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Thermoelastic plane problem for material with circular inclusions

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WE CONSIDER TWO-DIMENSIONAL thermoelastic composite materials in the case when the temperature is constant. Using complex potentials and applying a method of functional equations, we construct a simple algorithm to solve the corresponding boundary value problem. The stress tensor is written with the accuracy of up to the term $O\left(R^2\right)$, where $R = \max_{k,m} r_k d_{km}^{-1}$, r_k is the radius of the k-th inclusion, d_{km} is the distance between centers of the k-th and m-th inclusion $(k \neq m)$. The effective elastic constants and the coefficient of thermal expansion are written in analytic form up to $O\left(R^4\right)$.

1. Introduction

WE STUDY TWO-DIMENSIONAL problems of thermoelastic composite materials. A modern review devoted to homogenization and constructive formulae for such materials is due to WOJNAR *et al.* [1] and [2]. We consider the case of the plane strain. Such problems are extensively studied by the method of complex potentials. MUSKHELISHVILI [9], MIKHLIN [6] and others reduced the boundary value problem for materials with finite number of inclusions or holes to a system of singular integral equations or to an infinite set of linear algebraic equations. Integral equations and infinite sets of equations can be numerically solved, and the stress and displacement fields can be calculated. The method of complex potentials was extended to periodic problems by VAN FO FY [10] and GRIGILYUK and FIL'SHTINSKIJ [3]. In particular, integral equations and infinite systems were constructed and solved numerically for periodic problems.

The closed-form solution is preferable to numerical solution in mechanics of composite materials, since it allows us to obtain analytical formulae for the tensor of effective properties. There are special cases in the books cited above, when the stress and displacement fields were found in analytical form. In the present paper we study a new special case of such problems, the two-dimensional thermoelastic materials with circular inclusions. Each inclusion is modelled by a disk; the position of the center and the radius are arbitrary. Only one essential restriction is imposed, namely, these disks are mutually disjoint. Following [7, 8] we reduce the thermoelastic problem to a system of functional equations, which can be solved by the method of successive approximations. Let us note that the functional equations do not contain integral terms which are hard to calculate analytically. The right-hand sides of the equations involve combinations of functions and their derivatives. This allows us to express the first approximations of the stress and displacement fields in analytic forms.

Using the analytical formulae for the stress field, we can calculate the effective properties of the thermoelastic composite materials. As it is assumed in the homogenization theory (see [1] and [4]), the distribution of the inclusions on the plane is statistically uniform and ergodic. Hence, we can evaluate the effective properties using spatial averaging. Constructive formulae are obtained under the following additional assumptions. Composite material is divided into the same groups of the inclusions. All groups consists of a finite number of inclusions. A group is displayed in Fig. 1. Each group does not interact with others. Moreover, to simplify the calculations we assume that the considered material is isotropic in macroscale as a two-dimensional material. If the group contains only one inclusion, we arrive at the case of the dilute composite materials. In order to obtain a formula for the effective properties in the dilute case, it is sufficient to solve a boundary value problem (conjugate problem) for a single inclusion. This approach is called Maxwell's formalism in the literature. Our case (many inclusions in a group as displayed in Fig. 2) can be called the generalized dilute concentration of inclusions, because we take into account influence of each inclusion on the other ones in any fixed group, and inclusions in different groups do not strike each other. According to Maxwell's formalism, the area fraction of inclusions is a prescribed parameter. In the generalized dilute case we also assume the area concentration v_k of the $k-$th material ($k = 1, 2, ..., n$) as given parameters. Here n is the number of inclusions in each group. We cannot add all v_k since the inclusions have in general various sizes and thermoelastic properties.

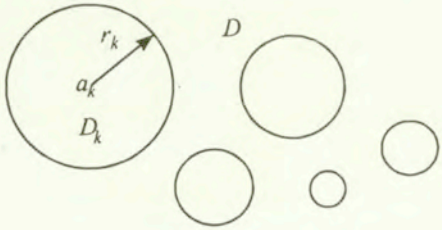

FIG. 1. Representative group of inclusions.

The paper is organized as follows. In Sec. 2 we reduce the problem to a system of functional equations. Section 3 is devoted to solution to this system. The local stress and displacement thermoelastic fields are given in Sec. 4. The

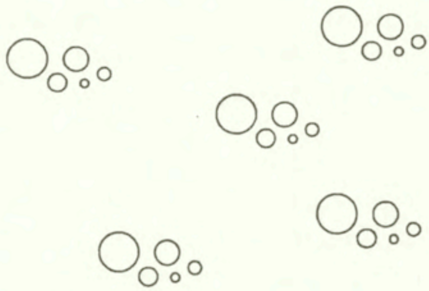

FIG. 2. Composite material with diluted groups of inclusions.

local fields presented in Sec. 5 concern the case of the pure elastic problem. In Sec. 6 the formulae of Sec. 5 are applied to deduce formulae for the effective elastic constants. Then the effective coefficient of thermal expansion is written in analytic form.

2. Complex potentials and conjugation problem

Let us consider a spatial variable (x, y, z). The (x, y)-plane thermoelasticity stress-strain relations for a linear isotropic material are given by the equations [9]

$$
\begin{aligned}
\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) + E\alpha^T T &= E\frac{\partial u}{\partial x}, \\
\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}) + E\alpha^T T &= E\frac{\partial v}{\partial y}, \\
\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}) &= 0, \\
\sigma_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}
\tag{2.2}
$$

is the stress tensor, $(u, v, 0)$ is the displacement vector, ν is the Poisson ratio, E is the Young modulus, $\mu = \frac{E}{2}(1+\nu)$ is the shear modulus, α^T is the coefficient of thermal expansion, T is the temperature distribution. Here all coefficients depend on the coordinates (x, y). The form of the third Eq. (2.1) is obtained from equation $\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}) + E\alpha^T T = f$, where the external pressure $f = E\alpha^T T$ is applied along the $z-$direction in such a way that the plane strain holds [9].

