, 2, 3. \end{aligned}$$

$\mathbf{W}$, $\mathbf{k}$ are our state variables, and $\mathbf{U}$ — control parameters. The background dynamics is described by the following vector fields:

$$(3.6)\quad \hat{Y}_0 = \frac{I_2 - I_3}{I_1} W_2 W_3 \frac{\partial}{\partial W_1} + \frac{I_3 - I_1}{I_2} W_3 W_1 \frac{\partial}{\partial W_2} + \frac{I_1 - I_2}{I_3} W_1 W_2 \frac{\partial}{\partial W_3} + \sum_{ij} C_{ij} W_j \frac{\partial}{\partial k_i}.$$

The basic vector fields of control have the form

$$(3.7)\qquad Y_i = \frac{1}{I_i} \frac{\partial}{\partial W_i} \qquad \text{(no summation over } i\text{)}.$$

Distribution spanned by vectorfields (Y_0, Y_1, Y_2, Y_3) is nonintegrable. Extending the family of vector fields Y_μ, $\mu = 0, 1, 2, 3$ in the sense described in Sect. 1, i.e., completing it with all iterated Lie brackets, we obtain the family of vector fields generating at all points of P the total 6-dimensional tangent spaces, i.e., the rank of the matrix $||Y(f)^i_{\mu(f)}||$ introduced in Eq. (1.16) equals $p = 6$ all over the state manifold.

Thus the system is dimensionally-controllable. The particular form of Y_0 implies that, as a matter of fact, it is globally controllable.

The controllability of Eqs. (3.5) can also be justified with the help of the following rough, but geometrically convincing arguments: The Euler subsystem is separately solvable because it does not contain **k**-variables. It is a three-dimensional dynamical system with 3 control parameters occurring in a linear nonhomogeneous way; thus its controllability is rather obvious (provided that both signs of $\mathbf{U}^i$ are technically admissible). The controllability and possibility of inducing all possible trajectories in the space of angular velocities **W** implies that motion in the state space of all (**k**, **W**) is also controllable. This is a fundamental geometric consequence of the very relationship between Lie groups and their Lie algebras (roughly speaking, angular velocities are elements of the Lie algebra of the rotation group).

It turns out that the number of control parameters may be reduced to 2 without violating the local controllability. Namely, let us put

$$\mathbf{U}(t) = u_1(t)\mathbf{n}_1 + u_2(t)\mathbf{n}_2, \tag{3.8}$$

$\mathbf{n}_1, \mathbf{n}_2$ being two constant vectors fixed within the body.

For example, if $\mathbf{n}_1 = (1, 0, 0)^T$ and $\mathbf{n}_2 = (0, 1, 0)^T$, then $\mathbf{U}(t) = (u_1(t), u_2(t), 0)$. Applying to the triple of vector fields $Y'_0 := Y_0$, $Y'_1 := \sum\limits_i n^i_1 Y_i$, $Y'_2 := \sum\limits_i n^i_2 Y_i$ the same reasoning as previously, we can show that the system (3.5) with **U** given by Eq. (3.8) is dimensionally-controllable. This follows from the fact that the three-dimensional rotation group is simple. Any pair of one-parameter subgroups generates the total group. In other words, all possible rotations can be constructed as iterations of rotations about two fixed axes (e.g., "x-axis" and "y-axis") although the total group is three-dimensional.

Is it possible to reduce the number of control parameters to one, i.e., to put

$$\mathbf{U}(t) = u(t)\mathbf{n}, \tag{3.9}$$

n being a body-fixed versor? Even without any particular calculations, it is obvious that the system will be incontrollable if **n** coincides with one of the inertial axes. Indeed, in this case, the controls (3.9) can produce only rotations about this fixed axis. Let us put

$$Y''_0 := Y_0, \quad Y''_1 := \sum_i n^i Y_i. \tag{3.10}$$

If **n** is a principal direction of **I**, then calculating carefully the extended system of Y''_0, Y''_1, we confirm the above intuitive statements about incontrollability. However, it is perhaps surprising that if **n** is not directed along a principal axis of inertia, then the extended system of vectors has also rank 6, i.e., the system is dimensionally-controllable with a possible exception of pathological values of constants **n**, **I**. As a price paid for this reduction of control agents, we must use sophisticated shapes of the control function $\mathbf{u}(t)$ and wait longer to obtain the desired results of control.

Let us notice that if the inertial tensor is spherical, then the system based on Eq. (3.9) is incontrollable because all possible directions are degenerate principal axes of inertia.

