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Irreducible representations for constitutive equations of anisotropic solids III: crystal and quasicrystal classes D_{2m+1h} and D_{2md}

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A SIMPLE, UNIFIED PROCEDURE is applied to derive irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric second order tensor-valued anisotropic constitutive equations involving any finite number of vector variables and second order tensor variables. In this part, our concern is for the crystal classes and quasicrystal classes D_{2m+1h} and D_{2md} for all integers $m \geq 1$.

1. Introduction

WE CONTINUE OUR STUDY of irreducible nonpolynomial representations for anisotropic constitutive equations involving any finite number of vector variables and second order tensor variables. In the final part, we are concerned with the crystal and quasicrystal classes D_{2m+1h} and D_{2md} for all $m \geq 1$. These classes are more complicated than those considered before, since we have to draw distinction between the reflection symmetry with respect to a plane and the two-fold rotation symmetry with respect to an axis.

As it has been done in the preceding two parts, we shall apply the unified procedure outlined in Sec. 3 in Part I of this series (see XIAO, BRUHNS and MEYERS [21]. Henceforth the just-mentioned reference will simply be referred to as Part I) to derive the desired results. For notations and preliminaries and for some relevant references, refer to Sec. 1 – 3 in Part I for detail.

2. Crystal and quasicrystal classes D_{2m+1h}

The classes at issue take forms

$$D_{2m+1h}(\mathbf{n},\mathbf{e}) = \left\{(-1)^k\mathbf{R}_{\mathbf{n}}^{\theta_k},\ (-1)^k\mathbf{R}_{\boldsymbol{\tau}_k}^{\pi} \mid \theta_k = \frac{2k\pi}{4m+2},\right.$$
$$\left.\boldsymbol{\tau}_k = \mathbf{R}_{\mathbf{n}}^{\theta_k/2}\mathbf{e},\ k = 0, 1, \cdots, 4m+1\right\}. \tag{2.1}$$

They include the crystal class D_{3h} as the particular case when $m = 1$. Note that each $\boldsymbol{\tau}_{2r+1}$ and each $\boldsymbol{\tau}_{2r}$, respectively, correspond to the reflection with respect to the $\boldsymbol{\tau}_{2r+1}$-plane and the two-fold rotation about an axis in the direction of $\boldsymbol{\tau}_{2r}$, where $r = 0, 1, \cdots, 2m$. Accordingly, we shall call each $\boldsymbol{\tau}_{2r+1}$ and each $\boldsymbol{\tau}_{2r}$ a *reflection axis vector* and a *two-fold rotation axis vector* of the group D_{2m+1h}. In particular, the two orthonormal vectors $\mathbf{e}$ $(= \boldsymbol{\tau}_0)$ and $\mathbf{e}'$ $(= \boldsymbol{\tau}_{2m+1})$ are a two-fold rotation axis vector and a reflection axis vector of D_{2m+1h}, respectively. Throughout this section, $\boldsymbol{\nu}$ is used to represent a two-fold rotation axis vector of D_{2m+1h}, i.e. $\boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 0, 1 \cdots, 2m\}$.

Throughout, for any given vector $\mathbf{z}$ we shall use $\mathbf{z}'$ to denote the vector $\mathbf{n} \times \mathbf{z}$, i.e.

$$\mathbf{z}' = \mathbf{n} \times \mathbf{z}.$$

A useful fact for the group D_{2m+1h} is: if $\boldsymbol{\tau}$ is a two-fold rotation (resp. reflection) axis vector, then $\boldsymbol{\tau}'$ is a reflection (resp. two-fold rotation) axis vector.

Let $Y = (\mathbf{J}_1, \cdots, \mathbf{J}_s)$, where each $\mathbf{J}_\alpha$ is a skewsymmetric tensor or a symmetric tensor. Then the identities

$$(2.2) \qquad f(\mathbf{Q}_0 Y \mathbf{Q}_0^T) = f(Y), \qquad \mathbf{F}(\mathbf{Q}_0 Y \mathbf{Q}_0^T) = \mathbf{Q}_0 \mathbf{F}(Y) \mathbf{Q}_0^T, \qquad \mathbf{Q}_0 = \pm \mathbf{I},$$

for every scalar-valued function $f(Y)$ and every skewsymmetric and every symmetric tensor-valued function $\mathbf{F}(Y)$. From this fact and the fact that the group D_{2m+1h} and the central inversion $-\mathbf{I}$ generate the centrosymmetrical group D_{4m+2h}, each invariant $f(Y)$ and each form-invariant function $\mathbf{F}(Y)$ under the group D_{2m+1h} turns out to be an invariant and a form-invariant function under the larger group D_{4m+2h} $(\supset D_{2m+1h})$. As a result, for the five sets of variables, $(\mathbf{W})$, $(\mathbf{A})$, $(\mathbf{W}, \boldsymbol{\Omega})$, $(\mathbf{W}, \mathbf{A})$ and $(\mathbf{A}, \mathbf{B})$, results for functional bases and skewsymmetric and symmetric tensor generating sets relative to the group D_{2m+1h}, as well as related invariants from the scalar products, can be obtained from the corresponding results given in Sec. 4 in Part I by the replacement of m with $2m + 1$. Thus, in what follows, for the foregoing five sets of variables, we only need to derive vector generating sets and their related invariants from the scalar products. Moreover, according to Sec. 4 (**xiii**) in Part I, we can omit the set $(\mathbf{u}, \mathbf{v}, \mathbf{r})$ of three vector variables. Finally, for each of the sets of three variables, $(\mathbf{u}, \mathbf{W}, \boldsymbol{\Omega})$ and $(\mathbf{u}, \mathbf{A}, \mathbf{B})$, from Theorem 2.1 and Theorem 3.3 in XIAO [20] we know that it suffices to supply a vector generating set and a functional basis.

2.1. Single variables

(**i**) A single vector $\mathbf{u}$

$V \quad \{(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \mathring{\mathbf{u}}, \eta_{2m}(\mathring{\mathbf{u}})\} \; (\equiv V''_{2m+1}(\mathbf{u}))$

Skw $\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge \overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge \eta_{2m}(\overset{\circ}{\mathbf{u}}), \beta_{2m+1}(\overset{\circ}{\mathbf{u}})\mathbf{N}\}$ $(\equiv \mathrm{Skw}''_{2m+1}(\mathbf{u}))$

Sym $\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\vee\eta_{2m}(\overset{\circ}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\eta_{2m}(\overset{\circ}{\mathbf{u}})\}$

$(\equiv \mathrm{Sym}''_{2m+1}(\mathbf{u}))$

R $(\mathbf{r}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n}), \overset{\circ}{\mathbf{r}}\cdot\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{r}}\cdot\eta_{2m}(\overset{\circ}{\mathbf{u}});$

$\beta_{2m+1}(\overset{\circ}{\mathbf{u}})(\mathrm{tr}\mathbf{HN}), (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\mathbf{Hn}, (\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Hn};$

$\mathrm{tr}\mathbf{C}, \mathbf{n}\cdot\mathbf{Cn}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}, \eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{C}}\mathbf{n};$

$\{(\mathbf{u}\cdot\mathbf{n})^2, |\overset{\circ}{\mathbf{u}}|^2, \alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\}$ $(\equiv I''_{2m+1}(\mathbf{u}))$.

The proof for the above results is as follows. According to Theorem 3 in XIAO [17], isotropic functional bases and generating sets of the extended variables $(\mathbf{u}, \eta_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\otimes\mathbf{n})$ supply anisotropic functional bases and generating sets of the vector variable $\mathbf{u}$ under the group D_{2m+1h}. Then, applying the related results for isotropic functions, we know that the three presented sets $I''_{2m+1}(\mathbf{u})$, $V''_{2m+1}(\mathbf{u})$ and $\mathrm{Skw}''_{2m+1}(\mathbf{u})$ supply a desired functional basis, a desired vector generating set and a desired skewsymmetric tensor generating set, respectively. Moreover, a desired symmetric tensor generating set is formed by the six generators in the presented set $\mathrm{Sym}''_{2m+1}(\mathbf{u})$ as well as the generator $\mathbf{G}=\eta_{2m}(\overset{\circ}{\mathbf{u}})\otimes\eta_{2m}(\overset{\circ}{\mathbf{u}})$. Here the decomposition formula $\mathbf{u}=(\mathbf{u}\cdot\mathbf{n})\mathbf{n}+\overset{\circ}{\mathbf{u}}$ (see (2.15) in Part I) is used.

We need to show that the generator $\mathbf{G}$ is redundant. In fact, $\mathbf{G}$ is a 2-dimensional symmetric tensor defined on the $\mathbf{n}$-plane. When the two vectors $\overset{\circ}{\mathbf{u}}$ and $\eta_{2m}(\overset{\circ}{\mathbf{u}})$ on the $\mathbf{n}$-plane are linearly independent, the three symmetric tensors $\overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}$, $\overset{\circ}{\mathbf{u}}\vee\eta_{2m}(\overset{\circ}{\mathbf{u}})$ and $\mathbf{I}-\mathbf{n}\otimes\mathbf{n}$ constitute a basis of the space of all symmetric tensors defined on the $\mathbf{n}$-plane. Hence $\mathbf{G}$ is redundant. $\mathbf{G}$ is obviously redundant when $\overset{\circ}{\mathbf{u}}$ and $\eta_{2m}(\overset{\circ}{\mathbf{u}})$ are linearly dependent.

(ii) A single skewsymmetric tensor $\mathbf{W}$

V $\{\alpha_{2m+1}(\mathbf{Wn})\mathbf{n}, \eta_{2m}(\mathbf{Wn}), \mathbf{W}\eta_{2m}(\mathbf{Wn}), \mathbf{W}^2\eta_{2m}(\mathbf{Wn})\}$ $(\equiv V''_{2m+1}(\mathbf{W}))$

R $(\mathbf{r}\cdot\mathbf{n})\alpha_{2m+1}(\mathbf{Wn}), \overset{\circ}{\mathbf{r}}\cdot\eta_{2m}(\mathbf{Wn}), \eta_{2m}(\mathbf{Wn})\cdot\mathbf{Wr}, \eta_{2m}(\mathbf{Wn})\cdot\mathbf{W}^2\mathbf{r}.$

According to Theorem 3 in XIAO [17], a vector generating set for form-invariant vector-valued functions of the skewsymmetric tensor variable $\mathbf{W}$ under D_{2m+1h} is obtainable from an isotropic vector generating set for the extended variables $(\eta_{2m}(\mathbf{Wn}), \mathbf{W}, \mathbf{n}\otimes\mathbf{n})$. By applying the related results for isotropic functions we know that the latter is just given by the presented set $V''_{2m+1}(\mathbf{W})$. Moreover, by considering $\mathbf{W}_1=\mathbf{E}(\mathbf{n}+\mathbf{e})$ and $\mathbf{W}_2=\mathbf{E}(\mathbf{n}+\mathbf{e}')$, we deduce that each of the four generators in the set $V''_{2m+1}(\mathbf{W})$ are irreducible.

(iii) A single symmetric tensor $\mathbf{A}$

$$V \quad \{\eta_m(\mathbf{q}(\mathbf{A})), \mathbf{A}_e\eta_m(\mathbf{q}(\mathbf{A})), \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$
$$|\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^2\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n} + J(\mathbf{A})\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\} \; (\equiv V''_{2m+1}(\mathbf{A}))$$

$$R \quad \overset{\circ}{\mathbf{r}}\cdot\eta_m(\mathbf{q}(\mathbf{A})), \eta_m(\mathbf{q}(\mathbf{A}))\cdot\mathbf{A}_e\,\overset{\circ}{\mathbf{r}}, (\mathbf{r}\cdot\mathbf{n})\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \overset{\circ}{\mathbf{r}}\cdot\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$
$$|\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^2\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})(\overset{\circ}{\mathbf{r}}\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + J(\mathbf{A})\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})(\mathbf{r}\cdot\mathbf{n}).$$

We show that the presented set $V''_{2m+1}(\mathbf{A})$ obeys the criterion (2.3) given in Part I. The case when $\overset{\circ}{\mathbf{A}} = \mathbf{O}$ can be treated easily. Let $\overset{\circ}{\mathbf{A}} \neq \mathbf{O}$ and $\overset{\circ}{\mathbf{A}}\,\mathbf{n} = \mathbf{0}$. Then we have

$$D = \mathrm{rank}V''_{2m+1}(\mathbf{A}) \geq \begin{cases} \mathrm{rank}\{\eta_m(\mathbf{q}(\mathbf{A})), \mathbf{A}_e\eta_m(\mathbf{q}(\mathbf{A}))\} \\ \qquad\qquad = 2 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0\,, \\ \mathrm{rank}\{\eta_m(\mathbf{q}(\mathbf{A}))\} = 1 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0\,, \end{cases}$$

$$\Gamma(\mathbf{A}) \cap D_{2m+1h} = \begin{cases} C_{1h}(\mathbf{n}) \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0\,, \\ C_{2v}(\mathbf{v}, \mathbf{n}, \mathbf{n}\times\mathbf{v}) \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0\,. \end{cases}$$

Let $\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) \neq 0$ and let $D = \mathrm{rank}V''_{2m+1}(\mathbf{A})$. Then we have

$$D \geq \begin{cases} \mathrm{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\} = 3 \;\text{ if }\; \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) \neq 0\,, \\ \mathrm{rank}\{\mathbf{n}, \eta_m(\mathbf{q}(\mathbf{A})), \mathbf{A}_e\eta_m(\mathbf{q}(\mathbf{A}))\} = 3 \;\text{ if }\; \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0, \\ \qquad\qquad\qquad\qquad \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0\,, \\ \mathrm{rank}\{\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \eta_m(\mathbf{q}(\mathbf{A}))\} = 3 \;\text{ if }\; \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) \\ \qquad\qquad\qquad = \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0,\; J(\mathbf{A}) \neq 0\,, \\ \mathrm{rank}\{\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\} = 2 \text{ if } \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = J(\mathbf{A}) = 0\,, \end{cases}$$

$$\Gamma(\mathbf{A}) \cap D_{2m+1h} = C_{1h}(\mathbf{n}\times\mathbf{v}) \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0},\; \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = J(\mathbf{A}) = 0.$$

Let $\overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0}$ and $\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0$. Then, we have

$$\mathrm{rank}V''_{2m+1}(\mathbf{A}) \geq \begin{cases} \mathrm{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{n}, \eta_m(\mathbf{q}(\mathbf{A}))\} = \;3 \text{ if }\; J(\mathbf{A}) \neq 0\,, \\ \mathrm{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} = 1 \;\text{ if }\; J(\mathbf{A}) = 0\,, \end{cases}$$

and

$$\Gamma(\mathbf{A}) \cap D_{2m+1} = C_2(\mathbf{v}) \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0},\ J(\mathbf{A}) = \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0.$$

It is readily understood that the four cases for **A** considered above exhaust all cases for **A**. Thus, from the above results and Table 1 in Sec. 2 in Part I, we conclude that the presented set $V''_{2m+1}(\mathbf{A})$ obeys the criterion (2.3) in Part I and hence supplies a desired vector generating set. Further, by considering the two tensors $\mathbf{A}_1 = \mathbf{n} \vee (\mathbf{e} + \boldsymbol{\tau}_1)$ and $\mathbf{A}_2 = \mathbf{e} \vee \boldsymbol{\tau}_1$, we deduce that each of the five generators in the set $V''_{2m+1}(\mathbf{A})$ are irreducible.

2.2. D_{2m+1h}-irreducible sets of two variables

(**iv**) The D_{2m+1h}-irreducible set $(\mathbf{u}, \mathbf{v})$ of two vectors

V $\quad V''_{2m+1}(\mathbf{u}) \cup V''_{2m+1}(\mathbf{v})\ (\equiv V''_{2m+1}(\mathbf{u}, \mathbf{v}))$

Skw $\quad \mathrm{Skw}''_{2m+1}(\mathbf{u}) \cup \mathrm{Skw}''_{2m+1}(\mathbf{v}) \cup \{\mathbf{u} \wedge \mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge \eta_{2m}(\overset{\circ}{\mathbf{v}})$

$\quad +|\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\mathbf{n} \wedge \eta_{2m}(\overset{\circ}{\mathbf{u}})\}\ (\equiv \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{v}))$

Sym $\quad \mathrm{Sym}''_{2m+1}(\mathbf{u}) \cup \mathrm{Sym}''_{2m+1}(\mathbf{v}) \cup \{\mathbf{u} \vee \mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee \eta_{2m}(\overset{\circ}{\mathbf{v}})$

$\quad +|\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\mathbf{n} \vee \eta_{2m}(\overset{\circ}{\mathbf{u}})\}\ (\equiv \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{v}))$

R $\quad \mathbf{r}\cdot V''_{2m+1}(\mathbf{u}, \mathbf{v}), \mathbf{H} : \mathrm{Skw}''_{2m+1}(\mathbf{z}), \mathbf{C} : \mathrm{Sym}''_{2m+1}(\mathbf{z}), \mathbf{z} = \mathbf{u}, \mathbf{v};$

$\quad \mathbf{u}\cdot\mathbf{H}\mathbf{v}; \mathbf{u}\cdot \overset{\circ}{\mathbf{C}}\,\mathbf{v};$

$\quad |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{v}})\cdot\mathbf{H}\mathbf{n} + |\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{H}\mathbf{n};$

$\quad |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{v}})\cdot \overset{\circ}{\mathbf{C}}\,\mathbf{n} + |\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot \overset{\circ}{\mathbf{C}}\,\mathbf{n};$

$\quad I''_{2m+1}(\mathbf{u}) \cup I''_{2m+1}(\mathbf{v}) \cup \{(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{v}}\}\ (\equiv I''_{2m+1}(\mathbf{u}, \mathbf{v})).$

To prove the above results, we first work out the D_{2m+1h}-irreducible set $(\mathbf{u}, \mathbf{v})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq \Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{u}, \mathbf{v}$. Evidently, we have $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq C_1$ for $\mathbf{z} = \mathbf{u}, \mathbf{v}$. The latter implies that either $-\mathbf{R}^{\pi}_{\mathbf{n}}$ or $\mathbf{R}^{\pi}_{\mathbf{v}}$ or $-\mathbf{R}^{\pi}_{\mathbf{v}'}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ of the vector **z**. Hence we derive: $\mathbf{z} = a\mathbf{e} + b\mathbf{e}'$; $\mathbf{z} = b\mathbf{v}$; and $\mathbf{z} = a\mathbf{n} + b\mathbf{v}$ for each $\mathbf{z} \in \{\mathbf{u}, \mathbf{v}\}$. These cases are equivalent to the three disjoint cases:

$$a\mathbf{n},\ a \neq 0; \quad a\mathbf{e} + b\mathbf{e}',\ a^2 + b^2 \neq 0; \quad a\mathbf{n} + b\mathbf{v},\ ab \neq 0\,. \tag{2.3}$$

Considering the combinations of the above forms and excluding the cases

$$\mathbf{u} = a\mathbf{n},\ \mathbf{v} = c\mathbf{n};\ \mathbf{u} = a\mathbf{n} + b\mathbf{v},\ \mathbf{v} = c\mathbf{n} + d\mathbf{v};$$

$$\mathbf{u} = a\mathbf{e} + b\mathbf{e}',\ \mathbf{v} = c\mathbf{e} + d\mathbf{e}',\ \beta_{2m+1}(\mathbf{z}) \neq 0,\ \mathbf{z} = \mathbf{u} \text{ or } \mathbf{z} = \mathbf{v};$$

$$\mathbf{u} = a\mathbf{n},\ \mathbf{v} = c\mathbf{n} + d\boldsymbol{\nu};\ \mathbf{u} = a\boldsymbol{\nu},\ \mathbf{v} = c\mathbf{n} + d\boldsymbol{\nu};$$

which violate the D_{2m+1h}-irreducibility condition for $(\mathbf{u}, \mathbf{v})$, we derive the following four disjoint cases for D_{2m+1h}-irreducible set $(\mathbf{u}, \mathbf{v})$:

(c1) $\mathbf{u} = a\mathbf{n},\ \mathbf{v} = b\mathbf{e} + c\mathbf{e}',\ a(b^2 + c^2) \neq 0;$

(c2) $\mathbf{u} = a\mathbf{e},\ \mathbf{v} = b\boldsymbol{\nu},\ \boldsymbol{\nu} \neq \mathbf{e},\ ab \neq 0;$

(c3) $\mathbf{u} = a\mathbf{e} + b\mathbf{e}',\ \mathbf{v} = c\mathbf{n} + d\mathbf{e},\ bcd \neq 0;$

(c4) $\mathbf{u} = a\mathbf{n} + b\mathbf{e},\ \mathbf{v} = c\mathbf{n} + d\boldsymbol{\nu},\ \boldsymbol{\nu} \neq \mathbf{e},\ abcd \neq 0.$

With cases (c1) – (c4) we show that the two presented sets $\mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{v})$ and $\mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{v})$ obey the criterion (2.3) in Part I separately, and therefore they supply the desired skewsymmetric and symmetric tensor generating sets. In fact, for case (c1) we have

$$\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2m+1h} = C_{1h}(\mathbf{n} \times \boldsymbol{\nu}) \text{ if } \beta_{2m+1}(\mathring{\mathbf{v}}) = 0,$$

$$\mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \begin{cases} \mathrm{rank}\{\mathbf{N}, \mathbf{u} \wedge \mathbf{v}, \mathbf{n} \wedge \eta_{2m}(\mathring{\mathbf{v}})\} = 3 \\ \qquad\qquad \text{if } \beta_{2m+1}(\mathring{\mathbf{v}}) \neq 0\,, \\ \mathrm{rank}\{\mathbf{u} \wedge \mathbf{v}\} = 1 \text{ if } \beta_{2m+1}(\mathring{\mathbf{v}}) = 0\,, \end{cases}$$

$$\mathrm{rank}\ \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \begin{cases} \mathrm{rank}(\mathrm{Sym}''_{2m+1}(\mathbf{v}) \cup \{\mathbf{u} \vee \mathbf{v}, \mathbf{n} \vee (\mathbf{n} \times \mathring{\mathbf{v}})\}) = 6 \\ \qquad\qquad \text{if } \beta_{2m+1}(\mathring{\mathbf{v}}) \neq 0\,, \\ \mathrm{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{v}} \otimes \mathring{\mathbf{v}}, \mathbf{u} \vee \mathbf{v}\} = 4 \\ \qquad\qquad \text{if } \beta_{2m+1}(\mathring{\mathbf{v}}) = 0\,. \end{cases}$$

For case (c2) we have

$$\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2m+1h} = C_{1h}(\mathbf{n}), \quad \mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \mathrm{rank}\{\mathbf{u} \wedge \mathbf{v}\} = 1,$$

$$\mathrm{rank}\ \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \mathrm{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{u}} \otimes \mathring{\mathbf{u}}, \mathring{\mathbf{v}} \otimes \mathring{\mathbf{v}}\} = 4;$$

For case (c3) we have

$$\mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \mathrm{rank}\{\beta_{2m+1}(\mathring{\mathbf{u}})\mathbf{N}, \mathbf{n} \wedge \mathring{\mathbf{v}}, \mathbf{u} \wedge \mathbf{v}, \mathbf{n} \wedge \eta_{2m}(\mathring{\mathbf{u}})\} = 3,$$

$$\mathrm{rank}\ \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \mathrm{rank}(\mathrm{Sym}''_{2m+1}(\mathbf{v}) \cup \{\mathring{\mathbf{u}} \vee \eta_{2m}(\mathring{\mathbf{u}}), \mathbf{u} \vee \mathbf{v}\}) = 6;$$

Finally, for case (c4), by using the formula (2.4) in Part I we have

$$\operatorname{rank} \operatorname{Skw}''_{2m+1}(\mathbf{u},\mathbf{v}) \geq \operatorname{rank}\{\mathbf{n}\wedge\eta_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\wedge\eta_{2m}(\overset{\circ}{\mathbf{v}}), \mathbf{u}\wedge\mathbf{v}\} = 3,$$

$$\begin{aligned} \operatorname{rank} \operatorname{Sym}''_{2m+1}(\mathbf{u},\mathbf{v}) &\geq \operatorname{rank}(\operatorname{Sym}''_{2m+1}(\mathbf{u}) \cup \operatorname{Sym}''_{2m+1}(\mathbf{v})) \\ &= \operatorname{rank}(\operatorname{Sym}(C_{1h}(\mathbf{e}')) \cup \operatorname{Sym}(C_{1h}(\mathbf{n}\times\mathbf{v}))) = 6\,. \end{aligned}$$

In deriving the last equality, Eq. (2.4) given in Part I has been used. From the above results and Tables 2 – 3 given in Sec. 2 in Part I we deduce that the foregoing facts concerning the two sets $\operatorname{Skw}''_{2m+1}(\mathbf{u},\mathbf{v})$ and $\operatorname{Sym}''_{2m+1}(\mathbf{u},\mathbf{v})$ are true. Moreover, by considering the pair $\mathbf{u}_1 = \mathbf{e}'$ and $\mathbf{v}_1 = \mathbf{n}$ we infer that the last two generators in the set $\operatorname{Skw}''_{2m+1}(\mathbf{u},\mathbf{v})$ are irreducible. Besides, by considering the pair $\mathbf{u}_2 = \mathbf{n}$ and $\mathbf{v}_2 = \mathbf{e}'$ we infer that the last two generators in the set $\operatorname{Sym}''_{2m+1}(\mathbf{u},\mathbf{v})$ are also irreducible.

Next, consider the two presented sets $I''_{2m+1}(\mathbf{u},\mathbf{v})$ and $V''_{2m+1}(\mathbf{u},\mathbf{v})$. The former set includes as a subset the set $\{(\mathbf{u}\cdot\mathbf{n})^2, |\overset{\circ}{\mathbf{u}}|^2, (\mathbf{v}\cdot\mathbf{n})^2, |\overset{\circ}{\mathbf{v}}|^2, (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{v}}\}$. The latter is a functional basis of $(\mathbf{u},\mathbf{v})$ under the group $D_{\infty h}(\mathbf{n})$. Using this fact and following the same procedure used in Sec. 4 (**vi**) in Part I, we infer that the set $I''_{2m+1}(\mathbf{u},\mathbf{v})$ provides a desired functional basis for $(\mathbf{u},\mathbf{v})$. Next, an anisotropic vector generating set for the variables $(\mathbf{u},\mathbf{v})$ is obtainable from an isotropic vector generating set for the extended variables $(\mathbf{u},\mathbf{v},\eta_{2m}(\overset{\circ}{\mathbf{u}}), \eta_{2m}(\overset{\circ}{\mathbf{v}}), \mathbf{n}\otimes\mathbf{n})$ (see Theorem 3 in XIAO [17]). By using the related result for isotropic functions we know that the former is just given by the presented set $V''_{2m+1}(\mathbf{u},\mathbf{v})$.

(**v**) The D_{2m+1h}-irreducible set $(\mathbf{W},\mathbf{\Omega})$ of two skewsymmetric tensors

$$\begin{aligned} V \quad & V''_{2m+1}(\mathbf{W}) \cup V''_{2m+1}(\mathbf{\Omega}) \cup \{\mathbf{W}\eta_{2m}(\mathbf{\Omega}\mathbf{n}), \mathbf{\Omega}\eta_{2m}(\mathbf{W}\mathbf{n}), \\ & |\mathbf{W}|^{2m}\beta_{2m+1}(\mathbf{\Omega}\mathbf{n})(\mathbf{E}:\mathbf{W}) + |\mathbf{\Omega}|^{2m}\beta_{2m+1}(\mathbf{W}\mathbf{n})(\mathbf{E}:\mathbf{\Omega})\} \\ & \qquad\qquad (\equiv V''_{2m+1}(\mathbf{W},\mathbf{\Omega})) \end{aligned}$$

$$\begin{aligned} R \quad & \mathbf{r}\cdot V''_{2m+1}(\mathbf{W}), \mathbf{r}\cdot V''_{2m+1}(\mathbf{\Omega}), \eta_{2m}(\mathbf{W}\mathbf{n})\cdot\mathbf{\Omega}\mathbf{r}, \eta_{2m}(\mathbf{\Omega}\mathbf{n})\cdot\mathbf{W}\mathbf{r}, \\ & |\mathbf{W}|^{2m}\beta_{2m+1}(\mathbf{\Omega}\mathbf{n})\mathbf{r}\cdot(\mathbf{E}:\mathbf{W}) + |\mathbf{\Omega}|^{2m}\beta_{2m+1}(\mathbf{W}\mathbf{n})\mathbf{r}\cdot(\mathbf{E}:\mathbf{\Omega});. \end{aligned}$$

To prove the above result, we first work out the D_{2m+1h}-irreducible set $(\mathbf{W},\mathbf{\Omega})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq \Gamma(\mathbf{W},\mathbf{\Omega}) \cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{W}, \mathbf{\Omega}$. Evidently, $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq C_1$ for $\mathbf{z} = \mathbf{W}, \mathbf{\Omega}$. The latter means that either $-\mathbf{R}^{\pi}_{\mathbf{n}}$ or $\mathbf{R}^{\pi}_{\mathbf{v}}$ or $-\mathbf{R}^{\pi}_{\mathbf{v}'}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ for each $\mathbf{z} \in \{\mathbf{W},\mathbf{\Omega}\}$. Hence each $\mathbf{z} \in \{\mathbf{W},\mathbf{\Omega}\}$ takes one of the forms

$$c\mathbf{E}\mathbf{n},\ c\neq 0;\quad c\mathbf{E}\mathbf{v},\ c\neq 0;\quad c\mathbf{n}\wedge\mathbf{v},\ c\neq 0\,. \tag{2.4}$$

Thus, we derive the following five disjoint cases for the D_{2m+1h}-irreducible set $(\mathbf{W}, \mathbf{\Omega})$:

(c1) $\mathbf{W} = a\mathbf{E}\mathbf{n}$, $\mathbf{\Omega} = b\mathbf{E}\mathbf{e}$, $ab \neq 0$;

(c2) $\mathbf{W} = a\mathbf{E}\mathbf{n}$, $\mathbf{\Omega} = b\mathbf{n} \wedge \mathbf{e}$, $ab \neq 0$;

(c3) $\mathbf{W} = a\mathbf{E}\mathbf{e}$, $\mathbf{\Omega} = b\mathbf{E}\boldsymbol{\nu}$, $\boldsymbol{\nu} \neq \mathbf{e}$, $ab \neq 0$;

(c4) $\mathbf{W} = a\mathbf{E}\mathbf{e}$, $\mathbf{\Omega} = b\mathbf{n} \wedge \boldsymbol{\nu}$, $ab \neq 0$;

(c5) $\mathbf{W} = a\mathbf{n} \wedge \mathbf{e}$, $\mathbf{\Omega} = b\mathbf{n} \wedge \boldsymbol{\nu}$, $\boldsymbol{\nu} \neq \mathbf{e}$, $ab \neq 0$.

Then, corresponding to the above five cases, we have

$$\mathrm{rank}V''_{2m+1}(\mathbf{W}, \mathbf{\Omega}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\mathbf{\Omega}\mathbf{n}), \mathbf{W}\boldsymbol{\eta}_{2m}(\mathbf{\Omega}\mathbf{n}), \mathbf{G}\} = 3,$$

$$\mathrm{rank}V''_{2m+1}(\mathbf{W}, \mathbf{\Omega}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\mathbf{\Omega}\mathbf{n}), \alpha_{2m+1}(\mathbf{\Omega}\mathbf{n})\mathbf{n}, \mathbf{W}\boldsymbol{\eta}_{2m}(\mathbf{\Omega}\mathbf{n})\} = 3,$$

$$\mathrm{rank}V''_{2m+1}(\mathbf{W}, \mathbf{\Omega}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), \boldsymbol{\eta}_{2m}(\mathbf{\Omega}\mathbf{n}), \mathbf{\Omega}\boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n})\} = 3,$$

$$\mathrm{rank}V''_{2m+1}(\mathbf{W}, \mathbf{\Omega}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), \alpha_{2m+1}(\mathbf{\Omega}\mathbf{n})\mathbf{n}, \mathbf{G}\} = 3,$$

$$\mathrm{rank}V''_{2m+1}(\mathbf{W}, \mathbf{\Omega}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), \boldsymbol{\eta}_{2m}(\mathbf{\Omega}\mathbf{n}), \alpha_{2m+1}(\mathbf{W}\mathbf{n})\mathbf{n}\} = 3,$$

where $\mathbf{G}$ is used to denote the last generator in the set $V''_{2m+1}(\mathbf{W}, \mathbf{\Omega})$. From the above results we infer that the presented set $V''_{2m+1}(\mathbf{W}, \mathbf{\Omega})$ obeys the criterion (2.3) in Part I and therefore supplies a desired vector generating set. Further, by considering the two pairs $\mathbf{W}_1 = \mathbf{E}\mathbf{n}$ and $\mathbf{\Omega}_1 = \mathbf{E}\mathbf{e}$, $\mathbf{W}_2 = \mathbf{E}\mathbf{e}$ and $\mathbf{\Omega}_2 = \mathbf{E}\mathbf{n}$, we deduce that the last three generators in the set $V''_{2m+1}(\mathbf{W}, \mathbf{\Omega})$ are irreducible.

(vi) The D_{2m+1h}-irreducible set $(\mathbf{W}, \mathbf{A})$ of a skewsymmetric tensor and a symmetric tensor

$$V \quad V''_{2m+1}(\mathbf{W}) \cup V''_{2m+1}(\mathbf{A}) \cup \{\mathbf{W}(\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})) + (-1)^m \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})),$$
$$(\mathrm{tr}\mathbf{W}\mathbf{N})\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n} + \mathbf{W}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})),$$
$$|\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\mathbf{W}\mathbf{n})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n} + |\mathbf{W}\mathbf{n}|^{2m}\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{W}\mathbf{n}\}$$
$$(\equiv V''_{2m+1}(\mathbf{W}, \mathbf{A}))$$

$$R \quad \mathbf{r} \cdot V''_{2m+1}(\mathbf{W}), \mathbf{r} \cdot V''_{2m+1}(\mathbf{A}), (\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})) + (-1)^m \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})) \cdot \mathbf{W}\mathbf{r},$$
$$(\mathbf{r} \cdot \mathbf{n})(\mathrm{tr}\mathbf{W}\mathbf{N})\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) - \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) \cdot \mathbf{W}\mathbf{r},$$
$$|\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\mathbf{W}\mathbf{n})(\overset{\circ}{\mathbf{r}} \cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n}) + |\mathbf{W}\mathbf{n}|^{2m}\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})(\overset{\circ}{\mathbf{r}} \cdot \mathbf{W}\mathbf{n}).$$

Here and henceforth, $\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))$ and $\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A}))$ are the two polynomial vector-valued functions of $\mathbf{A}$ given by (2.4) and (2.5) in Part II (see XIAO, BRUHNS and MEYERS [22]).

To prove the above result, we first work out the D_{2m+1h}-irreducible set $(\mathbf{W}, \mathbf{A})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq \Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{W}$, $\mathbf{A}$. As has been shown in (v), $\mathbf{W}$ takes one of the forms given by (2.4). Moreover, we have $\Gamma(\mathbf{A}) \cap D_{2m+1h} \neq C_1$. This implies that either $-\mathbf{R}^{\pi}_{\mathbf{n}}$ or $-\mathbf{R}^{\pi}_{\boldsymbol{\nu}'}$ or $\mathbf{R}^{\pi}_{\boldsymbol{\nu}}$ pertains to the symmetry group $\Gamma(\mathbf{A})$ of $\mathbf{A}$. Hence, we deduce that $\overset{\circ}{\mathbf{A}}$ takes one of the forms

$$(2.5) \qquad a\mathbf{D}_1 + b\mathbf{D}_2,\ a^2 + b^2 \neq 0; \quad a(\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \boldsymbol{\nu}' \otimes \boldsymbol{\nu}') + b\mathbf{n} \vee \boldsymbol{\nu},\ b \neq 0;$$
$$a(\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \boldsymbol{\nu}' \otimes \boldsymbol{\nu}') + b\mathbf{n} \vee \boldsymbol{\nu}',\ b \neq 0\,.$$

Considering the combinations of the above forms for $\mathbf{A}$ and the forms (2.4) for $\mathbf{W}$ and excluding the cases

$$\mathbf{W} = c\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0;$$
$$\mathbf{W} = f\mathbf{E}\boldsymbol{\tau},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}'),\ \boldsymbol{\tau} = \boldsymbol{\tau}_k;$$
$$\mathbf{W} = c\mathbf{E}\boldsymbol{\tau},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}') + b\mathbf{n} \vee \boldsymbol{\tau}',\ \boldsymbol{\tau} = \boldsymbol{\tau}_k;$$

which violate the D_{2m+1h}-irreducibility condition for $(\mathbf{W}, \mathbf{A})$, we derive the following nine disjoint cases for the D_{2m+1h}-irreducible set $(\mathbf{W}, \mathbf{A})$:

(c1) $\mathbf{W} = c\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1,\ ac \neq 0;$

(c2) $\mathbf{W} = c\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_i,\ i = 3,\ 4,\ bc \neq 0;$

(c3) $\mathbf{W} = c\mathbf{E}\mathbf{z},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \mathbf{z} \in \{\mathbf{e}, \mathbf{e}'\},\ bc \neq 0;$

$$\text{(c4)} \begin{cases} \mathbf{W} = c\mathbf{E}\boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_i,\ i = 3,\ 4,\ bc \neq 0\,, \\ i = 3:\ \boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 0, 1, \cdots, 2m\}\,, \\ i = 4:\ \mathbf{e} \neq \boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 1, \cdots, 2m\}\,. \end{cases}$$

$$\text{(c5)} \begin{cases} \mathbf{W} = c\mathbf{n} \wedge \boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_i,\ i = 3,\ 4,\ bc \neq 0\,, \\ i = 3: \quad \mathbf{e} \neq \boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 1, \cdots, 2m\}\,, \\ i = 4: \quad \boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 0, 1, \cdots, 2m\}\,. \end{cases}$$

Then, for case (c1) we have

$$\Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2m+1h} = C_{1h}(\mathbf{n}), \quad \operatorname{rank} V''_{2m+1}(\mathbf{W}, \mathbf{A}) \geq \operatorname{rank}\{\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \mathbf{W}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\} = 2.$$

For case (c2), we have

$$\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = (a^{2m} + b^{2m})\mathbf{e} \neq \mathbf{0},$$
$$\rho_m(\mathbf{q}(\mathbf{A})) = a^{2m+1}\mathbf{e},$$

for $i = 3,\ 4$, and hence with $D = \mathrm{rank} V''_{2m+1}(\mathbf{W}, \mathbf{A})$ we have

$$D \geq \begin{cases} \mathrm{rank}\{\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{W}(\pi_m(\mathbf{q}(\mathbf{A})) \\ \qquad\qquad +(-1)^m\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} = 3 \text{ if } i = 3\,, \\ \mathrm{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{W}(\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})), \\ \qquad\qquad \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\} = 3 \text{ if } i = 4\,. \end{cases}$$

For case (c3) we have (note that $b \neq 0$, i.e. $\psi(\mathbf{A}) =< \mathbf{q}(\mathbf{A}), \mathbf{e} > \neq k\pi$)

$$\begin{aligned} \mathrm{rank} V''_{2m+1}(\mathbf{W}, \mathbf{A}) \ &\geq\ \mathrm{rank}\{\eta_{2m}(\mathbf{W}\mathbf{n}), \eta_m(\mathbf{q}(\mathbf{A})), \mathbf{A}_\mathbf{e}\eta_m(\mathbf{q}(\mathbf{A})), \\ &\qquad \mathbf{W}\pi_m(\mathbf{q}(\mathbf{A})), \mathbf{W}\rho_m(\mathbf{q}(\mathbf{A}))\} \\ &=\ \mathrm{rank}\{\mathbf{e}, \mathbf{z}\sin m\psi(\mathbf{A}), \\ &\qquad \mathbf{z}\sin(m+1)\psi(\mathbf{A}) \mid \mathbf{z} = \mathbf{e}', \mathbf{n}\} = 3\,. \end{aligned}$$

For case (c4) we have (see the first equality for case (2))

$$D \geq \begin{cases} \mathrm{rank}\{\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{W}\mathbf{n}\} = 3 \text{ if } i = 3\,, \\ \mathrm{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \eta_{2m}(\mathbf{W}\mathbf{n}), \mathbf{W}(\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} \\ \qquad\qquad = 3 \ \text{ if } \ i = 4\,, \end{cases}$$

where $D = \mathrm{rank} V''_{2m+1}(\mathbf{W}, \mathbf{A})$. Finally, for case (c5) we have

$$\mathrm{rank} V''_{2m+1}(\mathbf{W}, \mathbf{A}) \geq \begin{cases} \mathrm{rank}\{\alpha_{2m+1}(\mathbf{W}\mathbf{n})\mathbf{n}, \eta_{2m}(\mathbf{W}\mathbf{n}), \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} \\ \qquad\qquad = 3 \ \text{ if } \ i = 3\,, \\ \mathrm{rank}\{\alpha_{2m+1}(\mathbf{W}\mathbf{n})\mathbf{n}, \eta_{2m}(\mathbf{W}\mathbf{n}), \\ \qquad\qquad \alpha_{2m+1}(\mathbf{W}\mathbf{n})\ \overset{\circ}{\mathbf{A}}\,\mathbf{n}\} = 3 \ \text{ if } \ i = 4\,. \end{cases}$$

From the above results and Table 1 in Sec. 2 in Part I, we infer that the presented set $V''_{2m+1}(\mathbf{W}, \mathbf{A})$ obeys the criterion (2.3) in Part I and hence is a desired vector generating set. Further, by considering the two pairs $\mathbf{W}_1 = \mathbf{En}$ and $\mathbf{A}_1 = \mathbf{n} \vee \mathbf{e}'$, $\mathbf{W}_2 = \mathbf{n} \wedge \mathbf{e}$ and $\mathbf{A}_2 = \mathbf{n} \vee \mathbf{e}'$, we deduce that the last three generators in the set $V''_{2m+1}(\mathbf{W}, \mathbf{A})$ are irreducible.

(**vii**) The D_{2m+1h}-irreducible set $(\mathbf{A}, \mathbf{B})$ of two symmetric tensors

$$V \quad V''_{2m+1}(\mathbf{A}) \cup V''_{2m+1}(\mathbf{B}) \cup \{((\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{B})) + (-1)^m \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}))\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n},$$

$$((\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})) + (-1)^m \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\cdot \overset{\circ}{\mathbf{B}}\,\mathbf{n})\mathbf{n},$$

$$|\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m} \alpha_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n} + |\overset{\circ}{\mathbf{B}}\,\mathbf{n}|^{2m} \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{B}}\,\mathbf{n},$$

$$(\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{B})) + \beta_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{A})))\mathbf{n}\}$$

$$(\equiv V''_{2m+1}(\mathbf{A}, \mathbf{B}))$$

$$R \quad \mathbf{r} \cdot V''_{2m+1}(\mathbf{A}), \mathbf{r} \cdot V''_{2m+1}(\mathbf{B}),$$

$$(\mathbf{r} \cdot \mathbf{n})((\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{B})) + (-1)^m \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}))\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$

$$(\mathbf{r} \cdot \mathbf{n})((\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})) + (-1)^m \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\cdot \overset{\circ}{\mathbf{B}}\,\mathbf{n}),$$

$$|\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m} \alpha_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\,\overset{\circ}{\mathbf{r}} \cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n} + |\overset{\circ}{\mathbf{B}}\,\mathbf{n}|^{2m} \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{r}} \cdot \overset{\circ}{\mathbf{B}}\,\mathbf{n},$$

$$(\mathbf{r} \cdot \mathbf{n})(\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{B})) + \beta_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{A}))).$$

Here and henceforth, $\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{D}))$ is the polynomial vector-valued function of the symmetric tensor $\mathbf{D}$ given by (2.5) in Part II (see XIAO, BRUHNS and MEYERS [22]), with the replacement of $\mathbf{A}$ by $\mathbf{D}$ therein.

We proceed to work out the D_{2m+1h}-irreducible set $(\mathbf{A}, \mathbf{B})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq \Gamma(\mathbf{A}, \mathbf{B}) \cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{A}$, $\mathbf{B}$. From (**vi**) we know that both $\overset{\circ}{\mathbf{A}}$ and $\overset{\circ}{\mathbf{B}}$ take the forms given by (2.5). Considering the combinations of the forms given by (2.5) and excluding the cases

$$\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \overset{\circ}{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_2,\ \beta_{2m+1}(\mathbf{q}(\mathbf{z})) \neq 0,\ \mathbf{z} = \mathbf{A} \text{ or } \mathbf{z} = \mathbf{B};$$

$$\overset{\circ}{\mathbf{A}} = a(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}') + b\mathbf{n} \vee \boldsymbol{\tau},\ \overset{\circ}{\mathbf{B}} = c(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}') + d\mathbf{n} \vee \boldsymbol{\tau},\ \boldsymbol{\tau} = \boldsymbol{\tau}_k;$$

$$\overset{\circ}{\mathbf{A}} = a(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}'),\ \overset{\circ}{\mathbf{B}} = c(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}') + d\mathbf{n} \vee \boldsymbol{\tau},\ \boldsymbol{\tau} = \boldsymbol{\tau}_k;$$

which violate the D_{2m+1h}-irreducibility condition for $(\mathbf{A}, \mathbf{B})$, we derive the following six disjoint cases for D_{2m+1h}-irreducible set $(\mathbf{A}, \mathbf{B})$:

(c1) $\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1$, $\overset{\circ}{\mathbf{B}} = c(\boldsymbol{\nu}\otimes\boldsymbol{\nu} - \boldsymbol{\nu}'\otimes\boldsymbol{\nu}')$, $\boldsymbol{\nu}\neq\mathbf{e}$, $ac\neq 0$;

(c2) $\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2$, $\overset{\circ}{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_3$, $bd\neq 0$;

(c3) $\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2$, $\overset{\circ}{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_4$, $bd\neq 0$;

$$\text{(c4)}\begin{cases} \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu}\otimes\boldsymbol{\nu} - \boldsymbol{\nu}'\otimes\boldsymbol{\nu}') + b\mathbf{n}\vee\boldsymbol{\nu},\ \overset{\circ}{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_i,\ i=3,\ 4,\ bd\neq 0\,, \\ i=3: \quad \mathbf{e}\neq\boldsymbol{\nu}\in\{\boldsymbol{\tau}_{2r}\mid r=1,\cdots,2r\}\,, \\ i=4: \quad \boldsymbol{\nu}\in\{\boldsymbol{\tau}_{2r}\mid r=0,1,\cdots,2r\}\,; \end{cases}$$

(c5) $\overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu}\otimes\boldsymbol{\nu} - \boldsymbol{\nu}'\otimes\boldsymbol{\nu}') + b\mathbf{n}\vee\boldsymbol{\nu}'$, $\overset{\circ}{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_4$, $bd\neq 0$, $\boldsymbol{\nu}\neq\mathbf{e}$.

Then, for case (c1) we have

$$\Gamma(\mathbf{A},\mathbf{B})\cap D_{2m+1h} = C_{1h}(\mathbf{n}), \quad \mathrm{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) \geq \{\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \\ \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{B}))\} = 2.$$

For case (c2), by using $\overset{\circ}{\mathbf{A}}\,\mathbf{n} = \mathbf{0}$ and $b\neq 0$, i.e. $\psi(\mathbf{A}) = <\mathbf{q}(\mathbf{A}),\mathbf{e}> \neq k\pi$, we have

$$\begin{aligned} \mathrm{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) &\geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}), \alpha_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\mathbf{n}, \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \\ &\quad \mathbf{A}_{\mathbf{e}}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\} \\ &= \mathrm{rank}\{\mathbf{n},\mathbf{e},\mathbf{e}'\sin m\psi(\mathbf{A}), \mathbf{e}'\sin(m+1)\psi(\mathbf{A})\} = 3\,. \end{aligned}$$

For case (c3), by using $\overset{\circ}{\mathbf{A}}\,\mathbf{n} = \mathbf{0}$ we have

$$\mathrm{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}), \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \\ \beta_{2m+1}(\mathbf{q}(\mathbf{A}))\beta_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\mathbf{n}\} = 3,$$

when $\beta_{2m+1}(\mathbf{q}(\mathbf{A}))\neq 0$, and

$$\begin{aligned} \mathrm{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) &\geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}), \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), (\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A}))\cdot\overset{\circ}{\mathbf{B}}\,\mathbf{n})\mathbf{n}\} \\ &= \mathrm{rank}\{\mathbf{e},\boldsymbol{\nu},\mathbf{n}\sin\frac{1}{2}(2m+1-(-1)^m)\psi(\mathbf{A})\} = 3\,, \end{aligned}$$

when $\beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0$, i.e. $\sin(2m+1)\psi(\mathbf{A}) = 0$ (note $\sin\psi(\mathbf{A})\neq 0$).

For case (c4) we have

$$\operatorname{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) \geq \begin{cases} \operatorname{rank}\{\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \\ \qquad\qquad \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\} = 3 \text{ if } i = 3\,, \\ \operatorname{rank}\{\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \\ \qquad\qquad \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{B}}\,\mathbf{n}\} = 3 \text{ if } i = 4\,. \end{cases}$$

Finally, for case (c5) we have

$$\begin{aligned} \operatorname{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) \ &\geq \ \operatorname{rank}\{\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}), ((\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{B})) \\ &\qquad\qquad + (-1)^m \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}))\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\} \\ &= \ \{\mathbf{e}, \boldsymbol{\nu}, b(c^{2m} + d^{2m})(\mathbf{e}\cdot\boldsymbol{\nu}')\mathbf{n}\} = 3\,. \end{aligned}$$

Thus, from the above results and Table 1 in Sec. 2 in Part I we infer that the presented set $V''_{2m+1}(\mathbf{A},\mathbf{B})$ obeys the criterion (2.3) in Part I, and therefore it is the desired vector generating set. Further, by considering the four pairs $(\mathbf{A}_i, \mathbf{B}_i)$ given by

$$\mathbf{A}_1 = \mathbf{n} \vee \boldsymbol{\tau}_1,\ \mathbf{B}_1 = \mathbf{D}_1; \qquad \mathbf{A}_2 = \mathbf{D}_1, \qquad \mathbf{B}_2 = \mathbf{n} \vee \boldsymbol{\tau}_1;$$

$$\mathbf{A}_3 = \boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 - \boldsymbol{\tau}'_1 \otimes \boldsymbol{\tau}'_1, \qquad \mathbf{B}_3 = \mathbf{n} \vee \mathbf{e}'\ ;$$

$$\boldsymbol{\nu}'_1 = \mathbf{n} \vee \mathbf{e}'; \qquad \mathbf{A}_4 = \mathbf{n} \vee \mathbf{e}, \qquad \mathbf{B}_4 = \mathbf{n} \vee \mathbf{e}',$$

we deduce that the last four generators in the set $V''_{2m+1}(\mathbf{A},\mathbf{B})$ are irreducible, respectively.