For the sake of simplicity we assume that T is constant. Then the equations of the steady heat conduction and the conditions of perfect thermal contact between materials are fulfilled. Moreover, we can take $T = 1$ because of the linear character of Eq. (2.1).

Let us consider mutually disjoint disks $D_k := \{\zeta \in \mathbb{C} : |\zeta - a_k| < r_k\}$ ($k = 1, 2, ..., n > 1$) in the complex plane $\mathbb{C}$ of the variable $\zeta = x + iy$. Let $D := \mathbb{C} \cup \{\infty\} \setminus (\cup_{k=1}^{n} D_k \cup T_k)$, where $T_k := \{t \in \mathbb{C} : |t - a_k| = r_k\}$. We assume that T_k are orientated in clockwise sense. Here and in the sequel we use the letter ζ for a complex variable in a domain, t – on the boundary of a domain. We study the thermoelasticity of the composite material, when the domains D and D_k are occupied by materials with the coefficients μ, α^T, $\kappa = 3 - 4\nu$ and μ_k, α_k^T, $\kappa_k = 3 - 4\nu_k$, respectively. We shall also use the constants E, ν and E_k, ν_k, respectively, to denote the elastic properties of the matrix and inclusions.

The component of the stress tensor can be determined by the Kolosov-Muskhelishvili formulae [9]

$$\sigma_{xx} + \sigma_{yy} = \begin{cases} 4\mathrm{Re}\,\phi_k'(\zeta), & \zeta \in D_k, \\ 4\mathrm{Re}\,\varphi'(\zeta), & \zeta \in D, \end{cases}$$

(2.3)

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = \begin{cases} -2\left[\zeta\overline{\phi_k''(\zeta)} + \overline{\psi_k'(\zeta)}\right], & \zeta \in D_k, \\ -2\left[\zeta\overline{\varphi''(\zeta)} + \overline{\psi'(\zeta)}\right], & \zeta \in D, \end{cases}$$

where Re denotes the real part. The functions $\phi_k(\zeta)$ and $\psi_k(\zeta)$, $\varphi(\zeta)$ and $\psi(\zeta)$ are analytical in D_k and D, respectively, and twice differentiable in the closures of the considered domains. The normal forces on T_k are given by the expression

$$\mathbf{T}_n^- = \phi_k(t) + t\overline{\phi_k'(t)} + \overline{\psi_k(t)}, \ \mathbf{T}_n^+ = \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}. \tag{2.4}$$

The plane displacements $U = (u, v)$ are expressed by the complex potentials

$$U = \begin{cases} \dfrac{1}{2\mu_k}\left[\kappa_k\phi_k(\zeta) - \zeta\overline{\phi_k'(\zeta)} - \overline{\psi_k(\zeta)} + 2\alpha_k^T\mu_k\zeta\right], & \zeta \in D_k, \\ \dfrac{1}{2\mu}\left[\kappa\varphi(\zeta) - \zeta\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)} + 2\alpha^T\mu\zeta\right], & \zeta \in D. \end{cases} \tag{2.5}$$

We assume that the contact between different materials is perfect, i.e., (2.6) $$\mathbf{T}_n^+ = \mathbf{T}_n^-,\ U^+ = U^- \ \text{ on } \ \partial D,$$

where ∂D is the boundary of D, $U^+(t) := \lim_{\zeta\to t,\,\zeta\in D} U(\zeta)$, $U^-(t) := \lim_{\zeta\to t,\,\zeta\in D_k} U(\zeta)$. Using the relation (2.4) – (2.5) we write the boundary condition (2.6) in the form

(2.7)
$$\phi_k(t) + t\overline{\phi_k'(t)} + \overline{\psi_k(t)} = \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)},$$
$$\frac{1}{\mu_k}\left[\kappa_k\phi_k(t) - t\overline{\phi_k'(t)} - \overline{\psi_k(t)}\right] = \frac{1}{\mu}\left[\kappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}\right] + 2\eta_k t,$$
$$|t-a_k| = r_k, \qquad k = 1,2,...,n,$$

where $\eta_k = \alpha^T - \alpha_k^T$. Introduce the new unknown functions

$$\Phi_k(\zeta) = \left(\frac{r_k^2}{\zeta - a_k} + \overline{a_k}\right)\phi_k'(\zeta) + \psi_k(\zeta),\ |\zeta - a_k| \le r_k.$$

Then condition (2.7) becomes

$$\phi_k(t) + \overline{\Phi_k(t)} = \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)},$$
$$\frac{\mu}{\mu_k}\left[\kappa_k\phi_k(t) - \overline{\Phi_k(t)}\right] = \kappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} + 2\eta_k\mu t, \quad |t-a_k| = r_k.$$

These relations can be written as follows:

(2.8) $$\left(1 + \frac{\mu}{\mu_k}\kappa_k\right)\phi_k(t) + \left(1 - \frac{\mu}{\mu_k}\right)\overline{\Phi_k(t)} = (1+\kappa)\,\varphi(t) + 2\mu\eta_k t,$$

(2.9) $$\left(\kappa - \frac{\mu}{\mu_k}\kappa_k\right)\overline{\phi_k(t)} + \left(\kappa + \frac{\mu}{\mu_k}\right)\Phi_k(t) = (1+\kappa)\left(\bar{t}\varphi'(t)\right.$$
$$\left. + \ \psi(t)\right) - 2\mu\eta_k\bar{t}.$$

The form of Eqs. (2.8) and (2.9) is similar to the $\mathbb{R}$-linear condition [7], since φ and ψ are analytic in D; ϕ_k is analytic in D_k, Φ_k is analytic in D_k except $\zeta = a_k$, where its principal part has the form $r_k^2(\zeta - a_k)^{-1}\phi_k'(a_k)$. Following [7], we reduce the problem (2.8), (2.9) to a system of functional equations.