As a technical realization of the control system (3.5), we can use, for example, three pairs of reaction motors or propeller motors fastened to the "deck" of the object. Any pair gives a control moment directed along a body-fixed versor $\mathbf{n}_i$. The total controlling moment is $\mathbf{U} = \sum\limits_i U_i(t)\mathbf{n}_i$, e.g., (U_1, U_2, U_3) when the versors $\mathbf{n}_i$ are principal axes of inertia. However, let us notice that in this particular technical realization the range of parameters U_i is restricted to the nonnegative semi-axis of R, what raises some questions concerning nondimensional, e.g., global, aspects of controllability.

4. Control based on gyroscopic forces. Angular momentum of rotors as a controlling agent

"Gyroscopic-like" control forces, i.e., control forces orthogonal to generalized velocities, are in some sense a priori suggested by variational problems with nonholonomic constraints [7].

In control problems concerning a rigid body, they are literally gyroscopic, i.e., they are based on the interaction of two angular velocities of mutually coupled rotating systems. Thus, to achieve this kind of control, we have to introduce additional, "steering" degrees of freedom.

The most natural way of generating gyroscopic controlling forces consists in using rotors. Let us assume that within the basic body there is a system of additional rigid bodies rotating about axes whose orientations and positions with respect to the carrying object are fixed once for all. Besides, we assume that all tops are symmetric, their rotation axes coincide with inertial symmetry axes, and there is no excentrity, i.e., mass centres of rotors are placed on their axes of rotation. This simplifies the treatment and makes the control effective because angular positions of rotors are cyclic variables and are not involved into the controlling interaction between two kinds of angular momenta (Fig. 2a).

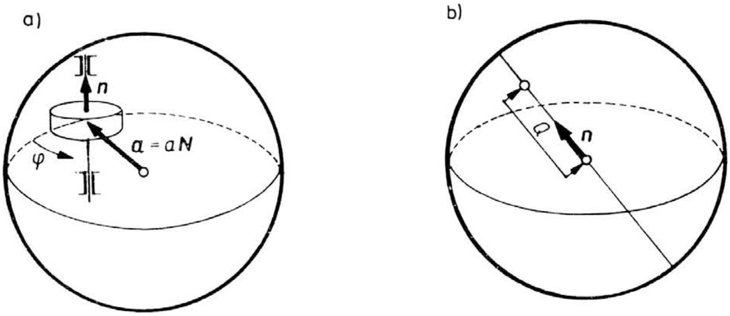

FIG. 2.

The rotation axis of the α-th top, $\alpha = 1 \ldots k$, is oriented along a constant body-fixed versor $\mathbf{n}_\alpha \in R^3$. The radius-vector of the mass centre of the α-th rotor with repect to the centre of mass of the carrying body will be denoted by $\mathbf{a}_\alpha = a_\alpha \mathbf{N}_\alpha$, where $\mathbf{N}_\alpha^T \mathbf{N}_\alpha = 1$, $a_\alpha = |\mathbf{a}_\alpha|$. Angular coordinates of rotors are denoted by φ_α, $\alpha = 1 \ldots k$. M_α denotes the mass of the α-th rotor, $I_\alpha^\perp$ — its moment of inertia with respect to the rotation axis, and I_α'' — the moment of inertia with respect to any straight-line perpendicular to the rotation axis and containing the centre of mass (as assumed, for a given α, all possible I_α'' are identical).

The co-moving inertial tensor of the total system (basic body+rotors) has the form

$$\mathbf{I}^{\text{tot}} = \mathbf{I} + \sum_{\alpha} (I''_{\alpha} + M_{\alpha} a_{\alpha}^{2})\mathbf{Id} + \sum_{\alpha} (I^{\perp}_{\alpha} - I''_{\alpha})\mathbf{n}_{\alpha} \otimes \mathbf{n}_{\alpha} - \sum_{\alpha} a_{\alpha}^{2} M_{\alpha} \mathbf{N}_{\alpha} \otimes \mathbf{N}_{\alpha}, \tag{4.1}$$

where **I** denotes the inertial tensor of the carrying body. It is convenient to use the modified inertial tensor

$$\mathbf{J} := \mathbf{I}^{\text{tot}} - \sum_{\alpha} I^{\perp}_{\alpha} \mathbf{n}_{\alpha} \otimes \mathbf{n}_{\alpha} = \mathbf{I} + \sum_{\alpha} (I''_{\alpha} + M_{\alpha} a_{\alpha}^{2})\mathbf{Id} - \sum_{\alpha} I''_{\alpha} \mathbf{n}_{\alpha} \otimes \mathbf{n}_{\alpha} - \sum_{\alpha} M_{\alpha} a_{\alpha}^{2} \mathbf{N}_{\alpha} \otimes \mathbf{N}_{\alpha}. \tag{4.2}$$