(viii) The D_{2m+1h}-irreducible set $(\mathbf{u}, \mathbf{W})$ of a vector and a skewsymmetric tensor

V $\quad V''_{2m+1}(\mathbf{u}) \cup V''_{2m+1}(\mathbf{W}) \cup \{\mathbf{Wu}\}\ (\equiv V''_{2m+1}(\mathbf{u},\mathbf{W}))$

Skw $\quad \mathrm{Skw}''_{2m+1}(\mathbf{u}) \cup \mathrm{Skw}_{4m+2}(\mathbf{W}) \cup \{\mathbf{u} \wedge \mathbf{Wu}, \mathbf{u} \wedge \boldsymbol{\eta}_{2m}(\mathbf{Wn}),$

$|\mathbf{u}|^{2m}\mathbf{Wu} \wedge \boldsymbol{\eta}_{2m}(\mathbf{Wn}) + |\mathbf{W}|^{2m}\ \overset{\circ}{\mathbf{u}} \wedge \mathbf{W}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}$

$(\equiv \mathrm{Skw}''_{2m+1}(\mathbf{u},\mathbf{W}))$

Sym $\quad \mathrm{Sym}''_{2m+2}(\mathbf{u}) \cup \mathrm{Sym}_{4m+2}(\mathbf{W}) \cup \{\mathbf{u} \vee \mathbf{Wu}, \mathbf{u} \vee \boldsymbol{\eta}_{2m}(\mathbf{Wn}),$

$|\mathbf{u}|^{2m}\mathbf{Wu} \vee \boldsymbol{\eta}_{2m}(\mathbf{Wn}) + |\mathbf{W}|^{2m}\ \overset{\circ}{\mathbf{u}} \vee \mathbf{W}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}(\equiv \mathrm{Sym}''_{2m+1}(\mathbf{u},\mathbf{W}))$

$R \quad \mathbf{r}\cdot V''_{2m+1}(\mathbf{u}), \mathbf{r}\cdot V''_{2m+1}(\mathbf{W}), \mathbf{r}\cdot \mathbf{Wu}; \mathrm{tr}\mathbf{HW};$

$\mathbf{C} : \mathrm{Sym}''_{2m+1}(\mathbf{u}), \overset{o}{\mathbf{u}} \cdot \overset{o}{\mathbf{C}}\, \mathbf{W}\, \overset{o}{\mathbf{u}};$

$I''_{2m+1}(\mathbf{u}) \cup I_{4m+2}(\mathbf{W}) \cup \{\mathbf{u}\cdot\mathbf{W}^2\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})(\overset{o}{\mathbf{u}}\cdot\mathbf{Wn})\}\ (\equiv I''_{2m+1}(\mathbf{u},\mathbf{W})).$

In the above table, the skewsymmetric tensor variable $\mathbf{H}$ is treated as having the form $\mathbf{H} = c\mathbf{W}$ with $c \neq 0$, which is derived from the condition (3.3) in Part I with $(\mathbf{z}_0, \mathbf{z}) = (\mathbf{W}, \mathbf{H})$ and $g = D_{2m+1h}$. As a result, of the invariants from the scalar products $\mathbf{H} : \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{W})$, we need only to retain the invariant $\mathrm{tr}\mathbf{HW}$. Moreover, consider the symmetric tensor variable $\mathbf{C}$. The D_{2m+1h}-irreducible set $(\mathbf{u}, \mathbf{W})$ is specified by cases (c1) – (c6) given below. Setting $\mathbf{z} = \mathbf{C}$, $X_0 = (\mathbf{u}, \mathbf{W})$ and $g = D_{2m+1h}$ in the condition (3.3) in Part I, we derive $\overset{o}{\mathbf{C}} = \mathbf{O}$ for case (c1); $\overset{o}{\mathbf{C}} = x\mathbf{D}_1 + y\mathbf{D}_2$ for cases (c2) and (c5); and $\mathbf{C} \in \mathrm{span}\ \mathrm{Sym}_{4m+2}(\mathbf{W})$ for cases (c3), (c4) and (c6). From these we deduce that, of the invariants from the scalar products $\mathbf{C} : \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{W})$, we need only to retain the invariants given by $\mathbf{C} : \mathrm{Sym}''_{2m+1}(\mathbf{u})$ and $\mathbf{C} : \mathrm{Sym}_{4m+2}(\mathbf{W})$, as well as $\overset{o}{\mathbf{u}} \cdot \overset{o}{\mathbf{C}}\, \mathbf{W}\, \overset{o}{\mathbf{u}}$, as has been done in the above table. Here we would mention that for cases (c2) and (c5), the subspace $\mathrm{Sym}(C_{2h}(\mathbf{n}))$ is generated by the three generators $\mathbf{I}$, $\mathbf{n}\otimes\mathbf{n}$ and $\overset{o}{\mathbf{u}} \otimes \overset{o}{\mathbf{u}}$ in the set $\mathrm{Sym}''_{2m+1}(\mathbf{u})$ as well as the generator $\overset{o}{\mathbf{u}} \vee \mathbf{W}\, \overset{o}{\mathbf{u}}$, and hence the invariant $\overset{o}{\mathbf{u}} \cdot \overset{o}{\mathbf{C}}\, \mathbf{W}\, \overset{o}{\mathbf{u}}$ is resulted in.

To prove the results in the table given, we first work out the D_{2m+1h}-irreducible set $(\mathbf{u}, \mathbf{W})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq \Gamma(\mathbf{u}, \mathbf{W}) \cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{u}$, $\mathbf{W}$. Considering the combinations of the forms for $\mathbf{u}$ and $\mathbf{W}$ given by (2.3) and (2.4) and excluding the cases

$$\mathbf{u} = a\mathbf{n},\ \mathbf{W} = c\mathbf{n}\wedge\mathbf{v};\ \mathbf{u} = a\mathbf{e} + b\mathbf{e}',\ \mathbf{W} = c\mathbf{En},\ \beta_{2m+1}(\mathbf{u}) \neq 0;$$

$$\mathbf{u} = a\mathbf{v},\ \mathbf{W} = c\mathbf{Ev};\ \mathbf{u} = a\mathbf{v},\ \mathbf{W} = c\mathbf{n}\wedge\mathbf{v};$$

$$\mathbf{u} = a\mathbf{n} + b\mathbf{v},\ \mathbf{W} = c\mathbf{n}\wedge\mathbf{v};$$

which violate the just-mentioned D_{2m+1h}-irreducibility condition, we derive the following eight disjoint cases for the D_{2m+1h}-irreducible set $(\mathbf{u}, \mathbf{W})$:

(c1) $\mathbf{u} = a\mathbf{n}$, $\mathbf{W} = c\mathbf{En}$, $ac \neq 0$;

(c2) $\mathbf{u} = a\mathbf{e}$, $\mathbf{W} = c\mathbf{En}$, $ac \neq 0$;

(c3) $\mathbf{u} = a\mathbf{n}$, $\mathbf{W} = c\mathbf{Ee}$, $ac \neq 0$;

(c4) $\mathbf{u} = a\mathbf{e} + b\mathbf{e}'$, $\mathbf{W} = c\mathbf{Ez}$, $\mathbf{z} \in \{\mathbf{e}, \mathbf{e}'\}$, $bc \neq 0$;

(c5) $\mathbf{u} = a\mathbf{n} + b\mathbf{e}$, $\mathbf{W} = c\mathbf{En}$, $abc \neq 0$;

$$(c6) \begin{cases} \mathbf{u} = a\mathbf{n} + b\mathbf{v},\ \mathbf{W} = c\mathbf{E}\mathbf{z},\ \mathbf{z} \in \{\mathbf{e},\ \mathbf{e}'\},\ abc \neq 0\,, \\ \mathbf{z} = \mathbf{e}: \quad \mathbf{v} \in \{\boldsymbol{\tau}_{2r} \mid r = 0, 1, \cdots, 2m, \}\,, \\ \mathbf{z} = \mathbf{e}': \quad \mathbf{e} \neq \mathbf{v} \in \{\boldsymbol{\tau}_{2r} \mid r = 1, \cdots, 2m\}\,. \end{cases}$$

With the above cases we show that the three presented sets $V''_{2m+1}(\mathbf{u}, \mathbf{W})$, $\mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{W})$ and $\mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{W})$ obey the criterion (2.3) in Part I. Case (c1) can be treated easily.

For case (c2) we have

$$\Gamma(\mathbf{u}, \mathbf{W}) \cap D_{2m+1h} = C_{1h}(\mathbf{n}), \quad \mathrm{rank} V''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{\mathring{\mathbf{u}}, \mathbf{W}\mathbf{u}\} = 2,$$

$$\mathrm{rank\ Skw}''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{\mathbf{W}\} = 1,$$

$$\mathrm{rank\ Sym}''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{u}} \otimes \mathring{\mathbf{u}}, \mathbf{u} \vee \mathbf{W}\mathbf{u}\} = 4.$$

For case (c3) we have

$$\mathrm{rank} V''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \eta_{2m}(\mathbf{W}\mathbf{n}), \mathbf{W}\mathbf{u}\} = 3,$$

$$\mathrm{rank\ Skw}''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{\mathbf{W}, \mathbf{u} \wedge \eta_{2m}(\mathbf{W}\mathbf{n}), \mathbf{W}\mathbf{u} \wedge \eta_{2m}(\mathbf{W}\mathbf{n})\} = 3,$$

$$\begin{aligned} \mathrm{rank\ Sym}''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{&\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathbf{W}^2, \mathbf{n} \vee \mathbf{W}\mathbf{n}, \mathbf{u} \vee \eta_{2m}(\mathbf{W}\mathbf{n}), \\ &\mathbf{W}\mathbf{u} \vee \eta_{2m}(\mathbf{W}\mathbf{n})\} = 6. \end{aligned}$$

For case (c4) we have

$$\begin{aligned} \mathrm{rank} V''_{2m+1}(\mathbf{u}, \mathbf{W}) &\geq \mathrm{rank}\{\mathring{\mathbf{u}}, \eta_{2m}(\mathbf{W}\mathbf{n}), \alpha_{2m+1}(\mathbf{W}\mathbf{n})\mathbf{n}, \mathbf{W}\mathbf{u}\} \\ &= \mathrm{rank}\{\mathbf{e}', \mathbf{e}, \alpha_{2m+1}(\mathbf{n} \times \mathbf{z})\mathbf{n}, a\mathbf{e} \times \mathbf{z} + b\mathbf{e}' \times \mathbf{z}\} = 3\,, \end{aligned}$$

$$\begin{aligned} \mathrm{rank\ Skw}''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{&\mathbf{W}, \mathbf{u} \wedge \eta_{2m}(\mathbf{W}\mathbf{n}), |\mathbf{u}|^{2m}\mathbf{W}\mathbf{u} \wedge \eta_{2m}(\mathbf{W}\mathbf{n}) \\ &+ |\mathbf{W}|^{2m}\, \mathring{\mathbf{u}} \wedge \mathbf{W}\eta_{2m}(\mathring{\mathbf{u}})\} = 3\,, \end{aligned}$$

$$\begin{aligned} \mathrm{rank\ Sym}''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{&\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathbf{W}^2, \mathbf{n} \vee \mathbf{W}\mathbf{n}, \mathbf{u} \vee \eta_{2m}(\mathbf{W}\mathbf{n}), \\ &\mathbf{u} \vee \mathbf{W}\mathbf{u}, |\mathbf{u}|^{2m}\, \mathring{\mathbf{u}} \vee \mathbf{W}\eta_{2m}(\mathbf{W}\mathbf{n}) \\ &+ |\mathbf{W}|^{2m}\, \mathring{\mathbf{u}} \vee \mathbf{W}\eta_{2m}(\mathring{\mathbf{u}})\} = 6\,. \end{aligned}$$

For case (c5) we have

$$\mathrm{rank} V''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{W}\mathbf{u}\} = 3,$$

$$\text{rank Skw}''_{2m+1}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge \overset{\circ}{\mathbf{u}}, \mathbf{W}, \mathbf{u}\wedge\mathbf{Wu}\} = 3,$$

$$\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}},$$
$$\mathbf{u}\vee\mathbf{Wu}, \overset{\circ}{\mathbf{u}}\vee\mathbf{W}\eta_{2m}(\overset{\circ}{\mathbf{u}})\} = 6.$$

Finally, for case (c6) we have the first two of the last three expressions above and, moreover,

$$\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{W}^2, \mathbf{n}\vee\mathbf{Wn},$$
$$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \mathbf{u}\vee\mathbf{Wu}\} = 6.$$

Then, from the above results and Tables 1 – 3 in Sec. 2 in Part I we infer that the three presented sets $V''_{2m+1}(\mathbf{u},\mathbf{W})$, $\text{Skw}''_{2m+1}(\mathbf{u},\mathbf{W})$ and $\text{Sym}''_{2m+1}(\mathbf{u},\mathbf{W})$ obey the criterion (2.3) in Part I, and therefore they supply desired vector, skewsymmetric tensor and symmetric tensor generating sets, respectively. Moreover, by considering the pair $\mathbf{u}_1 = \mathbf{n}$ and $\mathbf{W}_1 = \mathbf{Ee}$, we deduce that the generator $\mathbf{Wu}$ and the respective last two generators in the sets $\text{Skw}''_{2m+1}(\mathbf{u},\mathbf{W})$ and $\text{Sym}''_{2m+1}(\mathbf{u},\mathbf{W})$ are irreducible. By considering the pair $\mathbf{u}_2 = \mathbf{n}+\mathbf{e}$ and $\mathbf{W}_2 = \mathbf{En}$, we deduce that the two generators $\mathbf{u}\wedge\mathbf{Wu}$ and $\mathbf{u}\vee\mathbf{Wu}$ are also irreducible.

Finally, we show that the presented set $I''_{2m+1}(\mathbf{u},\mathbf{W})$ supplies a functional basis for the D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{W})$. Towards this goal it suffices to show that the former determines a functional basis for the D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{W})$ under the group $D_{\infty h}(\mathbf{n})$ (see the remark at the end of Sec. 4 (**vi**) in Part I). In fact, the just-mentioned functional basis is obtainable from an isotropic functional basis for $(\mathbf{u},\mathbf{W},\mathbf{n}\otimes\mathbf{n})$ (see BOEHLER [5]). The latter is formed by the invariants

$$(\mathbf{u}\cdot\mathbf{n})^2,\ |\overset{\circ}{\mathbf{u}}|^2,\ (\text{tr}\mathbf{WN})^2, |\mathbf{Wn}|^2, \mathbf{u}\cdot\mathbf{W}^2\mathbf{u},\ (\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn},$$
$$(\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn})(\mathbf{u}\cdot\mathbf{W}^2\mathbf{n}).$$

The first six invariants given above are included in the set $I''_{2m+1}(\mathbf{u},\mathbf{W})$. The last invariant is of the form $(\text{tr}\mathbf{W}^2)(\mathbf{u}\cdot\mathbf{n})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn}) + (\text{tr}\mathbf{WN})(\text{tr}\mathbf{W}(\mathbf{Eu}))(\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn})$. The first term is redundant and the second term vanishes for each of cases (c1) – (c6), and hence the invariant at issue is redundant.

(**ix**) The D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{A})$ of a vector and a symmetric tensor

$V \quad V''_{2m+1}(\mathbf{u}) \cup V''_{2m+1}(\mathbf{A}) \cup \{\overset{\circ}{\mathbf{A}}\,\mathbf{u}\}\ (\equiv V''_{2m+1}(\mathbf{u},\mathbf{A}))$

Skw $\mathrm{Skw}''_{2m+1}(\mathbf{u}) \cup \mathrm{Skw}_{4m+2}(\mathbf{A}) \cup \{\overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})),$

$\mathbf{u} \wedge \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + (-1)^m(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge \boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})),$

$\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\wedge \overset{\circ}{\mathbf{u}} + \overset{\circ}{\mathbf{A}}\,\mathbf{u} \wedge \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\}\ (\equiv \mathrm{Skw}''_{2m+1}(\mathbf{u},\mathbf{A}))$

Sym $\mathrm{Sym}_{2m+1}(\mathbf{u}) \cup \mathrm{Sym}_{4m+2}(\mathbf{A}) \cup \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})),$

$\mathbf{u} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + (-1)^m(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee \boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})),$

$\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\vee \overset{\circ}{\mathbf{u}} + \overset{\circ}{\mathbf{A}}\,\mathbf{u} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\}\ (\equiv \mathrm{Sym}''_{2m+1}(\mathbf{u},\mathbf{A}))$

R $\mathbf{r}\cdot V''_{2m+1}(\mathbf{u}), \mathbf{r}\cdot V''_{2m+1}(\mathbf{A}), \mathbf{r}\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{u}; \mathbf{H} : \mathrm{Skw}''_{2m+1}(\mathbf{u}); \overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}};$

$I''_{2m+1}(\mathbf{u}) \cup I_{4m+2}(\mathbf{A}) \cup \{\overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{A}}{}^2\, \overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\ \overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n}\}$

$(\equiv I''_{2m+1}(\mathbf{u},\mathbf{A})).$

In the above table, the skewsymmetric tensor variable $\mathbf{H}$ pertains to span $\mathrm{Skw}''_{2m+1}(\mathbf{u})$ or span $\mathrm{Skw}_{4m+2}(\mathbf{A})$. This fact can be derived from cases (c1) – (c6) for the D_{2m+1h}-irreducible set $(\mathbf{A},\mathbf{B})$ that will be given below, as well as the condition (3.3) in Part I with $\mathbf{z}_0 = \mathbf{H}$ and $\mathbf{z} \in \{\mathbf{u},\mathbf{A}\}$ and $g = D_{2m+1h}$. As a result, of the invariants given by the scalar products $\mathbf{H} : \mathrm{Skw}''_{2m+1}(\mathbf{u},\mathbf{A})$, we need only to retain those given by $\mathbf{H} : \mathrm{Skw}''_{2m+1}(\mathbf{u})$ and $\mathbf{H} : \mathrm{Skw}_{4m+2}(\mathbf{A})$. Moreover, by applying cases (c1) – (c6) for the D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{A})$ given below and the condition (3.3) in Part I with $X_0 = (\mathbf{u},\mathbf{A})$ and $(\mathbf{z}_0,\mathbf{z}) = (\mathbf{A},\mathbf{C})$ and $g = D_{2m+1h}$, we deduce that $\mathbf{C} \in \mathrm{Sym}(C_{2h}(\mathbf{n}))$ for case (c5) below and $\mathbf{C} \in$ span $\mathrm{Sym}_{4m+2}(\mathbf{A})$ for cases (c1) – (c4) and (c6) below. Accordingly, of the invariants given by the scalar products $\mathbf{C} : \mathrm{Sym}''_{2m+1}(\mathbf{u},\mathbf{A})$, we need only to retain $\mathbf{C} : \mathrm{Sym}_{4m+2}(\mathbf{A})$ and $\overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}$. The latter invariant results from the fact that for case (c5) below, the subspace $\mathrm{Sym}(C_{2h}(\mathbf{n}))$ is generated by the generator $\overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}}$ and the generators in the set $\mathrm{Sym}_{4m+2}(\mathbf{A})$.

To prove the results in the table given, we first work out the D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{A})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq \Gamma(\mathbf{u},\mathbf{A}) \cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{u}$, $\mathbf{A}$. Considering the combinations of the forms for $\mathbf{u}$ and $\overset{\circ}{\mathbf{A}}$ given by (2.3) and (2.5) and excluding the case

$$\mathbf{u} = c\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu}\otimes\boldsymbol{\nu} - \boldsymbol{\nu}'\otimes\boldsymbol{\nu}') + b\mathbf{n}\vee\boldsymbol{\nu};$$

$$\mathbf{u} = a\mathbf{e} + b\mathbf{e}',\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_2,\ \beta_{2m+1}(\mathbf{u}) \neq 0 \text{ or } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0;$$

$$\mathbf{u} = c\boldsymbol{\tau},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}') + b\mathbf{n} \vee \boldsymbol{\tau},\ \boldsymbol{\tau} = \boldsymbol{\tau}_k;$$

$$\mathbf{u} = c\mathbf{n} + d\boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \boldsymbol{\nu}' \otimes \boldsymbol{\nu}');$$

$$\mathbf{u} = c\mathbf{n} + d\boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \boldsymbol{\nu}' \otimes \boldsymbol{\nu}') + b\mathbf{n} \vee \boldsymbol{\nu};$$

which violate the just-mentioned D_{2m+1h}-irreducibility condition, we derive the following eight disjoint cases for the D_{2m+1h}-irreducible set $(\mathbf{u}, \mathbf{A})$:

(c1) $\mathbf{u} = a\mathbf{n},\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_2,\ a(c^2 + d^2) \neq 0;$

(c2) $\mathbf{u} = a\mathbf{n},\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_4,\ ad \neq 0;$

(c3) $\mathbf{u} = a\boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1,\ ac \neq 0,\ \mathbf{e} \neq \boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 1, \cdots, 2m\};$

(c4) $\mathbf{u} = a\mathbf{e} + b\mathbf{e}',\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_i,\ i = 3,\ 4,\ bd \neq 0;$

(c5) $\mathbf{u} = a\mathbf{n} + b\mathbf{e},\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_2,\ abd \neq 0;$

$$\text{(c6)} \begin{cases} \mathbf{u} = a\mathbf{n} + b\boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_i,\ i = 3,\ 4,\ abd \neq 0\,, \\ i = 3: \quad \mathbf{e} \neq \boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 1, \cdots, 2m\}\,, \\ i = 4: \quad \boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 0, 1, \cdots, 2m\}\,. \end{cases}$$

With the above cases we show that the three presented sets $V''_{2m+1}(\mathbf{u}, \mathbf{A})$, $\mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{A})$ and $\mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{A})$ obey the criterion (2.3) in Part I.

In fact, for case (c1) we have

$$\Gamma(\mathbf{u}, \mathbf{A}) \cap D_{2m+1h} = C_{1h}(\mathbf{n}) \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0,$$

$$\mathrm{rank} V''_{2m+1}(\mathbf{u}, \mathbf{A}) \geq \mathrm{rank}\{(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \mathbf{A}_{\mathbf{e}}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\}$$

$$= \begin{cases} 3 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0\,, \\ 2 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0\,, \end{cases}$$

$$\mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{A}) \geq \mathrm{rank}\{\beta_{2m+1}(\mathbf{q}(\mathbf{A}))\mathbf{N}, \mathbf{n} \wedge \boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})),$$

$$\mathbf{n} \wedge \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\} = \begin{cases} 3 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0\,, \\ 1 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0\,, \end{cases}$$

$$\begin{aligned}\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A}) &\geq \text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{A}},\Phi_{2m+1}(\mathbf{q}(\mathbf{A})),\mathbf{n}\vee\pi_m(\mathbf{q}(\mathbf{A})),\\ &\quad \mathbf{n}\vee\rho_m(\mathbf{q}(\mathbf{A}))\}\\ &= \begin{cases}\text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathbf{D}_1,\mathbf{D}_2,\mathbf{n}\vee\mathbf{e},\mathbf{n}\vee\mathbf{e}'\}=6 \\ \qquad\qquad \text{if } \beta_{2m+1}(\mathbf{q}(\mathbf{A}))\neq 0,\\ \text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{A}},\mathbf{n}\vee\pi_m(\mathbf{q}(\mathbf{A}))\}=4 \\ \qquad\qquad \text{if } \beta_{2m+1}(\mathbf{q}(\mathbf{A}))=0.\end{cases}\end{aligned}$$

For case (c2) we have

$$\text{rank}V''_{2m+1}(\mathbf{u})\geq\text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n},\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}),\mathring{\mathbf{A}}\,\mathbf{u}\}=3,$$

$$\begin{aligned}\text{rank Skw}''_{2m+1}(\mathbf{u}) &\geq \text{rank}\{\mathbf{n}\wedge\mathring{\mathbf{A}}\,\mathbf{n},\mathring{\mathbf{A}}\,\mathbf{u}\wedge\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}),\mathbf{n}\wedge\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\\ &\quad +(-1)^m\mathbf{n}\wedge\pi_m(\mathbf{q}(\mathbf{A}))\}\\ &= \text{rank}\{\mathbf{n}\wedge\mathbf{e}',\mathbf{e}\wedge\mathbf{e}',(c^{2m}+d^{2m})\mathbf{n}\wedge\mathbf{e}\}=3,\end{aligned}$$

$$\begin{aligned}\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A}) &\geq \text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{A}}\,\mathbf{n}\otimes\mathring{\mathbf{A}}\,\mathbf{n},\mathring{\mathbf{A}},\mathring{\mathbf{A}}\,\mathbf{u}\vee\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}),\\ &\quad \mathbf{n}\vee\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})+(-1)^m\mathbf{n}\vee\pi_m(\mathbf{q}(\mathbf{A}))\}\\ &= \text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathbf{e}'\otimes\mathbf{e}',\mathbf{n}\vee\mathbf{e}',\mathbf{e}\vee\mathbf{e}',\\ &\quad (d^{2m}+c^{2m})\mathbf{n}\vee\mathbf{e}\})=6.\end{aligned}$$

For case (c3) we have

$$\Gamma(\mathbf{u},\mathbf{A})\cap D_{2m+1h}=C_{1h}(\mathbf{n}),$$

$$\text{rank}V''_{2m+1}(\mathbf{u},\mathbf{A})\geq\text{rank}\{\mathring{\mathbf{u}},\eta_m(\mathbf{q}(\mathbf{A}))\}=2,$$

$$\text{rank Skw}''_{2m+1}(\mathbf{u},\mathbf{A})\geq\text{rank}\{\mathring{\mathbf{u}}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{u}}\}=1,$$

$$\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A})\geq\text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}},\mathring{\mathbf{A}}\}=4.$$

For case (c4) we have

$$\text{rank}V''_{2m+1}(\mathbf{u},\mathbf{A})\geq\begin{cases}\text{rank}\{\mathring{\mathbf{u}},\alpha_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{n},\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\}=3 \text{ if } i=3,\\ \text{rank}\{\mathring{\mathbf{u}},\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}),\mathring{\mathbf{A}}\,\mathbf{u}\}=3 \text{ if } i=4,\end{cases}$$

$$\text{rank Skw}''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{n}\wedge \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{u}\wedge\boldsymbol{\eta}, \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}\wedge\overset{\circ}{\mathbf{u}} + \overset{\circ}{\mathbf{A}}\mathbf{u}\wedge\boldsymbol{\eta}\} = 3,$$

$$\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}\otimes\overset{\circ}{\mathbf{A}}\mathbf{n}, \overset{\circ}{\mathbf{A}}, \mathbf{u}\vee\boldsymbol{\eta}, \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}} + \overset{\circ}{\mathbf{A}}\mathbf{u}\vee\boldsymbol{\eta}\} = 6,$$

where $\boldsymbol{\eta} = \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})$.

For case (c5) we have (note that $d \neq 0$, i.e., $\psi(\mathbf{A}) = <\mathbf{q}(\mathbf{A}), \mathbf{e}> \neq k\pi$)

$$\text{rank}V_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}}\mathbf{u}\} = 3,$$

$$\begin{aligned}\text{rank Skw}''_{2m+1}(\mathbf{u},\mathbf{A}) &\geq \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}},\\ &\qquad \mathbf{n}\wedge\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\wedge\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\}\\ &= \text{rank}\{\mathbf{n}\wedge\mathbf{e}, \mathbf{e}\wedge\mathbf{e}', \mathbf{n}\wedge\mathbf{e}'\sin m\psi(\mathbf{A}),\\ &\qquad \mathbf{n}\wedge\mathbf{e}'\sin(m+1)\psi(\mathbf{A})\} = 3\,,\end{aligned}$$

$$\begin{aligned}\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A}) &\geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}},\\ &\qquad \mathbf{n}\vee\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\vee\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\}\\ &= \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e}, \mathbf{n}\vee\mathbf{e}, \mathbf{e}\vee\mathbf{e}',\\ &\qquad \mathbf{n}\vee\mathbf{e}'\sin m\psi(\mathbf{A}), \mathbf{n}\vee\mathbf{e}'\sin(m+1)\psi(\mathbf{A})\} = 6\,.\end{aligned}$$

Finally, for case (c6) we have

$$\text{rank}V''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})\} = 3,$$

$$\text{rank Skw}''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \begin{cases} \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\overset{\circ}{\mathbf{u}}, \mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{u}\wedge\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})\} \\ \qquad = 3 \ \text{ if } \ \boldsymbol{\nu}\neq\mathbf{e}, \\ \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\overset{\circ}{\mathbf{u}}, \mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{u}\wedge\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})\} \\ \qquad = 3 \ \text{ if } \ i=4, \ \boldsymbol{\nu}=\mathbf{e}, \end{cases}$$

$$D \geq \begin{cases} \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}\otimes\overset{\circ}{\mathbf{A}}\mathbf{n}, \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}\} = 6 \text{ if } \boldsymbol{\nu}\neq\mathbf{e}, \\ \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\mathbf{u}\vee\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})\} \\ \qquad = 3 \ \text{ if } \ i=4, \ \boldsymbol{\nu}=\mathbf{e}, \end{cases}$$

where $D = \text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A})$.

From the above results we infer that the three presented sets $V''_{2m+1}(\mathbf{u},\mathbf{A})$, $\mathrm{Skw}''_{2m+1}(\mathbf{u},\mathbf{A})$ and $\mathrm{Sym}''_{2m+1}(\mathbf{u},\mathbf{A})$ obey the criterion (2.3) in Part I, and hence they supply the desired vector, skewsymmetric tensor and symmetric tensor generating sets, respectively. Further, by considering the pair $\mathbf{u}_1 = \mathbf{n}$ and $\mathbf{A}_1 = \mathbf{n}\vee\mathbf{e}'$, we deduce that the generators $\overset{\circ}{\mathbf{A}}\,\mathbf{u}$ and the respective last two generators in the sets $\mathrm{Skw}''_{2m+1}(\mathbf{u},\mathbf{A})$ and $\mathrm{Sym}''_{2m+1}(\mathbf{u},\mathbf{A})$ are irreducible. By considering the pair $\mathbf{u}_2 = \mathbf{n}$ and $\mathbf{A}_2 = \mathbf{e}\vee\mathbf{e}'$, we infer that the two generators $(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\rho_{2m}(\mathbf{q}(\mathbf{A}))$ and $(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\rho_{2m}(\mathbf{q}(\mathbf{A}))$ are irreducible. By considering the pair $\mathbf{u}_3 = \mathbf{v}_2$ and $\mathbf{A}_3 = \mathbf{D}_1$, we infer that the generator $\overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}$ is irreducible.

Finally, we show that the presented $I''_{2m+1}(\mathbf{u},\mathbf{A})$ supplies a functional basis for the D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{A})$ under the group D_{2m+1h}. Towards this goal it suffices to show that this set determines a functional basis for the set $(\mathbf{u},\mathbf{A})$ under the cylindrical group $D_{\infty h}(\mathbf{n})$ (see the remark at the end of Sec. 4 (**vi**) in Part I). In fact, the latter is obtainable from an isotropic functional basis for $(\mathbf{u},\overset{\circ}{\mathbf{A}},\mathbf{n}\otimes\mathbf{n})$ (see BOEHLER [5]), plus the invariants $\mathbf{n}\cdot\mathbf{A}\mathbf{n}$ and $\mathrm{tr}\mathbf{A}$. By using the related result for isotropic functions we know that the just-mentioned isotropic functional basis is formed by the invariants $\mathbf{u}\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{u}$, $\mathbf{u}\cdot\overset{\circ}{\mathbf{A}}{}^2\mathbf{u}$, $(\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n}$, as well as certain $D_{\infty h}$-invariants of $\mathbf{u}$ or $\overset{\circ}{\mathbf{A}}$. Each of the latter is determined by the basis $I''_{2m+1}(\mathbf{u})$ or $I_{4m+2}(\mathbf{A})$. The first three invariants yield the last three invariants in the set $I''_{2m+1}(\mathbf{u},\mathbf{A})$.

2.3. D_{2m+1h}-irreducible sets of three variables

(**x**) The D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{W})$ of two vectors and a skewsymmetric tensor

$$V \quad \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n},\overset{\circ}{\mathbf{u}},(\mathbf{v}\cdot\mathbf{n})\mathbf{n},\overset{\circ}{\mathbf{v}},\mathbf{W}\mathbf{u},\mathbf{W}\mathbf{v}\}$$

$$\mathrm{Skw} \quad \{\mathbf{W},\mathbf{u}\wedge\mathbf{v},(\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{n}\wedge(\mathbf{u}\times\mathbf{v})\}$$

$$\mathrm{Sym} \quad \{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}},\mathbf{u}\vee\mathbf{W}\mathbf{u},\mathbf{v}\vee\mathbf{W}\mathbf{v},\mathbf{u}\vee\mathbf{v},$$
$$(\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{n}\vee(\mathbf{u}\times\mathbf{v})\}$$

$$R \quad \{(\mathbf{u}\cdot\mathbf{n})^2,(\mathbf{v}\cdot\mathbf{n})^2,|\,\overset{\circ}{\mathbf{u}}\,|^2,|\,\overset{\circ}{\mathbf{v}}\,|^2,(\mathrm{tr}\mathbf{W}\mathbf{N})^2\}\,.$$

From the D_{2m+1h}-irreducibility condition for $(\mathbf{u},\mathbf{v},\mathbf{W})$ (see (3.2) in Part I) we know that $(\mathbf{x},\mathbf{W})$ is D_{2m+1h}-irreducible and $\Gamma(\mathbf{x},\mathbf{W})\cap D_{2m+1h}\neq C_1$ for each $\mathbf{x}\in\{\mathbf{u},\mathbf{v}\}$. Hence, case (c1) or case (c2) given in (**viii**) holds for each $\mathbf{x}\in\{\mathbf{u},\mathbf{v}\}$. It is evident that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. From these we deduce that the D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{W})$ is specified by $\mathbf{u}=a\mathbf{n}$, $\mathbf{v}=b\mathbf{e}$, $\mathbf{W}=c\mathbf{E}\mathbf{n}$ with $abc\neq 0$. With the aid of the latter, the presented results can be proved easily.

In the above table, the invariants from the scalar products have been omitted. The reason is as follows. First, the invariants $\mathbf{r}\cdot\boldsymbol{\psi}$, where $\boldsymbol{\psi}$ runs over the presented vector generating set, have been covered before. Second, the skewsymmetric tensor variable should be of the form $\mathbf{H} = c\mathbf{W}$, and hence here we need to consider only the invariant tr$\mathbf{HW}$, which has been covered before. The other form for $\mathbf{H}$ leads to $\Gamma(\mathbf{W}, \boldsymbol{\Omega}) \cap D_{2m+1h} = C_1$, which has been treated in (**iv**). Finally, the symmetric tensor variable $\mathbf{C}$ should be of the form $\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2$, and hence here we need to consider only the invariants tr$\mathbf{CG}$, where $\mathbf{G}$ is the first six symmetric tensor generators given. The six invariants have also been covered before. The other form of $\mathbf{A}$ leads to $\Gamma(\mathbf{W}, \mathbf{C}) \cap D_{2m+1h} = C_1$, which has been treated in (**vi**).

(**xi**) The D_{2m+1h}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{A})$ of two vectors and a symmetric tensor

$$V \quad \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{v}}, \eta_m(\mathbf{q}(\mathbf{A}))\}$$

$$\text{Skw} \quad \{\mathbf{u}\wedge\mathbf{v}, \overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\rho_m(\mathbf{q}(\mathbf{A})), \overset{\circ}{\mathbf{v}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{v}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\wedge\rho_m(\mathbf{q}(\mathbf{A}))\}$$

$$\text{Sym} \quad \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}}, \mathbf{u}\vee\mathbf{v}, \overset{\circ}{\mathbf{A}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\rho_m(\mathbf{q}(\mathbf{A})), (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\vee\rho_m(\mathbf{q}(\mathbf{A}))\}$$

$$R \quad \{(\mathbf{u}\cdot\mathbf{n})^2, (\mathbf{v}\cdot\mathbf{n})^2, |\overset{\circ}{\mathbf{u}}|^2, |\overset{\circ}{\mathbf{v}}|^2, |\mathbf{q}(\mathbf{A})|^2, \mathbf{n}\cdot\mathbf{A}\mathbf{n}, \mathrm{tr}\mathbf{A}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}\}.$$

From the D_{2m+1h}-irreducibility condition for $(\mathbf{u}, \mathbf{v}, \mathbf{A})$ (see (3.2) in Part I) we know that $(\mathbf{x}, \mathbf{A})$ is D_{2m+1h}-irreducible and $\Gamma(\mathbf{x}, \mathbf{A}) \cap D_{2m+1h} \neq C_1$ for each $\mathbf{x} \in \{\mathbf{u}, \mathbf{v}\}$. Hence, case (c1) with $\beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0$ or case (c3) given in (**ix**) holds for each $\mathbf{x} \in \{\mathbf{u}, \mathbf{v}\}$. It is evident that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. From these we deduce that the D_{2m+1h}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{A})$ is specified by $\mathbf{u} = a\mathbf{n}$, $\mathbf{v} = b\boldsymbol{\nu}$, $\overset{\circ}{\mathbf{A}} = c\mathbf{D}_1$ with $\boldsymbol{\nu} \neq \mathbf{e}$ and $abc \neq 0$. With the aid of the latter, the presented results can be proved easily.

In the above table, the invariants from the scalar products have been omitted. First, it is evident that the invariants $\mathbf{r}\cdot\boldsymbol{\psi}$ and tr$\mathbf{CG}$, where $\boldsymbol{\psi}$ and $\mathbf{G}$ run over the presented vector and symmetric tensor generating sets respectively, have been covered before. Next, the skewsymmetric tensor variable $\mathbf{H}$ should be of the form $\mathbf{H} = c\mathbf{E}\mathbf{n}$, and hence here we need to consider only the invariant $\mathbf{v}\cdot\mathbf{H}\overset{\circ}{\mathbf{A}}\mathbf{v}$, which has been given before. The other form of $\mathbf{H}$ leads to $\Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2m+1h} = C_1$, which has been treated in (**vi**).

(**xii**) The D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{W},\mathbf{A})$

$$
\begin{array}{ll}
V & \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n},\eta_m(\mathbf{q}(\mathbf{A})),\mathbf{W}\eta_m(\mathbf{q}(\mathbf{A}))\} \\
\mathrm{Skw} & \{\mathbf{W},(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\rho_m(\mathbf{q}(\mathbf{A})),(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{n}\wedge(\mathbf{n}\times\eta_m(\mathbf{q}(\mathbf{A})))\} \\
\mathrm{Sym} & \{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\overset{\circ}{\mathbf{A}},(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\rho_m(\mathbf{q}(\mathbf{A})),\overset{\circ}{\mathbf{A}}\,\mathbf{W}-\mathbf{W}\,\overset{\circ}{\mathbf{A}}, \\
 & (\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{n}\vee(\mathbf{n}\times\eta_m(\mathbf{q}(\mathbf{A})))\} \\
R & \{(\mathbf{u}\cdot\mathbf{n})^2,(\mathrm{tr}\mathbf{WN})^2,\mathbf{n}\cdot\mathbf{An},\mathrm{tr}\mathbf{A},|\mathbf{q}(\mathbf{A})|^2\}\,.
\end{array}
$$

From the D_{2m+1h}-irreducibility condition for $(\mathbf{u},\mathbf{W},\mathbf{A})$ we know that $(\mathbf{W},\mathbf{A})$ is D_{2m+1h}-irreducible and $\Gamma(\mathbf{W},\mathbf{A})\cap D_{2m+1h}\neq C_1$. Hence, case (c1) given in (**vi**), i.e. $\mathbf{W}=c\mathbf{En}$ and $\overset{\circ}{\mathbf{A}}=a\mathbf{D}_1$ with $ac\neq 0$, holds. Further, we derive $\mathbf{u}=b\mathbf{n}$ with $b\neq 0$. With the aid of these, the four presented results can be proved easily.

By virtue of the same argument used in (**x**), we have omitted the invariants from the scalar products.

(**xiii**) The D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{A},\mathbf{B})$ of a vector and two symmetric tensors

$$
\begin{array}{ll}
V & V''_{2m+1}(\mathbf{u},\mathbf{A})\cup V''_{2m+1}(\mathbf{u},\mathbf{B}) \\
\mathrm{Skw} & \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\rho_m(\mathbf{q}(\mathbf{A})),(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\rho_m(\mathbf{q}(\mathbf{B})),\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}-\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\} \\
\mathrm{Sym} & \mathrm{Sym}''_{2m+1}(\mathbf{u},\mathbf{A})\cup\mathrm{Sym}''_{2m+1}(\mathbf{u},\mathbf{B}) \\
R & \{(\mathbf{u}\cdot\mathbf{n})^2,\mathrm{tr}\mathbf{A},\mathbf{n}\cdot\mathbf{An},\mathrm{tr}\mathbf{B},\mathbf{n}\cdot\mathbf{Bn},\mathrm{tr}\mathbf{A}_\mathbf{e}\mathbf{B}_\mathbf{e}\}.
\end{array}
$$

From the D_{2m+1h}-irreducibility condition for $(\mathbf{u},\mathbf{A},\mathbf{B})$ we know that $(\mathbf{A},\mathbf{B})$ is D_{2m+1h}-irreducible and $\Gamma(\mathbf{A},\mathbf{B})\cap D_{2m+1h}\neq C_1$. Hence, case (c1) given in (**vii**) holds. Further, we derive $\mathbf{u}=b\mathbf{n}$ with $b\neq 0$. Thus, the D_{2m+1h}-irreducible set $(\mathbf{u},\mathbf{A},\mathbf{B})$ is of the form

$$
\begin{cases}
\mathbf{u}=a\mathbf{n},\ \overset{\circ}{\mathbf{A}}=b(\mathbf{e}\otimes\mathbf{e}-\mathbf{e}'\otimes\mathbf{e}'),\ \overset{\circ}{\mathbf{B}}=c(\boldsymbol{\nu}\otimes\boldsymbol{\nu}-\boldsymbol{\nu}'\otimes\boldsymbol{\nu}'),\ abc\neq 0\,, \\
\boldsymbol{\nu}'=\mathbf{n}\times\boldsymbol{\nu},\ \mathbf{e}\neq\boldsymbol{\nu}\in\{\tau_{2r}\mid r=1,\cdots,2m\}\,.
\end{cases}
$$

With the aid of the latter and the formula (2.4) in Part I, the presented results can be verified easily. All the invariants and generators given here have been covered before.

(**xiv**) The set $(\mathbf{u},\mathbf{W},\boldsymbol{\Omega})$ of a vector and two skewsymmetric tensors

Any given set $(\mathbf{u},\mathbf{W},\boldsymbol{\Omega})$ can not be a g-irreducible set for each subgroup $g\subset$Orth with $-\mathbf{I}\notin g$ and hence can be omitted. In fact, this fact is obviously true if $\mathbf{W}$ and $\boldsymbol{\Omega}$ are linearly dependent. If $\mathbf{W}$ and $\boldsymbol{\Omega}$ are linearly independent, then we have $\Gamma(\mathbf{W},\mathbf{\Omega}) \cap g = C_1$ for each just-mentioned subgroup g. The foregoing fact is also true.

2.4. The general result

Applying Theorem 2.1 in XIAO [20] and incorporating the fact indicated at the outset of this section, from (a) – (c) we obtain the following general result.

THEOREM 7. *The four sets given by*

$$I''_{2m+1}(\mathbf{u});\ I_{4m+2}(\mathbf{W});\ I_{4m+2}(\mathbf{A});\ I_{4m+2}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B},\mathbf{C});$$

$$(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{v}}, \overset{o}{\mathbf{u}}\cdot\eta_{2m}(\overset{o}{\mathbf{v}}), \overset{o}{\mathbf{v}}\cdot\eta_{2m}(\overset{o}{\mathbf{u}});$$

$$(\mathbf{u}\cdot\mathbf{n})\ \overset{o}{\mathbf{u}}\cdot\mathbf{Wn}, \overset{o}{\mathbf{u}}\cdot\mathbf{W}^2\ \overset{o}{\mathbf{u}}, \beta_{2m+1}(\overset{o}{\mathbf{u}})(\mathrm{tr}\mathbf{WN}), (\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{o}{\mathbf{u}})\cdot\mathbf{Wn},$$

$$(\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\mathbf{Wn}), \overset{o}{\mathbf{u}}\cdot\eta_{2m}(\mathbf{Wn}), \eta_{2m}(\mathbf{Wn})\cdot\mathbf{Wu}, \eta_{2m}(\mathbf{Wn})\cdot\mathbf{W}^2\mathbf{u};$$

$$\overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{A}}\overset{o}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\ \overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{A}}\ \mathbf{n}, \overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{A}}{}^2\ \overset{o}{\mathbf{u}}, \eta_{2m}(\overset{o}{\mathbf{u}})\cdot\overset{o}{\mathbf{A}}\overset{o}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{o}{\mathbf{u}})\cdot\overset{o}{\mathbf{A}}\ \mathbf{n},$$

$$\overset{o}{\mathbf{u}}\cdot\eta_m(\mathbf{q}(\mathbf{A})), \eta_m(\mathbf{q}(\mathbf{A}))\cdot\mathbf{A}_\mathbf{e}\ \overset{o}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\overset{o}{\mathbf{A}}\ \mathbf{n}), \overset{o}{\mathbf{u}}\cdot\eta_{2m}(\overset{o}{\mathbf{A}}\ \mathbf{n}),$$

$$|\ \overset{o}{\mathbf{A}}\ \mathbf{n}|^2\alpha_{2m+1}(\overset{o}{\mathbf{A}}\ \mathbf{n})(\overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{A}}\ \mathbf{n}) + J(\mathbf{A})\beta_{2m+1}(\overset{o}{\mathbf{A}}\ \mathbf{n})(\mathbf{u}\cdot\mathbf{n});$$

$$\mathbf{u}\cdot\mathbf{Wv}, |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{o}{\mathbf{v}})\cdot\mathbf{Wn} + |\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\eta_{2m}(\overset{o}{\mathbf{u}})\cdot\mathbf{Wn};$$

$$\mathbf{u}\cdot\overset{o}{\mathbf{A}}\ \mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{o}{\mathbf{v}})\cdot\overset{o}{\mathbf{A}}\ \mathbf{n} + |\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\eta_{2m}(\overset{o}{\mathbf{u}})\cdot\overset{o}{\mathbf{A}}\ \mathbf{n};$$

$$\eta_{2m}(\mathbf{Wn})\cdot\mathbf{\Omega u}, \eta_{2m}(\mathbf{\Omega n})\cdot\mathbf{Wu},$$

$$|\mathbf{W}|^{2m}\beta_{2m+1}(\mathbf{\Omega n})\mathbf{u}\cdot(\mathbf{E}:\mathbf{W}) + |\mathbf{\Omega}|^{2m}\beta_{2m+1}(\mathbf{Wn})\mathbf{u}\cdot(\mathbf{E}:\mathbf{\Omega});$$

$$(\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m\eta_{2m}(\overset{o}{\mathbf{A}}\ \mathbf{n}))\cdot\mathbf{Wu},$$

$$(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\beta_{2m+1}(\overset{o}{\mathbf{A}}\ \mathbf{n}) - \rho_m(\mathbf{q}(\mathbf{A}))\cdot\mathbf{Wu}, \overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{A}}\ \mathbf{W}\ \overset{o}{\mathbf{u}},$$

$$|\ \overset{o}{\mathbf{A}}\ \mathbf{n}|^{2m}\alpha_{2m+1}(\mathbf{Wn})(\overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{A}}\ \mathbf{n}) + |\mathbf{Wn}|^{2m}\alpha_{2m+1}(\overset{o}{\mathbf{A}}\ \mathbf{n})(\overset{o}{\mathbf{u}}\cdot\mathbf{Wn});$$

$$(\mathbf{u}\cdot\mathbf{n})(\pi_m(\mathbf{q}(\mathbf{B})) + (-1)^m\eta_{2m}(\overset{o}{\mathbf{B}}\ \mathbf{n}))\cdot\overset{o}{\mathbf{A}}\ \mathbf{n}),$$

$$(\mathbf{u}\cdot\mathbf{n})((\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m\eta_{2m}(\overset{o}{\mathbf{A}}\ \mathbf{n}))\cdot\overset{o}{\mathbf{B}}\ \mathbf{n}),$$

$$|\ \overset{o}{\mathbf{A}}\ \mathbf{n}|^{2m}\alpha_{2m+1}(\overset{o}{\mathbf{B}}\ \mathbf{n})\ \overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{A}}\ \mathbf{n} + |\ \overset{o}{\mathbf{B}}\ \mathbf{n}|^{2m}\alpha_{2m+1}(\overset{o}{\mathbf{A}}\ \mathbf{n})\ \overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{B}}\ \mathbf{n},$$

$$(\mathbf{u}\cdot\mathbf{n})(\beta_{2m+1}(\overset{o}{\mathbf{A}}\ \mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{B})) + \beta_{2m+1}(\overset{o}{\mathbf{B}}\ \mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{A})));$$

and

$$V''_{2m+1}(\mathbf{u}); V''_{2m+1}(\mathbf{W}); V''_{2m+1}(\mathbf{A}); \mathbf{Wu}; \overset{o}{\mathbf{A}}\ \mathbf{u};$$

$$\mathbf{W}\eta_{2m}(\mathbf{\Omega n}), \mathbf{\Omega}\eta_{2m}(\mathbf{Wn}),$$

$$|\mathbf{W}|^{2m}\beta_{2m+1}(\mathbf{\Omega n})(\mathbf{E}:\mathbf{W}) + |\mathbf{\Omega}|^{2m}\beta_{2m+1}(\mathbf{Wn})(\mathbf{E}:\mathbf{\Omega});$$

$$\mathbf{W}(\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})),$$

$$(\mathrm{tr}\mathbf{WN})\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n} + \mathbf{W}\rho_m(\mathbf{q}(\mathbf{A})),$$

$$|\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\mathbf{Wn})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n} + |\mathbf{Wn}|^{2m}\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{Wn};$$

$$((\pi_m(\mathbf{q}(\mathbf{B})) + (-1)^m\eta_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}))\cdot\,\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n},$$

$$((\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\cdot\,\overset{\circ}{\mathbf{B}}\,\mathbf{n})\mathbf{n},$$

$$|\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n} + |\,\overset{\circ}{\mathbf{B}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{B}}\,\mathbf{n},$$

$$(\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{B})) + \beta_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{A})))\mathbf{n};$$

and

$$\mathrm{Skw}''_{2m+1}(\mathbf{u}); \mathrm{Skw}_{4m+2}(\mathbf{W}); \mathrm{Skw}_{4m+2}(\mathbf{A});$$

$$\mathbf{u}\wedge\mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\eta_{2m}(\overset{\circ}{\mathbf{v}}) + |\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\mathbf{n}\wedge\eta_{2m}(\overset{\circ}{\mathbf{u}});$$

$$\mathbf{W\Omega} - \mathbf{\Omega W}; \overset{\circ}{\mathbf{A}}\,\mathbf{W} + \mathbf{W}\,\overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\,\mathbf{W}^2 - \mathbf{W}^2\,\overset{\circ}{\mathbf{A}}; \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}} - \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}},$$

$$\overset{\circ}{\mathbf{B}}\,\mathbf{n}\wedge\,\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\,\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\wedge\,\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\,\mathbf{n};$$

$$\mathbf{u}\wedge\mathbf{Wu}, \mathbf{u}\wedge\eta_{2m}(\mathbf{Wn}), |\mathbf{u}|^{2m}\mathbf{Wu}\wedge\eta_{2m}(\mathbf{Wn}) + |\mathbf{W}|^{2m}\,\overset{\circ}{\mathbf{u}}\wedge\mathbf{W}\eta_{2m}(\overset{\circ}{\mathbf{u}});$$

$$\overset{\circ}{\mathbf{u}}\wedge\,\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\rho_m(\mathbf{q}(\mathbf{A})), \mathbf{u}\wedge\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + (-1)^m(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\pi_m(\mathbf{q}(\mathbf{A})),$$

$$\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\wedge\,\overset{\circ}{\mathbf{u}} + \overset{\circ}{\mathbf{A}}\,\mathbf{u}\wedge\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n});$$

$$(\mathrm{tr}\mathbf{WN})\mathbf{n}\wedge(\mathbf{u}\times\mathbf{v}); (\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{n}\wedge(\mathbf{n}\times\eta_m(\mathbf{q}(\mathbf{A})));$$

and

$$\mathrm{Sym}''_{2m+1}(\mathbf{u}); \mathrm{Sym}_{4m+2}(\mathbf{W}); \mathrm{Sym}_{4m+2}(\mathbf{A});$$

$$\mathbf{u}\vee\mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\eta_{2m}(\overset{\circ}{\mathbf{v}}) + |\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\mathbf{n}\vee\eta_{2m}(\overset{\circ}{\mathbf{u}});$$

$$\mathbf{W\Omega} + \mathbf{\Omega W}, |\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})\mathbf{Wn}\vee\mathbf{NWn}$$

$$+|\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\mathbf{\Omega n}\vee\mathbf{N\Omega n};$$

$$\overset{\circ}{\mathbf{A}}\,\mathbf{W} - \mathbf{W}\,\overset{\circ}{\mathbf{A}}, (\mathrm{tr}\mathbf{WN})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}\vee\mathbf{N}\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}; \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}} + \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}};$$

$$\mathbf{u}\vee\mathbf{Wu}, \mathbf{u}\vee\eta_{2m}(\mathbf{Wn}),$$

$$|\mathbf{u}|^{2m}\mathbf{Wu}\vee\eta_{2m}(\mathbf{Wn}) + |\mathbf{W}|^{2m}\,\overset{\circ}{\mathbf{u}}\vee\mathbf{W}\eta_{2m}(\overset{\circ}{\mathbf{u}});$$

$$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\rho_m(\mathbf{q}(\mathbf{A})),\mathbf{u}\vee\eta_{2m}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})+(-1)^m(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\pi_m(\mathbf{q}(\mathbf{A})),$$
$$\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}+\overset{\circ}{\mathbf{A}}\ \mathbf{u}\vee\eta_{2m}(\overset{\circ}{\mathbf{A}}\ \mathbf{n});$$
$$(\mathrm{tr}\mathbf{WN})\mathbf{n}\vee(\mathbf{u}\times\mathbf{v});(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{n}\vee(\mathbf{n}\times\eta_m(\mathbf{q}(\mathbf{A})));$$

where $(\mathbf{u},\mathbf{v})=(\mathbf{u}_i,\mathbf{u}_j)$, $(\mathbf{W},\mathbf{\Omega},\mathbf{H})=(\mathbf{W}_\mu,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B},\mathbf{C})=(\mathbf{A}_L,\ \mathbf{A}_M,\ \mathbf{A}_N)$, $j>i=1,\cdots,a$, $\theta>\tau>\mu=1,\cdots,b$, $N>M>L=1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1,\cdots,\mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1,\cdots,\mathbf{A}_c$ *under the group* D_{2m+1h} *for each* $m\geq 1$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold rotation axis of the group* D_{2m+1h}.