Let $\zeta_{(k)}^* = r_k^2(\zeta - a_k)^{-1} + \overline{a_k}$ denote the inversion of ζ with respect to the circle T_k. If a function $f(\zeta)$ is analytic in $|\zeta - a_k| < r_k$, then $\overline{f(\zeta_{(k)}^*)}$ is analytic in $|\zeta - a_k| > r_k$. Basing on (2.8), we introduce the function analytic in D and all D_k:

$$\Omega(\zeta) := \begin{cases} \left(1+\frac{\mu}{\mu_k}\kappa_k\right)\phi_k(\zeta) - \sum\limits_{m\neq k}\left(1-\frac{\mu}{\mu_m}\right)\left[\overline{\Phi_m(\zeta^*_{(m)})}\right. \\ \left. -(\zeta-a_m)\,\overline{\phi'_m(a_m)}\right] - 2\mu\eta_k\zeta + \left(1-\frac{\mu}{\mu_k}\right)(\zeta-a_k)\,\overline{\phi'_k(a_k)}, \\ \qquad\qquad |\zeta-a_k|\le r_k, \quad k=1,2,...,n, \\ (1+\kappa)\,\varphi(\zeta) - \sum\limits_{m=1}^{n}\left(1-\frac{\mu}{\mu_m}\right)\left[\overline{\Phi_m(\zeta^*_{(m)})}\right. \\ \left. -(\zeta-a_m)\,\overline{\phi'_m(a_m)}\right], \quad \zeta\in D. \end{cases}$$

Let us calculate the jump of $\Omega(\zeta)$ across T_k

$$\Omega^+(t)-\Omega^-(t) = (1+\kappa)\,\varphi(t) - \left(1-\frac{\mu}{\mu_k}\right)\left[\overline{\Phi_k(t)} - (t-a_k)\,\overline{\phi'_k(a_k)}\right]$$
$$-\left(1+\frac{\mu}{\mu_k}\kappa_k\right)\phi_k(t) + 2\mu\left(\alpha^T - \alpha^T_k\right)t - \left(1-\frac{\mu}{\mu_k}\right)(t-a_k)\,\overline{\phi'_k(a_k)}.$$

It follows from Eq. (2.8) that this jump is zero. Hence, $\Omega(\zeta)$ is analytic in $\mathbb{C}\cup\{\infty\}$ by the principle of analytic continuation. Then the Liouville theorem implies that the function $\Omega(\zeta)=p_0$, where p_0 is a constant. It follows from the definition of $\Omega(\zeta)$ in D_k that

$$\left(1+\frac{\mu}{\mu_k}\kappa_k\right)\phi_k(\zeta) = \sum_{m\neq k}\left(1-\frac{\mu}{\mu_m}\right)\left[\overline{\Phi_m(\zeta^*_{(m)})} - (\zeta-a_m)\,\overline{\phi'_m(a_m)}\right] \tag{2.10}$$
$$+\,p_0 + 2\mu\eta_k\zeta - \left(1-\frac{\mu}{\mu_k}\right)(\zeta-a_k)\,\overline{\phi'_k(a_k)},$$
$$|\zeta-a_k|\le r_k, \quad k=1,2,...,n.$$

This is the first set of functional equations relating the unknown functions $\phi_k(\zeta)$ and $\Phi_k(\zeta)$. The definition of $\Omega(\zeta)$ in D yields

$$(1+\kappa)\,\varphi(\zeta) = \sum_{m=1}^{n}\left(1-\frac{\mu}{\mu_m}\right)\left[\overline{\Phi_m(\zeta^*_{(m)})} - (\zeta-a_m)\,\overline{\phi'_k(a_k)}\right] + p_0, \tag{2.11}$$
$$\zeta\in D\cup\partial D.$$

Hence, if $\phi_k(\zeta)$ and $\Phi_k(\zeta)$ are determined, φ can be calculated by (2.11).

We now proceed to deduce the second set of functional equations. First we differentiate (2.11) and substitute it in (2.9)

$$
\begin{aligned}
(2.12)\qquad & \left(\kappa-\frac{\mu}{\mu_k}\kappa_k\right)\overline{\phi_k(t)}+\left(\kappa+\frac{\mu}{\mu_k}\right)\Phi_k(t) \\
&= \left(\frac{r_k^2}{t-a_k}+\overline{a_k}\right)\sum_{m=1}^{n}\left(1-\frac{\mu}{\mu_m}\right)\left[\left(\overline{\Phi_m(t^*_{(m)})}\right)'-\overline{\phi_k'(a_k)}\right] \\
&\quad +(1+\kappa)\,\psi(t)-2\mu\eta_k\left(\frac{r_k^2}{t-a_k}+\overline{a_k}\right),\quad |t-a_k|=r_k.
\end{aligned}
$$