The total kinetic energy equals

$$T = \frac{1}{2}\mathbf{W}^{T}\mathbf{I}^{\text{tot}}\mathbf{W} + \frac{1}{2}\sum_{\alpha} I^{\perp}_{\alpha} \dot{\varphi}_{\alpha}^{2} + \sum_{\alpha} I^{\perp}_{\alpha} \dot{\varphi}_{\alpha} (\mathbf{W}^{T}\mathbf{n}_{\alpha}); \tag{4.3}$$

or, equivalently,

$$T = \frac{1}{2}\mathbf{W}^{T}\mathbf{J}\mathbf{W} + \frac{1}{2}\sum_{\alpha} I^{\perp}_{\alpha} (\dot{\varphi}_{\alpha} + \mathbf{W}^{T}\mathbf{n}_{\alpha})^{2}. \tag{4.4}$$

The interference term $\sum_{\alpha} I^{\perp}_{\alpha} \dot{\varphi}_{\alpha} (\mathbf{W}^{T}\mathbf{n}_{\alpha})$ is responsible for the control mechanism.

The total angular momentum with respect to the centre of mass is expressed in terms of the co-moving frame as follows:

$$\mathbf{S} = \mathbf{I}^{\text{tot}}\mathbf{W} + I^{\perp}_{\alpha} \dot{\varphi}_{\alpha} \mathbf{n}_{\alpha} = \mathbf{J}\mathbf{W} + \boldsymbol{\sigma}, \tag{4.5}$$

where

$$\boldsymbol{\sigma} := \sum_{\alpha} I^{\perp}_{\alpha} (\dot{\varphi}_{\alpha} + \mathbf{W}^{T}\mathbf{n}_{\alpha})\mathbf{n}_{\alpha} \tag{4.6}$$

is the total angular momentum of the system of rotors.

The simplest procedure of deriving equations of motion consits of the following steps:

(i) One takes the Lagrangian $L = T - V(\mathbf{k}, \dots \varphi_{\alpha} \dots)$ and performs nonholonomic Legendre transformation:

$$\mathscr{S}_{A} = \frac{\partial L}{\partial W_{A}}, \quad p_{\alpha} = \frac{\partial L}{\partial \dot{\varphi}_{\alpha}}. \tag{4.7}$$

(ii) One solves Eq. (4.7) with respect to velocities. Then, substituting the result to Eq. (4.4), one obtains the Hamiltonian

$$H = \mathscr{T}(\mathscr{S}, p) + V = T\big(W(\mathscr{S}, p), \varphi(\mathscr{S}, p)\big) + V(\mathbf{k}, \varphi). \tag{4.8}$$

(iii) Canonical equations of motion are obtained with the use of Poisson brackets:

$$(4.9)\quad \begin{aligned} \frac{d\mathcal{S}_A}{dt} &= \{\mathcal{S}_A, H\} = -\varepsilon_{ABC}\mathcal{S}_C\frac{\partial\mathcal{T}}{\partial\mathcal{S}_B} + \{\mathcal{S}_A, V\},\\ \frac{dp_\alpha}{dt} &= \{p_\alpha, H\} = -\frac{\partial H}{\partial\varphi_\alpha} = -\frac{\partial V}{\partial\varphi_\alpha},\\ \frac{d\varphi_\alpha}{dt} &= \{\varphi_\alpha, H\} = \frac{\partial H}{\partial p_\alpha} = \frac{\partial\mathcal{T}}{\partial p_\alpha},\\ \frac{dk_A}{dt} &= \{k_A, H\} = C_{AB}(\mathbf{k})W^B(\mathcal{S}, p). \end{aligned}$$

The quantities

$$(4.10)\quad \{\mathcal{S}_A, V\} = -C_{AB}(\mathbf{k})\frac{\partial V}{\partial k_B}$$

are identical with $M_A[V]$-co-moving components of the moment of V-potential forces with respect to the centre of mass.