In the above result, $I_{4m+2}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B},\mathbf{C})$ is used to represent the invariants depending on two or three symmetric and/or skewsymmetric tensors given in Theorem 1 in Part 1, with the replacement of m by $2m+1$ therein.

3. Crystal and quasicrystal classes D_{2md}

The classes at issue take the forms

$$D_{2md}=\{(-1)^k\mathbf{R}_{\mathbf{n}}^{\theta_k},\ (-1)^k\mathbf{R}_{\sigma_k}^{\pi}\mid\theta_k=\frac{2k\pi}{4m},\quad\sigma_k=\mathbf{R}_{\mathbf{n}}^{\theta_k/2}\mathbf{e},\quad k=1,\cdots,4m\}.\tag{3.1}$$

They include the crystal class D_{2d} as the particular case when $m=1$. Each σ_{2r} and each σ_{2r-1} are a two-fold rotation axis vector and a reflection axis vector of the group D_{2md}, respectively. In particular, both $\mathbf{e}$ $(=\sigma_{4m})$ and $\mathbf{e}'$ $(=\sigma_{2m})$ are two mutually orthogonal two-fold axis vectors of D_{2md}. Throughout this section, σ, μ and ν will be used to represent one of the vectors σ_t, one of the reflection axis vectors σ_{2r-1} and one of the two-fold rotation axis vectors σ_{2r}, respectively. A useful fact for the group D_{2md} is: if σ is a two-fold rotation (resp. reflection) axis vector, then $\tau'=\mathbf{n}\times\tau$ is also a two-fold rotation (resp. reflection) axis vector.

For the five sets of variables, $(\mathbf{W})$, $(\mathbf{A})$, $(\mathbf{W},\mathbf{\Omega})$, $(\mathbf{W},\mathbf{A})$ and $(\mathbf{A},\mathbf{B})$, it follows from the same argument indicated at the start of Sec. 2 that results for functional bases and skewsymmetric and symmetric tensor generating sets relative to the group D_{2md}, as well as related invariants from the scalar products, are obtainable from the corresponding results given in Sec. 4 in Part I by the replacement of m with $2m$, and hence we shall omit them in the process of derivation to come. As a result, in what follows, for the foregoing five sets of variables, we only need to derive vector generating sets and their related invariants from the scalar products. Moreover, according to Sec. 4 (**xiii**) in Part I and Sec. 2 (**xiv**) in this part, there is no need to take the set $(\mathbf{u}, \mathbf{v}, \mathbf{r})$ of three vector variables and the set $(\mathbf{u}, \mathbf{W}, \mathbf{\Omega})$ of a vector variable and two skewsymmetric tensor variables into account.

3.1. Single variables

(i) A single vector $\mathbf{u}$

V $\{\mathbf{u}, \beta_{2m}(\mathring{\mathbf{u}})\mathbf{n} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}}),$

$(\mathbf{u}\cdot\mathbf{n})^{4m-1}\mathbf{n} + \beta_{2m}(\mathring{\mathbf{u}})\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})\}\ (\equiv V''_{2m}(\mathbf{u}))$

Skw $\{\mathbf{E}\eta_{2m-1}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathring{\mathbf{u}})\mathbf{N} - \beta_{2m}(\mathring{\mathbf{u}})\mathbf{n}\wedge\mathring{\mathbf{u}},$

$\beta_{4m}(\mathring{\mathbf{u}})\mathbf{N} + (\mathbf{u}\cdot\mathbf{n})^{4m-1}\mathbf{n}\wedge\mathring{\mathbf{u}}\}\ (\equiv \mathrm{Skw}''_{2m}(\mathbf{u}))$

Sym $\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}}, \mathbf{n}\vee(\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})) + \delta_{1m}(\mathbf{u}\cdot\mathbf{n})\mathbf{D}_2,$

$(\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\vee(\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})) - \beta_{2m}(\mathring{\mathbf{u}})\mathbf{n}\vee\mathring{\mathbf{u}},$

$\beta_{2m}(\mathring{\mathbf{u}})\,\mathring{\mathbf{u}}\vee(\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})) + (\mathbf{u}\cdot\mathbf{n})^{4m-1}\mathbf{n}\vee\mathring{\mathbf{u}}\}\ (\equiv \mathrm{Sym}''_{2m}(\mathbf{u}))$

R $\mathbf{r}\cdot\mathbf{u}, (\mathbf{r}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{u}}) + (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathring{\mathbf{r}}, \eta_{2m-1}(\mathring{\mathbf{u}})],$

$(\mathbf{r}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n})^{4m-1} - \beta_{2m}(\mathring{\mathbf{u}})[\mathbf{n}, \mathring{\mathbf{r}}, \eta_{2m-1}(\mathring{\mathbf{u}})]; \mathrm{tr}\mathbf{H}(\mathbf{E}\eta_{2m-1}(\mathring{\mathbf{u}})),$

$(\mathrm{tr}\mathbf{HN})(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathring{\mathbf{u}}) + \beta_{2m}(\mathring{\mathbf{u}})\,\mathring{\mathbf{u}}\cdot\mathbf{Hn},$

$(\mathrm{tr}\mathbf{HN})\beta_{4m}(\mathring{\mathbf{u}}) - (\mathbf{u}\cdot\mathbf{n})^{4m-1}\,\mathring{\mathbf{u}}\cdot\mathbf{Hn};$

$\mathrm{tr}\mathbf{C}, \mathbf{n}\cdot\mathbf{Cn}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{C}}\mathring{\mathbf{u}}, [\mathbf{n}, \mathring{\mathbf{C}}\,\mathbf{n}, \eta_{2m-1}(\mathring{\mathbf{u}})] - \delta_{1m}(\mathbf{u}\cdot\mathbf{n})\mathrm{tr}\mathbf{CD}_2,$

$(\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathring{\mathbf{C}}\mathring{\mathbf{u}}, \eta_{2m-1}(\mathring{\mathbf{u}})] + \beta_{2m}(\mathring{\mathbf{u}})\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{C}}\,\mathbf{n},$

$\beta_{2m}(\mathring{\mathbf{u}})[\mathbf{n}, \mathring{\mathbf{C}}\mathring{\mathbf{u}}, \eta_{2m-1}(\mathring{\mathbf{u}})] - (\mathbf{u}\cdot\mathbf{n})^{4m-1}\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{C}}\,\mathbf{n};$

$\{(\mathbf{u}\cdot\mathbf{n})^2, |\,\mathring{\mathbf{u}}\,|^2, \alpha_{4m}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{u}})\}\ (\equiv I''_{2m}(\mathbf{u}))$

First, we prove that the presented set $I''_{2m}(\mathbf{u})$ supplies a desired functional basis. In fact, the latter is obtainable from an isotropic functional basis for the extended variables $(\mathbf{u}, \mathbf{G}, \mathbf{n}\otimes\mathbf{n})$ with $\mathbf{G} = \mathbf{n}\vee(\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})) + \delta_{1m}(\mathbf{u}\cdot\mathbf{n})\mathbf{D}_2$ (see

Theorem 4 and Theorem 6 in XIAO [17] ([1]). Here and henceforth, the Kronecker delta δ_{1m} takes values 1 and 0 when $m = 1$ and $m \geq 2$ respectively. Applying the related result for isotropic functions we know that the just-mentioned isotropic functional basis is formed by the invariants

$$\mathbf{u}\cdot\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})^2, \mathrm{tr}\mathbf{G}, \mathrm{tr}\mathbf{G}^2, \mathrm{tr}\mathbf{G}^3, \mathbf{n}\cdot\mathbf{G}^i\mathbf{u}, \mathbf{u}\cdot\mathbf{G}^i\mathbf{u}, \mathbf{u}\cdot\mathbf{G}(\mathbf{n}\otimes\mathbf{n})\mathbf{u},$$

where $i = 1, 2$. From the above invariants we derive the set $I''_{2m}(\mathbf{u})$.

Next, we prove that the three presented sets $V''_{2m}(\mathbf{u})$, $\mathrm{Skw}''_{2m}(\mathbf{u})$ and $\mathrm{Sym}''_{2m}(\mathbf{u})$ supply the desired vector, skewsymmetric tensor and symmetric tensor generating sets, respectively. Towards this goal we show that each of them obeys the criterion (2.3) in Part I. In fact, the case when $\mathbf{u} = \mathbf{0}$ is trivial. In what follows, suppose $\mathbf{u} \neq \mathbf{0}$. The respective last two generators in the foregoing three sets produce $\mathbf{n}$ and $\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})$, $\beta_{2m}(\mathring{\mathbf{u}})\mathbf{N}$ and $\mathbf{n}\wedge\mathring{\mathbf{u}}$, $\mathring{\mathbf{u}}\vee(\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}}))$ and $\mathbf{n}\vee\mathring{\mathbf{u}}$, when $\Delta = \begin{vmatrix} \beta_{2m}(\mathring{\mathbf{u}}) & -\mathbf{u}\cdot\mathbf{n} \\ (\mathbf{u}\cdot\mathbf{n})^{4m-1} & \beta_{2m}(\mathring{\mathbf{u}}) \end{vmatrix} = (\mathbf{u}\cdot\mathbf{n})^{4m} + (\beta_{2m}(\mathring{\mathbf{u}}))^2 \neq 0$. Hence, we have

$$\mathrm{rank}V''_{2m}(\mathbf{u}) \geq \begin{cases} \mathrm{rank}\{\mathbf{n}, \mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}}), \mathbf{u}\} = 3 \text{ if } \alpha_{2m}(\mathring{\mathbf{u}})\Delta \neq 0, \\ \mathrm{rank}\{\mathbf{n}, \mathring{\mathbf{u}}\} = 2 \text{ if } \alpha_{2m}(\mathring{\mathbf{u}}) = 0,\ \mathring{\mathbf{u}} \neq \mathbf{0}, \\ \mathrm{rank}\{\mathbf{u}\} = 1 \text{ if } \Delta = 0 \text{ or } \mathring{\mathbf{u}} = \mathbf{0}, \end{cases}$$

$$\mathrm{rank}\ \mathrm{Skw}''_{2m}(\mathbf{u}) \geq \begin{cases} \mathrm{rank}\{\mathbf{N}, \mathbf{n}\wedge\mathring{\mathbf{u}}, \mathbf{E}\eta_{2m-1}(\mathring{\mathbf{u}})\} = 3 \text{ if } \alpha_{2m}(\mathring{\mathbf{u}})\Delta \neq 0, \\ \mathrm{rank}\{\mathbf{n}\wedge\mathring{\mathbf{u}}\} = 1 \text{ if } \alpha_{2m}(\mathring{\mathbf{u}})\Delta = 0,\ \mathring{\mathbf{u}} \neq 0, \\ \qquad 0 \text{ if } \mathring{\mathbf{u}} = \mathbf{0}, \end{cases}$$

$$\mathrm{rank}\ \mathrm{Sym}''_{2m}(\mathbf{u}) \geq \begin{cases} \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}}, \mathbf{n}\vee\eta', \mathbf{n}\vee\mathring{\mathbf{u}}, \mathring{\mathbf{u}}\vee\eta'\} = 6 \\ \qquad \text{if } \alpha_{2m}(\mathring{\mathbf{u}})\Delta \neq 0, \\ \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}}, \mathbf{n}\vee\mathring{\mathbf{u}}\} = 4 \text{ if } \alpha_{2m}(\mathring{\mathbf{u}})\Delta = 0, \\ \qquad \mathring{\mathbf{u}} \neq \mathbf{0}, \\ \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \delta_{1m}\mathbf{D}_2\} = 2 + \delta_{1m} \text{ if } \mathring{\mathbf{u}} = \mathbf{0}, \end{cases}$$

[1]) In (3.21) and (3.30) therein, some corrigenda should be made. Here and later they are: the vector-valued functions $\eta_{2m-1}(\mathbf{z})$, $\mathbf{z} = \mathring{\mathbf{u}}$, $\mathbf{Wn}$, $\mathring{\mathbf{A}}\,\mathbf{n}$, appearing in (3.21) and (3.30) (for $m = 1$) and $\mathbf{D}_1$ in (3.31) should be changed to $\mathbf{N}\eta_{2m-1}(\mathbf{z})$ and $\mathbf{D}_2$, respectively.

where $\eta' = \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{u}})$. From these results and

$$\Gamma(\mathbf{u}) \cap D_{2md} = \begin{cases} C_{1h}(\mu) \text{ if } \alpha_{2m}(\overset{\circ}{\mathbf{u}}) = 0, \ \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ C_2(\nu) \text{ if } \Delta = 0, \ \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ C_{2mv}(\mathbf{n}) \text{ if } \overset{\circ}{\mathbf{u}} = \mathbf{0}, \ \text{ i.e. } \mathbf{u} = c\mathbf{n}, \end{cases}$$

as well as from Tables 1 – 3 in Sec. 2 in Part I, we deduce that the three sets at issue obey the criterion (2.3) in Part I, respectively. The three presented generating sets are minimal.

(ii) A single skewsymmetric tensor $\mathbf{W}$

V $\{\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn}), \mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn})), \mathbf{W}^2(\mathbf{n} \times \eta_{2m-1}(\mathbf{n} \times \mathbf{Wn})),$
$\beta_{2m}(\mathbf{Wn})\mathbf{n}\} \ (\equiv V''_{2m}(\mathbf{W}))$

R $[\mathbf{n}, \overset{\circ}{\mathbf{r}}, \eta_{2m-1}(\mathbf{Wn})], [\mathbf{n}, \mathbf{Wr}, \eta_{2m-1}(\mathbf{Wn})], [\mathbf{n}, \mathbf{W}^2\mathbf{r}, \eta_{2m-1}(\mathbf{Wn})],$
$(\mathbf{r} \cdot \mathbf{n})\beta_{2m}(\mathbf{Wn}).$

We show that the presented set $V''_{2m}(\mathbf{A})$ supplies a desired vector generating set. In fact, an anisotropic vector generating set for $\mathbf{W}$ under the group D_{2md} is obtainable from an isotropic vector generating set for the extended variables $(\mathbf{W}, \mathbf{n} \times \eta_{2m-1}(\mathbf{Wn}), \mathbf{n} \otimes \mathbf{n})$ (see Theorem 4 and Theorem 6 in XIAO [17] and the preceding footnote). From the related result for isotropic functions we know that the latter is just given by the presented set $V''_{2m}(\mathbf{W})$. Further, by considering the two tensors $\mathbf{W}_1 = \mathbf{E}(\mathbf{n} + \mathbf{e})$ and $\mathbf{W}_2 = \mathbf{En} + \mathbf{n} \wedge \sigma_1$, we infer that each of the four generators in the set $V''_{2m}(\mathbf{W})$ is irreducible.

(iii) A single symmetric tensor $\mathbf{A}$

V $\{\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \overset{\circ}{\mathbf{A}}\,(\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})), \beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}, \beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n},$
$J(\mathbf{A})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\} \ (\equiv V''_{2m}(\mathbf{A}))$

R $[\mathbf{n}, \overset{\circ}{\mathbf{r}}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})], [\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{r}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})], \beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{r}} \cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n},$
$(\mathbf{r} \cdot \mathbf{n})\beta_m(\mathbf{q}(\mathbf{A})), (\mathbf{r} \cdot \mathbf{n})J(\mathbf{A})\alpha_m(\mathbf{q}(\mathbf{A})).$

We show that the presented set $V''_{2m}(\mathbf{A})$ supplies a desired vector generating set. To this end we show that this set obeys the criterion (2.3) in Part I. The case when $\overset{\circ}{\mathbf{A}} = \mathbf{O}$ is trivial. In what follows, suppose $\overset{\circ}{\mathbf{A}} \neq \mathbf{O}$. First, utilizing the equality

$$\overset{\circ}{\mathbf{A}}\,(\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})) = -\beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n} + \mathbf{A}_{\mathbf{e}}(\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})), \tag{3.2}$$

we have

$$\mathrm{rank}V''_{2m}(\mathbf{A}) \geq \begin{cases} \mathrm{rank}\{\overset{\circ}{\mathbf{A}}\,\mathbf{n}, \boldsymbol{\eta}(\mathbf{A})', \beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\} = 3 \text{ if } \beta_{4m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) \neq 0\,, \\ \mathrm{rank}\{\mathbf{n}, \boldsymbol{\eta}(\mathbf{A})', \mathbf{A}_{\mathbf{e}}\boldsymbol{\eta}(\mathbf{A})'\} = 3 \text{ if } \beta_{4m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0, \\ \qquad J(\mathbf{A}) \neq 0, \end{cases}$$

where $\boldsymbol{\eta}(\mathbf{A})' = \mathbf{n} \times \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$. Second, we have

$$\mathrm{rank}V''_{2m}(\mathbf{A}) \geq \begin{cases} \mathrm{rank}\{\mathbf{n} \times \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{n}\} = 2 \text{ if } J(\mathbf{A}) = \alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0\,, \\ \mathrm{rank}\{\mathbf{n} \times \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} = 1 \text{ if } J(\mathbf{A}) = \beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0\,, \end{cases}$$

with $\overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0}$. Third, we have

$$\mathrm{rank}V''_{2m}(\mathbf{A}) \geq \begin{cases} \mathrm{rank}\{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\} = 1 \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} = \mathbf{0},\ \beta_m(\mathbf{q}(\mathbf{A})) \neq 0\,, \\ 0 \text{ if } |\overset{\circ}{\mathbf{A}}\,\mathbf{n}| = \beta_m(\mathbf{q}(\mathbf{A})) = 0\,. \end{cases}$$

The cases for $\overset{\circ}{\mathbf{A}} \neq \mathbf{O}$ involved in the above results cover all possible cases. Thus, from the above results and from

$$\varGamma(\mathbf{A}) \cap D_{2md} = \begin{cases} C_{1h}(\boldsymbol{\mu}) \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0},\ J(\mathbf{A}) = \alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0\,, \\ C_2(\boldsymbol{\nu}) \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0},\ J(\mathbf{A}) = \beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0\,, \\ C_2(\mathbf{n}) \text{ or } C_{2v}(\mathbf{n}, \boldsymbol{\mu}, \mathbf{n} \times \boldsymbol{\mu}) \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} = \mathbf{0},\ \beta_m(\mathbf{q}(\mathbf{A})) \neq 0\,, \\ D_2(\mathbf{n}, \boldsymbol{\nu}, \mathbf{n} \times \boldsymbol{\nu}) \text{ if } |\overset{\circ}{\mathbf{A}}\,\mathbf{n}| = \beta_m(\mathbf{q}(\mathbf{A})) = 0\,, \end{cases}$$

for $\overset{\circ}{\mathbf{A}} \neq \mathbf{O}$, as well as Table 1 in Sec. 2 in Part I, we infer that the presented set obeys the criterion (2.3) in Part I. Further, by considering the three tensors $\mathbf{A}_1 = \mathbf{n} \vee (\mathbf{e} + \boldsymbol{\sigma}_1)$ and $\mathbf{A}_2 = \boldsymbol{\sigma}_1 \otimes \boldsymbol{\sigma}_1$ and $\mathbf{A}_3 = \mathbf{D}_1 + \mathbf{n} \vee \boldsymbol{\sigma}_1$ we deduce that the five generators given are irreducible respectively.

3.2. D_{2md}-irreducible sets of two variables

(**iv**) The D_{2md}-irreducible set $(\mathbf{u}, \mathbf{v})$ of two vectors

$$V \quad \{\mathbf{u}, \mathbf{v}, \alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{v} \times \overset{\circ}{\mathbf{u}}, \alpha_{2m}(\overset{\circ}{\mathbf{v}})\mathbf{u} \times \overset{\circ}{\mathbf{v}}\} \ (\equiv V''_{2m}(\mathbf{u}, \mathbf{v}))$$

Skw $\mathrm{Skw}''_{2m}(\mathbf{u}) \cup \mathrm{Skw}''_{2m}(\mathbf{v}) \cup \{\mathbf{u} \wedge \mathbf{v},$

$((\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{v}}) + (\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{u}}))\mathbf{N}\}$

$(\equiv \mathrm{Skw}''_{2m}(\mathbf{u},\mathbf{v}))$

Sym $\mathrm{Sym}''_{2m}(\mathbf{u}) \cup \mathrm{Sym}''_{2m}(\mathbf{v}) \cup \{\mathbf{u} \vee \mathbf{v},$

$(\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\overset{\circ}{\mathbf{v}})\ \overset{\circ}{\mathbf{v}} \vee(\mathbf{n}\times\ \overset{\circ}{\mathbf{v}})$

$+(\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\overset{\circ}{\mathbf{u}})\ \overset{\circ}{\mathbf{u}} \vee(\mathbf{n}\times\ \overset{\circ}{\mathbf{u}})\}$ $(\equiv \mathrm{Sym}''_{2m}(\mathbf{u},\mathbf{v}))$

R $\mathbf{r}\cdot\mathbf{u}, \mathbf{r}\cdot\mathbf{v}, \alpha_{2m}(\overset{\circ}{\mathbf{v}})[\mathbf{r},\mathbf{u},\overset{\circ}{\mathbf{v}}], \alpha_{2m}(\overset{\circ}{\mathbf{u}})[\mathbf{r},\mathbf{v},\overset{\circ}{\mathbf{u}}];$

$\mathbf{H} : \mathrm{Skw}''_{2m}(\mathbf{u}), \mathbf{H} : \mathrm{Skw}''_{2m}(\mathbf{v}), \mathbf{u}\cdot\mathbf{H}\mathbf{v}; \mathbf{C} : \mathrm{Sym}''_{2m}(\mathbf{u}),$

$\mathbf{C} : \mathrm{Sym}''_{2m}(\mathbf{v}), \mathbf{u}\cdot \overset{\circ}{\mathbf{C}}\ \mathbf{v}; (\mathrm{tr}\mathbf{HN})((\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{v}})$

$+(\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{u}})); (\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\overset{\circ}{\mathbf{v}})[\mathbf{n},\overset{\circ}{\mathbf{v}},\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{v}}]$

$+(\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\overset{\circ}{\mathbf{u}})[\mathbf{n},\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}];$

$I''_{2m}(\mathbf{u}) \cup I''_{2m}(\mathbf{v}) \cup \{(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \mathbf{u}\cdot\mathbf{v}\}$ $(\equiv I''_{2m}(\mathbf{u},\mathbf{v}))$

To prove the above results, we first work out the D_{2md}-irreducible set $(\mathbf{u},\mathbf{v})$, which is specified by (see (3.1) in Part I): $\Gamma(\mathbf{u},\mathbf{v}) \cap D_{2md} \neq \Gamma(\mathbf{z}) \cap D_{2md}$, $\mathbf{z} = \mathbf{u}, \mathbf{v}$. Evidently, we have $\Gamma(\mathbf{z}) \cap D_{2md} \neq C_1$. The latter implies that $\mathbf{R}^{\pi}_{\mathbf{n}}$ or $-\mathbf{R}^{\pi}_{\boldsymbol{\mu}}$ or $\mathbf{R}^{\pi}_{\boldsymbol{\nu}}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ for each $\mathbf{z} = \mathbf{u}, \mathbf{v}$. Hence, we deduce that each vector $\mathbf{z} \in \{\mathbf{u},\mathbf{v}\}$ takes one of the forms:

$$c\mathbf{n},\ c\neq 0; \quad a\mathbf{n}+b\boldsymbol{\mu},\ b\neq 0; \quad c\boldsymbol{\nu},\ c\neq 0\,. \tag{3.3}$$

Considering the combinations of the above forms and excluding the cases

$$\mathbf{u}=a\mathbf{n},\ \mathbf{v}=b\mathbf{n};\ \mathbf{u}=a\mathbf{n},\ \mathbf{v}=c\mathbf{n}+d\boldsymbol{\mu};$$

$$\mathbf{u}=a\mathbf{n}+b\boldsymbol{\mu},\ \mathbf{v}=c\mathbf{n}+d\boldsymbol{\mu};\ \mathbf{u}=a\boldsymbol{\nu},\ \mathbf{v}=c\boldsymbol{\nu};$$

which violate the D_{2md}-irreducibility condition for $(\mathbf{u},\mathbf{v})$, we derive the following four disjoint cases for D_{2md}-irreducible sets $(\mathbf{u},\mathbf{v})$:

(c1) $\mathbf{u}=a\mathbf{n},\ \mathbf{v}=b\mathbf{e},\ ab\neq 0;$

(c2) $\mathbf{u}=a\mathbf{e},\ \mathbf{v}=b\boldsymbol{\nu},\ \boldsymbol{\nu}\neq\mathbf{e},\ ab\neq 0;$

(c3) $\mathbf{u}=c\mathbf{e},\ \mathbf{v}=a\mathbf{n}+b\boldsymbol{\mu},\ bc\neq 0;$

(c4) $\mathbf{u}=a\mathbf{n}+b\sigma_1,\ \mathbf{v}=c\mathbf{n}+d\boldsymbol{\mu},\ \boldsymbol{\mu}\neq\sigma_1,\ bd\neq 0.$

Then, for case (c1) we have

$$\mathrm{rank} V''_{2m}(\mathbf{u},\mathbf{v}) \geq \mathrm{rank}\{\mathbf{u},\mathbf{v},\mathbf{u}\times\ \overset{\circ}{\mathbf{v}}\} = 3\,, \tag{3.4}$$

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{v}) \geq \text{rank}\{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{v}}), \mathbf{u}\wedge\mathbf{v}, \mathbf{N}\} = 3,$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{v}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}}, \mathbf{n}\vee(\mathbf{n}\vee\eta_{2m-1}(\overset{\circ}{\mathbf{v}})),$$
$$\mathbf{u}\vee\mathbf{v}, \overset{\circ}{\mathbf{v}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{v}})\} = 6;$$

and for cases (c2) – (c4) we have (3.4) and

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{v}) \geq \text{rank}\{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{v}}), \mathbf{u}\wedge\mathbf{v}\} = 3,$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{v}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}},$$
$$\mathbf{n}\vee(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}})), \mathbf{n}\vee(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{v}})), \mathbf{u}\vee\mathbf{v}\} = 6.$$

From the above results we deduce that the three sets at issue obey the criterion (2.3) in Part I, respectively, and hence they supply the desired vector, skewsymmetric tensor and symmetric tensor generating sets. Further, by considering the pair $\mathbf{u}_0 = \mathbf{n}$ and $\mathbf{v}_0 = \boldsymbol{\mu}_1$ we infer that the generator $\beta_{2m}(\overset{\circ}{\mathbf{v}})\mathbf{u}\times\overset{\circ}{\mathbf{v}}$ and the respective last two generators in the two sets $\text{Skw}''_{2m}(\mathbf{u},\mathbf{v})$ and $\text{Sym}''_{2m}(\mathbf{u},\mathbf{v})$ are irreducible, respectively. Moreover, by exchanging $\mathbf{u}_0$ and $\mathbf{v}_0$ we know that the generator $\beta_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{v}\times\overset{\circ}{\mathbf{u}}$ is also irreducible.

By means of the relevant arguments used in (**iv**), (**viii**) and (**ix**) in Sec. 2, it can be proved that the set $I''_{2m}(\mathbf{u},\mathbf{v})$ given here and the sets $I''_{2m}(\mathbf{u},\mathbf{W})$ and $I''_{2m}(\mathbf{u},\mathbf{A})$ given later supply the desired functional bases for the D_{2md}-irreducible sets $(\mathbf{u},\mathbf{v})$, $(\mathbf{u},\mathbf{W})$ and $(\mathbf{u},\mathbf{A})$, respectively. Henceforth, this procedure will not be repeated.

(**v**) The D_{2md}-irreducible set $(\mathbf{W},\boldsymbol{\Omega})$ of two skewsymmetric tensors

$$V \quad V''_{2m}(\mathbf{W}) \cup V''_{2m}(\boldsymbol{\Omega}) \cup \{\boldsymbol{\Omega}(\mathbf{n}\times\eta_{2m-1}(\mathbf{Wn})), \mathbf{W}(\mathbf{n}\times\eta_{2m-1}(\boldsymbol{\Omega}\mathbf{n})),$$
$$((\text{tr}\mathbf{WN})|\text{tr}\mathbf{WN}|^{2m-1}\alpha_{2m}(\boldsymbol{\Omega}\mathbf{n}) + (\text{tr}\boldsymbol{\Omega}\mathbf{N})|\text{tr}\boldsymbol{\Omega}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{Wn}))\mathbf{n}\}$$
$$(\equiv V''_{2m}(\mathbf{W},\boldsymbol{\Omega}))$$

$$R \quad \mathbf{r}\cdot V''_{2m}(\mathbf{W}), \mathbf{r}\cdot V''_{2m}(\boldsymbol{\Omega}), [\mathbf{n}, \boldsymbol{\Omega}\mathbf{r}, \eta_{2m-1}(\mathbf{Wn})], [\mathbf{n}, \mathbf{Wr}, \eta_{2m-1}(\boldsymbol{\Omega}\mathbf{n})],$$
$$(\mathbf{r}\cdot\mathbf{n})((\text{tr}\mathbf{WN})|\text{tr}\mathbf{WN}|^{2m-1}\alpha_{2m}(\boldsymbol{\Omega}\mathbf{n}) + (\text{tr}\boldsymbol{\Omega}\mathbf{N})|\text{tr}\boldsymbol{\Omega}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{Wn})).$$

To prove the presented results, we first work out the D_{2md}-irreducible set $(\mathbf{W},\boldsymbol{\Omega})$, specified by (see (3.1) in Part I): $\Gamma(\mathbf{W},\boldsymbol{\Omega})\cap D_{2md} \neq \Gamma(\mathbf{z})\cap D_{2md}$, $\mathbf{z} = \mathbf{W}, \boldsymbol{\Omega}$. Evidently, we have $\Gamma(\mathbf{z})\cap D_{2md} \neq C_1$. The latter implies that $\mathbf{R}^{\pi}_{\mathbf{n}}$ or $-\mathbf{R}^{\pi}_{\boldsymbol{\mu}}$ or $\mathbf{R}^{\pi}_{\boldsymbol{\nu}}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ of $\mathbf{z}$ for each $\mathbf{z} = \mathbf{W}, \boldsymbol{\Omega}$. Hence, we deduce that each skewsymmetric tensor $\mathbf{z}\in\{\mathbf{W},\boldsymbol{\Omega}\}$ takes one of the forms:

$$(3.5) \qquad c\mathbf{En},\ c\neq 0; \quad c\mathbf{E}\boldsymbol{\mu},\ c\neq 0; \quad c\mathbf{E}\boldsymbol{\nu},\ c\neq 0\,.$$

Considering the combinations of the above forms and excluding the case $\mathbf{W} = c\mathbf{\Omega}$ which violates the D_{2md}-irreducibility condition for $(\mathbf{W}, \mathbf{\Omega})$, we derive the following five cases for D_{2md}-irreducible set $(\mathbf{W}, \mathbf{\Omega})$:

(c1) $\mathbf{W} = a\mathbf{En}$, $\mathbf{\Omega} = b\mathbf{E}\sigma_1$, $ab \neq 0$;

(c2) $\mathbf{W} = a\mathbf{En}$, $\mathbf{\Omega} = b\mathbf{Ee}$, $ab \neq 0$;

(c3) $\mathbf{W} = a\mathbf{Ee}$, $\mathbf{\Omega} = b\mathbf{E}\mathbf{\nu}$, $\mathbf{\nu} \neq \mathbf{e}$, $ab \neq 0$;

(c4) $\mathbf{W} = a\mathbf{E}\sigma_1$, $\mathbf{\Omega} = b\mathbf{E}\mu$, $\mu \neq \sigma_1$, $ab \neq 0$;

(c5) $\mathbf{W} = a\mathbf{E}\mu$, $\mathbf{\Omega} = b\mathbf{Ee}$, $ab \neq 0$.

With the above five cases we prove that the presented set $V''_{2m}(\mathbf{W}, \mathbf{\Omega})$ obeys the criterion (2.3) in Part I. We have

$$\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{\Omega}) \geq \operatorname{rank}\{\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega n}), \mathbf{\Omega}(\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega n})), \mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega n}))\} = 3,$$

$$\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{\Omega}) \geq \operatorname{rank}\{\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega n}), \mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega n})), \alpha_{2m}(\mathbf{\Omega n})\mathbf{n}\} = 3,$$

$$\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{\Omega}) \geq \operatorname{rank}\{\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn}), \mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega n}), \mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega n}))\} = 3,$$

for cases (c1) – (c3), respectively, and

$$\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{\Omega}) \geq \operatorname{rank}\{\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn}), \mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn})), \mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega n})\} = 3,$$

for cases (c4) – (c5).

Thus, we infer that the presented set $V''_{2m}(\mathbf{W}, \mathbf{\Omega})$ supplies a desired vector generating set. Further, from case (c1) we know that the last two generators in the set $V''_{2m}(\mathbf{W}, \mathbf{\Omega})$ are irreducible. By exchanging $\mathbf{W}$ and $\mathbf{\Omega}$ in case (c1) we know that the generator $\mathbf{\Omega}\eta_{2m-1}(\mathbf{Wn})$ is also irreducible.

(vi) The D_{2md}-irreducible set $(\mathbf{W}, \mathbf{A})$ of a skewsymmetric and a symmetric tensors

$$\begin{aligned} V \quad & V''_{2m}(\mathbf{W}) \cup V''_{2m}(\mathbf{A}) \cup \{(\operatorname{tr}\mathbf{WN})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \\ & (\alpha_{2m}(\mathbf{Wn}) + (\operatorname{tr}\mathbf{WN})|\operatorname{tr}\mathbf{WN}|^{2m-1})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, \\ & (\operatorname{tr}\mathbf{WN})^{2m-2}\mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})) + |\,\overset{\circ}{\mathbf{A}}\,|^{2m-2}\,\overset{\circ}{\mathbf{A}}\,(\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn}))\} \\ & \qquad (\equiv V''_{2m}(\mathbf{W}, \mathbf{A})) \end{aligned}$$

$$R \quad \mathbf{r}\cdot V_{2m}(\mathbf{W}), \mathbf{r}\cdot V_{2m}(\mathbf{A}), (\mathbf{r}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$
$$(\mathbf{r}\cdot\mathbf{n})(\alpha_{2m}(\mathbf{Wn}) + (\mathrm{tr}\mathbf{WN})|\mathrm{tr}\mathbf{WN}|^{2m-1})\alpha_m(\mathbf{q}(\mathbf{A})),$$
$$(\mathrm{tr}\mathbf{WN})^{2m-2}[\mathbf{n}, \mathbf{Wr}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})] - |\overset{\circ}{\mathbf{A}}|^{2m-2}[\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{r}, \eta_{2m}(\mathbf{Wn})].$$

To prove the above result, we first work out the D_{2md}-irreducible set $(\mathbf{W}, \mathbf{A})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2md} \neq \Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2md}$ for $\mathbf{z} = \mathbf{W}, \mathbf{A}$. Evidently, $\Gamma(\mathbf{z}) \cap D_{2md} \neq C_1$ for $\mathbf{z} = \mathbf{W}, \mathbf{A}$. Hence, the skewsymmetric tensor $\mathbf{W}$ takes one of the forms given by (3.5). In a similar way we deduce that the symmetric tensor $\mathbf{A}$ takes one of the forms

$$(3.6)\qquad \begin{cases} \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ a^2 + b^2 \neq 0\,; \\ \overset{\circ}{\mathbf{A}} = a(\mu\otimes\mu - \mu'\otimes\mu') + b\mathbf{n}\vee\mu',\ \mu' = \mathbf{n}\times\mu,\ b\neq 0\,; \\ \overset{\circ}{\mathbf{A}} = a(\nu\otimes\nu - \nu'\otimes\nu') + b\mathbf{n}\vee\nu',\ \nu' = \mathbf{n}\times\nu,\ b\neq 0\,. \end{cases}$$

Considering the combinations of the forms given by (3.5) – (3.6) and excluding the cases

$$\mathbf{W} = c\mathbf{En},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \beta_{2m}(\mathbf{q}(\mathbf{A})) \neq 0;$$
$$\mathbf{W} = c\mathbf{E}\sigma,\ \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma');$$
$$\mathbf{W} = c\mathbf{E}\sigma,\ \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma') + b\mathbf{n}\vee\sigma;$$

which violate the D_{2md}-irreducibility condition for $(\mathbf{W}, \mathbf{A})$, we derive the following five disjoint cases for D_{2md}-irreducible set $(\mathbf{W}, \mathbf{A})$:

(c1) $\mathbf{W} = c\mathbf{En},\ \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma'),\ ac\neq 0;$

(c2) $\mathbf{W} = c\mathbf{En},\ \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma') + b\mathbf{n}\vee\sigma',\ bc\neq 0;$

(c3) $\mathbf{W} = c\mathbf{E}\sigma,\ \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma') + b\sigma\vee\sigma',\ bc\neq 0;$

(c4) $\mathbf{W} = c\mathbf{E}\sigma_1,\ \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma') + b\mathbf{n}\vee\sigma',\ \sigma\neq\sigma_1,\ bc\neq 0;$

(c5) $\mathbf{W} = c\mathbf{Ee},\ \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma') + b\mathbf{n}\vee\sigma',\ \sigma\neq\mathbf{e},\ bc\neq 0.$

With the above cases we prove that the presented set $V''_{2m}(\mathbf{W}, \mathbf{A})$ obeys the criterion (2.3) in Part I. First, using the formula (2.4) in Part I and the equalities

$$(3.7)\qquad \mathrm{rank}(V(C_{1h}(\mathbf{a})) \cup V(C_{1h}(\mathbf{b}))) = 3,\ \ \mathrm{rank}(V(C_{1h}(\mathbf{a})) \cup V(C_2(\mathbf{b}))) = 3\,,$$

for any two noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$, we have

$$\mathrm{rank}V''_{2m}(\mathbf{W}, \mathbf{A}) \geq \mathrm{rank}(V''_{2m}(\mathbf{W}) \cup V''_{2m}(\mathbf{A}))$$
$$= \mathrm{rank}(V(\Gamma(\mathbf{W}) \cap D_{2md}) \cup V(\Gamma(\mathbf{A}) \cap D_{2md})) = 3$$

for case (c4) and for case (c5) with $\sigma = \sigma_{2r-1}$. For case (c1) we have

$$\Gamma(\mathbf{W},\mathbf{A}) \cap D_{2md} = C_2(\mathbf{n}),$$
$$\mathrm{rank} V''_{2m}(\mathbf{W},\mathbf{A}) \geq \mathrm{rank}\{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, \alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\} = 1.$$

For case (c2) we have

$$\mathrm{rank} V''_{2m}(\mathbf{W},\mathbf{A}) \geq \begin{cases} \mathrm{rank}\{\eta(\mathbf{A})', \overset{\circ}{\mathbf{A}}\, \eta(\mathbf{A})', \mathbf{W}\eta(\mathbf{A})'\} = 3 \\ \qquad\qquad \text{if } \sigma = \sigma_{2r-1}, \\ \mathrm{rank}\{\eta(\mathbf{A})', (\mathrm{tr}\mathbf{W}\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})\mathbf{n}, \mathbf{W}\eta(\mathbf{A})'\} = 3 \\ \qquad\qquad \text{if } \sigma = \sigma_{2r}, \end{cases}$$

where $\eta(\mathbf{A})' = \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})$. For case (c3) we have

$$\mathrm{rank} V''_{2m}(\mathbf{W},\mathbf{A}) \geq \begin{cases} \mathrm{rank}\{\eta(\mathbf{W})', \mathbf{W}\eta(\mathbf{W})', \overset{\circ}{\mathbf{A}}\, \eta(\mathbf{W})'\} = 3 \\ \qquad\qquad \text{if } \sigma = \sigma_{2r-1}, \\ \mathrm{rank}\{\beta_m(\mathbf{q})\mathbf{n}, \alpha_m(\mathbf{q})\alpha_{2m}(\mathbf{W}\mathbf{n})\mathbf{n}, \eta(\mathbf{W})', \overset{\circ}{\mathbf{A}}\, \eta(\mathbf{W})'\} \\ \qquad\qquad = 3 \text{ if } \sigma = \sigma_{2r}, \end{cases}$$

where $\mathbf{q} = \mathbf{q}(\mathbf{A})$, $\eta(\mathbf{W})' = \mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n})$.

Finally, for case (c5) with $\sigma = \sigma_{2r}$ we have

$$\mathrm{rank} V''_{2m}(\mathbf{W},\mathbf{A}) \geq \mathrm{rank}\{\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}), \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n}), \overset{\circ}{\mathbf{A}}\, (\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}))\} = 3.$$

From the above results and Table 1 in Sec. 2 in Part I, we infer that the set $V''_{2m}(\mathbf{W},\mathbf{\Omega})$ obeys the criterion (2.3) in Part I, and hence it supplies a desired vector generating set. Further, by considering the two pairs: $\mathbf{W}_1 = \mathbf{E}\mathbf{n}$ and $\mathbf{A}_1 = \mathbf{n} \vee \mathbf{e}$, $\mathbf{W}_2 = \mathbf{E}\mathbf{n}$ and $\mathbf{A}_2 = \mathbf{e} \otimes \mathbf{e}$, we deduce that the last three generators in the set $V''_{2m}(\mathbf{W},\mathbf{A})$ are irreducible.

(vii) The D_{2md}-irreducible set $(\mathbf{A},\mathbf{B})$ of two symmetric tensor variables

$$V \quad V''_{2m}(\mathbf{A}) \cup V''_{2m}(\mathbf{B}) \cup \{\overset{\circ}{\mathbf{A}}\, (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{B}}\, \mathbf{n})), \overset{\circ}{\mathbf{B}}\, (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})),$$
$$(1-\delta_{1m})(\alpha_m(\mathbf{q}(\mathbf{B}))\alpha_{2m}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})| \cdot |\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{A}), \mathbf{q}(\mathbf{B})]\alpha_m(\mathbf{q}(\mathbf{A})))\mathbf{n},$$
$$(1-\delta_{1m})(\alpha_m(\mathbf{q}(\mathbf{A}))\alpha_{2m}(\overset{\circ}{\mathbf{B}}\, \mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})| \cdot |\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{B}), \mathbf{q}(\mathbf{A})]\alpha_m(\mathbf{q}(\mathbf{B})))\mathbf{n}\} \ (\equiv V''_{2m}(\mathbf{A},\mathbf{B}))$$

$$R \quad \mathbf{r}\cdot V''_{2m}(\mathbf{A}), \mathbf{r}\cdot V''_{2m}(\mathbf{B}), [\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{r}, \eta_{2m-1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})], [\mathbf{n}, \overset{\circ}{\mathbf{B}}\,\mathbf{r}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})],$$
$$(1-\delta_{1m})(\mathbf{r}\cdot\mathbf{n})(\alpha_m(\mathbf{q}(\mathbf{B}))\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{A}), \mathbf{q}(\mathbf{B})]\alpha_m(\mathbf{q}(\mathbf{A}))),$$
$$(1-\delta_{1m})(\mathbf{r}\cdot\mathbf{n})(\alpha_m(\mathbf{q}(\mathbf{A}))\alpha_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{B}), \mathbf{q}(\mathbf{A})]\alpha_m(\mathbf{q}(\mathbf{B}))).$$

We show that the presented set $V''_{2m}(\mathbf{A},\mathbf{B})$ supplies the desired vector generating set, i.e. it obeys the criterion (2.3) in Part I. To this end, we first work out the D_{2md}-irreducible set $(\mathbf{A},\mathbf{B})$. From the D_{2md}-irreducibility condition for $(\mathbf{A},\mathbf{B})$ (see (3.1) in Part I), we know that $\Gamma(\mathbf{z})\cap D_{2md}\neq C_1$ for $\mathbf{z}=\mathbf{A},\ \mathbf{B}$. Hence, considering the combinations of the forms given by (3.6) and excluding the cases

$$\overset{\circ}{\mathbf{A}}= a\mathbf{D}_1+b\mathbf{D}_2,\ \overset{\circ}{\mathbf{B}}= c\mathbf{D}_1+d\mathbf{D}_2,\ \beta_{2m}(\mathbf{q}(\mathbf{z}))\neq 0,\ \mathbf{z}=\mathbf{A} \text{ or } \mathbf{z}=\mathbf{B};$$

$$\overset{\circ}{\mathbf{A}}= a(\sigma\otimes\sigma-\sigma'\otimes\sigma')+b\mathbf{n}\vee\sigma',\ \overset{\circ}{\mathbf{B}}= c(\sigma\otimes\sigma-\sigma'\otimes\sigma')+d\mathbf{n}\vee\sigma';$$

$$\overset{\circ}{\mathbf{A}}= a(\sigma\otimes\sigma-\sigma'\otimes\sigma'),\ \overset{\circ}{\mathbf{B}}= c(\sigma\otimes\sigma-\sigma'\otimes\sigma')+d\mathbf{n}\vee\sigma';$$

which violate the D_{2md}-irreducibility condition for $(\mathbf{A},\mathbf{B})$, we derive the following five disjoint cases for the D_{2md}-irreducible set $(\mathbf{A},\mathbf{B})$:

(c1) $\overset{\circ}{\mathbf{A}}= a(\sigma_1\otimes\sigma_1-\sigma'_1\otimes\sigma'_1),\ \overset{\circ}{\mathbf{B}}= c(\sigma\otimes\sigma-\sigma'\otimes\sigma'),\ \sigma\neq\sigma_1,\ \sigma'_1,\ ac\neq 0;$

(c2) $\overset{\circ}{\mathbf{A}}= a(\mathbf{e}\otimes\mathbf{e}-\mathbf{e}'\otimes\mathbf{e}'),\ \overset{\circ}{\mathbf{B}}= c(\mathbf{v}\otimes\mathbf{v}-\mathbf{v}'\otimes\mathbf{v}'),\ \mathbf{v}=\sigma_{2r}\neq\mathbf{e},\ \mathbf{e}',\ ac\neq 0;$

(c3) $\overset{\circ}{\mathbf{A}}= a(\sigma\otimes\sigma-\sigma'\otimes\sigma')+b\sigma\vee\sigma',\ \overset{\circ}{\mathbf{B}}= c(\sigma\otimes\sigma-\sigma'\otimes\sigma')+d\mathbf{n}\vee\sigma',\ bd\neq 0;$

(c4) $\overset{\circ}{\mathbf{A}}= a(\sigma_1\otimes\sigma_1-\sigma'_1\otimes\sigma'_1)+b\mathbf{n}\vee\sigma'_1,\ \overset{\circ}{\mathbf{B}}= c(\sigma\otimes\sigma-\sigma'\otimes\sigma')$
$+d\mathbf{n}\vee\sigma',\ \sigma\neq\sigma_1,\ bd\neq 0;$

(c5) $\overset{\circ}{\mathbf{A}}= a\mathbf{D}_1+b\mathbf{D}_4,\ \overset{\circ}{\mathbf{B}}= c(\mathbf{v}\otimes\mathbf{v}-\mathbf{v}'\otimes\mathbf{v}')+d\mathbf{n}\vee\mathbf{v}',\ \mathbf{v}=\sigma_{2r}\neq\mathbf{e},\ ac\neq 0.$

Here and henceforth, we denote $\mathbf{u}'=\mathbf{n}\times\mathbf{u}$ for every vector $\mathbf{u}$.