Introduce the function analytic in D and meromorphic in D_k

$$
\omega(\zeta):=\begin{cases}
\left(\kappa+\frac{\mu}{\mu_k}\right)\Phi_k(\zeta)-\sum\limits_{m\neq k}\left(1-\frac{\mu}{\mu_m}\right)\left(\frac{r_k^2}{\zeta-a_k}+\overline{a_k}\right. \\
\left.-\frac{r_m^2}{\zeta-a_m}-\overline{a_m}\right)\left[\left(\overline{\Phi_m(\zeta^*_{(m)})}\right)'-\overline{\phi_m'(a_m)}\right] \\
-\sum\limits_{m\neq k}\left(\kappa-\frac{\mu}{\mu_m}\kappa_m\right)\overline{\phi_m(\zeta^*_{(m)})}+2\mu\left(\alpha^T-\alpha_k^T\right) \\
\times\left(\frac{r_k^2}{\zeta-a_k}+\overline{a_k}\right),\quad |\zeta-a_k|\le r_k,\quad k=1,2,...,n, \\
(1+\kappa)\,\psi(\zeta)+\sum\limits_{m=1}^{n}\left(\frac{r_m^2}{\zeta-a_m}+\overline{a_m}\right)\left(1-\frac{\mu}{\mu_m}\right) \\
\times\left[\left(\overline{\Phi_m(\zeta^*_{(m)})}\right)'-\overline{\phi_m'(a_m)}\right]-\sum\limits_{m=1}^{n}\left(\kappa-\frac{\mu}{\mu_m}\kappa_m\right)\overline{\phi_m(\zeta^*_{(m)})}, \\
\qquad\zeta\in D.
\end{cases}
$$

Let us calculate the jump of $\omega(\zeta)$ across T_k

$$
\begin{aligned}
\omega^+(t)-\omega^-(t) &= (1+\kappa)\,\psi(t)+\left(\frac{r_k^2}{t-a_k}+\overline{a_k}\right)\sum_{m=1}^{n}\left(1-\frac{\mu}{\mu_m}\right) \\
&\times\left[\left(\overline{\Phi_m(t^*_{(m)})}\right)'-\overline{\phi_k'(a_k)}\right]-\left(\kappa+\frac{\mu}{\mu_k}\right)\Phi_k(t)-2\mu\left(\alpha^T-\alpha_k^T\right) \\
&\times\left(\frac{r_k^2}{t-a_k}+\overline{a_k}\right)-\left(\kappa-\frac{\mu}{\mu_k}\kappa_k\right)\overline{\phi_k(t)}.
\end{aligned}
$$

It follows from Eq. (2.12) that $\omega^+(t)-\omega^-(t)=0$. Hence, by the principle of analytic continuation, $\omega(\zeta)$ is analytic in D and D_k $(k=1,2,...,n)$ except the points $\zeta=a_k$. The generalized Liouville theorem implies that

$$\omega(\zeta) = \sum_{k=1}^{n} \frac{r_k^2 q_k}{\zeta - a_k} + q_0, \tag{2.13}$$

where q_0 is a constant,

$$q_k = 4\mu\eta_k + \phi_k'(a_k)\left(\kappa + \frac{\mu}{\mu_k} - 1 - \frac{\mu}{\mu_k}\kappa_k\right) - \overline{\phi_k'(a_k)}\left(1 - \frac{\mu}{\mu_k}\right), \quad k = 1, 2, ..., n. \tag{2.14}$$

It follows from the definition of $\omega(\zeta)$ that

$$\begin{aligned}\left(\kappa + \frac{\mu}{\mu_k}\right)\Phi_k(\zeta) = \sum_{m \neq k} \Bigg\{ \left(\kappa - \frac{\mu}{\mu_m}\kappa_m\right) \overline{\phi_m(\zeta_{(m)}^*)} \\ + \left(\frac{r_k^2}{\zeta - a_k} + \overline{a_k} - \frac{r_m^2}{\zeta - a_m} + \overline{a_m}\right)\left(1 - \frac{\mu}{\mu_m}\right)\left[\left(\overline{\Phi_m(\zeta_{(m)}^*)}\right)' - \overline{\phi_m'(a_m)}\right]\Bigg\} \\ - 2\mu\eta_k\left(\frac{r_k^2}{\zeta - a_k} + \overline{a_k}\right) + \omega(\zeta), \quad |\zeta - a_k| \leq r_k, \quad k = 1, 2, ..., n.\end{aligned} \tag{2.15}$$

This is the second set of functional equations. $2n$ relations (2.10), (2.15) constitute a system of functional equations with respect to ϕ_k and Φ_k ($k = 1, 2, ..., n$), where $\phi_k(\zeta), \Phi_k(\zeta) - r_k^2 (\zeta - a_k)^{-1} \phi_k'(a_k)$ are analytic in D_k and continuously differentiable in $D_k \cup T_k$. The definition of $\omega(\zeta)$ in D yields the relation

$$\begin{aligned}(1 + \kappa)\psi(\zeta) = \omega(\zeta) - \sum_{m=1}^{n}\left(\frac{r_m^2}{\zeta - a_m} + \overline{a_m}\right)\left(1 - \frac{\mu}{\mu_m}\right) \\ \times \left[\left(\overline{\Phi_m(\zeta_{(m)}^*)}\right)' - \overline{\phi_m'(a_m)}\right] + \sum_{m=1}^{n}\left(\kappa - \frac{\mu}{\mu_m}\kappa_m\right)\overline{\phi_m(\zeta_{(m)}^*)}, \quad \zeta \in D.\end{aligned} \tag{2.16}$$

3. Solution to the functional equations

It is possible to solve (2.10), (2.15) by the method of successive approximations. Here we only note that this method is applied at least when $1 - \dfrac{\mu}{\mu_m}$ and $\kappa - \dfrac{\mu}{\mu_m}\kappa_m$ are sufficiently small. This case corresponds to weakly inhomogeneous materials, when $\mu \approx \mu_m$ and $\kappa \approx \kappa_m$.