Similarly, the Poisson brackets

$$(4.11)\quad \{p_\alpha, V\} = -\frac{\partial V}{\partial\varphi_\alpha}$$

are identical with $\mu_\alpha[V]$-one-component moments of V-forces acting on rotors and calculated with respect to their rotation axes. By analogy to Eq. (4.6), we introduce the vector representation of μ_α,

$$(4.12)\quad \boldsymbol{\mu}_\alpha = \mu_\alpha\mathbf{n}_\alpha,$$

and the total moment of rotor V-forces

$$(4.13)\quad \boldsymbol{\mu}[V] := \sum_\alpha \boldsymbol{\mu}_\alpha = \sum_\alpha \mu_\alpha\mathbf{n}_\alpha.$$

(iv) One returns to kinematical variables $(W, \dot{\varphi})$ by substituting the Legendre transformations (4.7) to Eq. (4.9). The potential moments $\mathbf{M}[V]$, $\boldsymbol{\mu}_\alpha[V]$, $\boldsymbol{\mu}[V]$ are replaced by general phenomenological moments $\mathbf{M}$, $\boldsymbol{\mu}_\alpha$, $\boldsymbol{\mu}$. This enables us to include dissipative forces into the treatment.

Our system has $(k+3)$ degrees of freedom. However, it turns out that it is not the total system of $2(k+3)$ first-order differential equations that is relevant for gyroscopic control but rather 6 equations equivalent to the balance laws for the total angular momentum $\mathbf{S}$ and the rotor momentum $\boldsymbol{\sigma}$. Namely, Eq. (4.9) expressed through $\mathbf{W}$ imply the following balance equations:

$$(4.14)\quad \begin{aligned} \mathbf{J}\frac{d\mathbf{W}}{dt} &= (\mathbf{J}\mathbf{W})\times\mathbf{W} + \boldsymbol{\sigma}\times\mathbf{W} - \boldsymbol{\mu},\\ \frac{d\boldsymbol{\sigma}}{dt} &= \boldsymbol{\mu}, \end{aligned}$$

thus, eliminating $\boldsymbol{\mu}$, we obtain

$$\mathbf{J}\frac{d\mathbf{W}}{dt} = (\mathbf{J}\mathbf{W})\times\mathbf{W}+\boldsymbol{\sigma}\times\mathbf{W}-\frac{d\boldsymbol{\sigma}}{dt}. \tag{4.15}$$

It is obvious from Eq. (4.15) that the system of rotors affects its gyroscopic controlling influence only through its resulting angular momentum $\boldsymbol{\sigma}$. Thus, if there are $k \leqslant 3$ rotors with linearly independent rotation versors $\mathbf{n}_\alpha$, then the control input is k-dimensional. If $k > 3$, and the rotation versors $\mathbf{n}_\alpha$ span the whole space R^3, then, independently of the particular value of k, the control input is always three-dimensional.

There is a temptation to consider the angular momentum of rotors as subject to our direct manipulation and interpret $\boldsymbol{\sigma}$ as a control vector u. However, when written in state-control terms, Eq. (4.15) has the form

$$\frac{dx}{dt} = f\left(x, u, \frac{du}{dt}\right), \tag{4.16}$$

which, although in principle acceptable (and, as a matter of fact, commonly used in the theory of linear systems), is formally different from the canonical form (1.2). Obviously, it is Eq. (1.2) that is optimally adapted to studying general problems of controllability and observability. The only possibility of returning to this standard form is to extend formally the state manifold, i.e., to use pairs $(\mathbf{W}, \boldsymbol{\sigma})$, or triples $(\mathbf{k}, \mathbf{W}, \boldsymbol{\sigma})$ as states x, and the resultant rotoric moment of forces $\boldsymbol{\mu}$ as controls u. This approach is even more physical because it is rather $\boldsymbol{\mu}$ that is directly manipulable, especially when one deals with electrically driven rotors. Besides, considering $\boldsymbol{\sigma}$ (and, consequently, $\dot{\varphi}_\alpha$) as controllable quantities has also obvious practical and economical aspects in theory of autonomously moving spatial or submarine objects.

Introducing orientations into the treatment, we obtain the following control system:

$$\begin{aligned}\mathbf{J}\frac{d\mathbf{W}}{dt} &= (\mathbf{J}\mathbf{W})\times\mathbf{W}+\boldsymbol{\sigma}\times\mathbf{W}-\boldsymbol{\mu},\\ \frac{d\boldsymbol{\sigma}}{dt} &= \boldsymbol{\mu},\\ \frac{d\mathbf{k}}{dt} &= \mathbf{C}(\mathbf{k})\mathbf{W}.\end{aligned} \tag{4.17}$$

The state manifold in Eqs. (4.14) is six-dimensional, and in Eq. (4.17)—nine-dimensional. At the same time, for $k \geqslant 3$ the control input is three-dimensional (and for $k \leqslant 3$ — k-dimensional), thus the problem of dimensional controllability is certainly nonacademic.