Then, for cases (c1) we have

$$\Gamma(\mathbf{A},\mathbf{B})\cap D_{2md}=C_2(\mathbf{n}),\quad \operatorname{rank}V''_{2m}(\mathbf{A},\mathbf{B})\geq\operatorname{rank}\{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\}=1.$$

Case (c2) does not hold for $m=1$, since the group D_{2d} has only two two-fold rotation axis vectors, i.e. $\mathbf{e}$ and $\mathbf{e}'$. For case (c2) with $m\geq 2$, we have the first expression above and

$$\operatorname{rank}V''_{2m}(\mathbf{A},\mathbf{B})\geq\operatorname{rank}\{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, [\mathbf{n},\mathbf{q}(\mathbf{A}),\mathbf{q}(\mathbf{B})]\alpha_m(\mathbf{q}(\mathbf{A})\mathbf{n}\}=1.$$

For case (c3) we have

$$\operatorname{rank} V''_{2m}(\mathbf{A},\mathbf{B}) \geq \begin{cases} \operatorname{rank}\{\eta(\mathbf{B})', \overset{\circ}{\mathbf{B}}\,\eta(\mathbf{A})', \overset{\circ}{\mathbf{A}}\,\eta(\mathbf{B})'\} = 3 \text{ if } \sigma = \sigma_{2r-1}, \\ \operatorname{rank}\{\eta(\mathbf{B})', \overset{\circ}{\mathbf{A}}\,\eta(\mathbf{B})', \alpha_m(\mathbf{q}(\mathbf{A}))\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\} = 3 \\ \qquad \text{if } m \geq 2,\ \sigma = \sigma_{2r}, \\ \operatorname{rank}\{\eta(\mathbf{B})', \overset{\circ}{\mathbf{A}}\,\eta(\mathbf{B})', \beta_1(\mathbf{q}(\mathbf{A}))\mathbf{n}\} = 3 \text{ if } m = 1, \\ \qquad \sigma \in \{\mathbf{e}, \mathbf{e}'\}, \end{cases}$$

where $\eta(\mathbf{D})' = \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{D}}\,\mathbf{n})$, $\mathbf{D} = \mathbf{A}, \mathbf{B}$. For case (c4), by using the formula (2.4) in Part I and (3.7) we have

$$\begin{aligned} \operatorname{rank} V''_{2m}(\mathbf{A},\mathbf{B}) &\geq \operatorname{rank}(V''_{2m}(\mathbf{A}) \cup V''_{2m}(\mathbf{B})) \\ &= \operatorname{rank}(V(C_{1h}(\sigma_1)) \cup V(\Gamma(\mathbf{B}) \cap D_{2md})) = 3. \end{aligned}$$

Finally, for case (c5) we have

$$\begin{aligned} \operatorname{rank} V''_{2m}(\mathbf{A},\mathbf{B}) \geq \operatorname{rank}\{&\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}), \\ &\overset{\circ}{\mathbf{B}}\,(\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} = 3. \end{aligned}$$

From the above results and Table 1 in Sec. 2 we deduce that the presented set $V''_{2m}(\mathbf{A},\mathbf{B})$ obeys the criterion (2.3) in Part I. Further, by considering the four pairs $(\mathbf{A}_i, \mathbf{B}_i)$ given by

$$\mathbf{A}_1 = \mathbf{e} \otimes \mathbf{e},\ \mathbf{B}_1 = \mathbf{n} \vee \sigma_1;\ \mathbf{A}_2 = \mathbf{n} \vee \sigma_1,\ \mathbf{B}_2 = \mathbf{e} \vee \mathbf{e};$$

$$\mathbf{A}_3 = \mathbf{e} \otimes \mathbf{e},\ \mathbf{B}_3 = \mathbf{n} \otimes \sigma_2,\ m \geq 2;\ \mathbf{A}_4 = \mathbf{n} \otimes \sigma_2,\ \mathbf{B}_4 = \mathbf{e} \otimes \mathbf{e},\ m \geq 2,$$

we infer that the last four generators in the set $V''_{2m}(\mathbf{A},\mathbf{B})$ are irreducible.

(viii) The D_{2md}-irreducible set $(\mathbf{u}, \mathbf{W})$ of a vector and a skewsymmetric tensor

V $\quad V''_{2m}(\mathbf{u}) \cup V''_{2m}(\mathbf{W}) \cup \{\mathbf{W}\mathbf{u}, (\operatorname{tr}\mathbf{W}\mathbf{N})\alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{n}\}\ (\equiv V''_{2m}(\mathbf{u},\mathbf{W}))$

Skw $\quad \{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{W}, (\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathbf{W}\mathbf{n})\mathbf{N},$

$$|\mathbf{u}|^{2m-2}\mathbf{u} \wedge (\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n})) + (\operatorname{tr}\mathbf{W}\mathbf{N})^{2m-1}\mathbf{n} \wedge \eta_{2m-1}(\overset{\circ}{\mathbf{u}})\}$$

$$(\equiv \operatorname{Skw}''_{2m}(\mathbf{u},\mathbf{W}))$$

Sym $\mathrm{Sym}''_{2m}(\mathbf{u}) \cup \mathrm{Sym}_{4m}(\mathbf{W}) \cup \{\delta_{1m}(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{D}_1,$

$\mathbf{W}\overset{\circ}{\mathbf{u}} \vee (\mathbf{n}\times \eta_{2m-1}(\mathbf{Wn})), (\mathrm{tr}\mathbf{WN})\overset{\circ}{\mathbf{u}} \vee (\mathbf{n}\times \overset{\circ}{\mathbf{u}}),$

$|\mathbf{u}|^{2m-2}\mathbf{u}\vee(\mathbf{n}\times\eta_{2m-1}(\mathbf{Wn})) + (\mathrm{tr}\mathbf{WN})^{2m-1}\mathbf{n}\vee\eta_{2m-1}(\overset{\circ}{\mathbf{u}})\}$

$(\equiv \mathrm{Sym}''_{2m}(\mathbf{u},\mathbf{W}))$

R $\mathbf{r}\cdot V''_{2m}(\mathbf{u}); \mathrm{tr}\mathbf{HW}; \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\overset{\circ}{\mathbf{u}};$

$I''_{2m}(\mathbf{u}) \cup I_{4m}(\mathbf{W}) \cup \{(\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn}, \overset{\circ}{\mathbf{u}}\cdot\mathbf{W}^2\overset{\circ}{\mathbf{u}}\}\ (\equiv I''_{2m}(\mathbf{u},\mathbf{W}))$.

Here, the skewsymmetric tensor variable $\mathbf{H}$ is of the form $\mathbf{H} = c\mathbf{W}$. The other case leads to $\Gamma(\mathbf{W},\mathbf{H})\cap D_{2md} = C_1$, which has been treated in (v) in this section. Moreover, the vector variable $\mathbf{r}$ pertains to the space span$V''_{2m}(\mathbf{u})$. In fact, for each of cases (c2) – (c6) for the D_{2md}-irreducible set $(\mathbf{u},\mathbf{W})$ that will be given, from the condition (3.3) in Part I with $(\mathbf{z}_0,\mathbf{z}) = (\mathbf{u},\mathbf{r})$ and $g = D_{2md}$, we derive the foregoing fact for $\mathbf{r}$. For case (c1), we derive $\mathbf{r} = a\mathbf{n} + b\sigma_{2r-1}$. The case when $b \neq 0$ is excluded, since the pair $(\mathbf{r},\mathbf{W})$ yields case (c5) that has just been covered. Finally, from the condition (3.3) in Part I with $(\mathbf{z}_0,\mathbf{z}) = (\mathbf{W},\mathbf{C})$ and $g = D_{2md}$, we derive that the symmetric tensor variable $\mathbf{C}$ pertains to the space span $\mathrm{Sym}_{4m}(\mathbf{W})$ for cases (c1) – (c4) and to the space $\mathrm{Sym}(C_{2h}(\mathbf{n}))$ for cases (c5) – (c6).

Owing to the facts shown above, of the invariants from the scalar products, we need only to retain those listed in the above table. Of them, the two invariants $\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}$ and $\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\overset{\circ}{\mathbf{u}}$ result from the fact that the four generators $\mathbf{I}$, $\mathbf{n}\otimes\mathbf{n}$, $\overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}$ and $\overset{\circ}{\mathbf{u}}\vee\mathbf{W}\overset{\circ}{\mathbf{u}}$ generate the space $\mathrm{Sym}(C_{2h}(\mathbf{n}))$ for either of cases (c5) – (c6). As has been indicated earlier, here we omit the invariants $\mathbf{C} : \mathrm{Sym}_{4m}(\mathbf{W})$.

We proceed to show that the three presented sets $V''_{2m}(\mathbf{u},\mathbf{W})$, $\mathrm{Skw}''_{2m}(\mathbf{u},\mathbf{W})$ and $\mathrm{Sym}''_{2m}(\mathbf{u},\mathbf{W})$ supply the desired vector, skewsymmetric tensor and symmetric tensor generating sets, i.e. each of them obeys the criterion (2.3) in Part I. Towards this goal we first work out the D_{2md}-irreducible set $(\mathbf{u},\mathbf{W})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z})\cap D_{2md} \neq \Gamma(\mathbf{u},\mathbf{W})\cap D_{2md}$ for $\mathbf{z} = \mathbf{u}$, $\mathbf{W}$. Hence, $\mathbf{u}$ and $\mathbf{W}$ take one of the forms given by (3.3) and (3.5), respectively. Considering the combinations of the forms given by (3.3) and (3.5) and excluding the cases

$$\mathbf{W} = c\mathbf{E}\mu,\ \mathbf{u} = a\mathbf{n};\ \mathbf{W} = c\mathbf{E}\mu,\ \mathbf{u} = a\mathbf{n} + b\mu';$$

$$\mathbf{W} = c\mathbf{E}\nu,\ \mathbf{u} = a\nu;$$

which violate the D_{2md}-irreducibility condition for $(\mathbf{u},\mathbf{W})$, we derive the following six disjoint cases for D_{2md}-irreducible set $(\mathbf{u},\mathbf{W})$:

(c1) $\mathbf{W} = c\mathbf{E}\mathbf{e},\ \mathbf{u} = a\mathbf{n},\ ac \neq 0;$

(c2) $\mathbf{W} = c\mathbf{E}\sigma$, $\mathbf{u} = a\mathbf{n} + b\sigma_1$, $\sigma \cdot \sigma_1 \neq 0$, $bc \neq 0$;

(c3) $\mathbf{W} = c\mathbf{E}\sigma$, $\mathbf{u} = a\mathbf{e}$, $\sigma \neq \mathbf{e}$, $ac \neq 0$;

(c4) $\mathbf{W} = c\mathbf{E}\mathbf{n}$, $\mathbf{u} = a\mathbf{n}$, $ac \neq 0$;

(c5) $\mathbf{W} = c\mathbf{E}\mathbf{n}$, $\mathbf{u} = a\mathbf{n} + b\sigma_1$, $bc \neq 0$;

(c6) $\mathbf{W} = c\mathbf{E}\mathbf{n}$, $\mathbf{u} = a\mathbf{e}$, $ac \neq 0$;

For case (c1) we have

$$\text{rank} V''_{2m}(\mathbf{u}, \mathbf{W}) \geq \text{rank}\{\mathbf{u}, \mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}), \mathbf{W}\mathbf{u}\} = 3\,, \tag{3.8}$$

$$\text{rank Skw}''_{2m}(\mathbf{u}, \mathbf{W}) \geq \text{rank}\{\mathbf{W}, \alpha_{2m}(\mathbf{W}\mathbf{n})\mathbf{N}, \mathbf{u} \wedge (\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}))\} = 3,$$

$$\text{rank Sym}''_{2m}(\mathbf{u}, \mathbf{W}) \geq \text{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathbf{W}^2, \mathbf{n} \vee \mathbf{W}\mathbf{n}, \mathbf{W} \overset{\circ}{\mathbf{u}} \vee \eta(\mathbf{W})', \\ \mathbf{u} \vee \eta(\mathbf{W})'\} = 6,$$

where $\eta(\mathbf{W})' = \mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n})$.

For case (c2) we have

$$\text{rank} V''_{2m}(\mathbf{u}, \mathbf{W}) \geq \text{rank}\{\mathbf{n}, \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}), \mathbf{W}\mathbf{u}\} = 3\,, \tag{3.9}$$

$$\text{rank Skw}''_{2m}(\mathbf{u}, \mathbf{W}) \geq \text{rank}\{\mathbf{W}, \mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \\ \mathbf{u} \wedge (\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}))\} = 3\,, \tag{3.10}$$

$$\text{rank Sym}''_{2m}(\mathbf{u}, \mathbf{W}) \geq \text{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}}, \mathbf{n} \vee (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{u}})), \\ \mathbf{n} \vee \mathbf{W}\mathbf{n}, \mathbf{u} \vee (\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}))\} = 6\,. \tag{3.11}$$

For case (c3), we have (3.8), (3.10) and (3.11). For case (c4) we have

$$\Gamma(\mathbf{u}, \mathbf{W}) \cap D_{2md} = C_{2m}(\mathbf{n}),\ \text{rank} V''_{2m}(\mathbf{u}, \mathbf{W}) = \text{rank Skw}''_{2m}(\mathbf{u}, \mathbf{W}) = 1,$$

$$\text{rank Sym}''_{2m}(\mathbf{u}, \mathbf{W}) = \text{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \delta_{1m}\mathbf{D}_1, \delta_{1m}\mathbf{D}_2\} = 2(1 + \delta_{1m}).$$

For case (c5) we have (3.9) and

$$\text{rank Skw}''_{2m}(\mathbf{u}, \mathbf{W}) \geq \text{rank}\{\mathbf{W}, \mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n} \wedge \eta_{2m-1}(\overset{\circ}{\mathbf{u}})\} = 3,$$

$$\text{rank Sym}''_{2m}(\mathbf{u}, \mathbf{W}) \geq \text{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}}, \mathbf{n} \vee (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{u}})), \\ \mathbf{n} \vee \eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \overset{\circ}{\mathbf{u}} \vee (\mathbf{n} \times \overset{\circ}{\mathbf{u}})\} = 6.$$

Finally, for case (c6) we have the last two expressions above and

$$\text{rank} V''_{2m}(\mathbf{u}, \mathbf{W}) \geq \text{rank}\{\mathbf{u}, \mathbf{W}\mathbf{u}, \alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{n}\} = 3.$$

From the above results and Table 1 in Sec. 2 we infer that the three presented sets of generators obey the criterion (2.3) in Part I, respectively. Further, by considering the pair $\mathbf{u}_1 = \mathbf{e}$ and $\mathbf{W}_1 = \mathbf{En}$, we infer that the respective last two generators in the two sets $V''_{2m}(\mathbf{u}, \mathbf{W})$ and $\mathrm{Sym}''_{2m}(\mathbf{u}, \mathbf{W})$ are irreducible. Moreover, by considering the pair $\mathbf{u}_2 = \mathbf{n}$ and $\mathbf{W}_2 = \mathbf{Ee}$, we deduce that the last two generators in the set $\mathrm{Skw}''_{2m}(\mathbf{u}, \mathbf{W})$ and the generator $\mathbf{W}\,\mathring{\mathbf{u}} \vee(\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn}))$ are also irreducible, respectively. Finally, by considering case (c4) we deduce that the generator $\delta_{1m}(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{D}_1$ is irreducible.

(x) The D_{2md}-irreducible set $(\mathbf{u}, \mathbf{A})$ of a vector and a symmetric tensor

$$V \quad V''_{2m}(\mathbf{u}) \cup V''_{2m}(\mathbf{A}) \cup \{\mathring{\mathbf{A}}\,\mathbf{u}, (1-\delta_{1m})\alpha_{2m}(\mathring{\mathbf{u}})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\} \qquad (\equiv V''_{2m}(\mathbf{u}, \mathbf{A}))$$

$$\begin{aligned}\mathrm{Skw} \quad & \mathrm{Skw}_{4m}(\mathbf{A}) \cup \{\mathbf{E}\eta_{2m-1}(\mathring{\mathbf{u}}), \mathring{\mathbf{u}} \wedge \mathring{\mathbf{A}}\mathring{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{N},\\ & (\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{N}, |\mathbf{u}|^{2m-2}\mathbf{u} \wedge (\mathbf{n} \times \eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}))\\ & +|\mathbf{q}(\mathbf{A})|^{2m-2}\mathbf{n}\wedge \mathring{\mathbf{A}}\,(\mathbf{n} \times \eta_{2m-1}(\mathring{\mathbf{u}}))\} \quad (\equiv \mathrm{Skw}''_{2m}(\mathbf{u}, \mathbf{A}))\end{aligned}$$

$$\begin{aligned}\mathrm{Sym} \quad & \mathrm{Sym}''_{2m}(\mathbf{u}) \cup \mathrm{Sym}_{4m}(\mathbf{A}) \cup \{(\mathbf{u}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))(\mathbf{A_e}\mathbf{N} - \mathbf{N}\mathbf{A_e}),\\ & (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{A}}\,\mathbf{n} \vee (\mathbf{n} \times \eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})),\\ & |\mathbf{u}|^{2m-2}\mathbf{u} \vee (\mathbf{n} \times \eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})) + |\mathbf{q}(\mathbf{A})|^{2m-2}\mathbf{n}\vee \mathring{\mathbf{A}}\,(\mathbf{n} \times \eta_{2m-1}(\mathring{\mathbf{u}}))\}\\ & \qquad (\equiv \mathrm{Sym}''_{2m}(\mathbf{u}, \mathbf{A}))\end{aligned}$$

$$\begin{aligned}R \quad & \mathbf{r}\cdot V''_{2m}(\mathbf{u});\\ & I''_{2m}(\mathbf{u}) \cup I_{4m}(\mathbf{A}) \cup \{(\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{u}}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}^{2}\,\mathring{\mathbf{u}}\}\\ & \qquad (\equiv I''_{2m}(\mathbf{u}, \mathbf{A})).\end{aligned}$$

For each nonvanishing skewsymmetric tensor $\mathbf{H}$ and each D_{2md}-irreducible set $(\mathbf{u}, \mathbf{A})$ (see cases (c1) – (c6) given later), we have $\Gamma(\mathbf{z}_0, \mathbf{H})\cap D_{2md} = \Gamma(\mathbf{u}, \mathbf{A}, \mathbf{H})\cap D_{2md}$ with $\mathbf{z}_0 \in \{\mathbf{u}, \mathbf{A}\}$. From this fact and the condition $(3.3)_2$ in Part I with $\mathbf{z} = \mathbf{H}$ and $g = D_{2md}$, we derive $\mathbf{H} = \mathbf{O}$. Moreover, by means of the relevant procedure used at the start of (ix), for the vector variable $\mathbf{r}$ and the symmetric tensor variable $\mathbf{C}$ we derive $\mathbf{r} \in \mathrm{span}V''_{2m}(\mathbf{u})$ and $\mathbf{C} \in \mathrm{span}\ \mathrm{Sym}_{4m}(\mathbf{A})$ from cases (c1) – (c6) given below and the condition (3.3) in Part I with $(\mathbf{z}_0, \mathbf{z}) = (\mathbf{u}, \mathbf{r})$, $(\mathbf{A}, \mathbf{C})$, and $g = D_{2md}$. Owing to these facts, of the invariants from the scalar products, we retain the invariants $\mathbf{r}\cdot V_{2m}(\mathbf{u})$ only. Besides, as has been indicated earlier, we omit the invariants $\mathbf{C} : \mathrm{Sym}_{4m}(\mathbf{A})$.

We proceed to show that the three presented sets $V''_{2m}(\mathbf{u}, \mathbf{A})$, $\mathrm{Skw}''_{2m}(\mathbf{u}, \mathbf{A})$ and $\mathrm{Sym}''_{2m}(\mathbf{u}, \mathbf{A})$ supply desired vector, skewsymmetric tensor and symmetric tensor generating sets, i.e. each of them obeys the criterion (2.3) in Part I. Towards this goal we first work out the D_{2md}-irreducible set $(\mathbf{u}, \mathbf{A})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2md} \neq \Gamma(\mathbf{u}, \mathbf{A}) \cap D_{2md}$ for $\mathbf{z} = \mathbf{u}, \mathbf{A}$. Hence, $\mathbf{u}$ and $\mathbf{A}$ take one of the forms given by (3.3) and (3.6), respectively. Considering the combinations of the forms given by (3.3) and (3.6) and excluding the cases

$$\mathbf{u} = c\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \beta_{2m}(\mathbf{q}(\mathbf{A})) \neq 0;$$

$$\mathbf{u} = c\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\mu} \otimes \boldsymbol{\mu} - \boldsymbol{\mu}' \otimes \boldsymbol{\mu}') + b\mathbf{n} \vee \boldsymbol{\mu}';$$

$$\mathbf{u} = a\mathbf{n} + b\boldsymbol{\mu},\ \overset{\circ}{\mathbf{A}} = c(\boldsymbol{\mu} \otimes \boldsymbol{\mu} - \boldsymbol{\mu}' \otimes \boldsymbol{\mu}');$$

$$\mathbf{u} = a\mathbf{n} + b\boldsymbol{\mu},\ \overset{\circ}{\mathbf{A}} = c(\boldsymbol{\mu} \otimes \boldsymbol{\mu} - \boldsymbol{\mu}' \otimes \boldsymbol{\mu}') + d\mathbf{n} \vee \boldsymbol{\mu};$$

$$\mathbf{u} = c\boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \boldsymbol{\nu}' \otimes \boldsymbol{\nu}');$$

$$\mathbf{u} = c\boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \boldsymbol{\nu}' \otimes \boldsymbol{\nu}') + b\mathbf{n} \vee \boldsymbol{\nu}';$$

which violate the D_{2md}-irreducibility condition for $(\mathbf{u}, \mathbf{A})$ (see (3.1) in Part I), we derive the following six disjoint cases for the D_{2md}-irreducible set $(\mathbf{u}, \mathbf{A})$:

(c1) $\mathbf{u} = a\mathbf{n},\ \overset{\circ}{\mathbf{A}} = c(\boldsymbol{\sigma} \otimes \boldsymbol{\sigma} - \boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'),\ ac \neq 0;$

(c2) $\mathbf{u} = a\mathbf{n},\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_4,\ ad \neq 0;$

(c3) $\mathbf{u} = a\mathbf{n} + b\boldsymbol{\sigma}_1,\ \overset{\circ}{\mathbf{A}} = c(\boldsymbol{\sigma}_1 \otimes \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1' \otimes \boldsymbol{\sigma}_1') + d\boldsymbol{\sigma}_1 \vee \boldsymbol{\sigma}_1',\ bd \neq 0;$

(c4) $\mathbf{u} = a\mathbf{n} + b\boldsymbol{\sigma}_1,\ \overset{\circ}{\mathbf{A}} = c(\boldsymbol{\sigma} \otimes \boldsymbol{\sigma} - \boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}') + d\mathbf{n} \vee \boldsymbol{\sigma},\ \boldsymbol{\sigma} \neq \boldsymbol{\sigma}_1,\ bd \neq 0;$

(c5) $\mathbf{u} = b\mathbf{e},\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_2,\ bd \neq 0;$

(c6) $\mathbf{u} = b\mathbf{e},\ \overset{\circ}{\mathbf{A}} = c(\boldsymbol{\sigma} \otimes \boldsymbol{\sigma} - \boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}') + d\mathbf{n} \vee \boldsymbol{\sigma}',\ \boldsymbol{\sigma} \neq \mathbf{e},\ bd \neq 0.$

For case (c1) we have

$$\Gamma(\mathbf{u}, \mathbf{A}) \cap D_{2md} = \begin{cases} C_{2v}(\mathbf{n}, \boldsymbol{\sigma}, \boldsymbol{\sigma}') & \text{if } \boldsymbol{\sigma} = \boldsymbol{\sigma}_{2r-1}\,, \\ C_2(\mathbf{n}) & \text{if } \boldsymbol{\sigma} = \boldsymbol{\sigma}_{2r}\,, \end{cases}$$

$$\mathrm{rank} V''_{2m}(\mathbf{u}, \mathbf{A}) \geq \mathrm{rank}\{\mathbf{u}\} = 1,$$

$$\mathrm{rank\ Skw}''_{2m}(\mathbf{u}, \mathbf{A}) \geq \begin{cases} 0 \text{ if } \boldsymbol{\sigma} = \boldsymbol{\sigma}_{2r-1}\,, \\ \mathrm{rank}\{(\mathbf{v} \cdot \mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{N}\} = 1 \text{ if } \boldsymbol{\sigma} = \boldsymbol{\sigma}_{2r}\,, \end{cases}$$

$$\mathrm{rank\ Sym}''_{2m}(\mathbf{u}, \mathbf{A}) \geq \begin{cases} \mathrm{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{A}}\} = 3 \text{ if } \boldsymbol{\sigma} = \boldsymbol{\sigma}_{2r-1}\,, \\ \mathrm{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{A}}, \mathbf{A}_{\mathbf{e}}\mathbf{N} - \mathbf{N}\mathbf{A}_{\mathbf{e}}\} = 4 \text{ if } \boldsymbol{\sigma} = \boldsymbol{\sigma}_{2r}\,. \end{cases}$$

For case (c2) we have

$$\text{rank} V''_{2m}(\mathbf{u}) \geq \text{rank}\{\mathbf{u}, \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \overset{\circ}{\mathbf{A}}\,\mathbf{u}\} = 3\,, \tag{3.12}$$

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{n}\wedge \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{N}, \mathbf{u}\wedge(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} = 3,$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\otimes \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\vee\eta(\mathbf{A})', \mathbf{u}\vee\eta(\mathbf{A})'\} = 6,$$

where $\eta(\mathbf{A})' = \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$.

For case (c3) we have

$$\text{rank} V''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{n}, \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \overset{\circ}{\mathbf{A}}\,\mathbf{u}\} = 3,$$

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\wedge \overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}})), \overset{\circ}{\mathbf{u}}\wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}\} = 3\,, \tag{3.13}$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}}, \mathbf{n}\vee \overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}))\} = 6\,. \tag{3.14}$$

For cases (c4) we have

$$\text{rank} V''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{n}, \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} = 3,$$

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\wedge \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \mathbf{u}\wedge(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} = 3\,, \tag{3.15}$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}}, \mathbf{u}\vee(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} = 6\,. \tag{3.16}$$

For case (c5) we have (3.13) – (3.14) and (note that $\beta_1(\mathbf{q}(\mathbf{A})) \neq 0$)

$$\text{rank} V''_{2m}(\mathbf{u},\mathbf{A}) \geq \{\mathbf{u}, \overset{\circ}{\mathbf{A}}\,\mathbf{u}, \beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, (1-\delta_{1m})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\} = 3.$$

Finally, for case (c6) we have (3.12) and (3.15) – (3.16).

Thus, from the above results and Tables 1-3 in Sec. 2 in Part I, we deduce that the three presented sets of generators obey the criterion (2.3) in Part I. Further, by considering the pair $\mathbf{u}_1 = \mathbf{n}$ and $\mathbf{A}_1 = \mathbf{n} \vee \mathbf{e}$, we infer that the generator $\overset{\circ}{\mathbf{A}}\,\mathbf{u}$ and the respective last two generators in the two sets $\mathrm{Skw}''_{2m}(\mathbf{u}, \mathbf{A})$ and $\mathrm{Sym}''_{2m}(\mathbf{u}, \mathbf{A})$ are irreducible. Moreover, from case (c1) we know that the two generators $(\mathbf{u}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{N}$ and $(\mathbf{u}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))(\mathbf{A_e N} - \mathbf{N A_e})$ are irreducible. By considering, respectively, the pairs $\mathbf{u}_2 = \sigma_1$ and $\mathbf{A}_2 = \mathbf{e}\otimes\mathbf{e}$, $\mathbf{u}_3 = \sigma_2$ and $\mathbf{A}_3 = \mathbf{e}\otimes\mathbf{e}$ (for $m \geq 2$), we know that the generator $\overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}$ and the last generator in the set $V''_{2m}(\mathbf{u}, \mathbf{A})$ are irreducible.

3.3. Sets of three variables

As indicated at the outset of this section, we need only to treat the four sets $(\mathbf{u}, \mathbf{v}, \mathbf{W})$, $(\mathbf{u}, \mathbf{v}, \mathbf{A})$, $(\mathbf{u}, \mathbf{W}, \mathbf{A})$ and $(\mathbf{u}, \mathbf{A}, \mathbf{B})$. We shall demonstrate that each set X_0 just mentioned is D_{2md}-reducible, i.e there is a proper subset $S \subset X_0$ such that $\Gamma(S) \cap D_{2md} = \Gamma(X_0) \cap D_{2md}$.

First, let $X_0 = (\mathbf{u}, \mathbf{v}, \mathbf{D})$ with $\mathbf{D} \in \{\mathbf{W}, \mathbf{A}\}$ a skewsymmetric or a symmetric tensor. Suppose that X_0 is D_{2md}-irreducible. Then $(\mathbf{u}, \mathbf{v})$ is D_{2md}-irreducible and $\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2md} \neq C_1$. From these and cases (c1) – (c4) derived in (**iv**), we know that $\mathbf{u} = a\mathbf{n}$ and $\mathbf{v} = b\mathbf{e}$ with $ab \neq 0$ (see case (c1)). Since the group $\Gamma(\mathbf{v}) \cap D_{2md}$, i.e. $C_2(\mathbf{e})$ has only two subgroups, i.e. C_1 and $C_2(\mathbf{e})$, we deduce $\Gamma(\mathbf{u}, \mathbf{D}) \cap D_{2md} = C_1$ or $\Gamma(\mathbf{v}, \mathbf{D}) \cap D_{2md} = C_2(\mathbf{e}) = \Gamma(\mathbf{v}) \cap D_{2md}$. Either of the two cases mentioned above indicates that the $(\mathbf{u}, \mathbf{v}, \mathbf{D})$ is D_{2md}-reducible, contradicting the foregoing presupposition.

Second, let $X_0 = (\mathbf{u}, \mathbf{D}, \mathbf{A})$ with $\mathbf{D} \in \{\mathbf{W}, \mathbf{B}\}$ a skewsymmetric or a symmetric tensor. Suppose that X_0 is D_{2md}-irreducible. Then, both the set $(\mathbf{u}, \mathbf{A})$ and the set $(\mathbf{D}, \mathbf{A})$ are D_{2md}-irreducible and $\Gamma(\mathbf{z}, \mathbf{A}) \cap D_{2md} \neq C_1$, $\mathbf{z} = \mathbf{u}, \mathbf{D}$. From these and cases (c1) – (c6) derived in (**x**) and cases (c1) – (c5) derived in (**vi**) (for $\mathbf{D} = \mathbf{W}$) and cases (c1) – (c6) derived in (**vii**) (for $\mathbf{D} = \mathbf{B}$), we know $\mathbf{u} = a\mathbf{n}$, $\overset{\circ}{\mathbf{A}} = b(\sigma\otimes\sigma - \sigma'\otimes\sigma')$ (see case (c1) in (**x**)), $\mathbf{D} = \mathbf{W} = c\mathbf{En}$ (see case (c1) in (**vi**)) and $\overset{\circ}{\mathbf{D}} = \mathbf{B} = d(\bar\sigma\otimes\bar\sigma - \bar\sigma'\otimes\bar\sigma')$ (see cases (c1) – (c2) in (**vii**)), where $abcd \neq 0$ and σ, $\bar\sigma \in \{\sigma_1, \cdots, \sigma_{4m}\}$ and $(\sigma\cdot\bar\sigma)\sigma\times\bar\sigma \neq \mathbf{0}$. Thus, we deduce $\Gamma(\mathbf{u}, \mathbf{D}, \mathbf{A}) = C_2(\mathbf{n}) = \Gamma(\mathbf{D}, \mathbf{A})$, in contradiction to the foregoing presupposition.

3.4. The general results

Applying Theorem 2.1 in XIAO [20] and incorporating the fact indicated at the outset of this section, from (a) – (c) we obtain the following general result.

THEOREM 8. *The four sets given by*

$I''_{2m}(\mathbf{u})$; $I_{4m}(\mathbf{W})$; $I_{4m}(\mathbf{A})$; $I_{4m}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B},\mathbf{C})$; $\mathbf{u}\cdot\mathbf{v}, (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}),$

$(\mathbf{v}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{u}}) + (\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\mathring{\mathbf{v}},\eta_{2m-1}(\mathring{\mathbf{u}})],$

$(\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{v}}) + (\mathbf{v}\cdot\mathbf{n})[\mathbf{n},\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{v}})],$

$(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})^{4m-1} - \beta_{2m}(\mathring{\mathbf{v}})[\mathbf{n},\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{v}})],$

$(\mathbf{v}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n})^{4m-1} - \beta_{2m}(\mathring{\mathbf{u}})[\mathbf{n},\mathring{\mathbf{v}},\eta_{2m-1}(\mathring{\mathbf{u}})];$

$(\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathbf{W}^2\,\mathring{\mathbf{u}}, \mathrm{tr}\mathbf{W}(\mathbf{E}\eta_{2m-1}(\mathring{\mathbf{u}})),$

$(\mathrm{tr}\mathbf{W}\mathbf{N})(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathring{\mathbf{u}}) + \beta_{2m}(\mathring{\mathbf{u}})\,\mathring{\mathbf{u}}\cdot\mathbf{W}\mathbf{n},$

$(\mathrm{tr}\mathbf{W}\mathbf{N})\beta_{4m}(\mathring{\mathbf{u}}) - (\mathbf{u}\cdot\mathbf{n})^{4m-1}\,\mathring{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}, [\mathbf{n},\mathring{\mathbf{u}},\eta_{2m-1}(\mathbf{W}\mathbf{n})],$

$[\mathbf{n},\mathbf{W}\mathbf{u},\eta_{2m-1}(\mathbf{W}\mathbf{n})], [\mathbf{n},\mathbf{W}^2\mathbf{u},\eta_{2m-1}(\mathbf{W}\mathbf{n})],$

$(\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathbf{W}\mathbf{n}); \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}^2\,\mathring{\mathbf{u}},$

$[\mathbf{n},\mathring{\mathbf{A}}\,\mathbf{n},\eta_{2m-1}(\mathring{\mathbf{u}})] - \delta_{1m}(\mathbf{u}\cdot\mathbf{n})\mathrm{tr}\mathbf{A}\mathbf{D}_2, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\mathring{\mathbf{A}}\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{u}})]$

$+\beta_{2m}(\mathring{\mathbf{u}})\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n}, \beta_{2m}(\mathring{\mathbf{u}})[\mathbf{n},\mathring{\mathbf{A}}\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{u}})]$

$-(\mathbf{u}\cdot\mathbf{n})^{4m-1}\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n}, [\mathbf{n},\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})], [\mathbf{n},\mathring{\mathbf{A}}\,\mathbf{u},\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})],$

$\beta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\beta_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})J(\mathbf{A})\alpha_m(\mathbf{q}(\mathbf{A}));$

$\alpha_{2m}(\mathring{\mathbf{v}})[\mathbf{r},\mathbf{u},\mathring{\mathbf{v}}], \alpha_{2m}(\mathring{\mathbf{u}})[\mathbf{r},\mathbf{v},\mathring{\mathbf{u}}];$

$\mathbf{u}\cdot\mathbf{W}\mathbf{v}, (\mathrm{tr}\mathbf{W}\mathbf{N})((\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\mathring{\mathbf{v}}) + (\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\mathring{\mathbf{u}}));$

$\mathbf{u}\cdot\mathring{\mathbf{A}}\,\mathbf{v}, (\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\mathring{\mathbf{v}})[\mathbf{n},\mathring{\mathbf{v}},\mathring{\mathbf{A}}\mathring{\mathbf{v}}]$

$+(\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\mathring{\mathbf{u}})[\mathbf{n},\mathring{\mathbf{u}},\mathring{\mathbf{A}}\mathring{\mathbf{u}}];$

$[\mathbf{n},\mathbf{\Omega}\mathbf{u},\eta_{2m-1}(\mathbf{W}\mathbf{n})], [\mathbf{n},\mathbf{W}\mathbf{u},\eta_{2m-1}(\mathbf{\Omega}\mathbf{n})],$

$(\mathbf{u}\cdot\mathbf{n})((\mathrm{tr}\mathbf{W}\mathbf{N})|\mathrm{tr}\mathbf{W}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{\Omega}\mathbf{n})$

$+(\mathrm{tr}\mathbf{\Omega}\mathbf{N})|\mathrm{tr}\mathbf{\Omega}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{W}\mathbf{n}));$

$(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{W}\mathbf{N})\alpha_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}), (\mathbf{u}\cdot\mathbf{n})(\alpha_{2m}(\mathbf{W}\mathbf{n})$

$+(\mathrm{tr}\mathbf{W}\mathbf{N})|\mathrm{tr}\mathbf{W}\mathbf{N}|^{2m-1})\alpha_m(\mathbf{q}(\mathbf{A})),$

$(\mathrm{tr}\mathbf{W}\mathbf{N})^{2m-2}[\mathbf{n},\mathbf{W}\mathbf{u},\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})]$

$-|\,\mathring{\mathbf{A}}\,|^{2m-2}[\mathbf{n},\mathring{\mathbf{A}}\,\mathbf{u},\eta_{2m}(\mathbf{W}\mathbf{n})], \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{W}\,\mathring{\mathbf{u}};$

$$[\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{u}, \eta_{2m-1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})], [\mathbf{n}, \overset{\circ}{\mathbf{B}}\,\mathbf{u}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})],$$
$$(1-\delta_{1m})(\mathbf{u}\cdot\mathbf{n})(\alpha_m(\mathbf{q}(\mathbf{B}))\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{A}), \mathbf{q}(\mathbf{B})]\alpha_m(\mathbf{q}(\mathbf{A}))),$$
$$(1-\delta_{1m})(\mathbf{u}\cdot\mathbf{n})(\alpha_m(\mathbf{q}(\mathbf{A}))\alpha_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{B}), \mathbf{q}(\mathbf{A})]\alpha_m(\mathbf{q}(\mathbf{B})));$$

and

$$V''_{2m}(\mathbf{u}), V''_{2m}(\mathbf{W}), V''_{2m}(\mathbf{A});$$
$$\alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{v}\times\overset{\circ}{\mathbf{u}}, \alpha_{2m}(\overset{\circ}{\mathbf{v}})\mathbf{u}\times\overset{\circ}{\mathbf{v}}; \boldsymbol{\Omega}(\mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n})), \mathbf{W}(\mathbf{n}\times\eta_{2m-1}(\boldsymbol{\Omega}\mathbf{n})),$$
$$((\mathrm{tr}\mathbf{W}\mathbf{N})|\mathrm{tr}\mathbf{W}\mathbf{N}|^{2m-1}\alpha_{2m}(\boldsymbol{\Omega}\mathbf{n}) + (\mathrm{tr}\boldsymbol{\Omega}\mathbf{N})|\mathrm{tr}\boldsymbol{\Omega}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{W}\mathbf{n}))\mathbf{n};$$
$$(\mathrm{tr}\mathbf{W}\mathbf{N})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, (\alpha_{2m}(\mathbf{W}\mathbf{n}) + (\mathrm{tr}\mathbf{W}\mathbf{N})|\mathrm{tr}\mathbf{W}\mathbf{N}|^{2m-1})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n},$$
$$(\mathrm{tr}\mathbf{W}\mathbf{N})^{2m-2}\mathbf{W}(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})) + |\,\overset{\circ}{\mathbf{A}}\,|^{2m-2}\,\overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n}));$$
$$\overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})), \overset{\circ}{\mathbf{B}}\,(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})),$$
$$(1-\delta_{1m})(\alpha_m(\mathbf{q}(\mathbf{B}))\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{A}), \mathbf{q}(\mathbf{B})]\alpha_m(\mathbf{q}(\mathbf{A})))\mathbf{n},$$
$$(1-\delta_{1m})(\alpha_m(\mathbf{q}(\mathbf{A}))\alpha_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{B}), \mathbf{q}(\mathbf{A})]\alpha_m(\mathbf{q}(\mathbf{B})))\mathbf{n};$$
$$\mathbf{W}\mathbf{u}, (\mathrm{tr}\mathbf{W}\mathbf{N})\alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{n}; \overset{\circ}{\mathbf{A}}\,\mathbf{u}, (1-\delta_{1m})\alpha_{2m}(\overset{\circ}{\mathbf{u}})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n};$$

and

$$\mathrm{Skw}''_{2m}(\mathbf{u}), \mathrm{Skw}_{4m}(\mathbf{W}), \mathrm{Skw}_{4m}(\mathbf{A});$$
$$\mathbf{u}\wedge\mathbf{v}, ((\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{v}}) + (\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{u}}))\mathbf{N};$$
$$\mathbf{W}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{W}; \overset{\circ}{\mathbf{A}}\,\mathbf{W} + \mathbf{W}\,\overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\,\mathbf{W}^2 - \mathbf{W}^2\,\overset{\circ}{\mathbf{A}}; \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}} - \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}},$$
$$\overset{\circ}{\mathbf{B}}\,\mathbf{n}\wedge\,\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\,\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\wedge\,\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\,\mathbf{n};$$
$$(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathbf{W}\mathbf{n})\mathbf{N},$$
$$|\mathbf{u}|^{2m-2}\mathbf{u}\wedge(\mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n})) + (\mathrm{tr}\mathbf{W}\mathbf{N})^{2m-1}\mathbf{n}\wedge\eta_{2m-1}(\overset{\circ}{\mathbf{u}});$$
$$\overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{N},$$
$$|\mathbf{u}|^{2m-2}\mathbf{u}\wedge(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})) + |\mathbf{q}(\mathbf{A})|^{2m-2}\mathbf{n}\wedge\,\overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}));$$

and

$$\mathrm{Sym}''_{2m}(\mathbf{u}), \mathrm{Sym}_{4m}(\mathbf{W}), \mathrm{Sym}_{4m}(\mathbf{A});$$

$$\mathbf{u} \vee \mathbf{v}, (\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\overset{\circ}{\mathbf{v}})\, \overset{\circ}{\mathbf{v}} \vee(\mathbf{n}\times \overset{\circ}{\mathbf{v}})$$

$$+(\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\overset{\circ}{\mathbf{u}})\, \overset{\circ}{\mathbf{u}} \vee(\mathbf{n}\times \overset{\circ}{\mathbf{u}});$$

$$\mathbf{W\Omega} + \mathbf{\Omega W},$$

$$|\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})\mathbf{Wn} \vee \mathbf{NWn} + |\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\mathbf{\Omega n} \vee \mathbf{N\Omega n};$$

$$\overset{\circ}{\mathbf{A}}\, \mathbf{W} - \mathbf{W}\, \overset{\circ}{\mathbf{A}}, (\mathrm{tr}\mathbf{WN})\, \overset{\circ}{\mathbf{A}}\, \mathbf{n} \vee \mathbf{N}\, \overset{\circ}{\mathbf{A}}\, \mathbf{n}; \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}} + \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}};$$

$$\delta_{1m}(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{D}_1, \mathbf{W}\, \overset{\circ}{\mathbf{u}} \vee(\mathbf{n}\times \eta_{2m-1}(\mathbf{Wn})),$$

$$(\mathrm{tr}\mathbf{WN})\, \overset{\circ}{\mathbf{u}} \vee(\mathbf{n}\times \overset{\circ}{\mathbf{u}}),$$

$$|\mathbf{u}|^{2m-2}\mathbf{u} \vee (\mathbf{n}\times \eta_{2m-1}(\mathbf{Wn})) + (\mathrm{tr}\mathbf{WN})^{2m-1}\mathbf{n} \vee \eta_{2m-1}(\overset{\circ}{\mathbf{u}});$$

$$(\mathbf{u}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))(\mathbf{A_e N} - \mathbf{NA_e}), (\mathbf{u}\cdot\mathbf{n})\, \overset{\circ}{\mathbf{A}}\, \mathbf{n} \vee (\mathbf{n}\times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})),$$

$$|\mathbf{u}|^{2m-2}\mathbf{u} \vee (\mathbf{n}\times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})) + |\mathbf{q}(\mathbf{A})|^{2m-2}\mathbf{n}\vee\, \overset{\circ}{\mathbf{A}}\, (\mathbf{n}\times \eta_{2m-1}(\overset{\circ}{\mathbf{u}}));$$

where $(\mathbf{u},\mathbf{v},\mathbf{r}) = (\mathbf{u}_i,\mathbf{u}_j,\mathbf{u}_k)$, $(\mathbf{W},\mathbf{\Omega},\mathbf{H}) = (\mathbf{W}_\mu,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B},\mathbf{C}) = (\mathbf{A}_L,\mathbf{A}_M,\mathbf{A}_N)$, $k > j > i = 1,\cdots,a$, $\theta > \tau > \mu = 1,\cdots,b$, $N > M > L = 1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1,\cdots,\mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1,\cdots,\mathbf{A}_c$ *under the group* D_{2md} *for each* $m \geq 1$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold rotation axis of the group* D_{2md}.

In the above theorem, $I_{4m}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B},\mathbf{C})$ is used to represent the invariants depending on two or three symmetric and/or skewsymmetric tensors given in Theorem 1 in Part I with the replacement of m by $2m$ therein.

4. Concluding remarks

Based upon symmetry-reduced decompositions of the domain of any finite number of vector variables and second order tensor variables in XIAO [16, 20], a simple, unified procedure for constructing both generating sets and the functional bases is designed and developed in the recent work (XIAO [18 – 19]) and this series of works. This unified procedure reduces the tough problem of determining irreducible representations for anisotropic functions of any finite number of vector variables and second order tensor variables to that of determining irreducible representations for anisotropic functions of certain sets consisting of not more than three vector and/or second order tensor variables. The g-irreducibility conditions for sets of two and three variables (see (3.1) – (3.2) in Part I) further provide a considerable simplification in dealing with representations for sets of two and three variables. The condition (3.3) in Part I is helpful to remove some redundant invariants in forming the scalar products of the variables $\mathbf{r}$, $\mathbf{H}$ and $\mathbf{C}$ and the presented generators. In addition, the notion of isotropic extension of anisotropic functions and the much well-known results for isotropic functions (see, e.g., SPENCER [13], WANG [14], SMITH [12], BOEHLER [3]) are essential. The former was originated earlier from LOKHIN and SEDOV [9] and independently introduced and successfully applied to derive systematic results for anisotropic functions in some cases for the first time by BOEHLER *et al.* [4 – 7] and developed later by many researchers, refer to, e.g., LIU [8], BETTEN, BOEHLER, SPENCER (see [6]), RYCHLEWSKI [10], ZHANG and RYCHLEWSKI [23], BETTEN [2], ZHENG and SPENCER [25], *et al.*; see also the reviews by BETTEN [1], RYCHLEWSKI and ZHANG [11] and ZHENG [24] for detail. A substantial generalization in this aspect has been given very recently by one of the authors (see XIAO [15, 17]).

By applying the above unified procedure, together with the results for isotropic extension of anisotropic functions and the much well-known results for isotropic functions, we have derived irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of any finite number of vector variables and second order tensor variables under all crystal classes and quasicrystal classes as subgroups of the cylindrical group $D_{\infty h}$.

Thus, of all kinds of material symmetry groups of solids, only the five cubic crystal classes and the two icosahedral classes have not yet been covered. According to Theorem 3.2 in XIAO [16], the unified procedure outlined in Sec. 3 in Part I also applies to these classes, except for the fact that the set $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ of three symmetric tensor variables should be added.

Although the procedure used merely involves irreducible representations for one, two and three variables, many details concerning these representations need to be examined. Such a situation results from the complexity of nonlinear anisotropic tensor functions. In reality, much effort and labour should be made even for isotropic tensor functions. Nevertheless, once the results for relevant representations are derived, their correctness can be guaranteed by checking the fulfilment of Criteria 1 and 2 given in Part I, as has been done. Towards this goal, it is crucial to work out the related g-irreducible sets of two or three variables from the conditions (3.1) or (3.2) in Part I. This crucial aspect has been treated in a definite and rigorous manner.

This series of papers is concerned with material symmetries of solids, which are described by finite and continuous infinite subgroups of the 3-dimensional full orthogonal group. Other kinds of material symmetries, such as those of liquid crystals etc., are characterized by subgroups of the 3-dimensional unimodular group $\mathcal{U}(3)$. It is expected that the procedure used may be extended to cover the latter kinds of material symmetry groups and other groups. The main basis in this more general aspect has been laid down by RYCHLEWSKI [10]. In the latter, the existence and reality of isotropic extension is proved to be true in the most general sense that an arbitrary group acts on an arbitrary set.

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Thermomechanics of viscoplastic large strains of solid polymers

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EXPERIMENTAL RESULTS concerning cyclic tension tests performed on polyamid based – *braiding samples* and on *usual samples* machined of the same PA66 polyamid are presented. The measurements are *thermomechanical*: the temperature of the surface of the sample is recorded, resulting in the estimation of the intrinsic internal heat supply. The materials are exhibiting not only the classical viscoelastoplastic behaviour, but also thermomechanical effects of the Kelvin and Taylor-Farren-Quinney type. Modelling based on a previously proposed constitutive formulation of cyclic viscoplastic isotropic behaviour is implemented.

Notations

t	absolute time,
. or $\partial/\partial t$	partial time derivative
$M(x^k)$	material point, of invariant convected co-ordinates x^k
$\rho(M,t)$	current density of the material point $M(x^k)$
$D(M,t)$	current strain rate
$T(M,t)$	current absolute temperature of the material point $M(x^k)$
$K_n(M,t)$	current stretches along $\mathbf{e}_n$ for the material point $M(x^k)$
$J_n(M,t)$	$1 + K_n(M,t)$
S	current Cauchy stress
${}^t_R S$	current Cauchy stress at a loading-unloading point $R(t_R < t)$
$\Delta^t_R S$	current "variation" of the stress defined as the $\mathbf{S}$ variation $\mathbf{S} -{}^t_R \mathbf{S}$ on $]t_R, t]$

$\Delta_R^t \varepsilon$	current "variation" of the strain
ε	usual Almansi strain ($\varepsilon = \Delta_0^t \varepsilon$)
P_i, p_i	current internal power at the material point M per unit volume, per unit mass, respectively ($P_i(M,t) = \rho(M,t) p_i(M,t)$)
Π, π	current reversible power at the material point M
Φ, ϕ	current intrinsic dissipation at the material element $M(\Phi = -P_i - \Pi; \phi = -p_i - \pi)$
E, e	current internal energy at the material point M
I	current inertia-like tensor of the material point M
K	current kinetic energy at the material point M
Q_i, q_i	current internal irreversible heat supply at the material point M
Q_e, q_e	current internal reversible heat supply at the material point M
I_D, I_s	first invariant of strain rate and stress, respectively
S_o	limit stress of the Huber-Mises-Henckly criterion (associated cylinder of radius $Q_o = \sqrt{2} S_o$)
λ, μ	Lamé parameters
θ_ν	relaxation time parameter of the Oldroyd consitutive model
η_1, η_2	viscosity parameters of the Oldroyd consitutive model.

1. Introduction

1.1. From the initial general motivation to the present preliminary study

THE INITIAL MOTIVATION of the study was to elucidate the problems of constitutive nature raised by a set of thermomechanical measurements obtained by cyclic tests on polyamid based samples. It is worth noting immediately that these samples were of fabric type (belts) and of solid one, in order to involve two different types of *polyamid microstructures*.

The definition of an effective modelling was unusually difficult because mechanical and thermal effects were *simultaneously* recorded. Moreover, purely mechanical measurements of second the order effects of *ratchet* type were also available. Consequently, the required viscoelastoplastic modelling ought to be, in principle, thermomechanical and also defined up to the second order effects. Owing to the importance of the second order effects and of the Taylor-Farren-Quinney effect in order to *state the principles and the methods of continuum mechanics*, the span of the difficulties involved in the task was rather enormous (and, as a matter of fact, the current state of the study is still incomplete).

In the first step, an approach has been proposed to guarantee the *simultaneous* treatment of the *second order* effects such as those of ratcheting type and of the *first order* features of the viscoelastoplastic behaviour [7]. Due to rather restrictive assumptions of this theory in its present form, the modelling of the behaviour of filaments and of the fabric samples is not in a very close relationship with the manufacturing processes. For example, manufacturing processes of filaments involve indeed large temperature variations (and large strains to be taken into account from now on) whereas the present pattern assumes a constant temperature.

It is therefore necessary to make the theory *basically free from the constant temperature assumption*. This presentation is an overview of the results already available in order to *proceed in that direction*, namely towards an effective *isotropic* thermomechanical theory. Owing to the aim of this piece of research, the paper cannot be viewed as a comprehensive elucidation of the *initial* matter[1]). It remains also incomplete with respect to the history of the matter[2]), with regard to the results already obtained for the benefit of engineers[3]) and with respect to the question of the microstructural origin of the basically irreversible behaviour of the solid polymers and rubber-like materials.

1.2. Main restrictive assumptions of the study

i. The behaviour is supposed to be isotropic;

ii. Fatigue effects as well as ageing effects are neglected: on the contrary, rate-independent hardening effects are taken into account, if necessary, through an effective modelling, remaining however sketchy due to a previously defined theoretical background [2, 3];

iii. The limit plastic behaviour is supposed to be of the Huber-Mises-Hencky type.