Now we would like to present another method, the method of undetermined coefficients, based on the addition theorems [7]. Put

$$\phi_k(\zeta) = \sum_{l=0}^{\infty} \alpha_{lk}(\zeta - a_k)^l, \ \Phi_k(\zeta) = \frac{r_k^2 \alpha_{1k}}{\zeta - a_k} + \sum_{l=0}^{\infty} \beta_{lk}(\zeta - a_k)^l, \quad k = 1, 2, ..., n, \tag{3.1}$$

where α_{lk}, β_{lk} are unknown constants. Then

$$\overline{\phi_m(\zeta^*_{(m)})} = \sum_{l=0}^{\infty} \overline{\alpha_{lm}} r_m^{2l} (\zeta - a_m)^{-l}, \quad \overline{\Phi_m(\zeta^*_{(m)})} - (\zeta - a_m)\, \overline{\phi'_m(a_m)} = \sum_{l=0}^{\infty} \overline{\beta_{lm}} r_m^{2l} (\zeta - a_m)^{-l}, \tag{3.2}$$

$$\left[\overline{\Phi_m(\zeta^*_{(m)})}\right]' - \overline{\phi'_m(a_m)} = -\sum_{l=1}^{\infty} l \overline{\beta_{lm}} r_m^{2l} (\zeta - a_m)^{-(l+1)}, \quad m = 1, 2, ..., n.$$

Re-expand the functions (3.2) in the powers of $(\zeta - a_k)$ with $m \neq k$, using the relation

$$\frac{1}{\zeta - a_m} = -\frac{1}{a_m - a_k} \sum_{j=0}^{\infty} \left(\frac{\zeta - a_k}{a_m - a_k} \right)^j,$$

which can be considered as an addition theorem. Then

$$\overline{\phi_m(\zeta^*_{(m)})} = \sum_{l=0}^{\infty} \overline{\alpha_{lm}} \frac{(-1)^l r_m^{2l}}{(a_m - a_k)^l} \left[\sum_{j=0}^{\infty} \left(\frac{\zeta - a_k}{a_m - a_k} \right)^j \right]^l,$$

$$\overline{\Phi_m(\zeta^*_{(m)})} - (\zeta - a_m)\, \overline{\phi'_m(a_m)} = \sum_{l=0}^{\infty} \overline{\beta_{lm}} \frac{(-1)^l r_m^{2l}}{(a_l - a_k)^l} \left[\sum_{j=0}^{\infty} \left(\frac{\zeta - a_k}{a_m - a_k} \right)^j \right]^l, \tag{3.3}$$

$$\left[\overline{\Phi_m(\zeta^*_{(m)})}\right]' - \overline{\phi'_m(a_m)} = \sum_{m=0}^{\infty} \overline{\beta_{mk}} \frac{(-1)^l l r_m^{2l}}{(a_m - a_k)^{l+1}} \left[\sum_{j=0}^{\infty} \left(\frac{\zeta - a_k}{a_m - a_k} \right)^j \right]^{l+1}.$$

Substitute these expansions in (2.10) and (2.15), select and equate the coefficients of the same powers of $\zeta - a_k$. As a result, we obtain a system of linear algebraic equations with respect to α_{lm} and β_{lm}.

This system can be solved by the method of reduction [5] which consists in replacing the infinite sum $\sum_{m=0}^{\infty}$ in Eqs. (3.1) – (3.3) by the finite one $\sum_{m=0}^{N}$. The number N is determined according to the desired accuracy. Here we consider the simple case $N = 1$. Then Eq. (3.1) becomes

$$\phi_k(\zeta) \approx \alpha_{0k} + \alpha_k (\zeta - a_k), \quad \Phi_k(\zeta) \approx \frac{r_k^2 \alpha_k}{\zeta - a_k} + \beta_{0k} + \beta_k (\zeta - a_k). \tag{3.4}$$

Substituting (3.4) in (2.10) and (2.15) and selecting the coefficients on $(\zeta - a_k)^s, (s = 0, 1)$, we obtain a system of $\mathbb{R}$-linear algebraic equations with respect to p_0, q_0, α_{0k}, $\alpha_k = \alpha_{1k}$, β_{0k}, $\beta_k = \beta_{1k}$. It is easy to check that the coefficient with $(\zeta - a_k)^{-1}$ in (2.15) gives an identity. Further we consider the case when the inclusions are sufficiently far away from each other, i.e., the value $R := \max_{k,m} r_m |a_k - a_m|^{-1}$ for $m \neq k$ is sufficiently small. The remaining equations up to $O(R^2)$ are

$$\left(1 + \frac{\mu}{\mu_k}\kappa_k\right)\alpha_k = -\sum_{m\neq k}\left(1 - \frac{\mu}{\mu_m}\right)\frac{r_m^2\overline{\beta_m}}{(a_k - a_m)^2} + 2\mu\eta_k - \left(1 - \frac{\mu}{\mu_k}\right)\overline{\alpha_k}, \tag{3.5}$$

$$\left(\kappa + \frac{\mu}{\mu_k}\right)\beta_k = -\sum_{m\neq k}\left(\kappa - \frac{\mu}{\mu_m}\kappa_m\right)\frac{r_m^2\overline{\alpha_m}}{(a_k - a_m)^2} - \sum_{m\neq k}\left(1 - \frac{\mu}{\mu_m}\right)\frac{2r_m^2}{(a_k - a_m)^3}\overline{\beta_m}\,\overline{a_k}. \tag{3.6}$$