If we use co-moving coordinates diagonalizing $\mathbf{J}$, then the noninfluenced dynamics in Eqs. (4.17) is described by

$$\begin{aligned}\hat{Y}_0 = \frac{1}{J_1}[(J_2-J_3)W_2W_3+(\sigma_2W_3-\sigma_3W_2)]\frac{\partial}{\partial W_1}\\ +\frac{1}{J_2}[(J_3-J_1)W_3W_1+(\sigma_3W_1-\sigma_1W_3)]\frac{\partial}{\partial W_2}\\ +\frac{1}{J_3}[(J_1-J_2)W_1W_2+(\sigma_1W_2-\sigma_2W_1)]\frac{\partial}{\partial W_3}+C_{ij}(\mathbf{k})W^j\frac{\partial}{\partial k_i}.\end{aligned} \tag{4.18}$$

Basic controlling fields are given by

$$\hat{Y}_1 = -\frac{\partial}{\partial W_1}+\frac{\partial}{\partial \sigma_1}, \quad Y_2 = -\frac{\partial}{\partial W_2}+\frac{\partial}{\partial \sigma_2}, \quad Y_3 = -\frac{\partial}{\partial W_3}+\frac{\partial}{\partial \sigma_3}. \tag{4.19}$$

When we consider the problem (4.14), the only difference is that there is no last term in $\hat{Y}_0$.

It is a priori clear even without any calculation of Lie brackets that there is no controllability of Eqs. (4.14) in the total six-dimensional space of states $(\mathbf{W}, \boldsymbol{\sigma})$ and, similarly, there is no controllability of Eq. (4.17) in the nine-dimensional space of $(\mathbf{k}, \mathbf{W}, \boldsymbol{\sigma})$. The reason is that the control is based on purely internal forces acting in the total system: carrying body+rotors. Therefore laboratory components of the total angular momentum, i.e.,components of $\mathbf{R}(\mathbf{JW}+\boldsymbol{\sigma})$ are constants of motion, independently of the shape controlling function $\boldsymbol{\mu}(t)$.

Thus, if we consider the control problem in the nine-dimensional space of variables $(\mathbf{k}, \mathbf{W}, \boldsymbol{\sigma})$, then all possible trajectories are placed on six-dimensional manifolds of the form

$$\mathbf{R}(\mathbf{k})(\mathbf{JW}+\boldsymbol{\sigma}) = \mathbf{a} = \text{const}, \tag{4.20}$$

and there are certainly no controls connecting the states $(\mathbf{k}_1, \mathbf{W}_1, \boldsymbol{\sigma}_1)$ $(\mathbf{k}_2, \mathbf{W}_2, \boldsymbol{\sigma}_2)$ for which

$$\mathbf{R}(\mathbf{k}_1)(\mathbf{JW}_1+\boldsymbol{\sigma}_1) \neq \mathbf{R}(\mathbf{k}_2)(\mathbf{JW}_2+\boldsymbol{\sigma}_2).$$

Among the variables $\mathbf{k}, \mathbf{W}, \boldsymbol{\sigma}$ there are at least 3 completely incontrollable.

The condition (4.20) implies that

$$(\mathbf{JW}+\boldsymbol{\sigma})^T(\mathbf{JW}+\boldsymbol{\sigma}) = a^2 = \text{const}. \tag{4.21}$$

This equation does not involve the variable $\mathbf{k}$. If we consider the control problem in the six-dimensional space of variables $(\mathbf{W}, \boldsymbol{\sigma})$, then all possible trajectories are placed on five-dimensional submanifolds of the form (4.21). There is at least one completely incontrollable parameter contained in $(\mathbf{W}, \boldsymbol{\sigma})$. We have at our disposal 3 controlling parameters $\boldsymbol{\mu}$ and 4 basic fields (Y_0, Y_1, Y_1, Y_3) on the five-dimensional manifold (4.21) However, extending the system (4.18)–(4.19) by introducing its Lie brackets, we can show that on the manifolds (4.21) the system is dimensionally-controllable. This means, in particular, that the angular velocity $\mathbf{W}$ is completely controllable with the help of 3 independent rotors. Two parameters of $\boldsymbol{\sigma}$ remain non-restricted, what seems to suggest that perhaps controllability could also be achieved with the use of only one, but properly oriented, rotor, by analogy to Eq. (3.9).