1.3. Two basic features of the study

i. During a cyclic loading program, the total amplitude of the temperature variation associated with the internal intrinsic heat supply (of reversible and irreversible nature) is smaller than ten Kelvin degrees, approximately. Consequently, no thermal corrections are applied: the constitutive and physical parameters are supposed to be constant, and the outline of the model may be summarised as follows:

1) On the one hand, the "small" global temperature variations ΔT are obtained by *direct Infrared Thermovision (IRT) measurements*, and the usual mechanical data are *simultaneously recorded*;

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[1]) including elucidation of issues such as those of the anisotropic modelling, of the hardening modelling and of classification of the second order effects.

[2]) including ancient elements which are not generally known [25, 1].

[3]) including elements of the theory of thermoplasticity [12, 2, 3].

2) On the other hand, the Kelvin-like contribution to the temperature variations ΔT_k, and the Taylor-like irreversible contribution to the temperature variations ΔT_a are *deduced from the Kelvin relation and from the mechanical data* through the consitutive theory, respectively.

3) Finally, the measured ΔT is compared with the sum $\Delta T_k + \Delta T_a$.

The *temperature dependence* of the constitutive parameters is implemented in the constitutive theory *only* when large temperature variations are *imposed from the surrounding*, as it is often the case for the filaments and threads involved in the manufacturing processes of braiding or fabrics.

ii. The formalism of the proposed modelling is that of finite strain. Note that this formalism *must be* implemented *in the fixed frame* in the case of *irrotational* homogeneous deformation of *isotropic* materials (cf. [7]). This is relevant for interpretation of the cyclic traction tests considered in this paper.

2. Description of experiment and results

2.1. Samples

The geometric features of the samples of type (a), (b), (c) are given in Figs. 1a, b, c, respectively. All the samples are polyamid-based (PA66). The (a) type samples are fabric – belts used in aviation transport, the others are machined. The machined samples of type (b) and (c) were cut from a polyamide plate, in the same direction. For (a) the gauge length, width and thickness are: 150 ± 2 mm; 44.4 ± 0.4 mm; 3.8 ± 0.1 mm, respectively. For (b) these parameters are: 60 mm; 15.90 ± 0.06 mm; 8.65 ± 0.11 mm. For (c) the gauge length is half the previous one (30 mm) and both the width and thickness are similar.

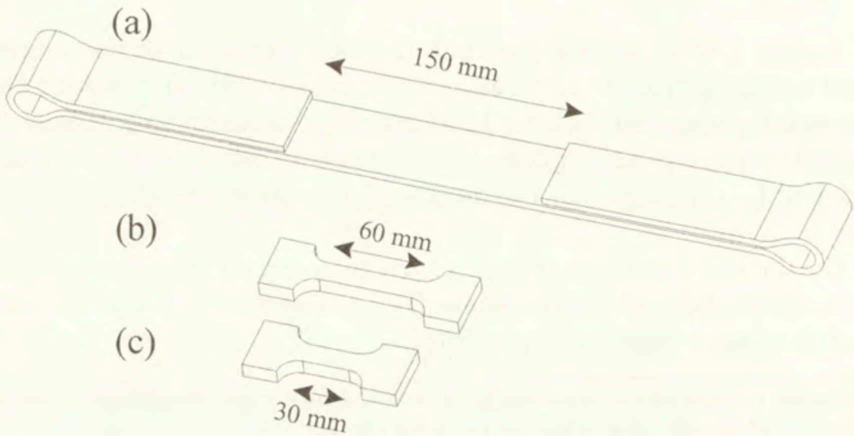


FIG. 1. Sketches of the fabric sample (a) and of the machined samples (b) and (c).

2.2. Experimental procedure

The belts were cycled according to the following procedure: first loading up to 10 kN and subsequent reloadings up to 20, 30, and 40 kN; minimum force at the end of the unloading is specified at a value of 0.2 kN. The value of elongation was measured with an extensometer with gauge length equal to 50 mm. The rate of deformation was equal to 1.3×10^{-2} s^{-1} for belts GV 40, and to 2.5×10^{-2} s^{-1} for belt GV 34 and GV 39.

The loading processes concerning the long samples (b) were chosen similarly to those implemented for the belts. Owing to the fact that the behaviour is rather of elastic-perfectly plastic type, the locations of the cycles are specified through a discrete set of specified strains, resulting in a typical procedure such as loading up to 3%, unloading towards 0.20 kN, and so on, with subsequent reloading up to 6, 9 and 12%. The strain rate value was 10^{-2} s^{-1} for all the tests.

The shorter samples (c) were subjected to symmetric push-pull loading of amplitude ±3%; 10 cycles were executed in each test. The rate of deformation was equal to 10^{-2} s^{-1}. The value of elongation of both kinds of samples was measured using the extensometer with gauge length of 25 mm.

During deformation, the load, the elongation and the distribution of infrared radiation, emitted by the sample, were continuously registered.

It is worth noting that some purely mechanical cyclic shear tests were also performed on the same material. These tests have been done at room temperature. The samples were rectangular sheets of PA66. The nominal overall dimensions were: thickness 3 mm, width 22 mm, and height 50 mm. The widths of the shearing zones were 5 mm, giving the ratio height to width equal to ten. Such a ratio is indeed generally admitted in order to minimise the error due to the non-homogeneity of the shear strain field at the ends of the sample [11, 24]. Using a two-way extensometer, both the shear strain and the lateral strain were measured locally, in the middle of the shearing zone. The shear device was designed to study the process of pure shear without lateral force (the force perpendicular to the shear direction), making the ratchetting effect conspicuous [13].

2.3. Temperature measurement

The distribution of infrared radiation emitted by the material surface was measured using a thermovision camera coupled with a computer system of data acquisition and conversion. The field of the sample surface was scanned through the optical system of the camera, focussed on a detector converting infrared radiation into a proportional electric signal. The power of the radiation emitted from the homogeneous surface obeys the Stefan – Boltzmann law, so it is a function of temperature only. Then, the measured signal of the material surface distribution can be processed in order to obtain the temperature distribution. This is done using a computer system of data acquisition and conversion. As a result one obtains the thermovision pictures (thermograms), registered one by one in the computer memory. The obtained pictures are of various precision. During 0.06 s the camera creates an image called frame. The frame contains few details, but owing to the short time of its creation it can be valuable only in monitoring the change of the temperature corresponding to the beginning of the process of elongation. Consequently, four such superimposed frames create a thermal picture, obtained during 0.24 s. The set of such pictures are the basic data to calculate the changes of the temperature during thermomechanical processes (see Fig. 2). The mean-square error of temperature evaluation was about 0.2 K.

The measuring system has no contact with the sample and, moreover, it does not involve inertial effects.

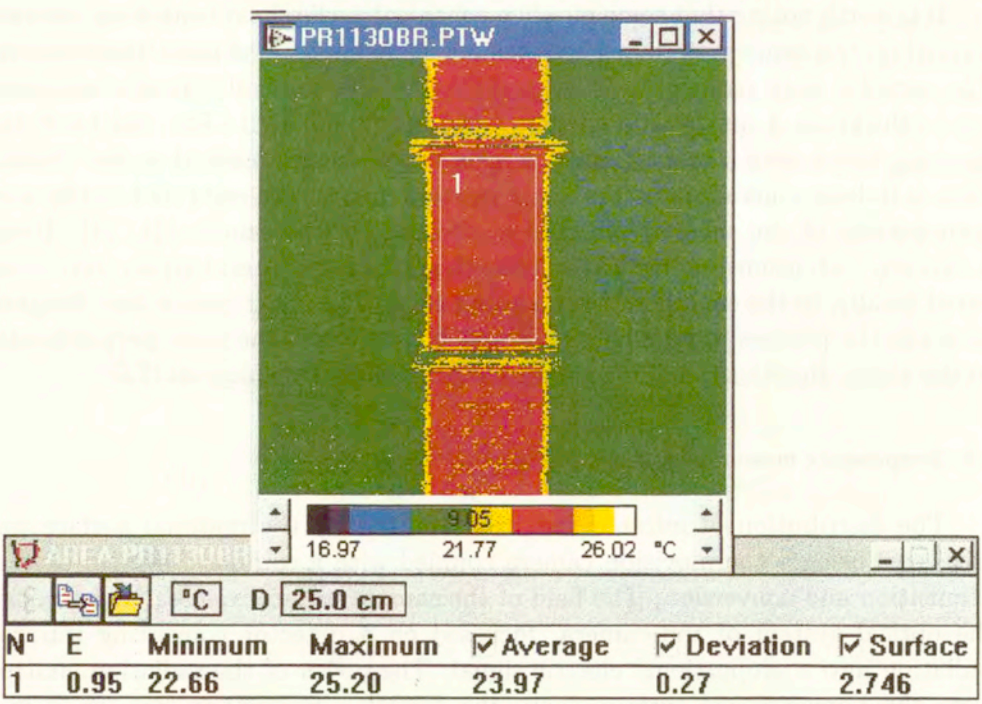


FIG. 2. Example of thermogram with the pointed area of temperature measurement.

2.4. Experimental results concerning belts

In the case of belts it was not possible to calculate the stresses accurately, because the dimensions of the cross-sections were not easily determined. Consequently, the mechanical characteristics of the belts are given as load – elongation relations. An example of those characteristics obtained during cycles of increasing amplitude of loading is shown in Fig. 3a.

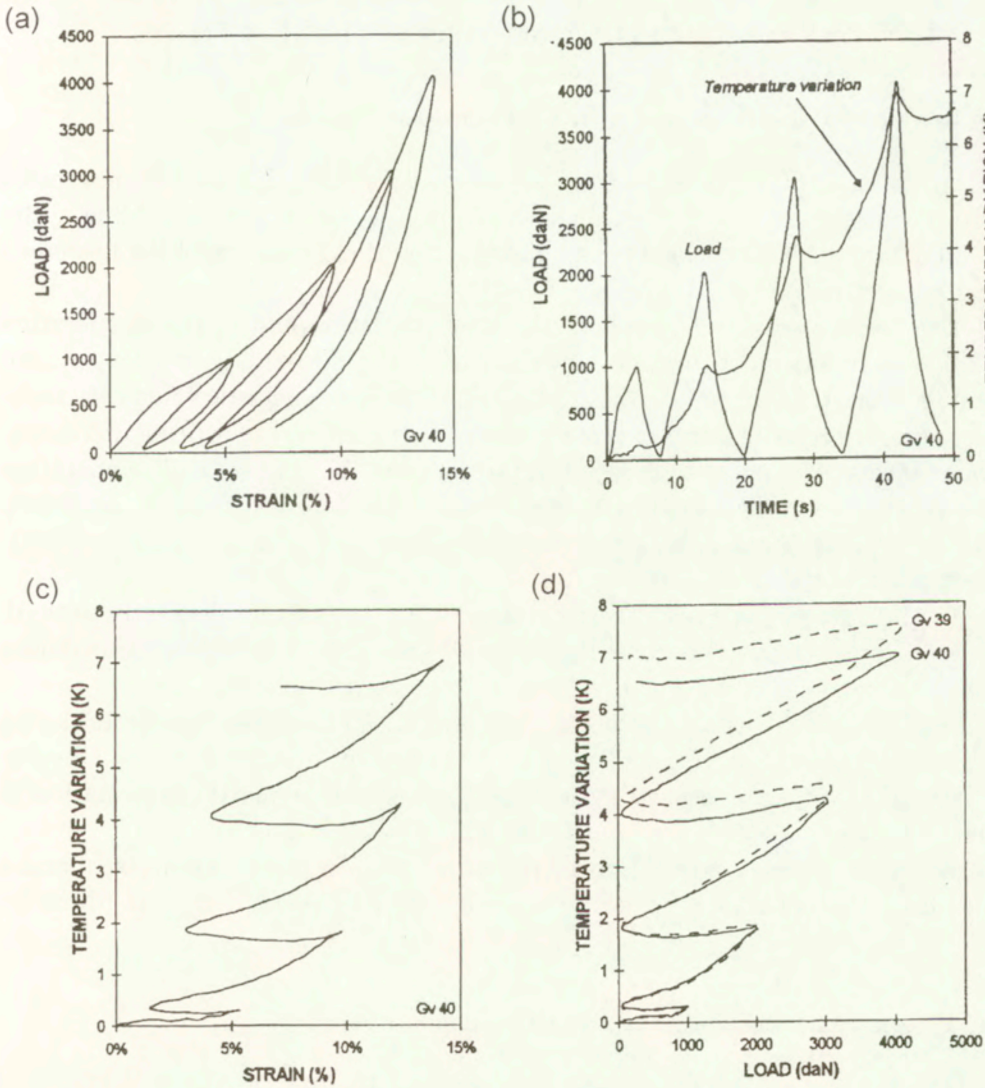


FIG. 3. Thermomechanical experimental result concerning the fabric sample: the rate of stretch is 1.3 s^{-1}, except for sample GV 39 (cf. (d) obtained with 2.5 s^{-1}).

These mechanical characteristics have been compared with the changes of temperature obtained during the tests (see Fig. 3b). It is seen that temperature was smoothly increasing during loading and less decreasing during unloading. During loading and unloading no thermoelastic effects were observed. These effects are more visible in Fig. 3c where temperature is shown as a function of strain and in Fig. 3d presenting the temperature versus the force graph. Temperature is not a linear function of strain or force in any periodic cycle. Fluctuations of temperature during the first cycles and during unloading are caused by measuring the errors and oscillations of the belt during unloading. An increase of the rate of deformation produces an increase of temperature (see Fig. 3d).

2.5. Experimental results concerning the PA66 samples of long type

An example of stress-strain relation obtained for the polyamid samples tested with increasing amplitude of strain is shown in Fig. 4a. In the succeeding cycles with an increase of viscoelastoplastic deformation the dimension of the hysteresis loop has been observed.

Fig. 4b presents an example of the stress variation and of the temperature changes as functions of time, obtained during the cyclic loading of a longer solid sample. Over the range of elastic deformation, the thermoelastic effects are registered, e.g. decrease of temperature during loading and increase during unloading. These effects are similar to those observed for steel [5]. The typical temperature change during the first loading is higher than that found for steel – exceeding 1 K. It is most observable during the two initial cycles, when the viscoplastic effect is not noticeable.

In the subsequent cycles thermoelastic effects are not so distinctly noticed. Temperature increments become higher and higher (due to heat generation during the viscoelastoplastic deformation and small heat conduction).

The same effect is shown in Figs. 4c and 4d. Thermoelastic effects during loading and unloading (Fig. 4d) can be described by linear functions. One can observe that after the second cycle there are regions of plastic deformation in which the temperature increase is faster than the linear one.

A viscous stress contribution of the order of 35% of the total stress and a relaxation time of 10 s is suggested by the results of relaxation tests (duration 300 s).

2.6. Experimental results concerning the PA66 samples of short type

Using sufficiently short sample it is possible to perform classical push-pull tests which are symmetric with respect to the origin in a reasonable strain range.

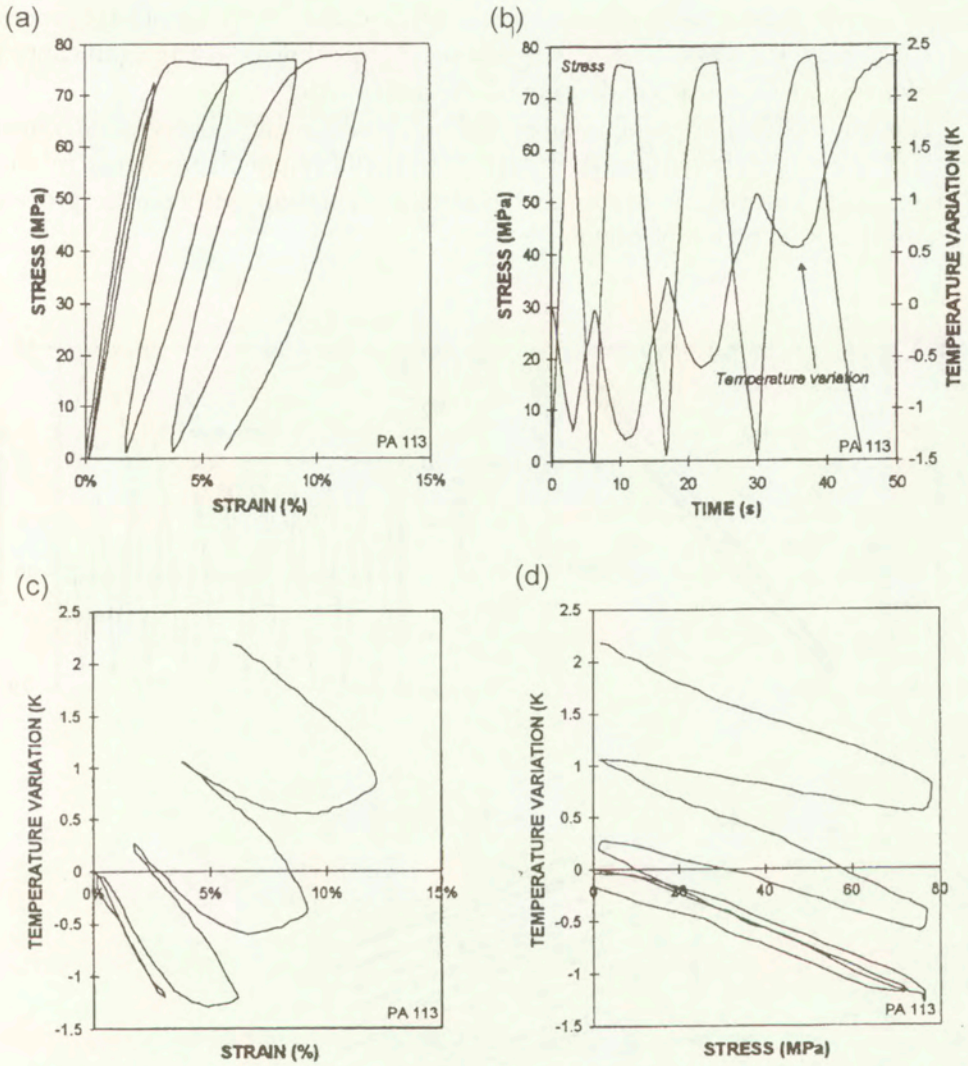


FIG. 4. Thermomechanical experimental results concerning the machined sample of long type.

The result is shown in Fig. 5a. The hysteresis loop was observed during this kind of loading. It was evaluated for each subsequent cycle, but the changes were smaller and smaller. The decrease of stresses was observed as well (Mullins effect).

Experimental temperature-time data and stress-time data obtained by means of this test are shown in Fig. 5b. In every cycle thermoelastic effects are seen during loading and unloading. These effects are modified along each next cycle in accordance with the decreasing stress variation. Average temperature of the sample increases as a result of heat accumulation during the viscoplastic process. After stopping the deformation, the temperature changes are insignificant, in accordance with low heat conduction of this polyamide.

The same effects are more visible in Fig. 5c. One can observe that during the first cycle of deformation (bottom curves), the temperature-stress relation deviates from linearity. It means that plastic deformation (viscoplastic process) takes part in this region of cycling.

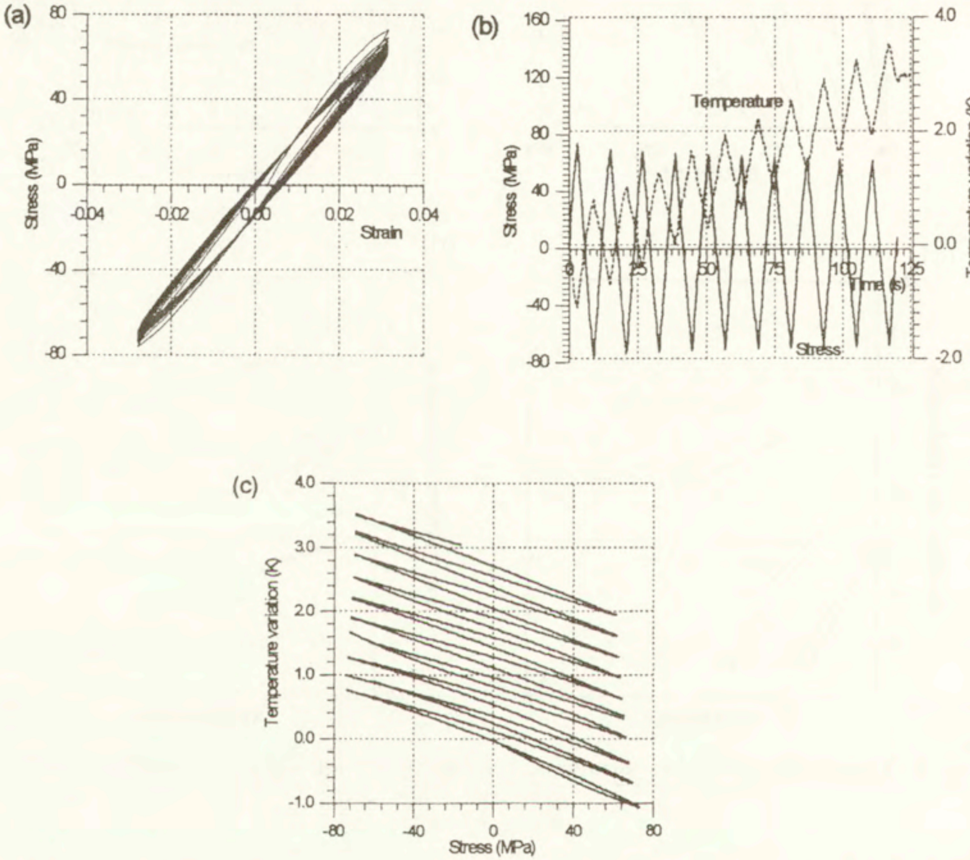


Fig. 5. Thermomechanical experimental results concerning the machined sample of short type.

2.7. Experimental results of the shearing test

The results obtained for short samples are indeed confirmed in a much larger strain range through shear tests *exhibiting ratchet effects* K_3 under vanishing axial force F_3 (cf. the provisional results shown in Fig. 6, where the evolution of

K_3 at the apexes of the cycles is, for the moment, much more reliable than the details concerning the shapes of the branches).

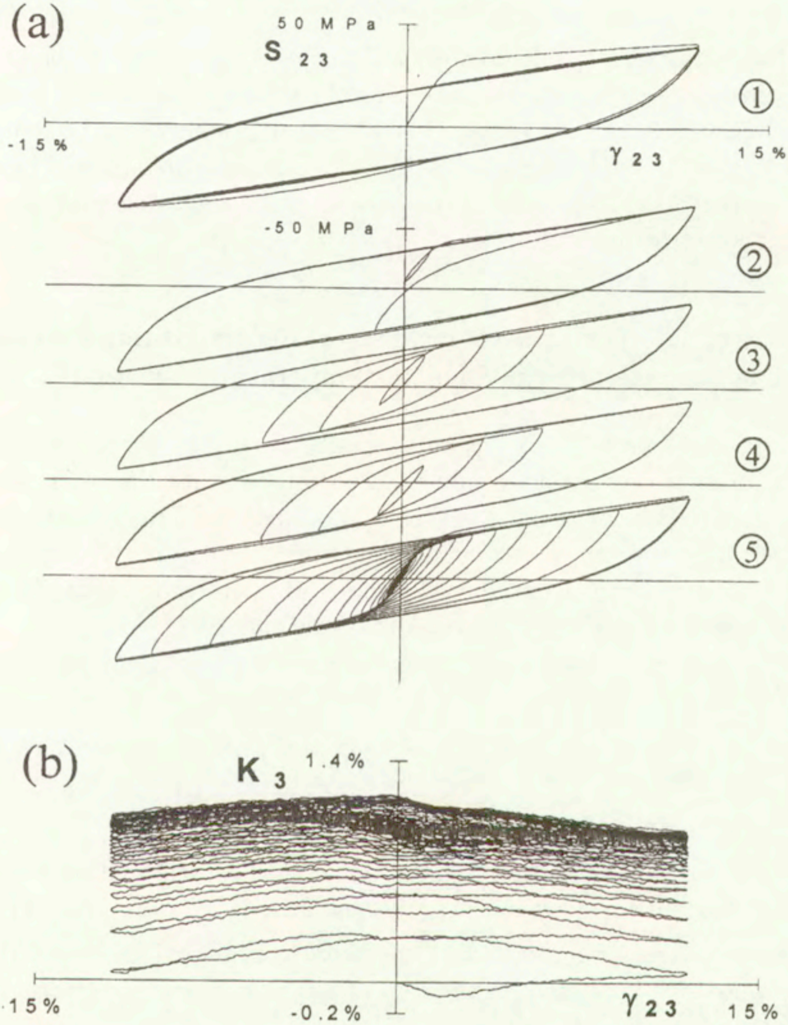


FIG. 6. Typical experimental results concerning the ratchet effect (b) exhibited during a long and intricate cyclic shear test with no normal force (stages 1 to 5). The sample is machined in the PA66 polymer studied in tension tests.

2.8. Summary of experimental results

The thermomechanical coupling observed during cyclic deformation of solid polyamide samples is similar to that observed in metals. Over the elastic range of deformation, temperature decreases during tension and increases during unloading and compression. These effects are higher than those established for steel

– exceeding 1 K. Average temperature of the sample cycled in elastic range of deformation increases as a result of heat accumulation during viscoelastic process.

Over the plastic range of deformation, the temperature of samples always increases.

The thermomechanical effects observed during loading and unloading of the belts are different – temperature was smoothly increasing during loading and slightly decreasing during unloading. The discrepancies between the temperature characteristics of the solid samples and the belts were probably caused by different properties of polyamide fibres in comparison with the solid material, and friction between the fibres of the belt.

3. Overview of the thermomechanical stress decomposition concept, including the case of large ambient temperature variations

Owing to the fact that an intricate amalgamation of processes is involved in the behaviour under consideration, it is useful to state the main features of the basic effective ingredients of the theoretical analysis. The point is even more important since the present paper cannot be self-contained (cf. [7] and [8 – 10, 14, 15, 20 – 22], if necessary). Consequently, some hints are given, *with illustrations*, concerning five of the six basic ingredients of the theory:

i. Pure hysteresis behaviour, allowing the description of *perfectly closed cycles* (cf.[8 – 10, 14, 15, 20 – 22]);

ii. Irreversible internal *rate of heat supply of pure hysteresis* type (cf. [8 – 10, 14, 15, 20 – 22]);

iii. Reversible internal rate of heat supply of *Kelvin* type;

iv. *Stress decomposition* thermomechanical rule, discriminating between the various contributions involved in the global balance of internal power P_i (cf. [7]);

v. Generalisation of the pure hysteresis pattern, and therefore of the whole stress decomposition approach, to the case of *large temperature variations* (cf. [2, 3]).

The generalisation allowing for the ratchet effects is not discussed here (cf. [7]).

3.1. Short reminder concerning the rate-independent ideal pure hysteresis behaviour

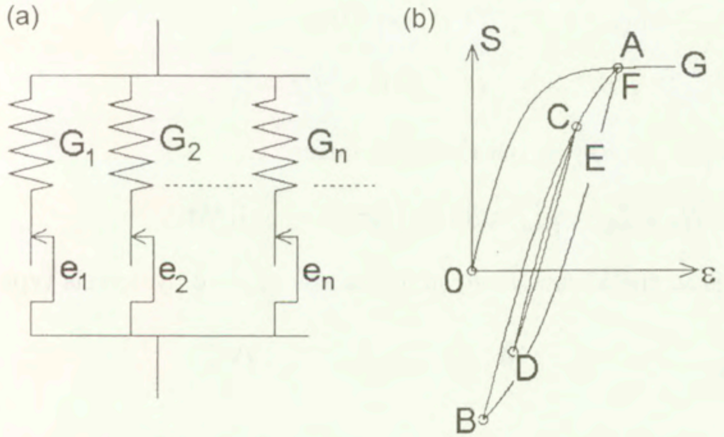
The parallel-series model of Fig. 7a is the heuristic symbolic model (Masing, 1926) of the always irreversible behaviour under consideration. It consists of a set of springs (stiffness G_i) and friction sliders (critical slipping at strain $e_i = S_i/G_i$) arranged in increasing *order* of e_i. If the number of elements becomes large and if the stiffness of the pairs having their deformation limit between e and $e + de$ is $g''(e)de$, then the mechanical analysis of the first loading OA (cf. Fig. 7b) results in:

$$(3.1) \qquad S(\varepsilon) = S_{\text{slipped}} + S_{\text{notslipped}} = S_s + S_{ns} = \int_0^{\varepsilon} \mathbf{e} g''(e)de + \int_{\varepsilon}^{\infty} \varepsilon g''(e)de.$$

For a branch (like AB) or an arc of branch (like EF), the general form is:

$$(3.2) \qquad \Delta^t_R S = S - {}^t_R S = \omega S[(\varepsilon - {}^t_R\varepsilon)/\omega], \quad \omega = 1 \text{ or } 2, \quad {}^t_R S = S(t_R),$$
$$ {}^t_R\varepsilon = \varepsilon(t_R), \quad t > t_R,$$

where ω is Masing similarity functional, and where ${}^t_R\mathbf{S}$ and ${}^t_R\varepsilon$ are the reference states of stress and strain (cf. table of Fig. 7). In Eq. (3.2), the process of convection along the branch is implicitly introduced through $\partial^t_R S/\partial t = 0$, $\partial^t_R\varepsilon/\partial t = 0$.



Path	${}^t_R S$	ω
OA	0	1
AB	S_A	2
BC	S_B	2
CD	S_C	2
DE	S_D	2
EF	S_B	2
FG	0	1

FIG. 7. A pure hysteresis symbolic model (Masing, 1926) and its behaviour.

The intrinsic dissipation (functional $\Phi \geq 0$) resulting firstly from the analysis of the quasi-reversibility to the right of the vicinity of the inversion point (Fig. 7): $\Pi = {}^t_R S\ D : t = t_{R+}$ (cf. [3, 7, 9, 14, 15]) and, secondly, from the conceptual departure from the classical thermostatics obtained by keeping the same form along the finite branch: $\Pi = {}^t_R S D : t \in]t_R, t_{R+}]$. Also the intrinsic dissipation is:

$$(3.3) \qquad \Phi = -P_i - \Pi = (S - {}^t_R S)D >= 0.$$

In paper [7], the notion of the "help functional" W_m in the form $\delta W_m = (2/\omega^2)\ \Phi\delta t$ is introduced. The algorithm $\mathcal{A}$ expressing the minimum increase of the intrinsic dissipation rate $\dot{\Phi}(R_+) = \min\{\dot{\Phi}$ already memorized at $R\}$ and practically defined in terms of W_m is introduced in paper [7] and described in details in paper [8]:

$$W_m \to \mathcal{A}(\delta W_m; [{}^t_m W]) \to (\omega; {}^t_m S; [{}^t_m S]; [{}^t_m W]). \tag{3.4}$$

The fundamental inequality concerning the discrete memory functional Φ results in the loading-unloading criterion (cf. Eq. (3.3)). The order discrete set $[{}^t_m W]$ of the still memorised "help-functional" W_m results in an algorithm $\mathcal{A}$ (cf. Eq. (3.4)), giving at the current time the value of the Masing functional ω, the value of the current reference state ${}^t_R\mathbf{S}$ actually implemented and also the *ordered* set of the still memorised reference states $[{}^t_m\mathbf{S}]$. The algorithm $\mathcal{A}$ is equivalent to answering the three following questions:

1. Is the evolution submitted to inversion?
2. Is the evolution monotonic?
3. Does the evolution lead to obtaining the maximum level of W_m? – cf. [8].

The rate of the obtained internal energy is, *by essence*, associated with the stress contribution S_{ns} (defined in Eq. (3.1)) resulting in:

$$\dot{E}_a = S_{ns}D = SD + [-(S - {}^t_R S)D + \dot{S}(\varepsilon - {}^t_R\varepsilon)]/\omega, \tag{3.5}$$

of discrete memory form, as well as the Gibbs equation:

$$\dot{E}_a = \Pi_a + \dot{I}_a; \quad \omega\dot{I}_a = \partial[(S - {}^t_R S)(\varepsilon - {}^t_R\varepsilon)]/\partial t. \tag{3.6}$$

Suffix "a" corresponds to the always irrversible process of pure hysteresis type.

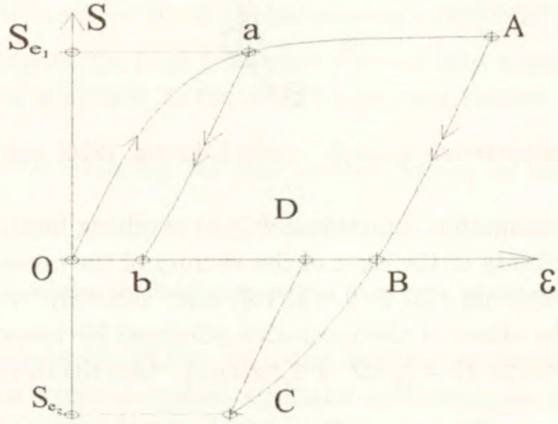


FIG. 8. The Bauschinger effect (1886) involved in the pure hysteresis pattern.

This general structure of the thermomechanical equations cannot be easily recalled under a comprehensive form in the present short section. For more detailed descriptions cf. [20], Sec. 3.5.1 or [8], Sec. 1, Eqs. (1.7), (1.8).

The Bauschinger effect ($S_{e1} > -S_{e2}$) is obvious (Fig. 8, where the offset is $0b = BD$).

3.2. The rate of heat supply of pure hysteresis type

It is, by essence, associated with the stress S_s (defined in Eq. (3.1)), resulting in the discrete memory form:

$$(3.7)\qquad -\dot{Q}_{ia} = S_s D = [(S - {}^t_R S)D - \dot{S}(\varepsilon - {}^t_R\varepsilon)]/\omega; \quad \dot{E}_a = -P_i + \dot{Q}_a$$

$$(\dot{K} = 0 = P_{\text{external}} + P_i).$$

The current value of $-\dot{Q}_{ia}$ is defined at any point A of the stress-strain diagram (Fig. 9) and its integral form results in a graphical interpretation easy to operate for the first loading as well as in the cyclic case (cf. the hatched lenses of Fig. 9 giving $\Delta_0^A Q_{ia}$ and $\Delta_A^B Q_{ia}$).

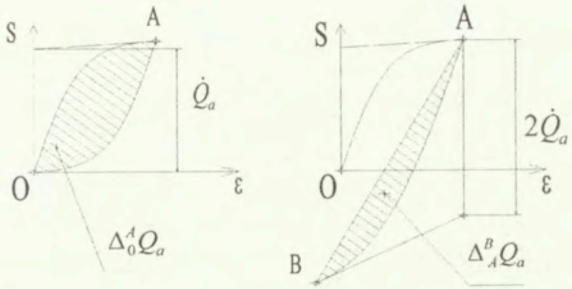


FIG. 9. The heat supply effect of the pure hysteresis behaviour and its associated rate [7].

The point should be illustrated for monotonous and cyclic loading. We have not a sufficient amount of information concerning energy transformations of solid polymers and we would like to presents our arguments on the example of two kinds of steel. In this case, the changes of internal energy and heat emitted during the process of tension are determined by the method prepared in the Institute of Fundamental Technological Research in Warsaw – cf. [5, 6, 16], for example.

Let us consider various qualitative features which may be extrapolated from an actual set of *thermomechanical* experimental results obtained concerning an XES steel (French Standard) through monotonic shear tests before rolling (see the curve S_1 in Fig. 10a) and after rolling (see the curve S_2 in Fig. 10a), at the same strain rate and for shear in the direction of rolling [15, 16]. Owing to the graphical interpretation of the heat supply rate $\dot{Q}_{ia}$ of pure hysteresis type (Fig. 9), it is obvious (see hatched zones in Fig. 10a) that the heat supply $\dot{Q}_2$ "after rolling" is greater than $\dot{Q}_1$ "before rolling". Consequently, $\Delta T_2 > \Delta T_1$, and $S_2 > S_1$ (Fig. 10b). If S_1 and S_2 have the same limit, ΔT_2 remains greater than ΔT_1 and the two ΔT variations are similar, resulting in parallel diagrams (Fig. 10b).

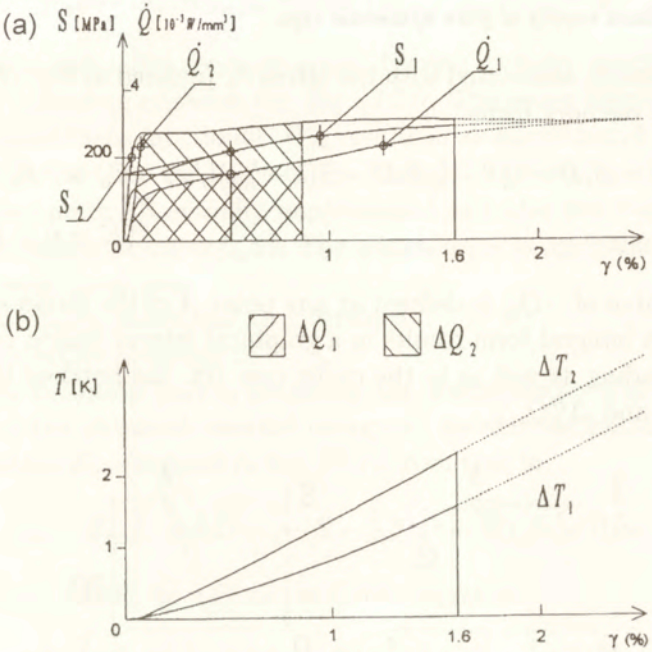


FIG. 10. In a strain range where experiments are rather easy, evidence of dissimilar qualitative mechanical features do not necessarily mean dissimilar qualitative thermal features.

Let us now consider a case of cyclic loading concerning the thermomechanical behaviour of the 00H19N17Pr (Polish Standard) stainless steel [6]. The qualitative features of the cyclic traction tests are sketched in Fig. 11 a, b, c. These tests are performed at a single moderate strain rate and they consist, basically, in a set of consecutive three-fold processes sketched in Fig. 11d: loading (O_n, A_n, B_n), then unloading (B_n, C_n), and finally cooling (C_n, O_{n+1}), allowing to recover the initial room temperature. The temperature variations can be plotted as a function of the stress (Fig. 11b) or as a function of the strain (Fig. 11c). Anticipating the reminder concerning the Kelvin effect ΔT_k associated with the variation ΔI_s, it is possible to describe this thermomechanical behaviour adding the Kelvin effect ΔT_k and the thermal effect ΔT_a of pure hysteresis, associated with $-\dot{Q}_{ia}$. The underlying assumption is that viscous effect and rate-independent hardening effects can be neglected. Moreover, it is possible to simplify the graphical

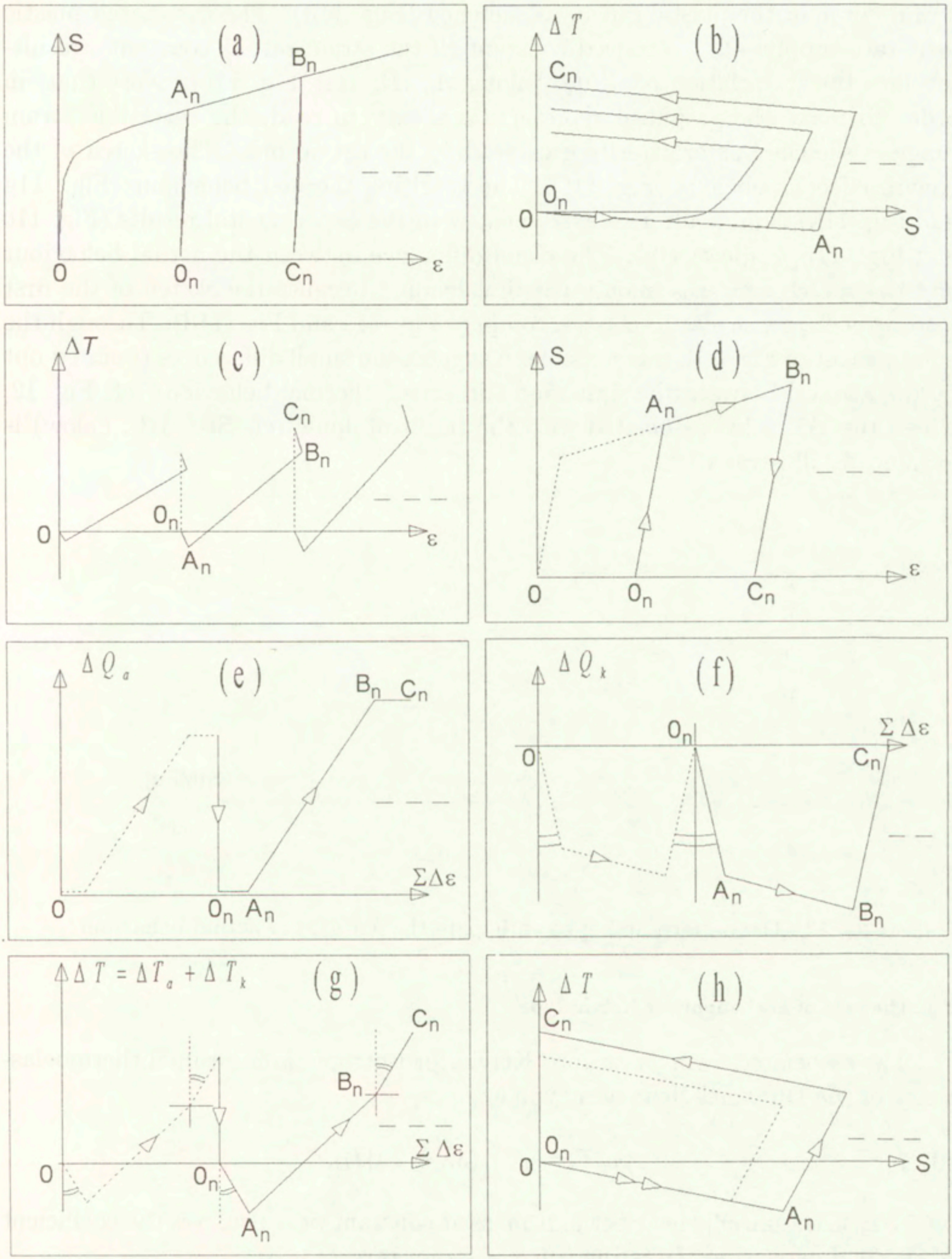


FIG. 11. The simplest sketch of the addition of Kelvin effect and Taylor effect.

features of the sketch using the piecewise rectilinear approximation of the stress-strain curve in the plastic range, as sketched (Fig. 11d). The associated plastic heat rate supply $-\dot{Q}_{ia}$ is indeed constant (if the strain rate is constant), resulting in a linear variation of $-\Delta Q_{ia}$ along A_n, B_n (cf. Fig. 11e). Note that, in order to make the graphical representation easy to read, the reversible strain range is sketched as arbitrarily greater than the actual one. The sketch of the Kelvin effect is obvious (Fig. 11f). The resulting thermal behaviour (Fig. 11g and Fig. 11h) is in qualitative agreement with the experimental results (Fig. 11c and Fig. 11b, respectively). The usual difference between the actual behaviour and the sketch concerns mainly the first loading because the sketch of the first loading is piecewise elastic-plastic (compare Fig. 11a and Fig. 11d). Through the enlargement of Fig. 11h it is possible to suggest the small differences (pointed out by the arrows) between the simplified and actual thermal behaviour (cf. Fig. 12, where the ΔT value associated with the result of Joule (cf. Sec. 3.3 i below) is graphically illustrated).

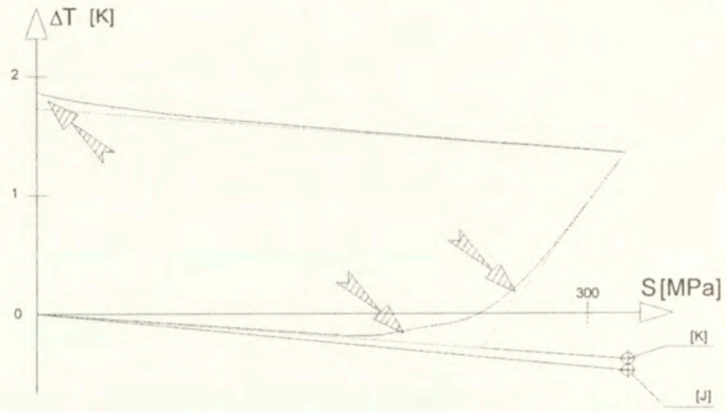


Fig. 12. The pattern acting to anticipate the features of actual behaviour.

3.3. The rate of heat supply of Kelvin type

The associated formula of Lord Kelvin for isotropic infinitesimal thermoelasticity of the Duhamel-Neumann type is:

$$\rho c_p \dot{T} = -T\alpha(3\lambda + 2\mu) I_D, \tag{3.8}$$

where c_p is the specific heat per unit mass at constant pressure, α is the coefficient of thermal expansion, I_D is the trace of strain rate D.

Owing to the experimental results currently available one may suppose, at least provisionally, that the Kelvin effect is related to the first order (isotropic) volume changes, but not to purely deviatoric effects without volume changes, which are able to produce second order variations of I_s: consequently, the simplest formalism is obtained by substituting the first order rate $\dot{I}_s$ for the product $(3\lambda + 2\mu)I_D$. Note that, when applied to the above traction case, the Kelvin equation results in

$$\Delta T_k = -(T\alpha/\rho c_p)\Delta I_s = -k\Delta S = 274.76 \times 1.68\ 10^{-5} \times 1.09\ 10^8/500 \times 7800 = 0.13\ \text{K}. \tag{3.9}$$

In the case of *cyclic* traction tests it is convenient to make use of the usual equation:

$$\Delta_0^t T = -kI_{\Delta s} = -k\Delta_0^t I_s \to T(t) - T(0) = -k[S(t) - S(0)] \tag{3.10}$$

in the rate form:

$$\dot{T} = -k\dot{I}_s. \tag{3.11}$$

3.4. The stress decomposition rule and its role in defining the effective constitutive patterns

i. Let us suppose that

$$S = S_a + S_\nu + S_r, \qquad P_i = P_a + P_\nu + P_r, \tag{3.12}$$

so that the associated internal powers associated with an *always irreversible* process of pure hysteresis type (suffix a for "algorithm"), with a viscous-viscoelastic process (suffix ν) and with the *reversible* (elastic) process (suffix r), are:

$$P_a = -S_aD, \quad P_\nu = -S_\nu D, \quad P_r = -S_rD, \tag{3.13}$$

respectively. Then the associated decomposition of the Gibbs equation and of the intrinsic dissipation Φ are:

$$\begin{aligned} \dot{E} &= [-P_a + \dot{Q}_{ia}] + [-P_\nu + \dot{Q}_{i\nu}] + [-P_r + \dot{Q}_{ek}], \\ \dot{E} &= (\Pi_a - P_r) + (\dot{I}_a + \dot{I}_\nu + \dot{Q}_{ek}), \\ \Phi &= \Phi_a + \Phi_\nu = (\dot{I}_a + \dot{I}_\nu) + (-\dot{Q}_{ia} - \dot{Q}_{i\nu}) = [\dot{I}_a - \dot{Q}_{ia}] + [\dot{I}_\nu + -\dot{Q}_{i\nu}]. \end{aligned} \tag{3.14}$$

The thermomechanical equations associated with the three types of processes are:

$$\begin{aligned} \dot{S} &= \dot{S}_\nu + \dot{S}_r + \dot{S}_a, \quad \dot{S}_\nu = f_\nu(D, \dot{D}, S_\nu), \quad \dot{S}_r = f_r(D, S_r, ...), \\ \dot{S}_a &= \Delta_R^t \dot{S}_a = f_a(D, \Delta_R^t S_a, \omega), \\ \dot{E}_a &= -P_a + \dot{Q}_{ia}, \quad \dot{E}_a = \Pi_a + \dot{I}_a, \\ \dot{E}_\nu &= -P_\nu + \dot{Q}_{i\nu}, \quad \dot{E}_\nu = 0 + \dot{I}_\nu, \\ \dot{E}_r &= -P_r + \dot{Q}_{ek}, \quad \dot{E}_r = -P_r + \dot{I}_r \to \dot{Q}_{ek} = \dot{I}_r. \end{aligned} \tag{3.15}$$

In this splitting-up the case of a possible isotropic-deviatoric decomposition is not made explicit. Note that the arguments of the reversible stress rate definition are provisionally not specified, and that, concerning the viscous stress contribution, a reasonable assumption may be $\dot{I}_\nu = 0$, resulting in: $\dot{E}_\nu = -P_\nu + \dot{Q}_{i\nu} = 0$ and $\Phi_\nu = -P_\nu = S_\nu D$.

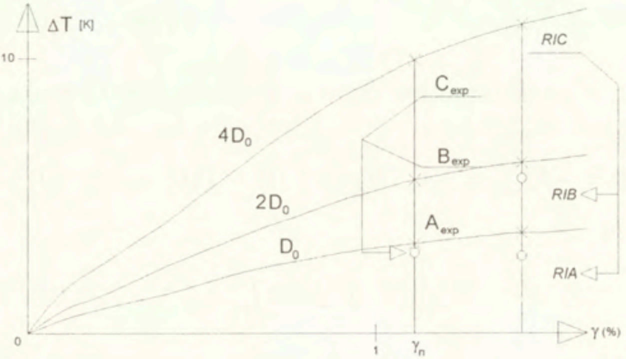


FIG. 13. Rate dependent stress-strain diagrams are not independent.

ii. The first illustration of the heuristic nature of the previous model is obtained by using the qualitative analysis of the thermomechanical behaviour of a 00H19N17Pr stainless steel (Polish Standard) during three monotonic simple shear tests performed at three different constant strain rates [16]. Let us consider three shear tests performed at three constant strain rates: $nD_o, n = 1, 2, 4$ (Fig. 13). The stress contributions of pure hysteresis type being rate-independent, one obtains first:

$$-\dot{Q}_{ia} = -n\dot{Q}_{iao} = n\rho c_p (dT_{ao}/d\varepsilon)(d\varepsilon/dt) = n\rho c_p (dT_{ao}/d\varepsilon) D_o \\ \rightarrow \Delta T_a = n\Delta T_{ao}, \tag{3.16}$$

and then, for the viscous contribution, a Newtonian approximation results obviously in:

$$dQ_{i\nu}/d\varepsilon = n^2 dQ\nu_o/d\varepsilon \rightarrow \Delta T_\nu = n^2 \Delta T\nu_o. \tag{3.17}$$

Therefore, for any fixed shear strain γ_n, the temperatures (and the stresses) associated with the states A, B and C are not independent, because one has (cf. Fig. 13):

$$T_C = 4T_a(A) + 16T_\nu(A), \qquad T_B = 2T_a(A) + 4T_A(A), \\ T_A = T_a(A) + T_\nu(A), \tag{3.18}$$

resulting in the relationship: $8T_A = 6T_B - T_C$. A fitting based on the experimental points C_{exp} and B_{exp} gives A, to be compared with A_{exp}. This fitting is much better than that obtained through the rate-independence assumption (points RIA, RIB, *given by* RIC). However, this does not mean that the Newtonian approximation is suitable to describe the viscous effects. It means mainly that the stress-power splitting assumption may be effective even with a rough estimate of the viscous stress contribution, provided that the range of strain rate is not too large.

iii. A second illustration may be given which concerns the purely mechanical cyclic features of a viscoelastoplastic behaviour resulting from the stress contributions S_ν , S_r and S_a (cf. Fig. 14a, b, c), involved "in parallel" in the stress splitting-up approach (cf. Fig. 14d). The behaviour is rather similar to that exhibited by the fabric samples when the cyclic loading is performed under constant strain rate intensity (Fig. 14e, where a small softening effect of Mullins type is shown).