The coefficients corresponding to the constant terms give equations for the values α_{0k}, β_{0k}, p_0, q_0. The values α_{0k}, β_{0k}, p_0, q_0 do no affect the stress distribution. In accordance with the general theory [9], some of them remain undetermined. Thus we concentrate our attention on the system (3.5), (3.6) and do not determine α_{0k}, β_{0k}, p_0, q_0.

The zero order approximation of R^2 for Eqs. (3.5), (3.6) yields

$$\alpha_k^{(0)} = \frac{2\mu\eta_k}{2 + \dfrac{\mu}{\mu_k}\kappa_k - \dfrac{\mu}{\mu_k}}, \qquad \beta_k^{(0)} = 0. \tag{3.7}$$

Then Eq. (2.14) implies that

$$q_k^{(0)} = 4\mu\eta_k + \mu X_k\left(\kappa + 2\frac{\mu}{\mu_k} - 2 - \frac{\mu}{\mu_k}\kappa_k\right).$$

Substituting $\alpha_k^{(0)}$, $\beta_k^{(0)}$ and $q_k^{(0)}$ in the right-hand part of Eqs. (3.5), (3.6), we obtain the first order approximation

$$\alpha_k^{(1)} = \alpha_k^{(0)},\ \beta_k^{(1)} = -\frac{4(1+\kappa)\mu}{\kappa + \dfrac{\mu}{\mu_k}}\sum_{m\neq k}\left(\frac{r_m}{a_k - a_m}\right)^2\eta_m\frac{1}{2 + \dfrac{\mu}{\mu_m}\kappa_m - \dfrac{\mu}{\mu_m}}, \tag{3.8}$$

since $\beta_m^{(0)} = 0$ up to $O(R^0)$. This process of the successive approximations can be extended and one can get the arbitrary approximation $O(R^{2N})$ for α_k, β_k. We stop at the formulae (3.8) which are valid up to $O(R^2)$.

It follows from (2.11) that

$$(1+\kappa)\,\varphi'(\zeta) = \sum_{m=1}^{n} \left(1 - \frac{\mu}{\mu_m}\right) \left(\frac{r_m}{\zeta - a_m}\right)^2 \overline{\beta_m}.$$

Hence,

$$\varphi'(\zeta) = 0 \quad \text{up to} \quad O(R^2), \tag{3.9}$$

since $\beta_m = 0$ up to $O(R^0)$. The function $\psi'(\zeta)$ is determined from the definition of $\omega(\zeta)$

$$\psi'(\zeta) = -\sum_{m=1}^{n} \gamma_m^{(1)} \left(\frac{r_m}{\zeta - a_m}\right)^2, \tag{3.10}$$

where

$$\gamma_m^{(1)} = \frac{4\eta_m \mu \eta_m}{2 + \dfrac{\mu}{\mu_m}\kappa_m - \dfrac{\mu}{\mu_m}}. \tag{3.11}$$

4. Local thermoelastic fields in the composite material

In the present section we gather the formulae concerning the local fields in the material discussed. Applying (2.3), (2.5) and the results of the previous section, we determine the local fields. The local stresses have the form

$$\sigma_{xx} = \begin{cases} 2\alpha_k^{(1)} - \operatorname{Re}\beta_k^{(1)}, & \zeta \in D_k, \\ \displaystyle\sum_{m=1}^{n} \operatorname{Re}\gamma_m^{(1)} \left(\frac{r_m}{\zeta - a_m}\right)^2, & \zeta \in D, \end{cases} \tag{4.1}$$

$$\sigma_{yy} = \begin{cases} 2\alpha_k^{(1)} + \operatorname{Re}\beta_k^{(1)}, & \zeta \in D_k, \\ \displaystyle -\sum_{m=1}^{n} \operatorname{Re}\gamma_m^{(1)} \left(\frac{r_m}{\zeta - a_m}\right)^2, & \zeta \in D, \end{cases} \tag{4.2}$$

$$\sigma_{xy} = \begin{cases} \operatorname{Im}\beta_k^{(1)}, & \zeta \in D_k, \\ \displaystyle -\sum_{m=1}^{n} \operatorname{Im}\gamma_m^{(1)} \left(\frac{r_m}{\zeta - a_m}\right)^2, & \zeta \in D, \end{cases} \tag{4.3}$$

$$\sigma_{xx} = \begin{cases} 4\alpha_k^{(1)} \dfrac{\mu}{\mu_k}, & \zeta \in D_k, \\ 0, & \zeta \in D, \end{cases} \tag{4.4}$$

where $\alpha_k^{(1)}$ and $\beta_k^{(1)}$ are expressed by Eqs. (3.7), (3.8).