If we consider the control problem (4.17), then, as mentioned, the basic fields $(Y_0, Y_i, i = 1, 2, 3)$, are tangent to six-dimensional manifolds given by Eqs. (4.20). Calculating restrictions of Y to those submanifolds, and the Poisson brackets of such restricted tangent fields, we can show that the problem is dimensionally-controllable on the set (4.20). Motion of the carrying body, i.e., time evolution in the space of variables $(\mathbf{k}, \mathbf{W})$ is controllable and there is no longer indeterminacy of $\boldsymbol{\sigma}$-variables. The system

$$\text{(4.22)}\qquad \begin{aligned} \mathbf{J}\frac{d\mathbf{W}}{dt} &= (\mathbf{J}\mathbf{W})\times\mathbf{W}+\boldsymbol{\sigma}\times\mathbf{W}-\frac{d\boldsymbol{\sigma}}{dt},\\ \frac{d\mathbf{k}}{dt} &= C(\mathbf{k})\mathbf{W} \end{aligned}$$

considered as a control system with the first-order differential input $\boldsymbol{\sigma}$ is dimensionally-controllable.

Let us notice that if we consider $\boldsymbol{\sigma}$ as a primary input data, then the control influence on the right-hand side of Eqs. (4.22) is a superposition of the gyroscopic force $\boldsymbol{\sigma}\times\mathbf{W}$ and the structure-less controlling moment — $\dot{\boldsymbol{\sigma}}$.

5. Inertia as a controlling agent

Variational problems with nonlinear nonholonomic constraints seem to suggest that inertia should be a promising physical agent in problems of control and programme motion. Obviously this way of control is nonrealistic when the translational motion in space is concerned because it would be rather hard to manipulate with masses of moving objects. On the contrary, inertial tensors of rigid bodies can be relatively easily subjected to our influence. This kind of control is achieved by introducing additional ("steering") degrees of freedom.

In this paper we restrict ourselves to presenting general ideas. The detailed analysis of the control through the inertia problem would be too difficult; dynamical equations are strongly nonlinear in control parameters.

The simplest scheme of manipulated inertia consists in using radially moving heavy sliders. Let us assume that within the carrying body there is a system of additional masses constrained to move along some body-fixed axes passing through the centre of mass of the basic body. Let $\mathbf{n}_\alpha$, $\alpha = 1 \ldots k$ denote body-fixed direction versors of those axes, and M_α-masses of sliders. Radius vector of the α-th slider will be denoted by $\mathbf{q}_\alpha = q_\alpha \mathbf{n}_\alpha$; $q_\alpha = |\mathbf{q}_\alpha|$. The co-moving inertial tensor of the controlled body is, as always, denoted by $\mathbf{I}$, and the total mass of the system — by M. Sliders are considered as material points; we neglect their own inertial moments (Fig. 2b).

Kinetic energy has the form

$$\text{(5.1)}\qquad T = \frac{1}{2}\mathbf{W}^T(\mathbf{J}+\mathbf{Q})\mathbf{W}+\frac{M}{2}\mathbf{V}^T\mathbf{V}+\frac{1}{2}\sum_\alpha M_\alpha \dot{q}_\alpha^2+\mathbf{V}^T(\mathbf{W}\times\mathbf{D})+\mathbf{V}^T\frac{d\mathbf{D}}{dt},$$

where:

$\mathbf{V} = (V_1, V_2, V_3)^T$ is the co-moving representation of the translational velocity (V_A are projections of the velocity of the centre of mass onto body-fixed coordinate axes),

$\mathbf{Q}(q)$ is the co-moving inertial tensor of the system of sliders,

$$\text{(5.2)}\qquad Q(q) = \sum_\alpha M_\alpha q_\alpha^2(\mathbf{Id}-\mathbf{n}_\alpha\otimes\mathbf{n}_\alpha);$$

D is the first-order moment (dipole) of the mass distribution of sliders, i.e., their centre of mass radius vector multiplied by their resultant mass M,

$$\mathbf{D} = \sum_{\alpha} M_\alpha q_\alpha \mathbf{n}_\alpha. \tag{5.3}$$

The total system has $(6+k)$ degrees of freedom. The explicit form of equations of motion is rather complicated; we do not quote them.

The qualitative structure of controlling interactions can be easily read from Eq. (5.1), namely, from its interference terms:

(i) **Q** is a manipulated part of the inertial tensor and describes the effect of sliders on rotational behaviour.