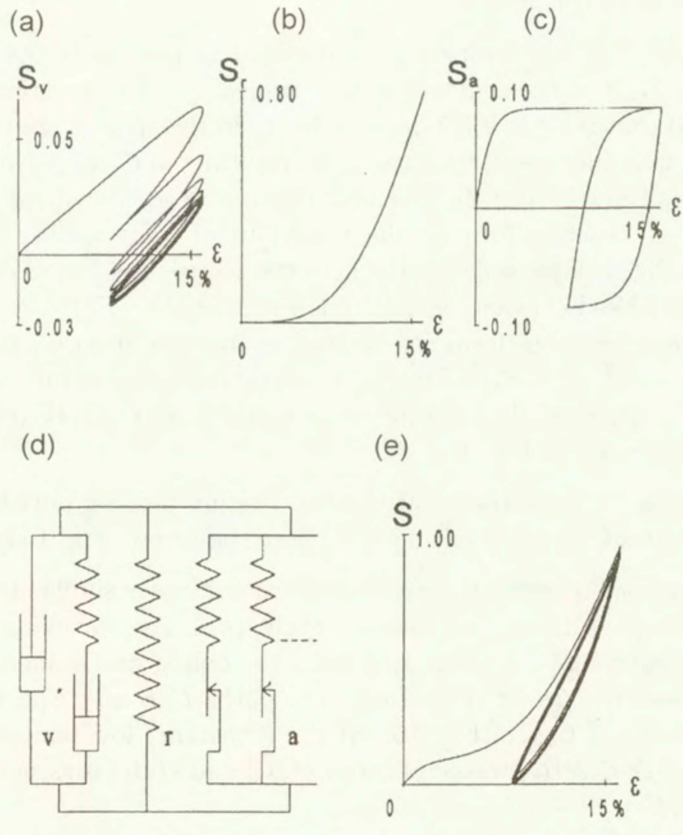


FIG. 14. Stress splitting-up approach in the viscoelastoplastic case.

3.5. Generalisation of the pattern to the case of large temperature variation

3.5.1. From the generalisation concerning the pure hysteresis stress contribution towards the modelling of rate-dependent isotropic and rubber-like behaviours without hardening. Owing to the previous illustration, the basic role of S_a is made conspicuous. The main issue is therefore to define the generalisation concerning the pure hysteresis stress contribution [8]. Let us consider the previously introduced basic symbolic model now endowed with the following temperature dependence: the stiffness G_i is constant and equal to G/n , but the critical slipping strain e_i is temperature-dependent and equal to $ie_oT_0/T = i(S_o/G)T_0/T$, $i = 1, ..., n$. The four constitutive parameters are G, e_o (or S_o), T_o and n. One considers two limit temperatures T_0 and $T_0/2$, resulting in stress-strain diagrams of specially simple reading. For example, the cycles symmetrical with respect to the origin are $C_1D_1E_1F_1G_1H_1B_1C_1$ at high temperature T_0 and $C_2D_2E_2....H_2B_2C_2$ at low temperature $T_0/2$ (Fig. 16a, obtained with the non-restrictive choice $n = 2$; $G = 2$; $e_o = 1$). Several thermomechanical loading processes should be considered.

i. From high to low temperature: the loading process is $OA_1B_1C_1$ at T_0; cooling up to $T_0/2$ under constant strain, resulting in the invariant state C_1 – no mechanical changes – at $T_0/2$ because forces in the springs are constant since there is no sliding; increasing strain at $T_0/2$, resulting in $C_1\alpha C_2$. Note that if this pure hysteresis behaviour at $T_0/2$ is added with a reversible stress contribution, it is possible to describe directly the behaviour of some rubber-like materials (cf. Fig. 15b, Fig. 1 of [4] and also [18]). In the sequel, one has of course slightly different rubber-like behaviour in view (cf. Fig. 15c and [19]).

If the strain decreases from C_1, instead to increase from C_1 to α, then the path is $C_1\beta\gamma$ (with $\gamma = E_2$). Finally, if strain and temperature variations are simultaneously specified, the final states are on the limit $C_2D_2E_2$ (cf. the smooth curves starting from C_1, Fig. 15).

ii. From low to high temperature: the loading process is $OA_2\ C$ at $T_0/2$; heating at constant strain, resulting in C_1 (see the arrow, Fig. 15a).

iii. Let us now consider thermomechanical processes similar to those which will be considered in the sequel, namely cyclic processes involving large strains at high temperature T_0 (cf. Fig. 16a) prior to cooling and standard loading at low temperature $T < T_0$ (cf. Fig. 16b). The paths E_1E and E_1F are similar to the paths C_1D_2 and $C_1\alpha$ of Fig. 15. At the (constant) low temperature T, the cycle $B_2C_2XD_2YE_2FB_2$ located between $\varepsilon(D_1)$ and $\varepsilon(B_1)$ is associated with the cycle $B_1C_1D_1E_1$.

3.5.2. From the viscoplastic case to the behaviour of a PA66 based filament.

i. Owing to the last remark (Sec. **iii** above), it is reasonable to admit that the cyclic behaviour at low temperature involves "plastic paths" such as C_2D_2 (cf. Fig. 16b). However, the manufacturing process may be neglected or more or less unknown resulting in the fact that the actual origin of the mechanical path is also unknown; for example, it may be X associated with the state x on C_1D_1, as well as Y associated with y on D_1E_1, or also E associated with E_1 (cf. Fig. 16b). Consequently, at $T < T_0$, the actual strain with respect to the genuine origin may be unknown.

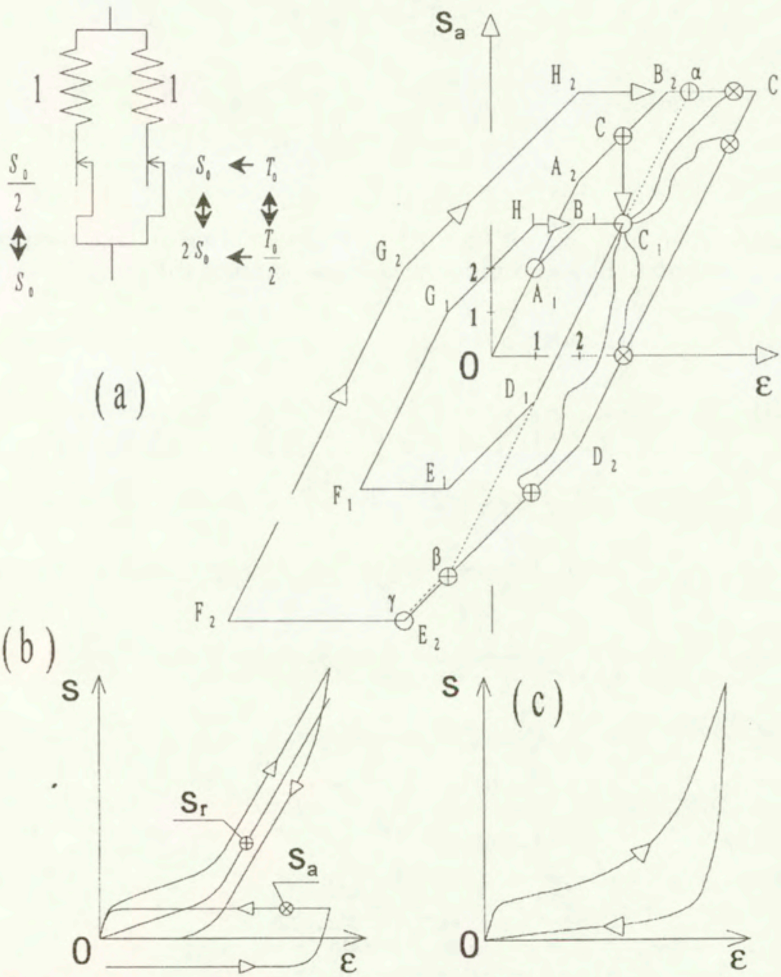


FIG. 15. Behaviour of a simple symbolic model [2] of temperature-dependent pure hysteresis (a); the resulting deviatoric behaviour of rubber-like elastic hysteresis at low temperature (b); an example of observed behaviour under hydrostatic loading (c).

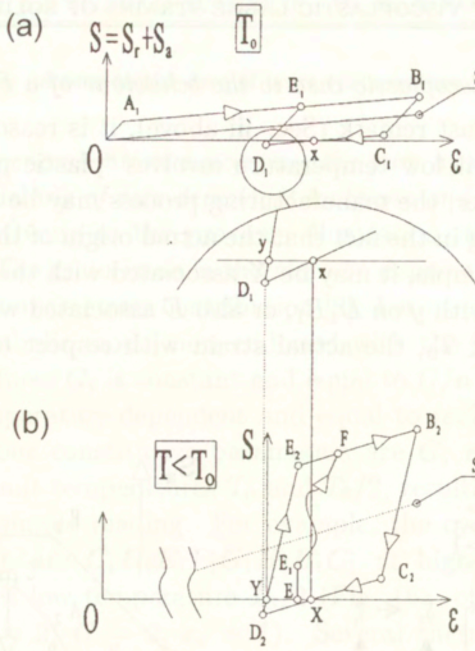


FIG. 16. The problem of the reference state: manufacturing at high temperature (a), implementation at low temperature (b).

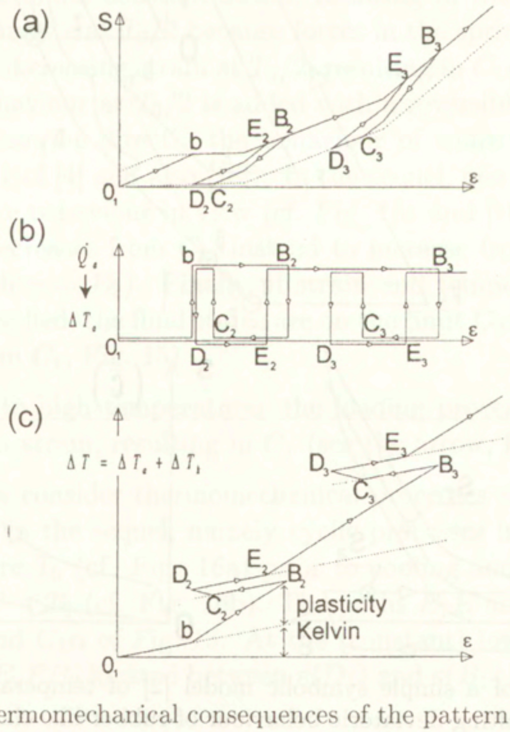


FIG. 17. Thermomechanical consequences of the pattern: $S = S_r + S_a$.

ii. If the material is implemented at T, a reasonable reference state may be located at Y in order to obtain the strain measure (cf. Fig. 16b). This conventional origin of the strain measure is implemented to obtain a cyclic behaviour endowed with *relevant features* as to the *dissipated energy* at low temperature T (cf. Fig. 17a, b). The heat supply of pure hysteresis type is directly obtained, as well as a Kelvin effect endowed a sign which is relevant owing to the polymers under consideration in the sequel (cf. Fig. 17b). The qualitative feature of the global temperature variation is then directly obtained (cf. Fig. 17c, compared with Fig. 3c).

iii. Concerning the measurement of the conventional elastic limit, it is worth noting that the implementation of the standard method with a reasonable offset, results in a rather puzzling situation: the stress S_n remains nearly linear with respect to the offset $0b_n$, and there is no reversible behaviour when $0b_n$ tends to zero (cf. Fig. 18). This feature may be explained, for example, as a lack of conventional elastic domain and in fact, such a reading is often commonly admitted for filaments and braiding.

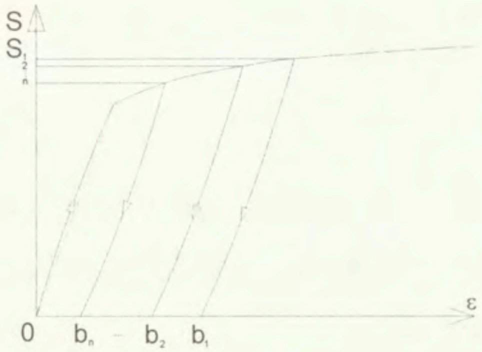


FIG. 18. The behaviour of a polymer filament may be read as always irreversible.

4. Implementation of the stress decomposition method

4.1. The differential-difference equations of the one-dimensional sketches

It is possible to define an *always irreversible* viscoelastoplastic pattern. Moreover, this pattern may be completed by an empirical transient hardening (resulting in the usual softening effect of the Mullins type) if some experimental evidence is actually existing. One obtains:

$$\dot{S} = \dot{S}_\nu + \dot{S}_r + \dot{S}_a + \dot{S}_h, \quad \dot{E} = \dot{E}_\nu + \dot{E}_r + \dot{E}_a, \quad \dot{Q} = \dot{Q}_\nu + \dot{Q}_r + \dot{Q}_a, \tag{4.1}$$

where

• irreversible process of pure hysteresis type

$$
\begin{aligned}
\dot{S}_a &= 2\mu_a\{1-(\Delta_R^t S_a/\omega S_{ao})^c\}D, \\
&\qquad\qquad \dot{W}_a = 2\Phi_a/\omega^2 = (2/\omega^2)\Delta_R^t S_a D > 0, \\
\dot{S}_{ah} &= 2\mu_a\{1-\Delta_R^t S_{ah}/\omega S_{hh})^c\}D, \quad \dot{S}_h = \dot{S}_{ah} - \dot{S}_a, \\
S_{hh} &= S_{ho} - (1/2)(S_{ho}-S_{ao})\{\tanh(\pi_1/\pi_2) - \tanh[(\pi_1-\pi)/\pi_2] \\
&\qquad\qquad \text{with } \dot{\pi} = \sqrt{D^2}, \\
\dot{E}_a &= S_a D - \Phi_a + \dot{S}_a \Delta_R^t \varepsilon, \\
-\dot{Q}_a &= \Phi_a - \dot{S}_a \Delta_R^t \varepsilon, \\
-\dot{Q}_h &= S_h D - \dot{I}_h, \quad \Pi_h = 0, \quad \dot{E}_h = \dot{I}_h = (1-K_h)S_h D,
\end{aligned}
\tag{4.2}
$$

(in Eq. (4.1) we have introduced a simple approximation of hardening effect by the additional term $\dot{S}_h$),

• viscous-viscoelastic process

$$
\begin{aligned}
\dot{S}_\nu &= [2(\eta_1+\eta_2)/\theta_\nu]D + 2\eta_2\dot{D} - S_\nu/\theta_\nu, \\
\dot{E}_\nu &= (\theta_\nu/2\eta_1)(S_\nu - 2\eta_2 D)(\dot{S}_\nu - 2\eta_2\dot{D}), \\
-\dot{Q}_\nu &= (1/2\eta_1)(S_\nu - 2\eta_2 D)^2 + 2\eta_2 D^2.
\end{aligned}
\tag{4.3}
$$

• reversible (elastic) process

$$
\begin{aligned}
\dot{S}_r &= \dot{S}_{r1} + \dot{S}_{r2} + \dot{S}_{r3}, \\
\dot{S}_{r1} &= 2\mu_{r1}\{1-(S_{r1}/S_{or1})^{c_{r1}}\}D, \quad \dot{S}_{r2} = 2\mu_{r2}D, \\
\dot{S}_{r3} &= -2\mu_{r2}\{1-(S_{r3}/S_{ro3})^{c_{r2}}\}D, \\
\dot{E}_r &= S_r D, \quad -\dot{Q}_r = 0,
\end{aligned}
\tag{4.4}
$$

and

• rate of heat supply of the Kelvin type

$$
\dot{T} = -\dot{Q}_{\nu a}/\rho c_p - K\dot{S} - \dot{Q}_h/\rho c_p, \quad \text{where } -\dot{Q}_{\nu a} = -\dot{Q}_\nu - \dot{Q}_a \tag{4.5}
$$

with

$$
D = \dot{J}/J, \quad \ddot{J} = J(\dot{D}+D^2), \quad 2\varepsilon = 1 - 1/J^2, \quad \dot{\varepsilon} = D/J^2 = \dot{J}/J^3.
$$

Fifteen constitutive parameters are involved, namely the three Oldroyd parameters η_1, η_2, θ_ν, the six elastic parameters μ_{r1}, μ_{r2}, S_{or1}, S_{ro2}, c_{r1}, c_{r2}, the three parameters of pure hysteresis μ_a (Lamé), S_{ao} (Huber-Mises-Hencky), c (Prager) and the three hardening parameters S_{ho}, π_1, π_2.

The always reversible Kelvin process (involving its proper energy balance) results in:

$$-\dot{Q}_k = K\dot{S} \quad (-\dot{Q} = -\dot{Q}_{\nu a} - \dot{Q}_k). \tag{4.6}$$

A weak point of the pattern is that concerning the distinction between the stress S and the force F. It is necessary to make use of a reasonable assumption. The isovoluminal choice results in $F_3 = S/J$, and suggesting therefore to make use of:

$$JF = S; \qquad \dot{J}F + J\dot{F} = \dot{S}. \tag{4.7}$$

The case of a loading control through the rate of strain rate may be interesting both at theoretical level and from the experimental point of view.

Let $\dot{D}$ be a given continuous function $f_d(t)$ which is equal to zero "almost everywhere", except for small intervals where it is a conventional curve. As a result, $(\mathrm{tr}\,\mathbf{D}^2)^{1/2}$ is "almost everywhere" piecewise constant. The pattern is:

$$\begin{aligned}
&\dot{S} = \dot{S}_\nu + \dot{S}_r + \dot{S}_a + \dot{S}_h, \quad \dot{E} = \dot{E}_\nu + \dot{E}_r + \dot{E}_a, \quad -\dot{Q}_{\nu a} = -\dot{Q}_\nu - \dot{Q}_a,\\
&\dot{S}_a = 2\mu_a[1 - (\Delta_R^t S_a/\omega S_{oa})^c]D = A_a D, \quad \dot{W}_a = (2/\omega^2)\Phi_a,\\
&\dot{S}_h = A_h D, \quad \dot{\Phi}_a = \dot{S}_a D + \Delta_R^t S_a f_d,\\
&\dot{S}_\nu = [2(\eta_1 + \eta_2)/\theta_\nu]D + 2\eta_2\dot{D} - S_\nu/\theta_\nu,\\
&\dot{E}_\nu = (\theta_\nu/2\eta_1)(S_\nu - 2\eta_2 f_d), \quad -\dot{Q}_\nu = (1/2\eta_1)(S_\nu - 2\eta_2 D)^2 + 2\eta_2 D^2,\\
&\dot{S}_r = A_r D (= (2/\omega^2)\Delta_R^t S D > 0),\\
&\dot{E}_r = S_r D, \quad -\dot{Q}_r = 0, \quad \dot{Q}_k = K\dot{S},\\
&\dot{E}_a = S_a D - \Phi_a + \dot{S}_a \Delta_R^t \varepsilon, \quad -\dot{Q}_a = \Phi_a - \dot{S}_a \Delta_R^t \varepsilon,\\
&\dot{T} = -\dot{Q}_{\nu a}/\rho c_p - K\dot{S} - K_g \dot{Q}_h/\rho c_p.
\end{aligned}$$

with

$$\ddot{J} = J(f_d + D^2), \quad \dot{D} = f_d, \quad \dot{J} = JD, \quad \varepsilon = D/J^2, \quad \dot{F} = \dot{S}/J - DF.$$

4.2. Simulations concerning the machined samples

The experimental result shown in Fig. 4 (obtained for: $\dot{J} = 10^{-2}\ \mathrm{s}^{-1}$) and the numerical simulation of the above one-dimensional pattern are compared (cf. Fig. 19a, b, c, d). The constitutive parameters are: $2\eta_1 = 2.35$ GPa.s, $2\eta_2 = 50$ MPa, $\theta_\nu = 6.5$ s and: $2\mu_a = 1.4$ GPa, $S_{ho} = 100$ MPa, $S_{ao} = 28$ MPa, $\pi_1 = 9\ 10^{-2}$, $\pi_2 = 10^{-3}$, $c = 1$. Note that the reversible stress contribution is reduced to that of type S_{r1}: $2\mu_{r1} = 1.8$ GPa, $S_{or1} = 33$ MPa, $c_{r1} = 0.8$. Moreover $\rho = 1150$ Kg/m^3, $c_p = 1700$ J/Kg.K , $K_h = 5.5$ and $K = -10.6 \times 10^{-3}$ K/MPa ($[273.16 + 23] \times 70\ 10^{-6}/1150 \times 1700$), as usually for the PA66 type polymer.

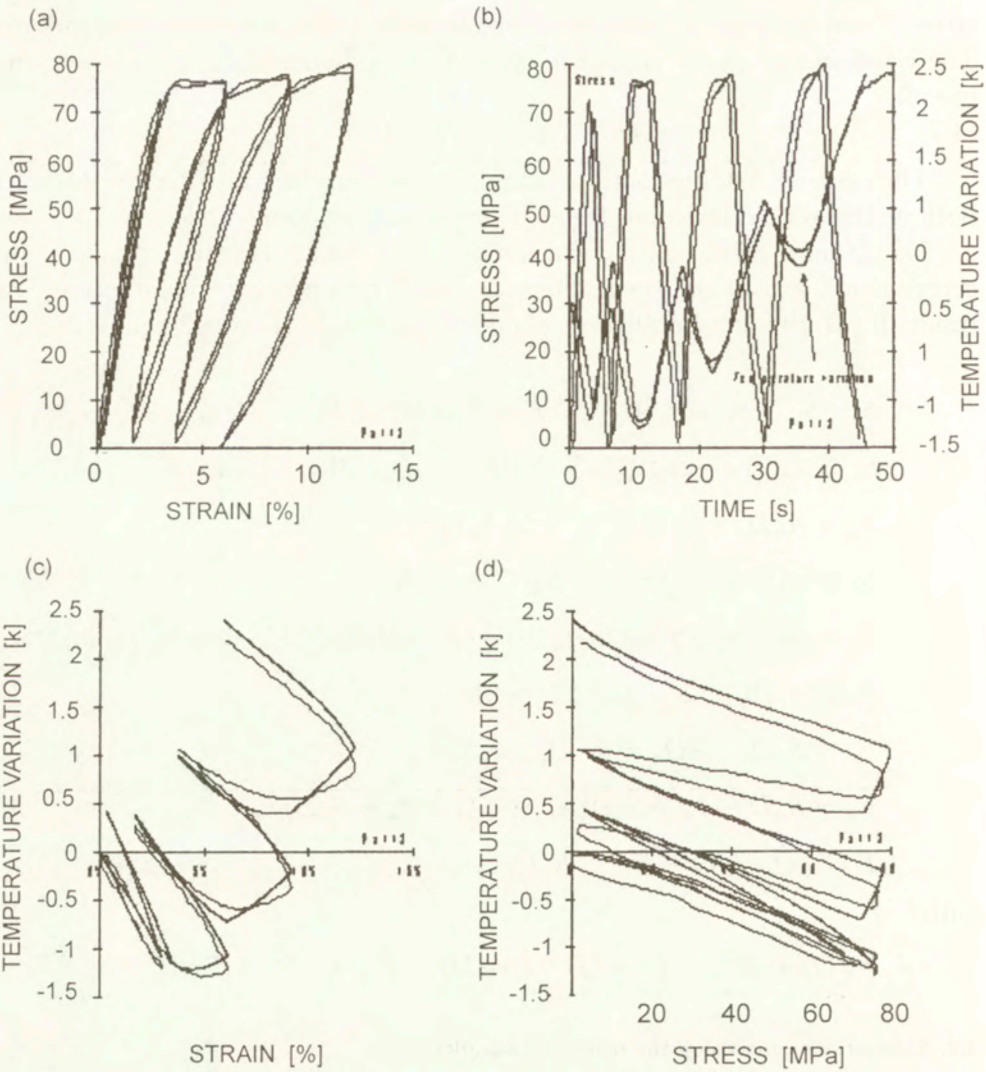


FIG. 19. Theory (thick solid lines) versus experiment (thin solid lines) on a machined sample: the Taylor and Kelvin displays are given in (c) and (d), respectively.

4.3. Simulations concerning the fabric samples

The simulation (cf. Fig. 20) concerning the result shown in Fig. 14 may be performed in the same general conditions than are given above. The constitutive parameters are: $2\eta_1 = 4$ GPa.s, $2\eta_2 = 20$ MPa.s, $\theta_\nu = 27$ s; $2\mu_{r1} = 1$ GPa, $S_{r1} = 0.5$ MPa, $c_{r1} = 2$, $2\mu_{r2} = 3.5$ GPa, $S_{r2} = 420$ MPa, $c_{r2} = 2.5$ and: $2\mu_a = 2.2$ GPa, $S_{oa} = 30$ MPa, $c = 1.5$; F [daN] $= S$ [Pa] $\times(144\ \text{mm}^2 = 0.8 \times 4.5\ \text{mm} \times 40\ \text{mm})$. Moreover, the specified strain rate is now: $\dot{J} = 1.29 \times 10^{-2}\ \text{s}^{-1}$ and: $K = 510^3$ K/MPa (but $\rho = 1150$ Kg/m^3 and $c_p = 1700$ J/Kg.K, as previo-

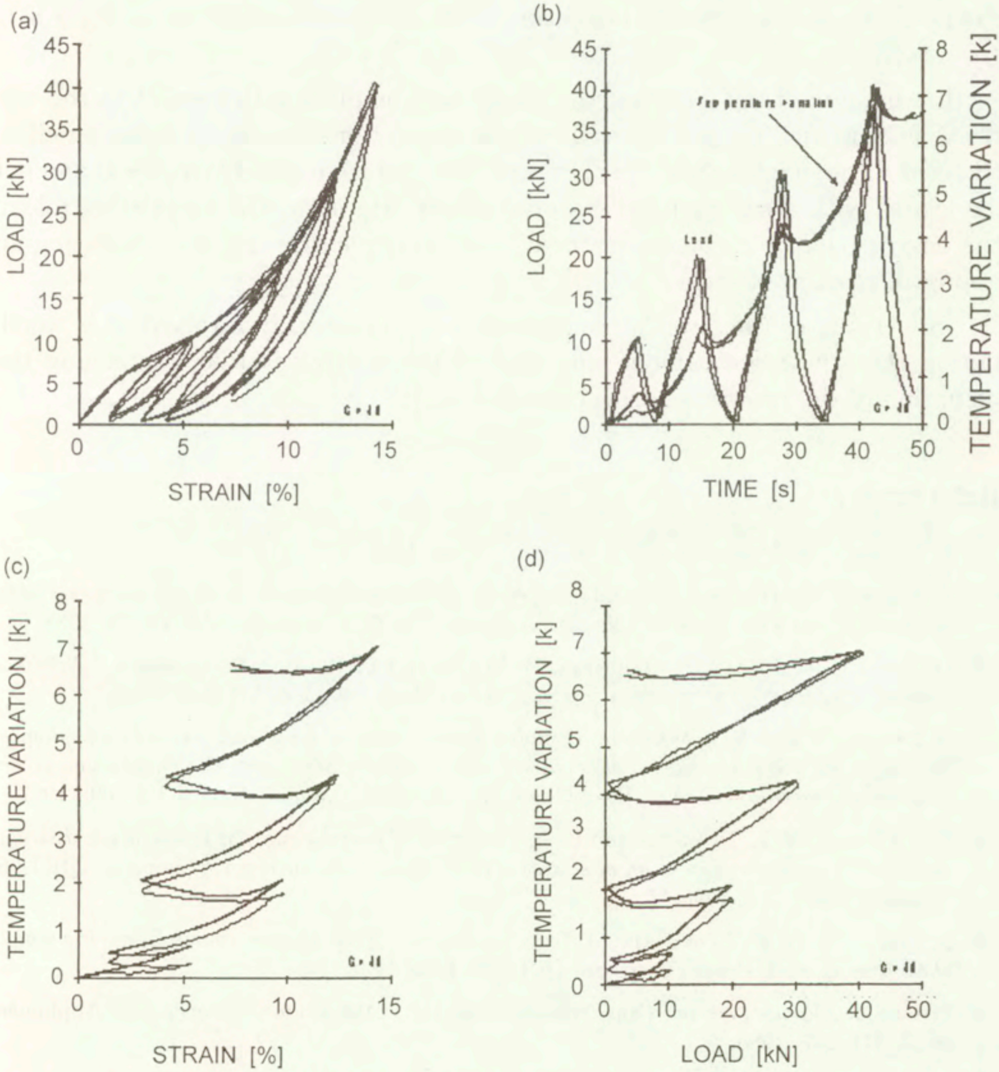


FIG. 20. Theory versus experiment on a fabric samples (the same display as for Fig. 19).

usly). Owing to the fact that the pattern is implemented in its temperature-independent form, the origin of the integration is associated with point Y in Fig. 16 (cf. Sec. 3.5.1 **iii** and 3.5.2 **i**). Note that the above provisional identification has been implemented as to the test of the sample GV $39(\dot{J} = 2.6 \times 10^{-2}\ s^{-1})$ introduced previously (cf. Fig. 3d). The result was satisfactory.

4.4. Remarks on the current state of the stress decomposition rule

i. Relaxation and creep have been simulated. The pattern is effective, specially in order to distinguish the difference between the relaxation time θ_ν and the non-unique characteristic creep times (which are dependent upon: $(\eta_1 + \eta_2)/(S - S_o)$).

ii. In spite of the fact that the study is incomplete with respect to the role of the microstructural processes, the above simulations involve a hardening effect (defined through a function $S_h(\pi)$ of tanh type) which is purely empirical and not associated with a well founded tensorial theory. However, the outsanding role of the energy balance remains preserved and, consequently, the stress splitting-up approach remains efficient.

iii. Owing to the fact that large strains are necessarily involved, it is worth noting that the more cumbersome part of the pattern is that concerning the definition of the reversible stress contribution [18, 19].

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Effect of fiber treatment on bending strength of aramid short fiber reinforced polyester

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Effect of fiber treatment on the bending strength of the aramid short fiber composite is studied in this paper. Interfacial shear strength and fiber strength are determined for different fiber treatments by performing single-fiber pull-out test. Effect of the fiber treatments on the property of the composite depends both on the interfacial shear strength and the fiber strength, simultaneously. Thus, to evaluate quantitatively the effect of the treatments on the strength of the composite a parameter "critical embedded length" is introduced that covers both the interfacial shear strength and the fiber strength. It is a parameter that defines the maximum embedded length during pull-out test, above which the shear force in the fiber/matrix interface exceeds the value of fracturing force of the fiber. The result shows that decrease of the critical embedded length causes increase in the bending strength of the composite, for the cases of the treatments employed in this research. Finally, an empirical equation is developed to calculate the bending strength of the aramid short fiber composite for the fiber treatments employed in this research.

1. Introduction

It is reported [1, 2] that reinforcement by 1 mm aramid short fiber does not cause an increase of the bending strength of the composite if compared to that of the plain Polyester resin. On the other hand, bending strength of the composite with 3 mm long fiber increases a little (only in the case of high fiber content) compared to that of the resin. Therefore in this research, we have tried to analyse interfacial property of the composite by performing several fiber treatments. We have tried to introduce an epoxy interface in between the fiber and matrix by treating the fiber with epoxy. In case of the fiber treatments applied in this paper, it is found that the treatment changes interfacial property and strength of the fiber.

Fiber/matrix interface is a crucial region in the composite and usually considered as responsible for transmitting the load from matrix to fiber. Therefore, the interface has great influence on mechanical properties of the aramid short fiber composites. In case of aramid fiber and the treatments employed in this research, it is found that the interfacial shear strength and fiber strength vary simultaneously with the treatments. So far, the change of the fiber strength due to fiber treatment is often neglected in many studies on composite interface. Therefore, conclusion on the effect of the interface in mechanical properties of composite cannot be drawn correctly, and thus a complete understanding of the effect of fiber treatment is not possible. The change of mechanical properties of the composite may not only depend on the interfacial property but also depends on the fiber strength. In an extreme case of very short fiber, the fiber strength might not be an important factor. In this research, experimental studies have been made to explain the effect of fiber treatment on the bending strength of the aramid short fiber composite.

Single-fiber pull-out test is a simple means to determine interfacial property of fiber and matrix. Here, the classical single-fiber pull-out test is employed to determine the interfacial property of the aramid fiber and polyester matrix and we can also determine the fiber strength during the pull-out test. It is found that the interfacial strength and the fiber strength are different for different fiber treatment. In this paper, a parameter "critical embedded length" is introduced that corresponds both the interfacial shear strength and the fiber strength. It is a parameter that defines the maximum embedded length during pull-out test, above which the shear force in the interface exceeds the value of fracturing force of the fiber. The result shows that decrease of the critical embedded length causes to increase the bending strength of the composite for the cases of the treatments employed in this research. Finally, an empirical equation is developed to calculate the bending strength of the aramid short fiber composite for the fiber treatments employed in this paper.

2. Experimental

Aramid (Kevlar 29 of type T950 and T965A, made by Toray Du Pont Co., Ltd., Japan) short fibers are used as filler. These short fibers are 1 and 3 mm in length. Unsaturated Polyester (8285AP type, made by Takeda Yakuhin Co., Ltd., Japan) resin is used as matrix. Permek-N (made by Nihon Yushi Co. Ltd., Japan) is used as hardener for the polyester resin.

Fabrication process of short fiber composite is similar to that employed in previous paper [1, 2]. Aramid short fiber is mixed into the polyester resin and degassed by a vacuum pump. After proper mixing of the hardener (1 wt% of the resin), the mixture is poured into a mold and compressive force is applied on it. The short fiber composite was cured under compression for 24 hr at room temperature (298 K). After releasing the composite from the mold, post-curing of the fabricated composite is done at 353 K for 3 hr in a drying oven.

Fabricated short fiber composite (without any machining) is used as test specimen for three point bending test. Figure 1 shows the nominal dimensions of the specimen and set-up of the bending test. Response of the load is measured and recorded by a $x - t$ recorder. The cross head speed is kept constant at 0.027 mm/sec during the test.

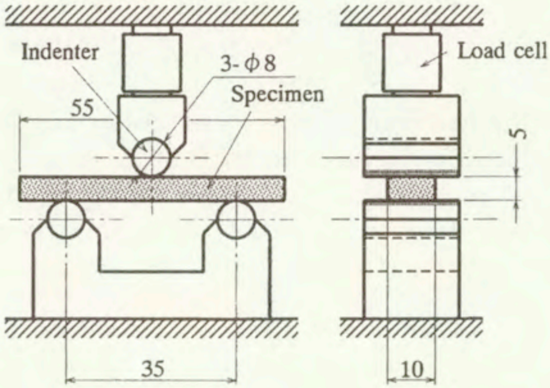

FIG. 1. Set-up of the three point bending test and nominal dimension of the specimen.

2.1. Fiber treatments

It is reported in previous paper [1, 2] that the bending strength of the composite of 1 mm short fresh aramid fibers does not increase compared to that of the resin even at very high fiber content. Only in case of high fiber content, bending strength of the composite of 3 mm short fresh aramid fiber increases a little compared to that of the resin. Therefore, in order to modify the interfacial property following fiber treatments are done.

Two kinds of solutions are prepared by mixing epoxy into water and toluene solvent where amount of the epoxy is 1% of the solution. Two types of Kevlar 29 fibers (T950 and T965A type) are immersed into the epoxy solutions and dried at 513 K. Another fiber treatment is done by heating Kevlar 29 (T950 type) fiber at 353 K for 60 days in a drying oven. In this paper, these above mentioned treatments are defined by EW-T950, EW-T965A, ET-T950, ET-T965A and H-T950 accordingly and are also shown in Table 1. In order to investigate the effect of the treatments on the strength of the composites, short fiber composites are produced by each treated fiber (1 and 3 mm long) and performed three-point bending test.

Table 1. Identification of the fiber treatments.

Type of fibers / Type of treatments	Kevlar 29 T950	Kevlar 29 T965A
Epoxy-Water solution treatment	EW-T950	EW-T965A
Epoxy-Toluene solution treatment	ET-T950	ET-T965A
Heating at 353 K for 60 days	H-T950	——

It is assumed that not only the interfacial properties affect the strength of the short fiber composite but tensile strength of the fiber also affects the strength property of the composite. Therefore, in order to investigate the effect of the treatments on the bending strength of the short fiber composite, interfacial property and fiber strength of each type of fibers are measured. Single-fiber pull-out test is a simple means for determining the interfacial property and in this research we have also performed this test to investigate the interfacial property.

2.2. Pull-out test

Figure 2 shows a schematic diagram of the specimen of single-fiber pull-out test. A construction paper was selected to prepare the pull-out test specimen. Two trapezoid shaped holes were cut off leaving a space in between them for embedment as shown in Fig. 2. Width of this space ultimately controls the embedded length of the specimen. A fiber was carefully placed along the center of the holes and was temporally fixed. A drop of polyester was placed on the space in between the trapezoids by a needle-like bow. Then the polyester was hardened for 24 hours. Embedded region was carefully observed and embedded length L was measured by an optical microscope. Enlarged drawing of the embedded region is also shown in Fig. 2. Middle portion of the construction paper is cut off after mounting the specimen on the test machine, so that, the embedded length would be directed towards the loading direction.

Response of the load cell is passed through an amplifier and is recorded by a $x - t$ recorder. P_t is maximum load required to pull-out the fiber from the resin. At a certain value of the embedded length, the load P_t required to pull out the fiber exceeds the fracturing load of the single fiber. This value of embedded length is defined as critical embedded length L_c. Most of the fibers are found to be fractured out when the embedded length is greater than the critical embedded length. This fracturing load of the fiber is used to calculate the strength of the fiber.

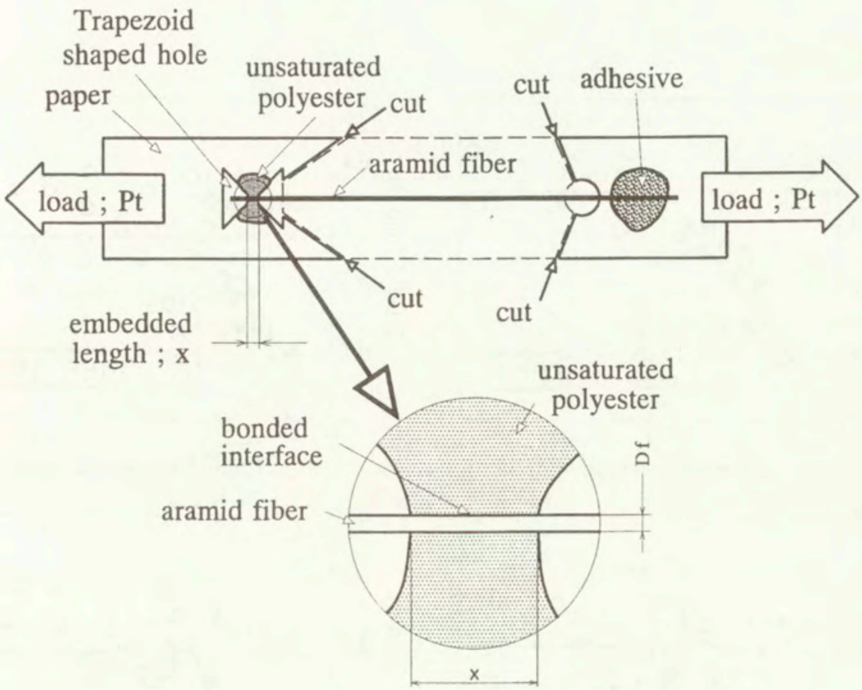

FIG. 2. A pull-out test specimen.

3. Results and discussion

Figure 3 shows the results of pull-out test of the aramid fresh fiber and the treated fibers. At small and large embedded length, fibers are usually found to be pulled out from the matrix and torn, respectively. From the results we could find a transient region where some fibers are torn and some are pulled out and usually this transient region is around the critical embedded length L_c. Figure 3 shows that the load required to pull out the fiber from the matrix increases with the embedded length. Here, it is assumed that the pull-out load increases linearly with the embedded length. Therefore, the data of pull-out region are fitted by the method of least squares and the line is defined as pull-out load line. On the other hand, mean value of the load required to fracture the fiber is also plotted in Fig. 3 and this line is defined as fiber fracture load line. However due to the presence of the transient region, it is difficult to find a unique critical embedded length from the results of the pull-out test. Therefore we have defined the critical embedded length L_c as the embedded length at which the pull-out load line and the fiber fracture load line intersect each other. High inclination (P_t/L) of the pull-out load line and mean fiber fracture load correspond high interface strength and

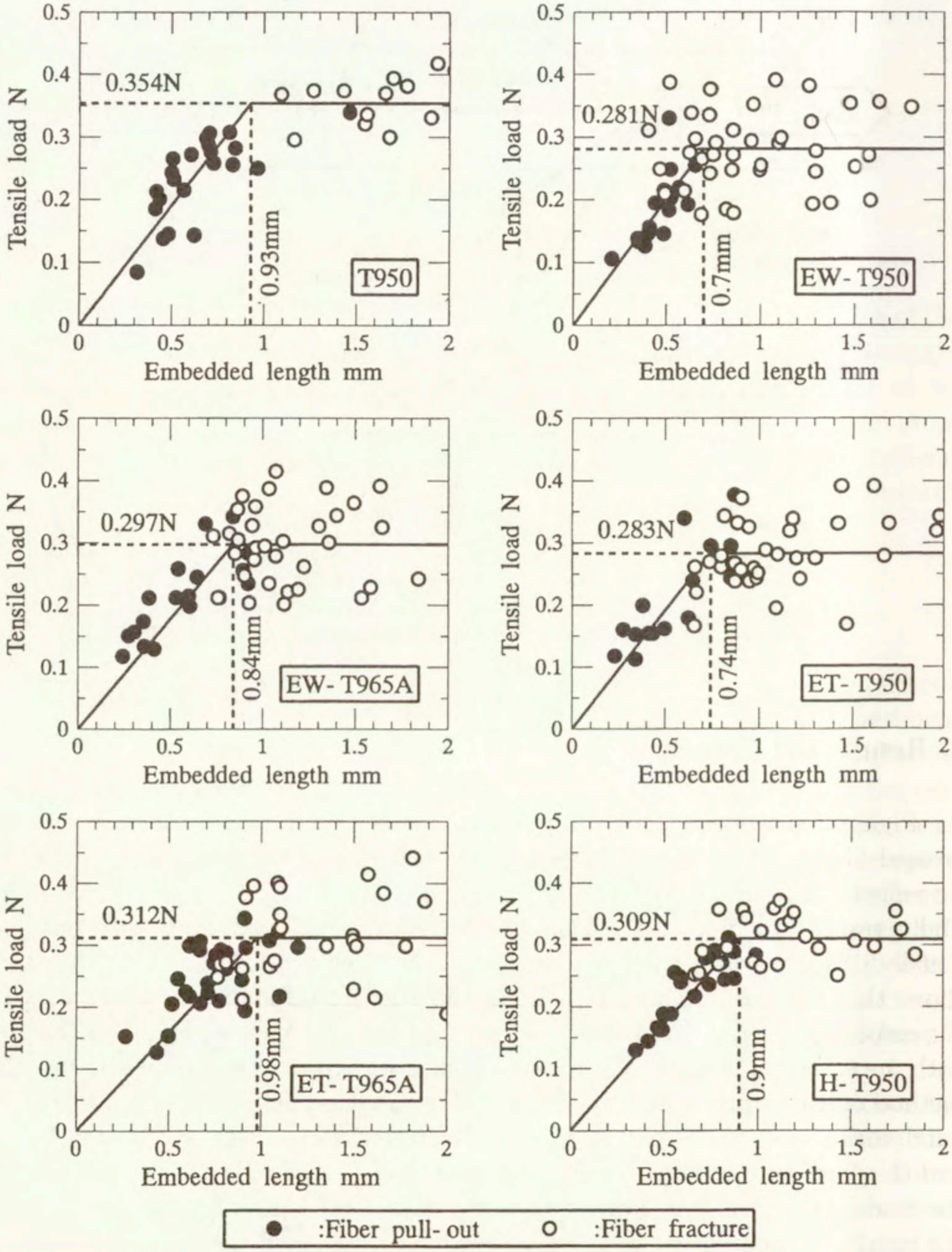

FIG. 3. Pull out test results of the aramid fresh fiber and the treated fibers.

fiber strength, respectively. From Fig. 3, we could find that inclination (P_t/L) of the pull-out load lines and mean value of the fiber fracture loads is different for different fiber treatment. Therefore, the critical embedded length L_c also differs from treatment to treatment. As this critical embedded length depends both on the pull-out and fiber fracture property, we have considered this length an important parameter in defining the fiber treatments.

Interfacial shear strength for each treatment was calculated using following Eq. (3.1), with assumption that shear stress is distributed uniformly in the part of fiber/matrix interface.

$$\tau_i = \frac{P_t}{\pi D_a L}, \tag{3.1}$$

where, τ_i and D_a is defined as interfacial shear strength and average measured diameter of each type of fiber, respectively. Diameters of the fiber are measured at more than 20 points along the fiber length by using Scanning Electron Microscope (SEM) and compared with the results obtained by an optical microscope. The standard deviation is obtained as about $0.9 \sim 1.2$ µm. In Eq. (3.1) P_t/L is the inclinations of the pull-out load lines for each fiber treatment. On the other hand, average fiber fracture load is divided by cross sectional area of the fiber to calculate average fiber strength σ_f. Result obtained from the pull-out test is summarised in Table 2.

Table 2. Fiber fracture strength and interfacial property of the aramid fresh fiber and the treated fibers.

Type of fiber	Average diameter D_a µm	Inclination P_t/L N/mm	Interfacial shear strength τ_i MPa	Critical embedded length L_c mm	Average fiber fracture load P_f N	Average fiber strength σ_f MPa
T950	13.31	0.381	9.10	0.93	0.354	2544.2
EW-T950	12.45	0.401	10.25	0.70	0.281	2308.2
EW-T965A	12.20	0.355	9.26	0.84	0.297	2540.7
ET-T950	12.62	0.383	9.66	0.74	0.283	2262.4
ET-T965A	12.35	0.319	8.22	0.98	0.312	2604.5
H-T950	12.17	0.344	9.00	0.90	0.309	2656.4

The critical embedded length L_c depend on the inclination of the pull-out load line as well as the interfacial shear strength and the fiber strength, for all the cases of the fiber treatments. Moreover, it can be concluded that decrease of the fiber strength reduces the critical embedded length and increase of that strength makes the length longer, and increase of the interfacial shear strength reduces the critical embedded length and decrease of that strength makes the length longer. However, H-T950 treatment seems somewhat different. Therefore, we assumed that the critical embedded length is a single parameter to define the fiber treatment for aramid fibers.

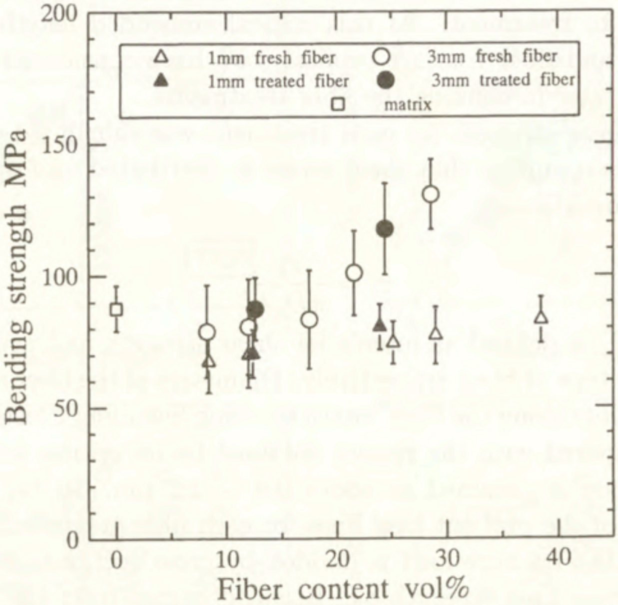

FIG. 4. Bending strength of the composite of treated and untreated fiber with respect to the content of the fiber.

Three point bending test is carried out in order to investigate the effect of the treatments on the bending strength of the short fiber composite. Figure 4 shows the bending strength of the composite of the aramid fresh fiber and the treated fibers. In case of the fresh fibers, we could find that the bending strength of the composite of 1 mm fiber does not increase compared to that of the matrix even at very high fiber content, but the strength of the composite of 3 mm fiber increases compared to that of the matrix when the fiber content is above 20 vol% [1]. Two typical fiber contents 12 vol% and 24 vol% are selected and investigated the effect of the fiber treatments on the composite in detail. Filled circle and triangle shown in Fig. 4 indicate the average strength of the composites of all the five fiber treatments. We could expect from Fig. 4 that the strength of the composite increases a maximum of about 10% with the treatments employed in this research, except in case of the composite of 1 mm fiber and about 12 vol%.

Therefore, the strength of the composite depends on the content and length of the fiber and the fiber treatment. If we concentrate our discussion on the individual treatment, we could find the bending strength of the composite of different treated fiber as shown in Table 3 and 4.

Table 3. Bending strength of composite of 1 mm fiber.

Type of fiber	Fiber content V_f vol%	Average bending strength σ_c MPa	Standard deviation MPa
T950	11.98	70.02	12.48
EW-T950	12.49	72.16	9.17
EW-T965A	12.02	68.72	8.74
ET-T950	11.68	68.02	8.34
ET-T965A	12.38	63.69	6.15
H-T950	12.24	79.23	13.35
T950	25.16	74.48	8.15
EW-T950	23.96	85.84	6.81
EW-T965A	23.24	69.49	6.69
ET-T950	24.31	91.35	12.95
ET-T965A	23.58	72.57	6.83
H-T950	24.52	83.69	9.66

Table 4. Bending strength of composite of 3mm fiber.

Type of fiber	Fiber content V_f vol%	Average bending strength σ_c MPa	Standard deviation MPa
T950	12.00	80.72	18.2
EW-T950	12.04	86.29	7.84
EW-T965A	12.04	85.30	12.88
ET-T950	12.29	84.35	8.34
ET-T965A	12.42	74.60	19.74
H-T950	14.18	107.37	10.32
T950	21.62	101.00	15.99
EW-T950	24.00	110.10	8.7
EW-T965A	24.20	115.56	22.1
ET-T950	23.98	125.6	13.55
ET-T965A	25.98	119.89	15.2
H-T950	24.00	116.96	26.7

Concerning to the composites of 1 mm fiber shown in Table 3, in case of the composite of fiber content of about 24 vol%, bending strength of the composite of EW-T950, ET-T950 and H-T950 fiber increase about 13% $\sim$ 17% compared to that of the T950 fiber.

On the other hand, bending strengths of the composite of 3 mm fiber are shown in Table 4. In case of the composite of fiber content of about 12 vol%, bending strength do not vary significantly with the treatments except H-T950 fiber. Bending strength of the composite of H-T950 fiber increase exceptionally about 33% compared to that of the T950 fiber. Moreover in case of the composite of fiber content of about 24 vol%, bending strength of the composite of all the treated fibers increase about 9% $\sim$ 16% compared to that of the T950 fiber. However, this increment and decrement of the strength depend not only on the treatments but also on the content of the fiber into the composite.

After all we can summarise that these variations of the strength of the composite of different fibers depend on the fiber content, fiber length, interfacial shear strength and fiber strength as well. Therefore, in order to define the effect of those factors and fiber treatments on the strength, we have developed an empirical equation.