The local displacement $U = u + iv$ is calculated with (2.5) up to an additive constant

$$U = \begin{cases} \dfrac{1}{2\mu_k}\left[\alpha_k^{(1)}(\kappa_k - 1)(\zeta - a_k) - \overline{\beta_k^{(1)}}\overline{(\zeta - a_k)}\right] + \alpha_k^T(\zeta - a_k), \\ \hfill \zeta \in D_k, \\ -\dfrac{1}{2\mu}\displaystyle\sum_{k=1}^{n}\overline{\gamma_k^{(1)}}\dfrac{r_k^2}{\overline{\zeta - a_k}} + \alpha^T\zeta, \qquad \zeta \in D. \end{cases}$$

This formula yields the following relations for the deformations:

$$\frac{\partial u}{\partial x} = \begin{cases} \dfrac{1}{2\mu_k}\left[\alpha_k^{(1)}(\kappa_k - 1) - \operatorname{Re}\beta_k^{(1)}\right] + \alpha_k^T, & \zeta \in D_k, \\ \dfrac{1}{2\mu}\displaystyle\sum_{k=1}^{n}\operatorname{Re}\gamma_k^{(1)}\left(\dfrac{r_k}{\zeta - a_k}\right)^2 + \alpha^T, & \zeta \in D. \end{cases} \tag{4.5}$$

$$\frac{\partial v}{\partial y} = \begin{cases} \dfrac{1}{2\mu_k}\left[\alpha_k^{(1)}(\kappa_k - 1) + \operatorname{Re}\beta_k^{(1)}\right] + \alpha_k^T, & \zeta \in D_k, \\ -\dfrac{1}{2\mu}\displaystyle\sum_{k=1}^{n}\operatorname{Re}\gamma_k^{(1)}\left(\dfrac{r_k}{\zeta - a_k}\right)^2 + \alpha^T, & \zeta \in D. \end{cases} \tag{4.6}$$

$$\frac{\partial u}{\partial y} = \begin{cases} \dfrac{1}{2\mu_k}\operatorname{Im}\beta_k^{(1)}, & \zeta \in D_k, \\ -\dfrac{1}{2\mu}\displaystyle\sum_{k=1}^{n}\operatorname{Im}\gamma_k^{(1)}\left(\dfrac{r_k}{\zeta - a_k}\right)^2, & \zeta \in D. \end{cases} \tag{4.7}$$

$$\frac{\partial v}{\partial x} = \begin{cases} \dfrac{1}{2\mu_k}\operatorname{Im}\beta_k^{(1)}, & \zeta \in D_k, \\ -\dfrac{1}{2\mu}\displaystyle\sum_{k=1}^{n}\operatorname{Im}\gamma_k^{(1)}\left(\dfrac{r_k}{\zeta - a_k}\right)^2, & \zeta \in D. \end{cases} \tag{4.8}$$

5. Local elastic fields in the composite material

It is convenient to determine the effective constants of the composite materials in two steps. First we calculate the pure elastic effective constants. Second we find the effective coefficient of thermal expansion. Let us recall that we discuss the isotropic material in macroscale.

Let us fix the external forces

$$\sigma_{xx}^{(\infty)} = 1,\ \sigma_{yy}^{(\infty)} = \sigma_{xy}^{(\infty)} = 0.$$

Then the complex potentials become [9]

$$\varphi^{(\infty)}(\zeta) = \frac{\zeta}{4}, \qquad \psi^{(\infty)}(\zeta) = -\frac{\zeta}{2},$$

and the disturbed potentials $\widetilde{\varphi}$ and $\widetilde{\psi}$ in the domain D take the form

$$\widetilde{\varphi}(\zeta) = \varphi(\zeta) + \frac{\zeta}{4}, \qquad \widetilde{\psi}(\zeta) = \psi(\zeta) - \frac{\zeta}{2},$$

where $\varphi(\zeta)$ and $\psi(\zeta)$ are bounded at infinity.

The contact condition (2.6) becomes

$$\begin{aligned} \phi_k(t) + t\overline{\phi_k'(t)} + \overline{\psi_k(t)} &= \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} + \frac{t+\bar{t}}{2}, \\ \frac{\mu}{\mu_k}\left[\kappa_k\phi_k(t) - t\overline{\phi_k'(t)} - \overline{\psi_k(t)}\right] &= \kappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} + \frac{\kappa-1}{4}t + \frac{\bar{t}}{2}, \\ |t-a_k| &= r_k,\ k=1,2,...,n. \end{aligned} \tag{5.1}$$

Equation (5.1) corresponds to Eq. (2.7). The difference between Eqs. (5.1) and (2.7) appears in the known terms. We repeat all arguments of Secs. 2 and 3 to solve Eq. (5.1). If we are looking for ϕ_k and Φ_k in the form (3.4), we obtain, with the accuracy of up to $O(R^2)$,

$$\alpha_k = \frac{1+\kappa}{4(2+\frac{\mu}{\mu_k}\kappa_k - \dfrac{\mu}{\mu_k})}, \tag{5.2}$$

$$\beta_k = \frac{1+\kappa}{\dfrac{\mu}{\mu_k}+\kappa}\sum_{m\neq k}\gamma_m\left(\frac{r_m}{a_m-a_k}\right)^2 - \frac{1+\kappa}{2\left(\dfrac{\mu}{\mu_k}+\kappa\right)}, \tag{5.3}$$

where

$$\gamma_k = \frac{1}{2}\frac{\kappa - 1 - \dfrac{\mu}{\mu_k}\kappa_k + \dfrac{\mu}{\mu_k}}{2+\dfrac{\mu}{\mu_k}\kappa_k - \dfrac{\mu}{\mu_k}}, \qquad k=1,2,...,n. \tag{5.4}$$

Here the coefficients α_k, β_k, γ_k are written for the pure elastic statement. They correspond to the coefficients $\alpha_k^{(1)}$, $\beta_k^{(1)}$, $\gamma_k^{(1)}$ for the thermoelastic statement (see Sec. 4). In this case, the complex potentials (up to additive constants) become

$$\phi_k(\zeta) = \alpha_k\left(\zeta - a_k\right),\ \psi_k(\zeta) = \beta_k\left(\zeta - a_k\right). \tag{5.5}$$

Moreover, we have

$$\varphi(\zeta) = 0, \ \psi(\zeta) = \sum_{k=1}^{n} \gamma_k \frac{r_k^2}{\zeta - a_k} \tag{5.6}$$

within the accuracy of an additive constant and $O(R^2)$.