(ii) the term $\mathbf{V}^T \frac{d\mathbf{D}}{dt}$ describes the very delicate, corrective effect of sliders on the translational motion of the basic body (in a small range of the order of the effective radius of the system).

(iii) $\mathbf{V}^T(\mathbf{W}\times\mathbf{D})$ represents the coupling between angular and translational velocities through the centre of mass of sliders.

The controlling effect of sliders is described by 9 parameters — 6 components of **Q** and 3 components of **D**. (Thus there is no need to use more than 9 independent sliders).

If the centre of mass is to be nonaffectable, then we impose holonomic constraints of balanced sliders, $\mathbf{D} = 0$ (simple realization: pairs of identical sliders moving along common axes in a symmetric way).

One can complicate the system by introducing excentrity, i.e., shifting the axes of sliders from the basic centre of mass by the vectors $\mathbf{a}_\alpha = a_\alpha \mathbf{N}_\alpha$ where $\mathbf{N}_\alpha^T\mathbf{N}_\alpha = 1$, $\mathbf{n}_\alpha^T\mathbf{N}_\alpha = 0$. The expression (5.1) is then modified in the following way:

(i) **Q**, **D** are modified by constant additive terms depending on a_α, $\alpha = 1, \ldots, k$.

(ii) There appears in T a new term, namely,

$$\mathbf{W}^T \sum_{\alpha} M_\alpha \dot{q}_\alpha a_\alpha^2 (\mathbf{N}_\alpha \times \mathbf{n}_\alpha). \tag{5.4}$$

The first modification is nonessential and the second one is equivalent to introducing torsionally vibrating rotors, cf. Eq. (4.3). Thus there are no physical reasons to introduce excentrity.

The question is what is to be reasonably chosen as a directly manipulated control input. By analogy to Eqs. (4.17), where we have used moments of forces acting on rotors, we could try to use as controlling parameters the co-moving components of forces acting on sliders. However, the resulting system of equations would be very complicated and would involve many intermediary parameters, namely, the coordinates q_α. It is formally simpler, although physically less correct, to use just the inertial tensor as a controlling quantity.

Let **J** denote the total time-dependent co-moving moment of inertia considered, in a phenomenological way, as a primary quantity. Thus the kinetic energy (5.1) and Lagrangian L become explicitly time-dependent quantities. We consider only rotational motion. It is ruled by the following equations:

(5.5)
$$\frac{d}{dt}(\mathbf{JW}) = (\mathbf{JW})\times\mathbf{W}+\mathbf{M},$$
$$\frac{d\mathbf{k}}{dt} = \mathbf{C}(\mathbf{k})\mathbf{W},$$

i.e.,

(5.6)
$$\frac{d\mathbf{W}}{dt} = \mathbf{J}^{-1}((\mathbf{JW})\times\mathbf{W})-\mathbf{J}^{-1}\frac{d\mathbf{J}}{dt}\mathbf{W}+\mathbf{J}^{-1}\mathbf{M},$$
$$\frac{d\mathbf{k}}{dt} = \mathbf{C}(\mathbf{k})\mathbf{W}.$$

These equations, just as Eq. (4.22), involve explicitly time derivatives of the controlling signal; thus they are not written in the canonical form (1.2). Let us notice, however, that they become formally free of time derivatives on the input if we use the state variables $(\mathbf{k}, \mathbf{S}) = (\mathbf{k}, \mathbf{JW})$, $\mathbf{S}$ denoting the co-moving representation of the angular momentum. Indeed Eq. (5.5) imply that

(5.7)
$$\frac{d\mathbf{S}}{dt} = -(\mathbf{J}^{-1}\mathbf{S})\times\mathbf{S}+\mathbf{M},$$
$$\frac{d\mathbf{k}}{dt} = \mathbf{C}(\mathbf{k})\mathbf{J}^{-1}\mathbf{S}.$$

Obviously this formulation is satisfactory if we are interested in configurational control problems, i.e., in controlling the $\mathbf{k}$-variable but it is rather useless in angular velocity control problems.