Tensile strength of the short fiber composites σ_t is usually calculated by the following equation [3].

$$\sigma_t = \Phi \eta \sigma_f V_f + (1 - V_f)\sigma_m, \tag{3.2}$$

where, σ_m is the strength of the matrix material. The factor η is considered the orientation efficiency of the reinforcing fibers and Φ is the length efficiency factor. Here, the first term of the Eq. (3.2) explains the effect of the short fiber on the strength of the composite. Interfacial property of the fiber and matrix is not directly considered in the Eq. (3.2), and thus the construction of this equation can not represent the effect of fiber treatment on the strength of the composite. Therefore, we have derived an empirical equation that can represent the effect of the fiber treatment on the strength.

As during the bending test, maximum tensile force would act in a very thin layer on the bottom surface and the direction of the tensile force is along the neutral axis, we could consider this thin layer a tensile test specimen. Fracture would occur in this thin layer at an imaginary plane perpendicular to the direction of the tensile force. This imaginary plane would divide the distributed fibers into two portions and smaller one would act as embedded length. When the embedded length is smaller than corresponding critical embedded length then the fiber would pull out. When the embedded length is longer than corresponding critical embedded length, then the fiber would fracture out. Therefore, it is assumed that the first term of Eq. (3.2) could be considered to define the effect of the short fibers on the bending strength of the composite.

However, in order to define the bending strength of the composite made by different fibers, we have introduced a parameter ξ as a ratio of the strength of the composite to the effect of the short fibers into the composites. Basic idea of the effect of the short fiber into the composite is taken from the first part of the Eq. (3.2). This parameter ξ might be called as relative bending strength of the composite with respect to the effect of the short fibers into the composite and is shown below

$$\xi = \frac{\sigma_c}{\sigma_f V_f \eta \left(\dfrac{L_f}{L_0}\right)}, \tag{3.3}$$

where, σ_c, σ_f, V_f and L_f are defined as experimental value of the bending strength of the composite, fiber strength, fiber volume fraction in the composite and length of the short fibers into the composite. Here, the term L_f/L_0 is considered as fiber length efficiency factor and L_0 is ideal length of the short aramid fiber for reinforcement in the composite. Then the value of length L_0 can be obtained as follows,

$$L_0 = 2\frac{\Sigma L_c}{N}, \tag{3.4}$$

where, L_c is the critical embedded length of fibers of each treatment that are shown in Table 2 and N is number of fiber treatments. Then value of the length L_0 is found approximately 1.7 mm. In considering the effect of the short fibers into the composite, it is assumed that both of the fiber content V_f and the length efficiency factor L_f/L_0 are directly contributing to the strength as shown in the denominator of the right hand side of the Eq. (3.3).

As described above, the maximum tensile force would be induced in a very thin layer on bottom surface under the bending loading. And the fracture or the pull-out of the reinforcing fiber is decided by the relationship between the embedded length and the critical embedded length. Further, the critical embedded length can well define the fiber treatments as described in the discussion of Table 2. Therefore, in this paper the critical embedded length is employed as a unique parameter to define effectiveness of the fiber treatments.

In order to define the fiber treatments, another parameter ζ is introduced and defined as a ratio of the critical embedded length of the treated fiber to that of the fresh fiber, that is, value of the parameter ζ is shown below

$$\zeta = L_c/L_{c0}, \tag{3.5}$$

where, L_{c0} is defined as critical embedded length of untreated fiber. The parameter ζ is a non-dimensional parameter that describes the fiber treatments.

Figure 5 shows the effect of the treatments on the relative bending strength of the composite with respect to the effect of the short fiber into the composite. From Fig. 5, we find that the data of 1 mm fiber show higher value of the relative bending strength than those of 3 mm fiber. Moreover, higher fiber content (24 vol%) also reduces the relative bending strength in both the cases of 1 and 3 mm fiber. These results mean that the increase of the length and content of the fibers do not cause sufficient increase of the bending strength of the composite. Decrease of the parameter ζ defines developed fiber treatment. Therefore, an increase of the parameter ζ causes a decrease of the relative bending strength for each of the cases of the fiber length and fiber content.

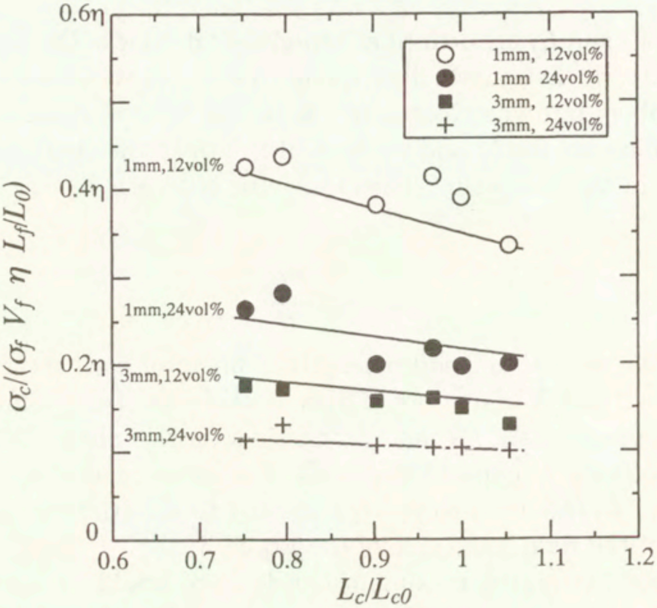

FIG. 5. Effect of the non-dimensional critical embedded length on the relative bending strength of the composite with respect to effect of the short fibers into the composite.

We have developed an equation to explain the result described in Fig. 5. Here, it is assumed that the relative bending strength of the composite decreases linearly with the parameter ζ, for each of the cases of the fiber length and fiber content. Then generalised linear equation can be expressed as follows,

$$\xi = A\zeta + B, \tag{3.6}$$

where, A and B are defined as constants depend on the length and content of the fiber. Value of A and B is calculated such that sum of square of error of each plot of Fig. 5 is minimum. Substituting the value of ξ and ζ from Eqs. (3.3) and (3.5) into the Eq. (3.6), the empirical equation of the bending strength of the composites can be derived as shown below,

$$\sigma_c = 0.075\sigma_f \left(V_f \frac{L_f}{L_0} \right)^{1/5} - 0.02\sigma_f \frac{L_c}{L_{c0}}. \tag{3.7}$$

The first term of the Eq. (3.7) explains the effect of the short fiber on the strength of the composite, and the second one of the Eq. (3.7) explains the development of the fiber treatment.

The fiber content V_f and length efficiency factor L_f/L_0 are powered by 1/5 in the first term. The constant 0.075 of the first term might depend on the distribution of the fiber. In some research works [3, 4], fiber orientation parameter f_p is introduced as follows,

$$f_p = 2\cos\phi - 1, \tag{3.8}$$

where, ϕ is the average angle between distributing fiber and the loading direction. If we consider the constant 0.075 of the first term of Eq. (3.7) as an orientation parameter f_p, then the average orientation angle ϕ would become 42.8 degree. On the other hand, we know that the average orientation angle is 45 degree, in case of random orientation [4]. As this average angle does not define adequately the distribution of the fibers into the composite, it might not be quite reasonable to say that the distribution is very close to random. However, the assumption that the distribution of the fibers is three dimensional proposed in previous discussions [1, 2] would be closely compromised by the average orientation angle obtained.

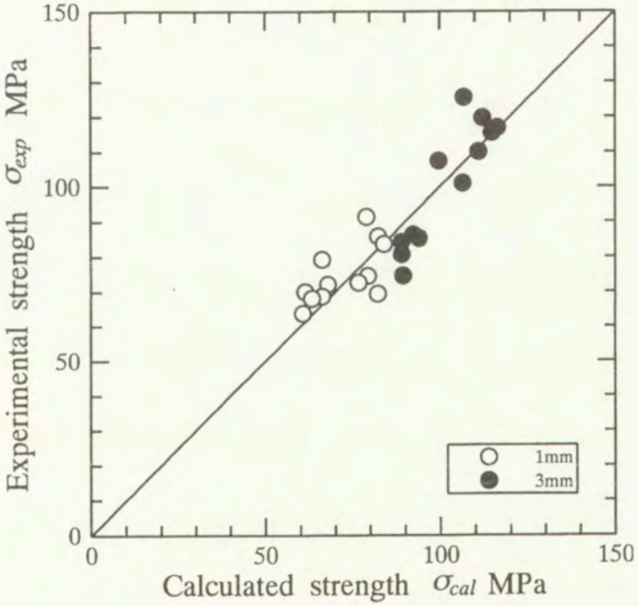

FIG. 6. Comparison of the experimental bending strength σ_{exp} to the calculated bending strength σ_{cal}.

The second part of the Eq. (3.7) defines the interfacial property as well as the fiber treatments. If L_c decreases keeping the fiber strength constant, that means an increase of interfacial shear strength, then the strength of the composite calculated by Eq. (3.7) would increase. This is the same conclusion obtained through the discussion about the results of Table 2 and 3, 4.

Figure 6 shows the comparison of the bending strength of the composite calculated by the empirical Eq. (3.7) to the practical bending strength obtained in our experiment. All the plots are almost very close to the diagonal line. Therefore, the empirical equation is justified well under the experimental conditions employed in this research.

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Application of a constitutive equation for softening, yield and permanent deformation to finite plane simple shear

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The finite homogeneous simple shear deformation of an incompressible material is considered. The response is modeled with a constitutive equation that reflects a continuous process of microstructural transformation as the deformation increases beyond a threshold value. The original and transformed portions of the material are both taken to respond as incompressible elastic solids. It is shown that the transformation can lead to softening of the response with increasing deformation and to a local maximum in the shear stress-shear strain curve. The existence of permanent deformation after release of the shearing traction is demonstrated. It is confirmed that a process of increasing deformation followed by decreasing deformation to the point of zero shear traction is a dissipative cycle. A special case is then considered in which both the original and transformed materials are assumed to respond as neo-Hookean solids. The critical volume fraction of transforming material at which the shear stress-shear strain curve loses monotonicity is found analytically. Representations are obtained for the dependence of the residual shear deformation on the fraction of transforming material; on the ratio of moduli of the original and transformed materials; and on the maximum shear reached before unloading.

1. Introduction

Considerable attention has been focused on the modeling of stress softening, yield and permanent set in polymeric materials. A constitutive equation was recently proposed by Wineman and Rajagopal [23] that assumes a continuous process of microstructural *conversion* as deformation increases beyond a threshold value. This conversion process entails the rupture of a stress-bearing *microstructural unit*, such as a chain molecule, a crosslink or an entanglement. Upon rupture, the microstructural unit cannot bear any stress. It is possible that a new microstructure forms in place of the original one, with a new unstressed reference configuration.

Several analytical and numerical studies have been conducted using the constitutive relation proposed by WINEMAN and RAJAGOPAL [23]. The constitutive equation has been applied to study the inflation of a circular membrane (WINEMAN and HUNTLEY [22]); the radial deformation of hollow spheres (HUNTLEY, WINEMAN and RAJAGOPAL [5, 7]); and the circumferential shear of a hollow cylinder (HUNTLEY, WINEMAN and RAJAGOPAL [6]). RAJAGOPAL and SRINIVASA [13, 14] have considerably generalized the model to describe the twinning of metals, traditional plasticity and solid-to-solid phase transition. HUNTLEY [3] has used the equation to model the Mullins effect and permanent deformation in vulcanized rubbers. HUNTLEY and WALDRON [4] have compared experimental results for polycarbonate from G'SELL and GOPEZ [1] with the response predicted by the constitutive theory suggested by WINEMAN and RAJAGOPAL [23] for finite plane simple shear. Agreement with measured pre-yield and post-yield response was excellent; a stress peak and subsequent drop was also predicted that conformed well to expectations of the events associated with yield.

In the present work, plane simple shear is studied within the context of the framework suggested by WINEMAN and RAJAGOPAL [23].

2. Constitutive equation

Consider a sample of material undergoing a homogeneous deformation described by $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, where $\mathbf{x}$ is the current position of a particle located at $\mathbf{X}$ in the undeformed reference configuration, when $t = 0$. The deformation gradient associated with this mapping is $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ and the left Cauchy-Green tensor is given by $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. Assume that there is a range of deformation for which the material behaves as an isotropic, incompressible, nonlinear Green elastic solid. It is well known (e.g., SPENCER [20]) that the Cauchy stress $\mathbf{T}$ for this material takes the form

$$\mathbf{T} = -p\mathbf{I} + 2[W_1^{(1)}\mathbf{B} - W_2^{(1)}\mathbf{B}^{-1}], \tag{2.1}$$

where $-p\mathbf{I}$ is an indeterminate part of the stress due to the constraint of incompressibility. It will be convenient to denote the extra stress by $\mathcal{T} = \mathbf{T} + p\mathbf{I}$. The strain energy per unit volume is $W^{(1)}(I_1, I_2)$, with $I_1 = \mathrm{tr}(\mathbf{B})$ and $I_2 = \mathrm{tr}(\mathbf{B}^{-1})$, the first two invariants of $\mathbf{B}$. Also, $W_1^{(1)} = \partial W^{(1)}/\partial I_1$ and $W_2^{(1)} = \partial W^{(1)}/\partial I_2$.

An activation criterion determines when the original material network begins to undergo microstructural change and form new networks. This criterion is taken to be expressed as a function of the deformation gradient $\mathbf{F}$ which vanishes when microstructural change begins. Material frame indifference, isotropy and incompressibility imply that the activation criterion can be expressed in terms of the invariants of $\mathbf{B}$: $A(I_1, I_2) = 0$.

In general, a proper, fully three-dimensional loading condition has to be considered. Here, however, only a restricted special deformation is considered. For such a deformation, the terms *increasing deformation* and *decreasing deformation* are meaningful, as there is a one-to-one relationship between the measure of the deformation and the scalar parameter s which is introduced below.

Transformation of the original microstructural network is assumed to be continuous with increasing deformation. Introduce a scalar deformation state parameter s whose value is determined by the extent of deformation. It is assumed that it can be expressed in terms of the stretch invariants: $s = s(I_1, I_2)$. The value of s increases as deformation increases. No unique definition of the term "increasing deformation" is proposed here. Instead, as in the previous applications of this constitutive equation (WINEMAN and RAJAGOPAL [23]; HUNTLEY [2, 3]; WINEMAN and HUNTLEY [22]; HUNTLEY, WINEMAN and RAJAGOPAL [5, 6, 7]; HUNTLEY and WALDRON [4]), an appropriate form of s is selected for the deformation process under consideration. Recasting the activation criterion in terms of the state parameter gives $A(I_1, I_2) = s(I_1, I_2) - s_a$. Microstructural conversion is initiated when the state parameter s first reaches the conversion-activation value s_a.

For $s < s_a$, no conversion has yet occurred; thus all material is original and the total stress is given by (2.1). At the current deformation state s, with $s \geq s_a$, stress in the remaining original material is also a function of the current deformation gradient $\mathbf{F}$.

Introduce the scalar-valued conversion rate function $a(s)$. As increasing deformation causes the state parameter to increase beyond $s = s_a$, the conversion rate function determines the amount of network transformation induced by additional deformation. The conversion rate function may have any form respecting the constraints $a(s) = 0$, $s < s_a$ and $a(s) \geq 0$, $s \geq s_a$. The function $a(s)$ must remain non-negative in order that an increase in the parameter s always be associated with additional microstructural change. It is assumed that a is a continuous function of s.

Consider a value of the deformation state parameter $\hat{s} \geq s_a$. It is assumed that a network is formed at this value of the deformation state parameter. Its reference configuration is the configuration of the original material at state $\hat{s}$. It is assumed to be an unstressed configuration for the newly formed network. The subsequent stress in such a material network is a function of the subsequent deformation of the network relative to this unstressed configuration. Define the relative deformation gradient for the material formed at state $\hat{s}$ as $\hat{\mathbf{F}} = \partial \mathbf{x} / \partial \hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$

is the position of the particle in the configuration corresponding to deformation state $\hat{s}$. This gradient compares the neighborhood of a particle in the configuration at state s with the configuration of the new network when it was formed at state $\hat{s}$. The associated left Cauchy-Green tensor is given by $\hat{\mathbf{B}} = \hat{\mathbf{F}}\hat{\mathbf{F}}^T$.

Let it be assumed that the material network formed at state $\hat{s}$ is elastic, isotropic and incompressible. The extra Cauchy stress at state s in a network formed at the deformation state $\hat{s}$ then becomes

$$\mathcal{T}^{(2)} = 2\left[W_1^{(2)}\hat{\mathbf{B}} - W_2^{(2)}\hat{\mathbf{B}}^{-1}\right]. \tag{2.2}$$

Here $W^{(2)} = W^{(2)}(\hat{I}_1, \hat{I}_2)$ is the strain energy density of the material formed at state $\hat{s}$ and subsequently deformed to the state s, while $\hat{I}_1$ and $\hat{I}_2$ are the appropriate invariants of $\hat{\mathbf{B}}$. The strain energy density functions $W^{(1)}$ and $W^{(2)}$ may each be of any form. It is assumed that the single function $W^{(2)}$ governs the strain energy density in each newly formed network. The material defined by (2.1) and (2.2) and having multiple reference configurations is not a simple material in the sense of NOLL [10] (see RAJAGOPAL [12]).

Total current stress in the material is taken as the superposition of the contribution from the remaining material of the original network and the contributions from all network formed at deformation states $\hat{s} \in [s_a, s]$. During a process of increasing deformation the total current stress is given by

$$\mathbf{T} = -p\mathbf{I} + b(s)\mathcal{T}^{(1)} + \int_{s_a}^{s} a(\hat{s})\mathcal{T}^{(2)} d\hat{s}. \tag{2.3}$$

The function $b(s)$ is the volume fraction of the original network material remaining at state s, with $b(s) = 1$, $s \leq s_a$, and $b(s) \in [0,1]$, $s \geq s_a$. The volume fraction $b(s)$ decreases as s increases. The stress $\mathcal{T}^{(1)}$, found from (2.1), is the current stress in the remaining original material. The quantity $a(\hat{s})d\hat{s}$ may be interpreted as the volume fraction of original material that ruptures and reforms as the deformation state increases from $\hat{s}$ to $\hat{s} + d\hat{s}$. The stress $\mathcal{T}^{(2)}$, given by (2.2), is the stress in that portion of newly formed material. With (2.1) and (2.2), Eq. (2.3) can be written in the form

$$\mathbf{T} = -p\mathbf{I} + 2b(s)\left[W_1^{(1)}\mathbf{B} - W_2^{(1)}\mathbf{B}^{-1}\right] + 2\int_{s_a}^{s} a(\hat{s})\left[W_1^{(2)}\hat{\mathbf{B}} - W_2^{(2)}\hat{\mathbf{B}}^{-1}\right] d\hat{s}. \tag{2.4}$$

Equations (2.3) and (2.4) are constitutive equations for incompressible materials and respect the requirements of frame indifference.

Assume that the material has undergone a process of deformation whereby s has increased monotonically, and that the deformation is subsequently reduced, so that s decreases monotonically. Two assumptions are made concerning the process of decreasing the parameter s: (a) there is no further conversion of the original material; (b) there is no reversal of microstructural transformation. These assumptions are made partly for analytical convenience. It may also be said, however, that any more complicated theory governing the reduction of deformation will only be useful when more information concerning real material behavior is available to guide its formulation.

The above requirements imply that $a(s) = 0$ as the parameter s is reduced. Thus the upper limit of the integral in (2.3) becomes fixed at $s = s^*$, the maximum value of the state parameter reached. The volume fraction of remaining original material undergoes no further change, so that $b(s) = b(s^*)$ as the parameter s is reduced. The stress during a reduction from $s = s^*$ then has the form

$$\mathbf{T} = -p\mathbf{I} + b(s^*)\mathcal{T}^{(1)} + \int\limits_{s_a}^{s^*} a(\hat{s})\mathcal{T}^{(2)} d\hat{s}, \tag{2.5}$$

where $\mathcal{T}^{(1)}$ is found from (2.1) and $\mathcal{T}^{(2)}$ is given by (2.2). Equation (2.5) can be written with (2.1) and (2.2) as

$$\mathbf{T} = -p\mathbf{I} + 2b(s^*)\left[W_1^{(1)}\mathbf{B} - W_2^{(1)}\mathbf{B}^{-1}\right] + 2\int\limits_{s_a}^{s^*} a(\hat{s})\left[W_1^{(2)}\hat{\mathbf{B}} - W_2^{(2)}\hat{\mathbf{B}}^{-1}\right] d\hat{s}. \tag{2.6}$$

Equations (2.1), (2.4) and (2.6) represent the complete set of constitutive equations for all deformation processes.

Unhatted kinematic quantities, such as quantities, such as $\mathbf{F}$, $\mathbf{B}$, I_1 and I_2, are referred to as "current" and compare the configuration at the current deformation state s with the initial reference configuration. Kinematic quantities bearing the hat notation ($\hat{}$), such as $\hat{\mathbf{F}}$, $\hat{\mathbf{B}}$, $\hat{I}_1$ and $\hat{I}_2$, are called "relative" quantities. They represent comparison of the configuration at the current state s with the configuration as state $\hat{s}$.

The superscript $(\ )^{(1)}$ appearing in stress quantities such as $\mathcal{T}^{(1)}$ indicates that the stress is in the material with the original microstructure. Such stresses are functions of the current left Cauchy-Green tensor $\mathbf{B}$. The superscript $(\ )^{(2)}$ appearing, for example, in $\mathcal{T}^{(2)}$ indicates stress in a material network formed at the deformation state $\hat{s}$. These stresses are functions of the relative left Cauchy-Green tensor $\hat{\mathbf{B}}$. Unsuperscribed stresses, such as $\mathbf{T}$, are total stresses following the superposition given by (2.3) of stresses in the original and newly formed networks. They are thus functions of the current tensor $\mathbf{B}$ and of the relative tensors $\hat{\mathbf{B}}$ relating the current configuration to each state $\hat{s} \in [s_a, s]$ during increasing s. For a process of increasing deformation, unsuperscribed stresses also depend explicity on the current value of the deformation state parameter s, which appears as the upper limit of integration and as the argument of $b(s)$. During reversal of deformation, unsuperscribed stresses depend explicity on s^*.

The function $W^{(1)}$ denotes the Helmholtz strain energy density in the material with the original network; it is a function of the current stretch invariants I_1 and I_2. The function $W^{(2)}$ is the strain energy density in the material of a subsequently formed network and is a function of the relative invariants $\hat{I}_1$ and $\hat{I}_2$.

Non-dimensionalized quantities bear the tilde notation $\sim$, as $\tilde{\mathbf{T}}$.

For purposes of notational simplicity, none of the functional dependences mentioned above is indicated explicitly when kinematic or stress quantities are written.

3. Formulation

3.1. Kinematics of deformation

Consider a particle of an isotropic, incompressible material undergoing simple shear. The material is subjected to the shearing and normal tractions necessary to induce the isochoric mapping

$$\begin{aligned} x_1 &= X_1, \\ x_2 &= X_2 + kX_1, \\ x_3 &= X_3. \end{aligned} \tag{3.1}$$

Here, $x_i (i = 1, 2, 3)$ is the current position of the particle located at $X_j (j = 1, 2, 3)$ in the initial reference configuration, and k is the measure of the current amount of shear deformation relative to the reference configuration. It serves as the deformation control parameter. The current deformation gradient is found from (3.1) to be

$$\mathbf{F}(k) = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.2}$$

The current left Cauchy-Green tensor is given by

$$\mathbf{B}(k) = \mathbf{F}(k)\mathbf{F}^T(k) = \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.3}$$

Its inverse is found to be

$$\mathbf{B}^{-1}(k) = \begin{bmatrix} 1+k^2 & -k & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.4}$$

The invariants of the left Cauchy-Green tensor are

$$I_1(k) = I_2(k) = 3 + k^2. \tag{3.5}$$

3.1.1. State parameter s. In order to evaluate the Cauchy stress tensor $\mathbf{T}$, specific forms will have to be chosen for the deformation state parameter $s(I_1, I_2)$, the conversion rate function $a(s)$ and the volume fraction of original material remaining $b(s)$. The only requirement yet imposed on $s(I_1, I_2)$ has been that it increase monotonically as the deformation increases. In general, $s(I_1, I_2)$ may be represented by any surface above or below the $I_1 - I_2$ plane which displays such monotonicity. It is not the intent of the present work to propose a form for $s(I_1, I_2)$ which would be valid over the entire $I_1 - I_2$ domain. Indeed, the development of such a form would first require the definition of a loading condition similar to that used in plasticity (see RAJAGOPAL and SRINIVASA [15, 16]). This definition is not needed for the present work, as the loading is a simple shear and s increases monotonically with the shear; thus no general definition will be proposed.

As the shear k increases monotonically from $k = 0$, I_1 and I_2, given by (3.5), increase monotonically from the undeformed state $I_1 = I_2 = 3$ along a path defined parametrically by (3.5). The associated point in the $I_1 - I_2$ plane associated with the current shear of the particle moves outward along the straight line $I_1 = I_2$. Thus in the specific case of simple shear, any choice for $s(I_1, I_2)$ which increases monotonically along the line $I_1 = I_2$ is valid. The deformation state parameter may thus be written as a monotonically increasing function of the current shear, $s = s(k)$.

3.2. Increasing deformation

3.2.1. $s < s_a$. For $s < s_a$, no microstructural transformation has yet occurred. The Cauchy stress tensor $\mathbf{T}$ in all the original material can thus be determined from (2.1), (3.3) and (3.4) as

$$\mathbf{T} = -p\mathbf{I} + 2\left\{ W_1^{(1)} \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - W_2^{(1)} \begin{bmatrix} 1+k^2 & -k & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \tag{3.6}$$

where p is an indeterminate scalar.

3.2.2. $s \geq s_a$. Let k_a denote the shear when $s = s_a$, i.e., $s_a = s(k_a)$. Consider the shear deformation to $\hat{k} > k_a$, which corresponds to state $\hat{s}$ by the relation $\hat{s} = s(\hat{k})$. Let the deformation gradient at state $\hat{s}$ be denoted by $\mathbf{F}(\hat{k})$. Subsequent deformation to the state $s > \hat{s}$ introduces the relative deformation gradient $\hat{\mathbf{F}}(k)$. The relative deformation gradient $\hat{\mathbf{F}}(k)$ of a network formed at state $\hat{s}$ relates its current configuration at state s to its new reference configuration at state $\hat{s}$. It is formed as $\hat{\mathbf{F}}(k) = \partial\mathbf{x}/\partial\hat{\mathbf{x}}$ or $\hat{F}_{ij}(k) = \partial x_i/\partial\hat{x}_j$. Here $\mathbf{x} = (x_i)$ is the current position vector of the particle, corresponding to the deformation state s; $\hat{\mathbf{x}} = (\hat{x}_j)$ is the new reference position vector of the particle, corresponding to state $\hat{s}$. The relative deformation gradient can be constructed as

$$\hat{F}_{ij}(k) = (\partial x_i/\partial X_k)(\partial X_k/\partial\hat{x}_j) \tag{3.7}$$

or

$$\hat{\mathbf{F}}(k) = \mathbf{F}(k)\mathbf{F}^{-1}(\hat{k}). \tag{3.8}$$

From (3.2) one can find

$$\mathbf{F}^{-1}(\hat{k}) = \begin{bmatrix} 1 & 0 & 0 \\ -\hat{k} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.9}$$

Equations (3.2), (3.7) and (3.9) then give the relative deformation gradient as

$$\hat{\mathbf{F}}(k) = \begin{bmatrix} 1 & 0 & 0 \\ k-\hat{k} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.10}$$

It should be noted that the quantity $(k-\hat{k})$ is the current shear deformation of a network formed at the state $\hat{s}$. The relative left Cauchy-Green tensor $\hat{\mathbf{B}}(k)$ and its inverse become

$$\hat{\mathbf{B}}(k) = \hat{\mathbf{F}}(k)\hat{\mathbf{F}}^T(k) = \begin{bmatrix} 1 & k-\hat{k} & 0 \\ k-\hat{k} & 1+(k-\hat{k})^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.11}$$

and

$$\hat{\mathbf{B}}^{-1}(k) = \begin{bmatrix} 1+(k-\hat{k})^2 & \hat{k}-k & 0 \\ \hat{k}-k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.12}$$

The invariants of the relative left Cauchy-Green tensor are

$$(3.13) \qquad \hat{I}_1 = \hat{I}_2 = 3 + (k - \hat{k})^2.$$

The current extra stress $\mathcal{T}^{(2)}(k-\hat{k})$ in a network element formed at the state of deformation $\hat{s}$ is determined from (2.2), (3.11) and (3.12) as

$$(3.14) \qquad \frac{\mathcal{T}^{(2)}(k-\hat{k})}{2} = W_1^{(2)} \begin{bmatrix} 1 & k-\hat{k} & 0 \\ k-\hat{k} & 1+(k-\hat{k})^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - W_2^{(2)} \begin{bmatrix} 1+(k-\hat{k})^2 & \hat{k}-k & 0 \\ \hat{k}-k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

At all states of deformation, the extra stress $\mathcal{T}^{(2)}(k-\hat{k})$ in any remaining original material follows (3.14). From (2.3), (3.6) and (3.14), the total stress during a process of increasing s is found to be

$$(3.15) \qquad \mathbf{T} = -p\mathbf{I} + 2b(s)\left\{ W_1^{(1)} \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - W_2^{(1)} \begin{bmatrix} 1+k^2 & -k & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} + 2\int_{s_a}^{s} a(\hat{s}) \left\{ W_1^{(2)} \begin{bmatrix} 1 & k-\hat{k} & 0 \\ k-\hat{k} & 1+(k-\hat{k})^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - W_2^{(2)} \begin{bmatrix} 1+(k-\hat{k})^2 & \hat{k}-k & 0 \\ \hat{k}-k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} d\hat{s}.$$

3.2.3. Universal relation. The stress tensor $\mathbf{T}$ in (3.15) reveals a distinction between the present constitutive model for materials undergoing microstructural change and models for purely elastic isotropic materials. Discussions can be found in WINEMAN and GANDHI [21] and RAJAGOPAL and WINEMAN [17] of universal relations for isotropic elastic materials. For materials subjected to simple shear deformation, one of these relations reduces to that of RIVLIN [19]:

$$T_{22} - T_{11} = kT_{12}. \tag{3.16}$$

Upon substitution of the appropriate stress components from (3.15) and simplification, the equality (3.16) is seen not to hold: in general,

$$\int_{s_a}^{s} a(\hat{s})(W_1^{(2)} + W_2^{(2)})(k-\hat{k})^2 d\hat{s} \neq k \int_{s_a}^{s} a(\hat{s})(W_1^{(2)} + W_2^{(2)})(k-\hat{k}) d\hat{s}. \tag{3.17}$$

Since the behavior described by the constitutive Eq. (2.3) is not purely elastic, it may be said in general that the applicability of universal relations from elasticity cannot be guaranteed. Indeed, as can be seen from (3.17), such assurance cannot be given even if the constituent extra stresses $\mathcal{T}^{(1)}(k)$ and $\mathcal{T}^{(2)}(k-\hat{k})$ are themselves purely elastic.

3.2.4. Shear stress. Consider the current shear stress $T_{12}(k)$ as the shear deformation increases. It is given by (3.15) as

$$\frac{T_{12}(k)}{2} = b(s)(W_1^{(1)} + W_2^{(1)})k + \int_{s_a}^{s} a(\hat{s})(W_1^{(2)} + W_2^{(2)})(k-\hat{k}) d\hat{s}. \tag{3.18}$$

Note that

$$W_i^{(1)} = W_i^{(1)}(I_1(k), I_2(k)) \qquad (i = 1, 2). \tag{3.19}$$

On using (3.5), this becomes

$$W_i^{(1)} = W_i^{(1)}(3 + k^2, 3 + k^2). \tag{3.20}$$

For simplicity of notation, introduce the quantity

$$\mu^{(1)} = \mu^{(1)}(k^2) = W_1^{(1)}(k^2) + W_2^{(1)}(k^2). \tag{3.21}$$

Similarly,

$$W_i^{(2)} = W_i^{(2)}(\hat{I}_1(k), \hat{I}_2(k)) \tag{3.22}$$

becomes

$$W_i^{(2)} = W_i^{(2)}(3 + (k - \hat{k})^2, 3 + (k - \hat{k})^2), \tag{3.23}$$

where the relative invariants are as given by (3.13). Introduce the notation

$$\mu^{(2)} = \mu^{(2)}((k - \hat{k})^2) = W_1^{(2)}((k - \hat{k})^2) + W_2^{(2)}((k - \hat{k})^2). \tag{3.24}$$

Here, $\mu^{(1)}$ is the deformation-dependent shear modulus of the original material, while $\mu^{(2)}$ is the deformation-dependent shear modulus of the material of all subsequently formed networks. It is assumed that these moduli are strictly positive:

$$\mu^{(1)} > 0; \qquad \mu^{(2)} > 0. \tag{3.25}$$

With the change of notation, (3.18) simplifies to

$$\frac{T_{12}(k)}{2} = b(s)\mu^{(1)}k + \int_{s_a}^{s} a(\hat{s})\mu^{(2)}(k - \hat{k})d\hat{s}. \tag{3.26}$$

The first term on the right-hand side of (3.26) represents the contribution to the total shear stress of the remaining material of the original network, while the second term is the contribution of all networks formed as the deformation increases. The change of notation given by (3.21) and (3.24) can also be applied to (3.26) to write

$$T_{12}^{(1)}(k) = 2\mu^{(1)}k \tag{3.27}$$

and

$$T_{12}^{(2)}(k - \hat{k}) = 2\mu^{(2)}(k - \hat{k}). \tag{3.28}$$

The shear stress in each constituent material is thus an odd function of its shear deformation.

3.3. Monotonicity of response

It is of interest to know whether there are conditions under which the shear response is non-monotonic. The monotonicity of the shear stresss-shear deformaton relation may be studied by inspection of the derivative of the shear stress with respect to the shear deformation. From (3.26), the current shear stress can be written as

$$T_{12}(k) = b(s)T_{12}^{(1)}(k) + \int_{s_a}^{s} a(\hat{s})T_{12}^{(2)}(k - \hat{k})d\hat{s}. \tag{3.29}$$

Differentiating the total shear stress (3.29) with respect to the current shear k gives

$$\frac{dT_{12}}{dk} = b(s)\frac{dT_{12}^{(1)}(k)}{dk} + \frac{db(s)}{dk}T_{12}^{(1)}(k) + \int_{s_a}^{s} a(\hat{s})\frac{dT_{12}^{(2)}(k-\hat{k})}{dk}d\hat{s}. \tag{3.30}$$

Note that the derivative of the integral in (3.29) with respect to its upper limit vanishes, as it is found from (3.28) that $T_{12}^{(2)}(k-\hat{k}) = 0$ when evaluated at $k = \hat{k}$.

The current volume fraction $b(s)$ of material composed of the original microstructural network is a positive number: the requirement that $b(s) \in [0,1]$ has been stated above. Recall also the requirement that $a(s) > 0$. The moduli $\mu^{(1)}$ and $\mu^{(2)}$ are assumed to be strictly positive, as stated in (3.25). The shear k is taken to be positive, as the material is sheared in the positive sense during loading. The quantity $k - \hat{k}$ must also be positive, as increasing deformation implies that the current shear k is greater than all previous values $\hat{k}$. It can then be seen from (3.27), (3.28) and (3.29) that the shear stresses $T_{12}^{(1)}(k)$ and $T_{12}^{(2)}(k-\hat{k})$ are positive.

It is the aim of the present work to isolate the effects of the microstructural conversion phenomenon from those of any constitutive assumptions implicit in the strain energy functions $W^{(1)}$ and $W^{(2)}$. Toward that end, let us confine attention to a certain class of strain energy density functions. Assume that the shear stress response is monotonic in the shear deformation, both for the original network and for the subsequently formed networks; that is,

$$\frac{dT_{12}^{(1)}(k)}{dk} > 0; \qquad \frac{dT^{(2)}(x)}{dx} > 0 \quad (x = k - \hat{k}). \tag{3.31}$$

Note that the argument of $T_{12}^{(2)}(x)$ is $x = k - \hat{k}$, the shear relative to the reference configuration at deformation state s. However, positiveness of the derivative with respect to this argument implies that the term $dT_{12}^{(2)}(k-\hat{k})/dk$ from (3.30) is also strictly positive.

The constitutive theory of microstructural change assumes that the volume fraction of original remaining network material decreases with increasing deformation:

$$\frac{db(s)}{dk} < 0. \tag{3.32}$$

It follows that

$$T_{12}^{(1)}(k)\frac{db(s)}{dk} < 0. \tag{3.33}$$

If the magnitude of this term becomes great enough to outweigh the positive terms in (3.30), then $dT_{12}/dk < 0$ and a local maximum in the shear stress-shear strain curve develops. Thus it can be seen from (3.30) that the process of network scission and healing introduces the possibility of a loss of monotonicity in the total stress response. Equations (3.29) and (3.30) indicate that a combination of factors could lead to this result: a state of relatively high current shear k; relatively great stiffness of the original network material; and rapid scission of the original material.

3.4. Reversal of deformation

For $s < s_a$, the extra stress is as found from (3.6) and assumes the same value for a given value of k whether shear deformation is increasing or decreasing. Thus when shear has been reversed to $k = 0$, all components of the total stress are returned to zero by appropriate choice of p.

Now consider a process of decreasing shear deformation after the state parameter has reached a value $s^* \geq s_a$. The expressions for $\mathcal{T}^{(1)}(k)$ in (3.6) and $\mathcal{T}^{(2)}(k - \hat{k})$ in (3.14) are still valid during reduction of the current shear from a state of maximum deformation s^*. From (2.5), (3.6) and (3.14), the stress tensor during this process then takes the form

$$
\mathbf{T}(k) = -p\mathbf{I} + 2b(s^*)\left\{W_1^{(1)}\begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - W_2^{(1)}\begin{bmatrix} 1+k^2 & -k & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}
$$
$$
+2\int_{s_a}^{s^*} a(\hat{s})\left\{W_1^{(2)}\begin{bmatrix} 1 & k-\hat{k} & 0 \\ k-\hat{k} & 1+(k-\hat{k})^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - W_2^{(2)}\begin{bmatrix} 1+(k-\hat{k})^2 & \hat{k}-k & 0 \\ \hat{k}-k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} d\hat{s}. \tag{3.34}
$$

Consider the shear stress $T_{12}(k)$ associated with current shear deformation k during reduction of deformation:

$$
\frac{T_{12}(k)}{2} = b(s^*)\mu^{(1)}k + \int_{s_a}^{s^*} a(\hat{s})\mu^{(2)}(k-\hat{k})d\hat{s}. \tag{3.35}
$$

This can be rewritten as

$$\frac{T_{12}(k)}{2} = \left[b(s^*)\mu^{(1)} + \int_{s_a}^{s^*} a(\hat{s})\mu^{(2)} d\hat{s} \right] k - \int_{s_a}^{s^*} a(\hat{s})\mu^{(2)} \hat{k} d\hat{s}. \tag{3.36}$$

The first term on the right-hand side of (3.36) represents the contribution to the total stress of the volume fraction $b(s^*)$ of original material remaining at state s^*. The second term is due to stresses in networks that were formed at states $\hat{s}$ during the loading process, $s \in [s_a, s^*]$. Both of these contributions to the current stress show a direct dependence on the current shear k. The final term on the right-hand side of (3.36) is also due to stresses in newly formed networks. However, the only effect that the current shear deformation can have on this term is through the evaluation of the shear modulus $\mu^{(2)}$ at the relative strain invariants $\hat{I}_1(k-\hat{k})$ and $\hat{I}_2(k-\hat{k})$. Even so, one may assume positiveness of the shear modulus at all levels of decreasing deformation. Assume further that the shear modulus is bounded. The conversion rate $a(s)$ has been defined as a non-negative function. The quantity $\hat{k}$ represents the amount of shear corresponding to deformation state $\hat{s} \in [s_a, s^*]$. Since any level of shear $\hat{k}$ is induced by a positive shearing load, it also is positive. Therefore,

$$\int_{s_a}^{s^*} a(\hat{s})\mu^{(2)} \hat{k} d\hat{s} > 0. \tag{3.37}$$

This inequality will be useful in the following discussion of the residual state.

3.5. Residual state

3.5.1. Normal tractions. Assume that a material sample, portions of which have undergone microstructural conversion during a process of increasing shear, is returned to its original reference configuration $k = 0$. Then (3.36) and (3.37) imply that negative shearing strains and stresses exist in those portions of the material which formed new networks as the deformation was increased. From (3.36) and (3.37), it is seen that a negative shearing traction whose absolute value is $2\int_{s_a}^{s^*} a(\hat{s})\mu^{(2)} \hat{k} d\hat{s}$, is necessary to maintain the specimen in the state $k = 0$. Similarly, it can be found from (3.34) that normal tractions $T_{11}(k)$ and $T_{22}(k)$ are also required when the material is returned to its initial configuration. The extra portions of these normal tractions when $k = 0$ are

$$
(3.38)\qquad
\begin{aligned}
\mathcal{T}_{11}(0) &= 2b(s^*)(W_1^{(1)} - W_2^{(1)}) + 2\int\limits_{s_a}^{s^*} a(\hat{s})[W_1^{(2)} - W_2^{(2)}(1+\hat{k}^2)]d\hat{s}\\
\mathcal{T}_{22}(0) &= 2b(s^*)(W_1^{(1)} - W_2^{(1)}) + 2\int\limits_{s_a}^{s^*} a(\hat{s})[W_1^{(2)}(1+\hat{k}^2) - W_2^{(2)}]d\hat{s}.
\end{aligned}
$$

3.5.2. Permanent set. Conversely, suppose now that the external shear traction is released, so that $T_{12}(k) = 0$. Equations (3.36) and (3.37) imply a residual positive shear deformation k^{res} of the specimen which satisfies

$$
(3.39)\qquad k^{\text{res}} = \frac{\int\limits_{s_a}^{s^*} a(\hat{s})\mu^{(2)}\hat{k}d\hat{s}}{b(s^*)\mu^{(1)} + \int\limits_{s_a}^{s^*} a(\hat{s})\mu^{(2)}d\hat{s}} > 0.
$$

The original material network is in a state of residual positive shear with respect to the original reference configuration, while material networks formed at deformation states $\hat{s}$ during loading are in various states of residual negative or positive shear with respect to the reference configurations when they were formed. The positive shear stresses in the original material are balanced by negative shear stresses in the subsequently formed material, so that the total shear stress is zero.

3.6. Work done

It is not the purpose of the present work to place the constitutive Eqs. (2.3) and (2.5) within a complete and general thermodynamic framework. It seems reasonable, nonetheless, to expect that the net work done on a specimen be greater than or equal to zero for a mechanical cycle of deformations. Positive net work done indicates a dissipative process. It should be confirmed that the constitutive equations for materials undergoing microstructural change conform to this requirement.

Due to the nature of the simple shear deformation, the only stress that does work is the shearing stress $T_{12}(k)$. Define $T_{12}^{\text{inc}}(k)$, given by (3.27) for $k < k_a$ or by (3.26) for $k \geq k_a$, as the shear stress during a process of increasing deformation; $T_{12}^{\text{dec}}(k)$, given by (3.35), is defined as the shear stress during a process of decreasing deformation. It will be taken as a sufficient condition for positiveness of the net work done that the shear stress $T_{12}^{\rm dec}(k)$ as deformation is reversed be less than the corresponding value of $T_{12}^{\rm inc}(k)$ as deformation increases. This must hold for all values of $k \in [0, k^*]$, where k^* is the level of shear corresponding to deformation state s^*.

For $k \in [0, k_a]$, the difference between $T_{12}^{\rm inc}(k)$ and $T_{12}^{\rm dec}(k)$ is formed from (3.27) and (3.35) as

$$\frac{T_{12}^{\rm inc}(k) - T_{12}^{\rm dec}(k)}{2} = [1 - b(s^*)]\mu^{(1)}k - \int_{s_a}^{s^*} a(\hat{s})\mu^{(2)}(k - \hat{k})d\hat{s}. \tag{3.40}$$

With $s^* > s_a$, it is known that $1 - b(s^*) > 0$. Strict positiveness of the deformation-dependent shear moduli $\mu^{(1)}$ and $\mu^{(2)}$ has been established in (3.25). All values of $\hat{k}$ in the integral in (3.40) satisfy $\hat{k} > k_a$; thus the term $k - \hat{k} < 0$. It follows that $T_{12}^{\rm inc}(k) - T_{12}^{\rm dec}(k) > 0$ for $k \in [0, k_a]$.

For $k \in [k_a, k^*]$, the difference between the two expressions for the shear stress can be formed from (3.26) and (3.35) as

$$\frac{T_{12}^{\rm inc}(k) - T_{12}^{\rm dec}(k)}{2} = [b(s) - b(s^*)]\mu^{(1)}k - \int_{s}^{s^*} a(\hat{s})\mu^{(2)}(k - \hat{k})d\hat{s}. \tag{3.41}$$

Since s^* describes the state of maximum deformation reached during the deformation cycle, any current state s satisfies $s < s^*$. Furthermore, any value of the deformation state parameter $\hat{s}$ in the integral in (3.41) satisfies $s < \hat{s}$. The corresponding values of shear give $k - \hat{k} < 0$. It is known that $\mu^{(1)}$ and $\mu^{(2)}$ are positive. Thus $\int_{s}^{s^*} a(\hat{s})\mu^{(2)}(k - \hat{k})d\hat{s} < 0$. The volume fraction of remaining original material $b(s)$ is assumed to decrease monotonically as s increases, so $b(s) - b(s^*) > 0$ and $[b(s) - b(s^*)]\mu^{(1)}k > 0$. It then follows from (3.41) that $T_{12}^{\rm inc}(k) - T_{12}^{\rm dec}(k) > 0$ for $k \in [k_a, k^*]$.

The preceding discussion shows that the stress-shear curve for reversal of the shear deformation lies below the curve for increasing shear for all $k \in [0, k^*]$. Thus the cycle of increasing simple shear followed by reduction of shear is a dissipative process when the constitutive equation for microstructural change is employed. This result holds for any valid strain energy density functions $W^{(1)}$ and $W^{(2)}$.

4. Example – Neo-Hookean structure before and after conversion

Let it be assumed that both the original material and the material that is newly formed as deformation increases beyond s_a, are neo-Hookean. Then

$$W^{(1)}(I_1, I_2) = c^{(1)}(I_1 - 3); \qquad W^{(2)}(\hat{I}_1, \hat{I}_2) = c^{(2)}(\hat{I}_1 - 3), \tag{4.1}$$

where $c^{(1)}$ and $c^{(2)}$ are constants. To highlight the role of these constants as moduli in shear, let the notation $\mu^{(1)} = c^{(1)}$ and $\mu^{(2)} = c^{(2)}$ be adopted. It should be emphasized that the restriction to neo-Hookean network response is not at all necessary. Both original and subsequently formed materials are taken as neo-Hookean in order to demonstrate as well as possible the effects of the conversion phenomenon itself on overall mechanical response. To assume differing forms of response in the constituent material networks would lead to mathematical complexity which would cloud this investigation. However, for reasons which are discussed below, the possibility is admitted that the original and newly formed materials have different moduli in shear, that is, that $\mu^{(1)}$ and $\mu^{(2)}$ may not be equal.

4.1. Increasing deformation

4.1.1. $s < s_a$. At levels of deformation satisfying $s < s_a$, no material network has undergone conversion. Thus overall material response is given by that of the original material. From (3.6) and (4.1), the current Cauchy stress for $s < s_a$ is determined as

$$\mathbf{T}^{(1)} = -p\mathbf{I} + 2\mu^{(1)} \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.2}$$

where p is an indeterminate scalar. The shear stress component when $s < s_a$ is seen from (4.2) to be

$$T_{12}(k) = T_{12}^{(1)}(k) = 2\mu^{(1)}k. \tag{4.3}$$

4.1.2. $s \geq s_a$. The stress in a material network formed at the deformation state $s \geq s_a$ depends on the current response of the remaining original material and on that of all newly formed networks. From (3.15) and (4.1) the current total Cauchy stress as deformation increases on $s \geq s_a$ is found to be

$$\mathbf{T} = -p\mathbf{I} + 2b(s)\mu^{(1)} \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu^{(2)} \int\limits_{s_a}^{s} a(\hat{s}) \begin{bmatrix} 1 & k-\hat{k} & 0 \\ k-\hat{k} & 1+(k-\hat{k})^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} d\hat{s}. \tag{4.4}$$

The current shear stress $T_{12}(k)$ is thus

$$\frac{T_{12}(k)}{2} = b(s)\mu^{(1)}k + \mu^{(2)} \int_{s_a}^{s} a(\hat{s})(k - \hat{k})d\hat{s}. \tag{4.5}$$

4.2. State parameter s

It has been stated that any choice for the deformation state parameter $s(k)$ which increases monotonically with k is valid. In a regime of positive current shear $k > 0$, the simplest function which satisfies this requirement is $s = k$. The present work is not concerned with the possible justification of any more complicated form. Therefore, let $s = k$ for the study of simple shear, so that the underformed state is represented by $s = k = 0$.

The forms for $a(s)$ and $b(s)$ discussed below have been introduced by RAJAGOPAL and WINEMAN [18]. The only restriction on the conversion rate function $a(s)$ has been stated in Sec. 2 and is repeated here for convenience:

$$a(s) = 0, \qquad s < s_a; \qquad a(s) \geq 0, \qquad s \geq s_a. \tag{4.6}$$

As in the case of the state parameter $s(I_1, I_2)$, $a(s)$ is chosen as a simple function which satisfies the requirements imposed. Let $s = s_c > s_a$ denote the value of the deformation state parameter at which microstructural transformation is completed. It is assumed that no further conversion occurs as s increases beyond s_c, regardless of the nature of the associated deformation. The maximum deformation must be finite, so s is assumed to vary on the finite domain $s \in [s_a, s_c]$. In examples studied in this work, deformations for values of $s > s_c$ are not considered. Let $a(s)$ be given as the quadratic polynominal

$$a(s) = \begin{cases} 0, & s < s_a \\ \alpha(s - s_a)(s - s_c), & s \in [s_a, s_c] \\ 0, & s > s_c \end{cases} \tag{4.7}$$

where α is a constant. A typical form of $a(s)$ is shown by the dotted line in Fig. 1.

Let C represent the total volume fraction of new network that has been formed when the conversion process is complete:

$$C = \int_{s_a}^{s_c} a(\hat{s})d\hat{s}. \tag{4.8}$$

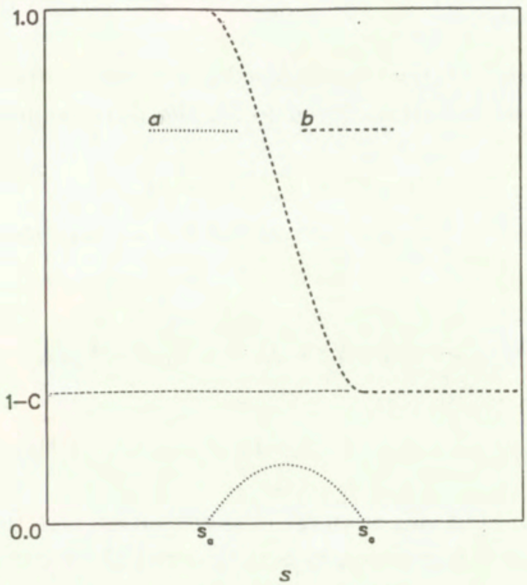

FIG. 1. Typical forms of conversion rate function and volume fraction of original material *versus* deformation state parameter.

It is emphasized that C represents the volume fraction of converted material when the deformation state reaches $s = s_c$. For any other value of $s \in [s_a, s_c]$, the current fraction of converted material is less than C. It follows from (4.7) and (4.8) that

$$\alpha = \frac{-6C}{(s_c - s_a)^3}. \tag{4.9}$$

A simple expression for the volume fraction $b(s)$ of material remaining in the original network can be formed in terms of the above quantities. First impose the restriction $C \leq 1$. Assume now that each original material network which undergoes scission is replaced by exactly one new network, in effect, that "conservation of network junctions" holds. This implies that

$$b(s) = 1 - \int\limits_{s_a}^{s} a(\hat{s})d\hat{s}. \tag{4.10}$$

From (4.7), (4.8) and (4.10), it can be seen that $b(s) = 1 - C$ for $s > s_c$. A typical form of $b(s)$ is shown by the dashed line in Fig. 1. The form (4.10) and the assumptions which underlie it are not necessary. The analysis that is carried out here can be easily redone when (4.10) does not hold.

With the substitution $s = k$, (4.7) through (4.10) can be rewritten to give expressions for $a(k)$, C, α and $b(k)$ in terms of k, $k_a = s_a$ and $k_c = s_c$.

4.3. Shear stress-shear relations

Let the shear stress be non-dimensionalized through division by the modulus $\mu^{(1)}$ of the original material. From (4.3), the dimensionless shear stress for $k < k_a$ is

$$\tilde{T}_{12}(k) = \tilde{T}_{12}^{(1)}(k) = 2k. \tag{4.11}$$

When the current level of deformation satisfies $k \geq k_a$, the non-dimensional shear stress is found from (4.5) to be

$$\tilde{T}_{12}(k) = 2b(s)k + 2\tilde{\mu} \int\limits_{s_a}^{s} a(\hat{s})(k - \hat{k})d\hat{s}. \tag{4.12}$$

Here $\tilde{\mu} = \mu^{(2)}/\mu^{(1)}$ is the ratio of the shear moduli of the newly formed and original materials.

For all of the results in this example, the activation criterion is considered to be satisfied when $k_a = 0.5$; conversion is considered to be complete at $k_c = 2.65$. These values are selected solely to facilitate demonstration of the effects implied by the constitutive equation over what is assumed to be a reasonable range of simple shear deformation for highly elastic materials.

Figure 2 shows plots of the shear stress $\tilde{T}_{12}$ *versus* current shear k for various values of C. The solid line corresponds to a standard neo-Hookean model with no microstructural transformation ($C = 0.0$), while the curves lying below it show the results for varying values of the conversion fraction C. Here $\tilde{\mu} = 1.0$ is assumed. All of the plots coincide for $k < k_a$. It can be seen from Fig. 2 that the microstructural transformation which begins when $k = k_a$ induces a softening of the overall mechanical response of the material for all shear deformations $k > k_a$. Moreover, this softening becomes more pronounced as C increases. When $C = 1.0$, a loss of monotonicity of response is evident. This effect is discussed in greater detail below.

Figure 3 shows plots of $\tilde{T}_{12}$ *versus* k at various values of the conversion fraction C when $\tilde{\mu} = 2.0$. These results demonstrate the effect of the conversion process when the shear modulus of the newly formed material networks is greater than that of the original material. The plots display a softening effect similar to that observed in Fig. 2. As was the case above for $\tilde{\mu} = 1.0$, overall response becomes softer with increasing C. However, it is clear from Figs. 2 and 3 that the total loss of stiffness in shear is not as great in the case of $\tilde{\mu} = 2.0$. The higher modulus of the newly formed material leads to higher stress $\tilde{T}_{12}^{(2)}(k - \tilde{k})$ in material formed at state $\hat{s} = \hat{k}$ than is the case when $\tilde{\mu} = 1.0$. This effect tends to counteract the relaxation of stress in the material element which occurs when it undergoes scission and healing at state $\hat{k}$. Hence the total stress $\tilde{T}_{12}(k)$ remains higher for $k > k_a$ when $\tilde{\mu} = 2.0$. Response remains monotonic for all values of C.

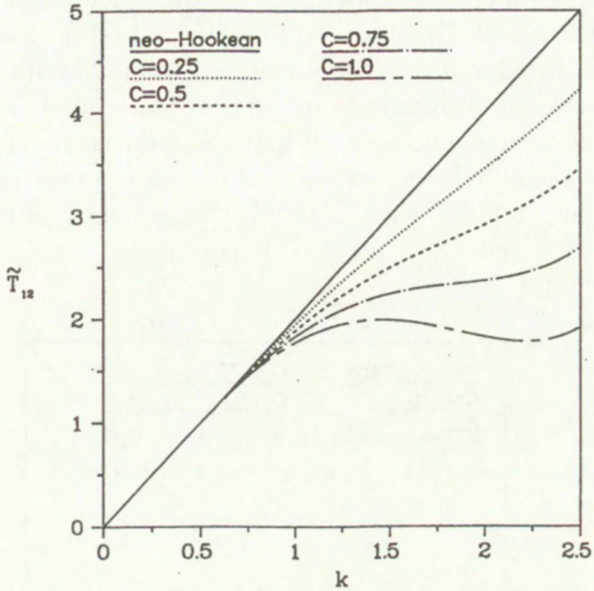

FIG. 2. Shear stress *versus* shear deformation as deformation increases for various conversion fractions, with $\tilde{\mu} = 1.0$, $k_a = 0.5$ and $k_c = 2.65$.

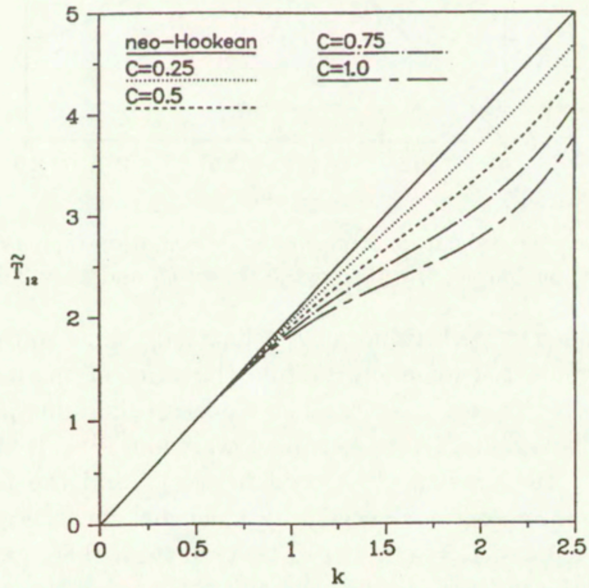

FIG. 3. Shear stress *versus* shear deformation as deformation increases for various conversion fractions, with $\tilde{\mu} = 2.0$, $k_a = 0.5$ and $k_c = 2.65$.

Figure 4 repeats the set of shear stress-shear plots with $\tilde{\mu} = 0.5$. The modulus of newly formed networks is now lower than that of the original network. The general form of the response is similar to that shown in Figs. 2 and 3. As may be anticipated, the reduced stiffness of the material formed at state $\hat{k}$ slows the regeneration of the stress that is released when that material undergoes conversion. Thus both the conversion process itself and the reduced modulus of the newly formed material contribute to the softening of overall response for $k > k_a$. As can be seen from a comparison of Figs. 2 and 4, the reduction of overall stiffness is greater when $\tilde{\mu} = 0.5$ than when $\tilde{\mu} = 1.0$. Furthermore, the loss of monotonicity occurs at a lower value of C for $\tilde{\mu} = 0.5$.

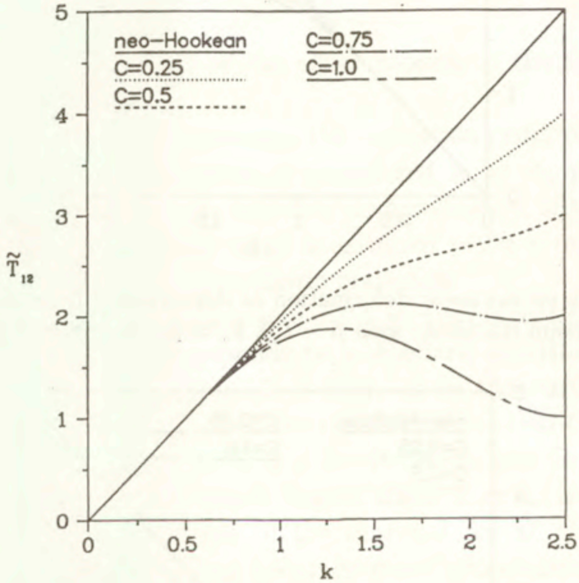

FIG. 4. Shear stress *versus* shear deformation as deformation increases for various conversion fractions, with $\tilde{\mu} = 0.5$, $k_a = 0.5$ and $k_c = 2.65$.

Figures 2 through 4, with values of $\tilde{\mu} \in [0.5, 2.0]$, show softening of response due to the conversion phenomenon. While the tangent modulus for $C = 1.0$ and $\tilde{\mu} = 2.0$ appears greater than the neo-Hookean modulus $\mu^{(1)}$ near $k = 2.5$ (Fig. 3), the secant modulus alvays remains lower than $\mu^{(1)}$. It should be pointed out, however, that the scission of original networks and the formation of new networks in their place does not necessarily have this effect. Figure 5 shows $\tilde{T}_{12}$ *versus* k for $C = 1.0$ and $\tilde{\mu} = 4.0$. It can be seen from this figure that response first softens after the initiation of conversion, as $\tilde{T}_{12}(k)$ is lower than the corresponding stress for the purely neo-Hookean material. As conversion proceeds, though, a hardening behavior becomes apparent, with the shear stress becoming greater than that for the neo-Hookean case. For k near k_a, the rupture of original networks entails release of the stress in those networks, which accounts for the early softening. As the deformation of subsequently formed networks relative to their new reference configurations increases at larger k, the much higher modulus $\mu^{(2)}$ of the new networks ultimately causes the effective stiffness of the material to be greater than in the neo-Hookean case. The assumption of $\tilde{\mu} \gg 1$ in the present constitutive equation may have application to the strain-dependent crystallization of polymers. PETERLIN [11] has studied this phenomenon; additional work has been done by NEGAHBAN [8] and NEGAHBAN and WINEMAN [9]. Henceforth, examples presented in this work will consider only cases of conversion-softening, whereby the secant modulus of the material is reduced by the conversion process.

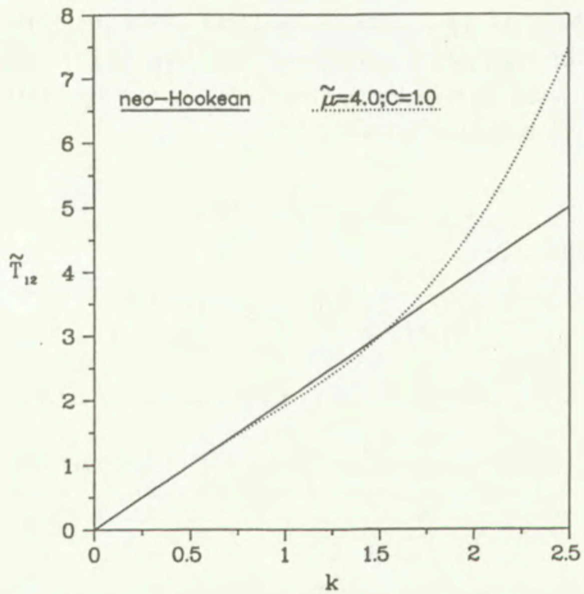

FIG. 5. Shear stress *versus* shear deformation as deformation increases for $\tilde{\mu} = 4.0$, with $C = 1.0$, $k_a = 0.5$ and $k_c = 2.65$.

4.4. Monotonicity of response

The plots of $\tilde{T}_{12}$ *versus* k for $C = 0.25$, $C = 0.5$ and $C = 0.75$ in Fig. 2 all display monotonic response. However, the curve for $C = 1.0$ clearly exhibits a local maximum. It may be assumed that a local maximum first appears at some value of $C \in (0.75, 1.0)$. That the rapid scission of original material networks could lead to such a loss of monotonicity has been discussed above. This situation arises if the negative quantity db/dk achieves a sufficiently great magnitude for some value of k. Since $db/dk = -a$ by (4.10), monotonicity may be lost if a

becomes large enough. When k_a and k_c are held fixed, (4.7) and (4.9) indicate that the fraction of total conversion C acts as a scaling factor of the conversion rate function a for any chosen value of $k \in [k_a, k_c]$. When the conversion fraction reaches a critical value $C = C^{\rm cr}$, a becomes large enough to produce a point of inflection in the stress-shear curve. A value satisfying $C > C^{\rm cr}$ then leads to a local maximum of T_{12} in k.

In order to study the conditions which cause the $\tilde{T}^{12} - k$ relation to become non-monotonic, consider the derivative of the shear stress (4.12) with respect to the shear k for $k \in [k_a, k_c]$:

$$\frac{d\tilde{T}_{12}}{dk} = 2[(1-\tilde{\mu})b(k) + \tilde{\mu} - ka(k)]. \tag{4.13}$$

Here a has the form of (4.7) and b the form of (4.10). To see more clearly the influence of the parameter C on the derivative (4.13), recall from (4.7) the definition of a for $k \in [k_a, k_c]$. The constant α is given by (4.9). When k_a and k_c are prescribed, $a(k)$ can be written as

$$a(k) = Ch_1(k), \tag{4.14}$$

where $h_1(k)$ is defined as

$$h_1(k) = -6\frac{(k-k_a)(k-k_c)}{(k_c-k_a)^3}. \tag{4.15}$$

With the use of (4.14) and (4.15), Eq. (4.13) can be written in the form

$$\frac{d\tilde{T}_{12}}{dk} = 2C\left[(\tilde{\mu}-1)\int_{k_a}^{k} h_1(\hat{k})d\hat{k} - kh_1(k)\right] + 2. \tag{4.16}$$

Equation (4.16) shows that $d\tilde{T}_{12}/dk$ depends on the conversion fraction C and the ratio of moduli $\tilde{\mu}$. Monotonicity is lost if $d\tilde{T}_{12}/dk < 0$ for some $k \in [k_a, k_c]$. Therefore, monotonicity of response depends on both the extent of microstructural transformation and the relation of the material properties $\mu^{(1)}$ and $\mu^{(2)}$.

In determining $C^{\rm cr}$ for different values of $\tilde{\mu}$, it will be convenient to examine three distinct ranges of $\tilde{\mu}$.

First let $\tilde{\mu} = 1$. Equation (4.16) then gives

$$\frac{d\tilde{T}_{12}}{dk} = 2[1 - Ckh_1(k)]. \tag{4.17}$$

The condition for the shear stress-shear deformation curve to have a negative slope for $k \in [k_a, k_c]$ becomes

$$1 - Ckh_1(k) < 0. \tag{4.18}$$

It can be seen from (4.15) that $h_1(k) > 0$ for $k \in (k_a, k_c)$. Thus the condition (4.18) can be written as

$$C > \frac{1}{kh_1(k)}. \tag{4.19}$$

Recall additionally the restriction $C \leq 1$, which has been imposed above.

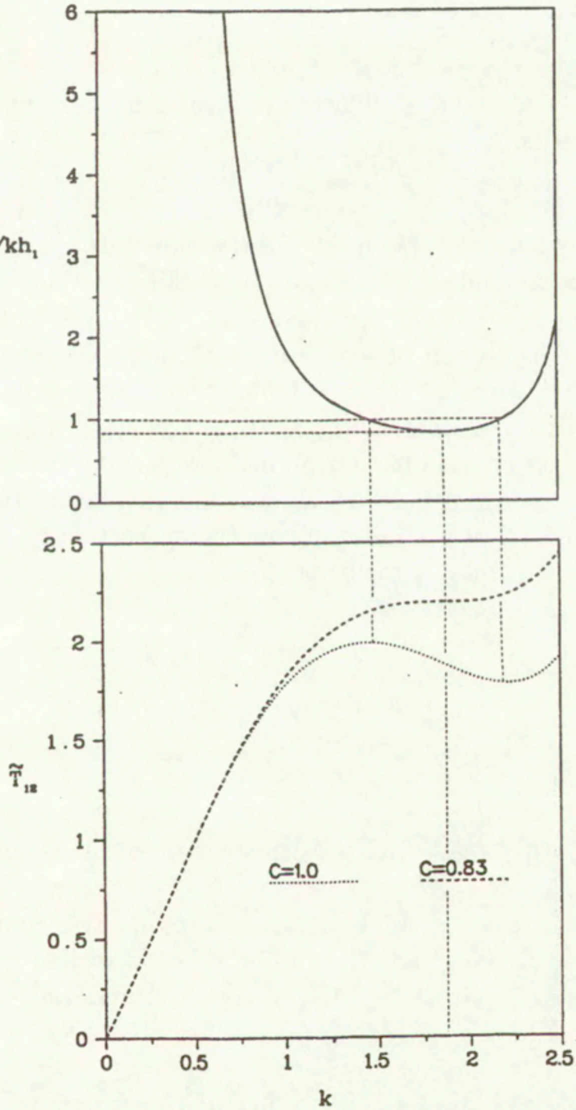

FIG. 6. Shear stress *versus* shear deformation as deformation increases for various conversion fractions and $1/kh_1$ vs. shear deformation, with $\tilde{\mu} = 1.0$, $k_a = 0.5$ and $k_c = 2.65$.

As an illustration of the condition (4.19), Fig. 6 shows $1/kh_1$ *versus* k and $\tilde{T}_{12}$ *versus* k. The quantity $1/kh_1$ is plotted only on the domain $k \in [0.7, 2.5]$; $\tilde{T}_{12}$ is represented by the dotted line for $C = 1.0$ and by the dashed line for $C = 0.83$. It can be seen that the domain on which (4.19) holds corresponds to the domain on which $d\tilde{T}_{12}/dk < 0$ when $C = 1.0$.

Given k_a and k_c and with $\tilde{\mu} = 1.0$, the critical value of C at which a point of inflection in the stress-shear curve first emerges can be found from (4.19) as

$$C^{\mathrm{cr}} = \frac{1}{k^{\mathrm{cr}} h_1(k^{\mathrm{cr}})}, \tag{4.20}$$

where k^{cr} is the value of k at which the local minimum in $1/kh_1$ occurs. It satisfies the condition

$$\frac{d[1/k^{\mathrm{cr}} h_1(k^{\mathrm{cr}})]}{dk} = 0. \tag{4.21}$$

Owing to the simplicity of $h_1(k)$ in (4.15), the numerator of the derivative (4.21) is a quadratic polynomial in k^{cr}. Equation (4.20) can thus be solved in closed form for k^{cr}. With $\tilde{\mu} = 1.0$, this value is found to be $k^{\mathrm{cr}} \approx 1.86$. Equation (4.20) then gives $C^{\mathrm{cr}} \approx 0.83$. It can be seen from the figure that the derivative $d\tilde{T}_{12}/dk$ vanishes at $k = k^{\mathrm{cr}}$ when $C = C^{\mathrm{cr}} = 0.83$. The critical values C^{cr} and k^{cr} are indicated by light dashed lines on the $1/kh_1 - k$ curve of Fig. 6.

For values of $\tilde{\mu} \neq 1.0$, a cubic polynomial arises and a closed-form solution for k^{cr} is not possible. Let $\tilde{\mu} < 1$. It is assumed that there exist values of k at which $d\tilde{T}_{12}/dk = 0$. The critical value of the conversion fraction at which $d\tilde{T}_{12}/dk = 0$ first emerges is found from (4.16) to be

$$C^{\mathrm{cr}} = \frac{1}{k^{\mathrm{cr}} h_1(k^{\mathrm{cr}}) + (1 - \tilde{\mu}) \int\limits_{k_a}^{k^{\mathrm{cr}}} h_1(\hat{k}) d\hat{k}}. \tag{4.22}$$

Since the term $(1 - \tilde{\mu}) \int\limits_{k_a}^{k} h_1(\hat{k}) d\hat{k} > 0$, comparison of (4.20) and (4.22) indicates that C^{cr} is smaller when $\tilde{\mu} < 1.0$ than when $\tilde{\mu} = 1.0$. Less conversion is required to cause the shear stress-shear deformation relation to lose monotonicity if the newly formed networks are softer than the original material.

Consider now the case $\tilde{\mu} > 1$. If $\tilde{\mu}$ is sufficiently large, it can be seen from (4.16) that the positive term $(\tilde{\mu} - 1) \int\limits_{k_a}^{k} h_1(\hat{k}) d\hat{k}$ may dominate, with the result that $d\tilde{T}_{12}/dk > 0$ for any conversion fraction C. On the other hand, if $\tilde{\mu}$ is near unity, a negative slope may be expected somewhere on $k \in [k_a, k_c]$. The existence of a local maximum, then, depends on $\tilde{\mu}$.

Figure 7 shows the shear stress-shear curves produced for $C = 0.75$ when the ratio of shear moduli has the values $\tilde{\mu} = 2.0$, $\tilde{\mu} = 1.0$ and $\tilde{\mu} = 0.5$. It can be seen from the figure that decreasing the shear modulus of the newly formed material relative to the modulus of the original material leads to softer overall response for $k > k_a$. It has been indicated that high stiffness of the original material relative to that of the newly formed networks could lead to a loss of monotonicity. This situation corresponds to low values of $\tilde{\mu}$. While the curves for $\tilde{\mu} = 2.0$ and $\tilde{\mu} = 1.0$ show monotonic response, a local maximum is evident in the stress-shear graph when $\tilde{\mu} = 0.5$. It may be assumed that this loss of monotonicity first occurs at some critical value $\tilde{\mu}^{\rm cr} \in (0.5, 1.0)$. At $\tilde{\mu} = \tilde{\mu}^{\rm cr}$, there is a value of $k > k_a$ at which the generation of stress in the relatively soft newly formed material is exactly matched by the relaxation of stress due to the conversion process.

When k_a, k_c and C are prescibed, an analysis analogous to that used to find $C^{\rm cr}$ in (4.13) through (4.22) can be performed to find $\tilde{\mu}^{\rm cr}$. Equation (4.13) can be rewritten as

$$\frac{d\tilde{T}_{12}}{dk} = 2\{b(k) - ka(k) + \tilde{\mu}[1 - b(k)]\}. \tag{4.23}$$

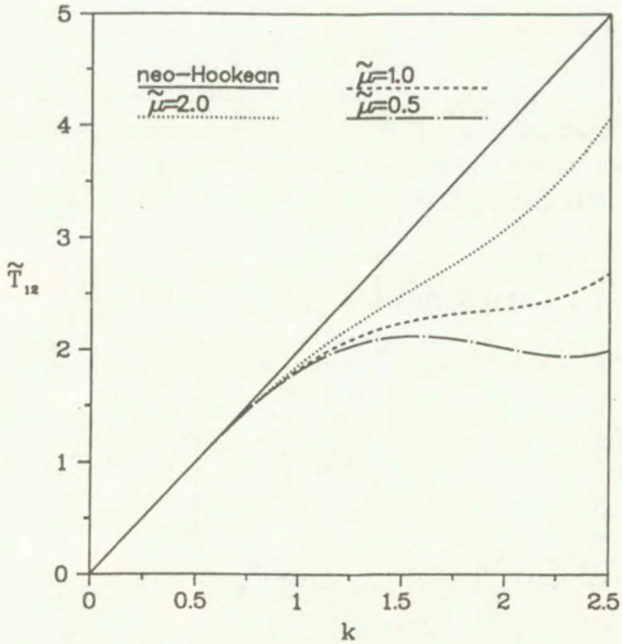

FIG. 7. Shear stress *versus* shear deformation as deformation increases for various shear modulus ratios, with $C = 0.75$, $k_a = 0.5$ and $k_c = 2.65$.

The condition for the stress-shear curve to have a negative slope becomes

$$b(k) - ka(k) + \tilde{\mu}[1 - b(k)] < 0, \tag{4.24}$$

or, since $1 - b > 0$,

$$\frac{ka(k) - b(k)}{1 - b(k)} > \tilde{\mu}. \tag{4.25}$$

For convenience, let

$$h_2(k) = \frac{ka(k) - b(k)}{1 - b(k)}, \tag{4.26}$$

so that (4.25) becomes

$$h_2(k) > \tilde{\mu}. \tag{4.27}$$

The critical value of $\tilde{\mu}$ at which monotonicity of the stress-shear relation is first lost, is given by $h_2(k^{\text{cr}}) = \tilde{\mu}^{\text{cr}}$. The critical shear k^{cr} is found as a solution of $dh_2(k^{\text{cr}})/dk = 0$. For $\tilde{\mu} < \tilde{\mu}^{\text{cr}}$, the release of stress due to network conversion dominates and there is a range of values of $k > k_a$ over which $d\tilde{T}_{12}/dk < 0$. When $a(k)$ has the form of (4.7) and $b(k)$ that of (4.10), $dh_2(k^{\text{cr}})/dk = 0$ cannot be solved for k^{cr} in closed form. A numerical solution is not carried out here.

4.5. Reversal of deformation

Assume that deformation is reversed, so that current shear k decreases after reaching a maximum value k^*. If $k^* < k_a$, the Cauchy stress given by (4.2) holds during the process of decreasing deformation. When $k^* > k_a$, the current stress is formed from (2.5) and (4.4) as

$$\tilde{\mathbf{T}}(k) = -\tilde{p}\mathbf{I} + 2b(k^*)\begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\tilde{\mu}\int_{k_a}^{k^*} a(\hat{k})\begin{bmatrix} 1 & k-\hat{k} & 0 \\ k-\hat{k} & 1+(k-\hat{k})^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} d\hat{k}. \tag{4.28}$$

In non-dimensional form, the current shear stress is then

$$\tilde{T}_{12}(k) = 2\left[b(k^*)k + \tilde{\mu}\int_{k_a}^{k^*} a(\hat{k})(k-\hat{k})d\hat{k}\right]. \tag{4.29}$$

Equations (4.28) and (4.29) hold regardless of the forms used for $a(k)$ and $b(k)$. The present theory of microstructural transformation assumes that no further conversion occurs during the reversal of deformation. Thus the shear stress in each material network is directly proportional to the shear deformation of the network relative to the state at which it is formed. For the original material, that deformation is the current shear k; in a newly formed element, it is given by $k - \hat{k}$. The roles of these two deformation measures can be seen from (4.28).

It is useful to rewrite (4.29) in the form

$$\tilde{T}_{12}(k) = 2\left\{\left[b(k^*) + \tilde{\mu}\int\limits_{k_a}^{k^*} a(\hat{k})d\hat{k}\right]k - \tilde{\mu}\int\limits_{k_a}^{k^*} a(\hat{k})\hat{k}d\hat{k}\right\}. \tag{4.30}$$

From (4.30), it is clear that there is a straight-line relation between the shear stress $\tilde{T}_{12}$ and the current shear k during reversal of deformation. Making use of the specific forms for $a(k)$ and $b(k)$ proposed in (4.7) and (4.10), Eq. (4.30) can be written as

$$\tilde{T}_{12}(k) = 2\left\{[(1-\tilde{\mu})b(k^*) + \tilde{\mu}]k - \tilde{\mu}\int\limits_{k_a}^{k^*} a(\hat{k})\hat{k}d\hat{k}\right\}. \tag{4.31}$$

It is evident from (4.30) that the slope of the $\tilde{T}_{12} - k$ curve as k decreases depends on the material parameter $\tilde{\mu}$. It can be shown from (4.7), (4.9) and (4.10) that the fraction of original material $b(k^*)$ decreases as C increases; thus the slope of the shear stress-shear deformation curve also depends on C.

Inspection of (4.31) reveals that the slope increases with C when $\tilde{\mu} > 1.0$. A greater value of C means that as deformation increases, a larger volume fraction of material undergoes conversion to newly formed networks with shear modulus $\tilde{\mu}^{(2)} > \tilde{\mu}^{(1)}$. Thus the effective stiffness $\tilde{T}_{12}/k$ increases. The slope of $\tilde{T}_{12}$ *versus* k decreases for larger C when $\tilde{\mu} < 1.0$. As the fraction of conversion C is increased, more original material converts to new networks with modulus $\mu^{(2)} < \mu^{(1)}$. The effective material stiffness decreases and with it the slope of the shear stress-shear deformation curve for decreasing deformation. For $\tilde{\mu} = 1.0$, (4.31) reduces to

$$\tilde{T}_{12}(k) = 2\left[k - \int\limits_{k_a}^{k^*} a(\hat{k})\hat{k}d\hat{k}\right]. \tag{4.32}$$

The slope of the stress-shear curve during reversal of deformation is thus independent of the conversion fraction C when the moduli of original and newly formed networks are equal.

Figure 8 replicates the plots from Fig. 2 of the dimensionless shear stress $\tilde{T}_{12}$ *versus* k for different values of the conversion fraction C, with $\tilde{\mu} = 1.0$. Figure 8 also plots $\tilde{T}_{12}$ *versus* k for the various values of C as deformation is reversed from $k^* = 2.0$ to the residual shear state denoted by k^{res}, where total current shear stress $\tilde{T}_{12} = 0$. Let this process of deformation increasing to $k = k^*$ and subsequently decreasing to $k = k^{\text{res}}$ be referred to as the deformation cycle. The straight-line relation between $\tilde{T}_{12}$ and k during reversal of deformation is evident from the figure. It can also be seen that, for $\tilde{\mu} = 1.0$, the slope of this line is unaffected by C.

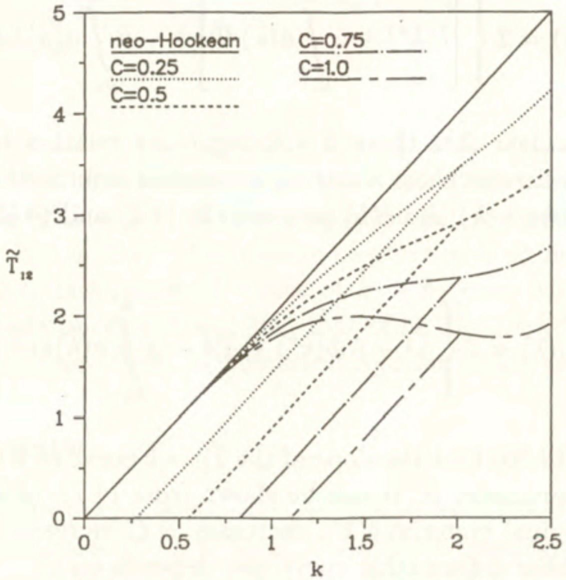

FIG. 8. Shear stress *versus* shear deformation as deformation increases and subsequently decreases for various conversion fractions, with $\tilde{\mu} = 1.0$, $k_a = 0.5$ and $k_c = 2.65$.

Figure 9 shows $\tilde{T}_{12}$ *versus* k at various values of C for the same deformation cycle when $\tilde{\mu} = 2.0$. It can be seen from the figure that the shear stress-shear curve for decreasing deformation is still a straight line, but that the slope is now steeper when the conversion fraction is greater.

Figure 10 shows the shear stress $\tilde{T}_{12}$ *versus* k for the deformation cycle, with various C and with $\tilde{\mu} = 0.5$. Here the slope of the $\tilde{T}_{12} - k$ lines for reversal of deformation is seen to decrease as the conversion fraction C is increased.

4.6. Permanent set

When $k^* < k_a$, all response is elastic and there is no residual deformation when the net external shear traction is returned to zero. Setting $\tilde{T}_{12} = 0$ in (4.30) gives the residual shear deformation for $k^* \geq k_a$ as

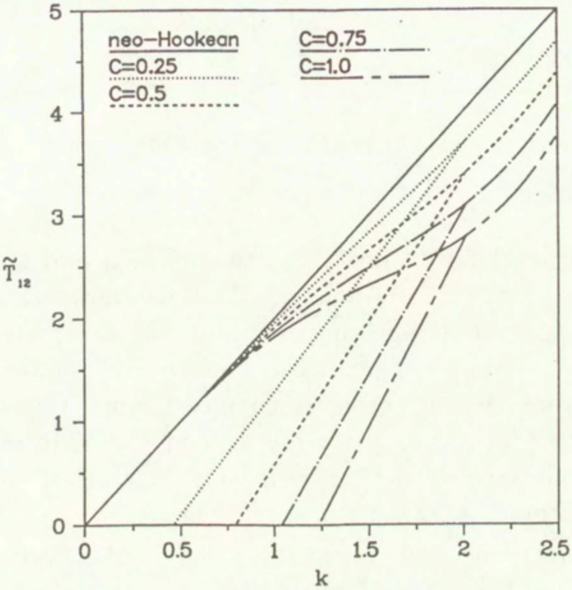

FIG. 9. Shear stress *versus* shear deformation as deformation increases and subsequently decreases for various conversion fractions, with $\tilde{\mu} = 2.0$, $k_a = 0.5$ and $k_c = 2.65$.

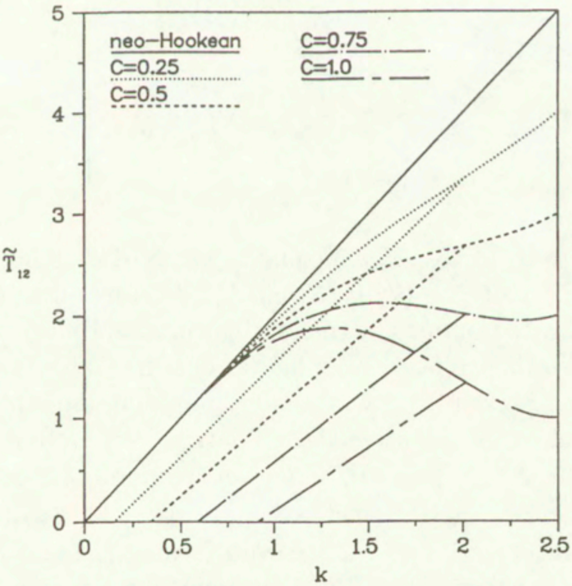

FIG. 10. Shear stress *versus* shear deformation as deformation increases and subsequently decreases for various conversion fractions, with $\tilde{\mu} = 0.5$, $k_a = 0.5$ and $k_c = 2.65$.

$$k^{\mathrm{res}} = \frac{\tilde{\mu} \int\limits_{k_a}^{k^*} a(\hat{k})\hat{k}d\hat{k}}{b(k^*) + \tilde{\mu} \int\limits_{k_a}^{k^*} a(\hat{k})d\hat{k}}. \tag{4.33}$$

Equation (4.33) holds for all admissible forms of $a(k)$ and $b(k)$. Now consider the specific forms given by (4.7) and (4.10). It is known that with k_a and k_c specified, the only parameter affecting $a(k)$ and $b(k)$ at a prescibed value of k is the fraction of conversion C. Thus it can be seen from (4.33) that the residual shear deformation k^{res} depends on three coupled factors: the conversion fraction C; the ratio of original and new moduli $\tilde{\mu}$; and the state of maximum shear k^* reached before reversal of deformation. The partial derivatives of k^{res} with respect to each of these quantities are studied below.

Let $a(k)$ in (4.33) be given by (4.14). The partial derivative of k^{res} with respect to C is found from (4.33) to be

$$\frac{\partial k^{\mathrm{res}}}{\partial C} = \frac{\tilde{\mu} \int\limits_{k_a}^{k^*} h_1(\hat{k})\hat{k}d\hat{k}}{\left[1 + C(\tilde{\mu} - 1) \int\limits_{k_a}^{k^*} h_1(\hat{k})d\hat{k}\right]^2}, \tag{4.34}$$

where $h_1(k)$ is given by (4.15). The inequality $\partial k^{\mathrm{res}}/\partial C > 0$ holds for all admissible values of $\tilde{\mu}$ and for all $k^* \in [k_a, k_c]$. Thus k^{res} increases with C. Greater values of C imply that larger fractions of the original material have undergone conversion and adopted as their reference configurations the states of shear $k \in [k_a, k^*]$. As deformation is reversed from $k = k^*$, an increasing amount of the converted material is sheared in the negative sense relative to its reference configuration, with $k - \hat{k} < 0$. If $k < k_a$ is reached, all converted material elements are in states of negative shear. The negative shear stress $\tilde{T}_{12}^{(2)}(k - \hat{k}) = 2\tilde{\mu}(k - \hat{k})$ associated with the relative deformation of the network formed at state $\hat{k}$ tends to reduce the total positive shear stress $\tilde{T}_{12}(k)$ as current shear k is reduced. Thus the condition $\tilde{T}_{12}(k) = 0$ is satisfied at a higher value of residual shear k^{res}.

The partial derivative of the residual shear k^{res} with respect to $\tilde{\mu}$ is found from (4.32) as

$$\text{(4.35)} \qquad \frac{\partial k^{\mathrm{res}}}{\partial \tilde{\mu}} = \frac{b(k^*) \int\limits_{k_a}^{k^*} a(\hat{k})\hat{k}d\hat{k}}{\left[b(k^*) + \tilde{\mu} \int\limits_{k_a}^{k^*} a(\hat{k})d\hat{k} \right]^2}.$$

Inspection of (4.35) reveals that $\partial k^{\mathrm{res}}/\partial \tilde{\mu} > 0$. As in the case of increasing C, the physical reason is the more rapid generation of negative shear stress $\tilde{T}_{12}^{(2)}(k - \hat{k})$ in the converted material as current shear k is reduced from k^*. In the case of increasing $\tilde{\mu}$, however, $\tilde{T}_{12}^{(2)}(k - \hat{k})$ is greater in each newly formed network due to greater stiffness of the newly formed material relative to the original material. Thus $\tilde{T}_{12}(k) = 0$ is satisfied at a larger k^{res} as $\tilde{\mu}$ increases.

Differentiation of k^{res}, given by (4.33), with respect to the maximum shear k^* gives

$$\text{(4.36)} \qquad \frac{\partial k^{\mathrm{res}}}{\partial k^*} = \frac{\tilde{\mu}(k^*) \left[k^* + (\tilde{\mu} - 1) \int\limits_{k_a}^{k^*} a(\hat{k})(k^* - \hat{k})d\hat{k} \right]}{\left[1 + (\tilde{\mu} - 1) \int\limits_{k_a}^{k^*} a(\hat{k})d\hat{k} \right]^2}.$$

In the integrand in the numerator of (4.36), the term $k^* - \hat{k} > 0$, as $k^* > \hat{k}$ for all $\hat{k} \in [k_a, k^*]$. The term $\tilde{\mu} - 1$ may be either positive or negative, depending on the value of $\tilde{\mu}$. All other terms in (4.36) can be shown to be positive. It is thus possible that $\partial k^{\mathrm{res}}/\partial k^* < 0$ for some k^* if $\tilde{\mu}$ is sufficiently small. The larger the value of $k^* \in [k_a, k_c]$, the more total conversion occurs for a given C during the process of increasing deformation. For sufficiently large $\tilde{\mu}$, this causes the condition $\tilde{T}_{12}(k^{\mathrm{res}}) = 0$ to be satisfied at a larger k^{res} as shear is reduced from k^* and hence causes $\partial k^{\mathrm{res}}/\partial k^* > 0$.

The variation of the residual shear k^{res} with each of the three parameters discussed above is presented in Figs. 11 through 14. For all of the figures, shear deformation has been reversed from a maximum of $k^* = 2.0$. Figure 11 shows plots of k^{res} *versus* C for various values of $\tilde{\mu}$. It is evident from the figure that k^{res} increases monotonically with C for all $C \in [0.0, 1.0]$, as given by (4.34). While $\tilde{\mu}$ clearly influences the results shown in Fig. 11, the general trend of increasing residual shear resulting from increasing total microstructural conversion holds for all values of $\tilde{\mu}$ shown.

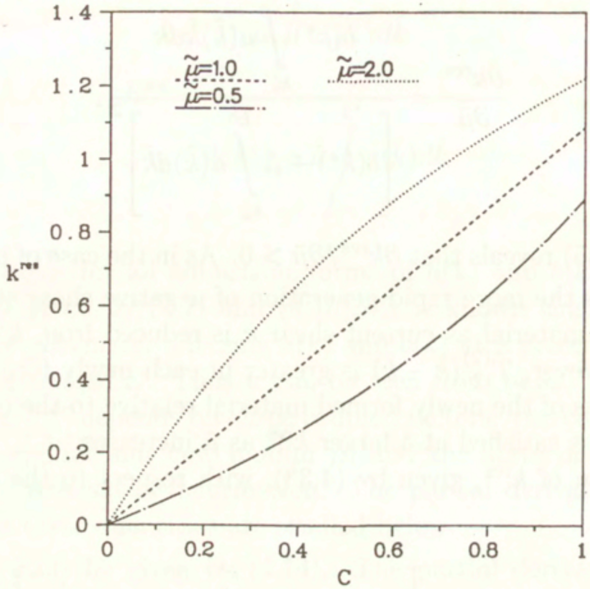

FIG. 11. Residual shear deformation *versus* conversion fraction for various shear modulus ratios, with $k^* = 2.0$, $k_a = 0.5$ and $k_c = 2.65$.

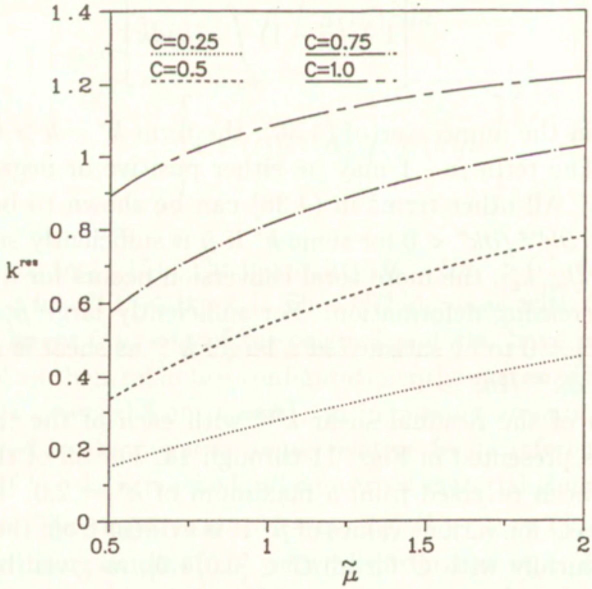

FIG. 12. Residual shear deformation *versus* shear modulus ratio for various conversion fractions, with $k^* = 2.0$, $k_a = 0.5$ and $k_c = 2.65$.

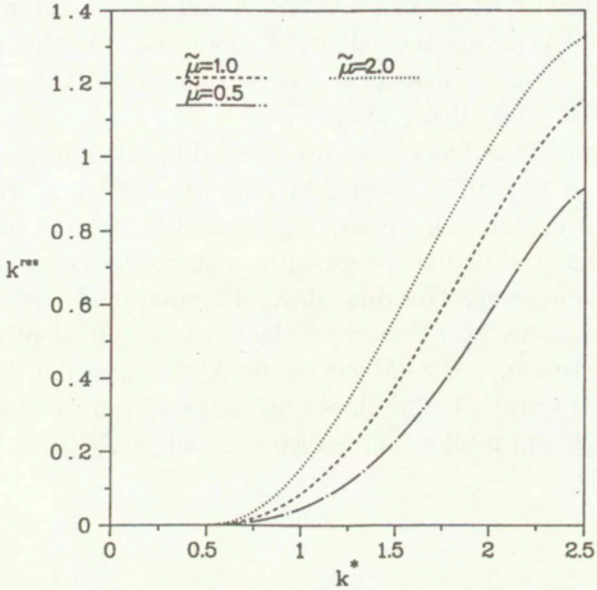

FIG. 13. Redidual shear deformation *versus* maximum shear deformation for various shear modulus ratios, with $C = 0.75$, $k_a = 0.5$ and $k_c = 2.65$.

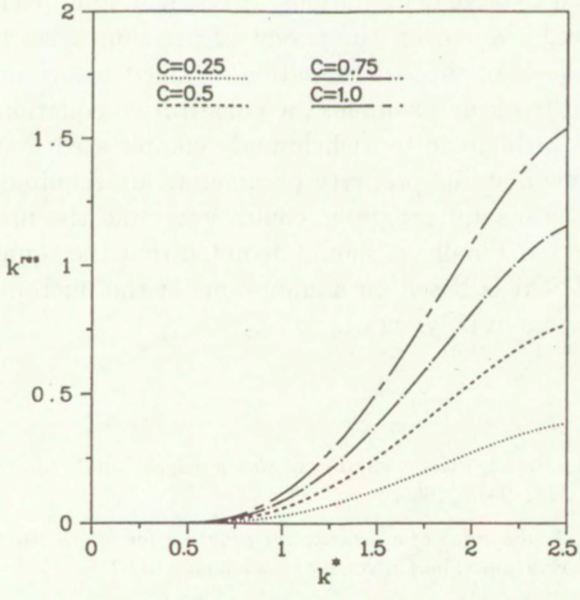

FIG. 14. Residual shear deformation *versus* maximum shear deformation for various conversion fractions, with $\tilde{\mu} = 1.0$, $k_a = 0.5$ and $k_c = 2.65$; when $k^* < k_a$, $k^{\text{res}} = 0.0$.

Figure 12 shows $k^{\rm res}$ *versus* $\tilde{\mu}$ for different values of C. At all values of C considered, the residual shear can be seen to increase monotonically with $\tilde{\mu}$, as indicated by (4.35). Varying the value of C does not alter this general trend, as seen in the figure: greater values of C serve largely to shift the entire relation to a higher range of $k^{\rm res}$ for all $\tilde{\mu} \in [0.5, 2.0]$.

Figure 13 plots $k^{\rm res}$ *versus* k^* for various moduli ratios $\tilde{\mu}$ when the conversion fraction is taken as $C = 0.75$. Figure 14 shows the $k^{\rm res} - k^*$ curves for various C when $\tilde{\mu} = 1.0$ is chosen. The figures show the development of greater residual shear when the material is first deformed to a greater maximum shear k^*. They also show the region of zero residual shear deformation for $k^* < k_a$. It is thus apparent that the values of $\tilde{\mu}$ chosen for the plots are sufficiently large to ensure monotonically increasing $k^{\rm res} - k^*$ curves on $k^* \in [k_a = 0.5, 2.5]$. It should be noted that Figs. 13 and 14 also show the range of purely elastic deformation $k \in [0, k_a = 0.5]$ on which all strain is recovered when $\tilde{T}_{12}(k) = 0$.

5. Conclusion

The constitutive equation proposed by Wineman and Rajagopal [23] for materials undergoing microstructural change has proven to be successful in describing qualitatively some of the important responses exhibited by polymeric materials subjected to large deformations. Stress softening, yield and permanent set are all predicted. Moreover, the extent of permanent set is seen to depend on the maximum level of shear deformation attained before unloading. As evidenced by the neo-Hookean example, the constitutive equation can predict this complex response without an overwhelmingly complicated mathematical structure. Relatively few material-property parameters are required, indicating hope that practical experimental programs could determine the necessary constants for a specific material. Finally, it should be noted that the equation is not purely phenomenological, but is based on assumptions of the micromechanics of finite deformation processes in polymers.

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Announcement

14th U.S. National Congress of Applied Mechanics

The 14th U.S. National Congress of Applied Mechanics will be hosted by the Department of Engineering Science and Mechanics, Virginia Tech on June 23-28, 2002, and will be held on the campus of Virginia Polytechnic Institute and State University (Virginia Tech) in Blacksburg, VA. Virginia Tech is the largest comprehensive university (with approximately 25,780 students) in the Commonwealth of Virginia. The College of Engineering awards nearly 840 B.S. degrees, 390 M.S. degrees and 175 Ph.D. degrees every year. The Department of Engineering Science and Mechanics enrolled 89 graduate students in 1998-99, awarded 15 Masters and 12 Doctoral degrees in 1998-99, and had a research expenditure of $6.8 million dollars in 1998-99. The graduate program of the College of Engineering has been ranked 25th in the country in the 1999 U.S. News & World Report.

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31 Jan 2002 (Submission of Abstracts)
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Corrigenda:

In the paper entitled *Laminar Dispersed two-phase flows at low concentration – I. Generalised system of equations*, Arch. Mech., **2**, 1, 25–53, 2000 by J.L. Achard and A. Cartellier : The superscripts 2, 3 etc... associated with the averaged quantities are of upright type when the test particle(s) position(s) only is (are) fixed, and appear in italic when the whole phase(s) of the test particle(s) is (are) fixed. Consequently, italic superscripts must be used everywhere in Sec. 4.2, while upright ones must be used exclusively thereafter (in the text as well as in equations) except in the definitions (4.15), (4.16) and (4.21) (4.22) where italic superscripts remain in the R.H.S. of first equalities. Upright superscripts are to also be used in page 35 for the cross-correlation $\mathbf{u}\boldsymbol{\omega}^{\circ}$. f_i must be replaced by f_1 in (4.15) and (4.16), and f_2 must be replaced by f_2 in (4.21), (4.22) and in the denominator of (4.11). In Eq. (3.6), the ensemble average applies to the indicator function X_i^c multiplied by $\varphi_i(\mathbf{x}, t)$. The d at the beginning of Eq. (4.5) must be cancelled. Page 39, the reference to Eq. (5.15) must be replaced by (4.10). Coma are missing in (4.24) where the average applies to $\varphi_i(\mathbf{u}_i - \overline{\mathbf{u}}^1)(\mathbf{u}_i - \overline{\mathbf{u}}^1)$, and before the equality in the first line of Eq. (5.10). In Eq. (4.26) and the next line, the agitation tensor is $\mathbb{A}^2_{u^{\circ}u^{\circ}}(\mathbf{x}^{\circ}|\mathbf{x})$. In (3.4), (3.5), (4.5), (4.31), (4.34) and (4.15), $\boldsymbol{\omega}$ is of bold type. In Eq. (5.4), the averaging operator E applies to the bracket term in the last line. Page 47, the divergence operator is applied to the extra-deformation tensor discussed in the text.

In the paper entitled *Laminar Dispersed two-phase flows at low concentration – II. Disturbance equations*, Arch. Mech., **52**, 2, 275–302, 2000 by J. L. Achard and A. Cartellier : In the whole paper, $\boldsymbol{\omega}$ must be of bold type. Despite a change from bold to normal weight, the various domains $\mathcal{V}$ are identical in parts I and II. Page 276, the shortened notation mentioned in line 14 is $\mathbf{v}^{\circ *}$ instead of $\mathbf{v}^{\circ}$. In Eq. (1.6), $\tilde{\mathbf{x}}^{c4}$ must be replaced by $\tilde{\mathbf{v}}^{c4}$. In the expression of α^{**} Eq. (2.12), the integration is performed over $|\tilde{\mathbf{x}} - \mathbf{x}^{\circ\circ}| \le a$. In Eq. (2.17), the contraction sign : applies between the alternator tensor ε and the first integral. In Eq. (3.3), a^2 replaces a_2. In Eq. (3.8), a bracket is missing in the R.H.S of the first equality. In Eq. (3.11) the bracket just before the second equality must be dropped out. In Eq. (4.1), $\mathbb{A}$ must be eliminated. In Eq. (4.18), $u^{\circ *}$ is of bold type. In Eq. (4.21), an overbar is missing above $\boldsymbol{\omega}$. In Eq. (5.10) and (5.11), the derivative tensor is $\partial^m/\partial\mathbf{x}^m$. The definitions of C_v^* and C_v^{**} must be modified all the agitation contributions must be eliminated from the definitions (4.4) and (4.6), and they must be included in the l.h.s. of the corresponding momentum Eq. (4.3) and (4.5).

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