It is easy to see that, really, control problems in $\mathbf{S}$ and $\mathbf{W}$-spaces are different. Namely, like in Eqs. (4.17), the controlling forces are internal and do not influence the balance of S. Thus, in particular, if $\mathbf{M} = 0$, $\mathbf{RS} = \mathbf{RJW}$ is a conserved quantity and therefore

(5.8)
$$S^2 := \mathbf{S}^T\mathbf{S}$$

is a constant of motion in the $\mathbf{S}$-space, and

(5.9)
$$(\mathbf{JW})^T\mathbf{JW}$$

is a constant of motion in the W-space. Thus, in the $\mathbf{S}$-space there is at least one incontrollable parameter, namely S. On the contrary, in the $\mathbf{W}$-space there are no a priori obstacles against the controllability, with the exception of the rest state $W = 0$ which, independently of the used control $\mathbf{J}(t)$, cannot be transformed into any other state (and conversely).

Let us notice that for the generic $\mathbf{J}$ we have 6 independent controlling parameters and at the most 6 variables [3 in the $\mathbf{W}$-space, 6 in the $(\mathbf{k}, \mathbf{W})$-space and 5 on the value-surfaces of (5.9) in the $(\mathbf{k}, \mathbf{W})$-space]. We have sufficiently many independent Y in Eq. (5.6), and the problem is always controllable, with the above-mentioned exception. Reducing the number of parameters of $\mathbf{J}$ (more rigorously-of $\mathbf{Q}$), we can destroy controllability in a variety of ways. For example, if $\mathbf{J}$ is constrained to be diagonal (e.g., if sliders move along principal axes of the carrying body), then rotations about principal axes of inertia are nontransformable into any more general situation, i..e, if $\mathbf{IW} = \mathbf{W}$ ($\mathbf{I}$ being the background inertial tensor), then all admissible controls of this type result only in multiplying $\mathbf{W}$ by scalar factors.

Note added in proof

When preparing this paper I was not aware of papers by P. C. MÜLLER, K. POPP, W. SCHIEHLEN, and N. I. WEBER devoted to controlled motion of rigid bodies, and of certain coincidences with them (cf. for example ZAMM **54**, 695–702, 1974, ZAMM **54**, T58–T59, 1974, Automatica **8**, 237–246, 1972, and literature mentioned there). I am very grateful to professor P. C. MÜLLER for interesting discussions on this topic during the Polish-German Workshop ,,Dynamical Problems in Mechanical Systems" in Mądralin near Warsaw, March 1989, for sending me the mentioned papers and for other, very profitable for me, references to literature.

References

1. V. I. ARNOLD, *Mathematical methods of classical mechanics*, Springer Grad. Texts in Math., 60, Springer-Verlag, New York 1978.
2. P. CIFFET, *Les robots; modelisation et commande*, 1, Hermes, 1981.
3. V. JURDJEVIC, I. KUPKA, *Control systems on semisimple Lie groups and their homogeneous spaces*, Ann. Inst. Fourier. **31**, 151, 1981.
4. S. KOBAYASHI, K. NOMIZU, *Foundations of differential geometry*, Interscience, New York 1963/69.
5. W. W. KOZLOV, W. P. MAKARYTSHEV, A. W. TIMOFIEJEW, R. J. JURJEVITSH, *Dynamics of steering of robots*, Nauka, Moscow 1984 [in Russian].
6. D. Q. MAYNE, R. W. BROCKET [Eds.], *Geometric methods in system theory*; Proceedings of the NATO Advanced Study Institute, held at London, England, August 27-September 7, 1973, D. Reider Publ. Company, Dordrecht-Holland/Boston USA, 1973.
7. J. J. SŁAWIANOWSKI, *Algorithms for reactions of non-holonomic constraints and servo-constraints*, Arch. Mech., **39**, 615, 1987.
8. J. J. SŁAWIANOWSKI, *Nonlinear torsional vibrations of rigid bodies*, Arch. Mech., **39**, 663, 1987.
9. J. T. TOU, *Modern control theory*, Mc Graw-Hill Book Comp., New York, San Francisco, Toronto, London 1964.

References added in proof

A.1. P. C. MÜLLER, K. POPP, *Zur Theorie der ersten Integrale bei gesteuerten mechanischen Systems*, ZAMM, **54**, 695, 1974.

A.2. P. C. MÜLLER, H. I. WEBER, *Analysis and optimization of certain qualities of controllability and observability for linear dynamical systems*, Automatica, **8**, 237, 1972.

A.3. H. I. WEBER, W. SCHIEHLEN, *Attitude controllability of a satellite with flywheels*, J. Spacecr, Rockets, **7**, 501, 1970.

A.4. H. I. WEBER, *Optimale Lageregelung von Satelliten mit simultan entdrallten Rotoren*, Doctoral Thesis, Technical University of Munich, 1971.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received June 26, 1988.