Applying the second Kolosov-Muskhelishvili formula (2.3), we obtain

$$\sigma_{xx} - \sigma_{yy} = \begin{cases} -2\mathrm{Re}\,\beta_k, & \zeta \in D_k, \\ 2\mathrm{Re} \displaystyle\sum_{k=1}^{n} \gamma_k \left(\frac{r_k}{\zeta - a_k}\right)^2 + 1, & \zeta \in D. \end{cases} \tag{5.7}$$

It follows also from Eq. (2.3) that

$$\sigma_{yy} = \begin{cases} 2\alpha_k + \mathrm{Re}\,\beta_k, & \zeta \in D_k, \\ -\mathrm{Re} \displaystyle\sum_{k=1}^{n} \gamma_k \left(\frac{r_k}{\zeta - a_k}\right)^2, & \zeta \in D. \end{cases} \tag{5.8}$$

The displacement vector $U = u + iv$ is calculated (up to an additive constant) by

$$U = \begin{cases} \dfrac{1}{2\mu_k}\left[\kappa_k \phi_k(\zeta) - \zeta\overline{\phi_k'(\zeta)} - \overline{\psi_k(\zeta)}\right], & \zeta \in D_k, \\ \dfrac{1}{2\mu}\left[\kappa\varphi(\zeta) - \zeta\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)} + \dfrac{\kappa - 1}{4}\zeta + \dfrac{\overline{\zeta}}{2}\right], & \zeta \in D. \end{cases} \tag{5.9}$$

Substituting Eqs. (5.4) and (5.6) in (5.9) we calculate

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \begin{cases} -\dfrac{\mathrm{Re}\,\beta_k}{\mu_k}, & \zeta \in D_k, \\ \dfrac{1}{\mu}\mathrm{Re} \displaystyle\sum_{k=1}^{n} \gamma_k \left(\frac{r_k}{\zeta - a_k}\right)^2 + \frac{1}{2\mu}, & \zeta \in D, \end{cases} \tag{5.10}$$

$$\frac{\partial v}{\partial y} = \begin{cases} \dfrac{1}{2\mu_k}\left[\alpha_k(\kappa_k - 1) + \mathrm{Re}\,\beta_k\right], & \zeta \in D_k, \\ -\dfrac{1}{2\mu}\mathrm{Re} \displaystyle\sum_{k=1}^{n} \gamma_k \left(\frac{r_k}{\zeta - a_k}\right)^2 + \frac{1}{2\mu}\frac{\kappa - 3}{4}, & \zeta \in D, \end{cases} \tag{5.11}$$

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \begin{cases} \dfrac{\kappa_k - 1}{\mu_k}\alpha_k, & \zeta \in D_k, \\ \dfrac{\kappa - 3}{4\mu}, & \zeta \in D. \end{cases} \tag{5.12}$$

6. Effective constants

We apply Maxwell's formalism discussed in Sec. 1 to calculate the effective constants in the generalized dilute case. We assume that the composite material contains groups of circular inclusions and all the groups are the same, one of them is displayed in Fig. 1. The composite material is shown in Fig. 2.

Using the formulae from the previous section concerning the purely elastic field, we first calculate the effective elastic constants. We shall use the following equations of elasticity:

$$\sigma_{xx} = \lambda\theta + 2\mu\frac{\partial u}{\partial x}, \tag{6.1}$$

$$\sigma_{yy} = \lambda\theta + 2\mu\frac{\partial v}{\partial y}. \tag{6.2}$$

Here we use the Lamé coefficients

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \ \mu = \frac{E}{2(1+\nu)}$$

in Eqs. (6.1) and (6.2) which are understood as piece-wise constant functions.

Let us subtract (6.2) from (6.1)

$$\sigma_{xx} - \sigma_{yy} = 2\mu\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$$

and average the expression in the latter equation applying (5.7) and (5.10),

$$\langle\sigma_{xx} - \sigma_{yy}\rangle = -2\sum_{k=1}^{n} \mathrm{Re}\,\beta_k v_k + 2\mathrm{Re}\,\Psi_0 + v, \tag{6.3}$$

$$\left\langle\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right\rangle = -\sum_{k=1}^{n}\frac{1}{\mu_k}\mathrm{Re}\,\beta_k v_k + \frac{1}{\mu}\mathrm{Re}\,\Psi_0 + \frac{1}{2\mu}v, \tag{6.4}$$

where $v = 1 - \sum_{k=1}^{n} v_k$ is the area fraction of the matrix,

$$\Psi_0 = \frac{1}{|U|}\int\int_D \sum_{k=1}^{n}\gamma_k\left(\frac{r_k}{\zeta - a_k}\right)^2 dxdy = -\sum_{k=1}^{n}\sum_{m\neq k}\gamma_k\left(\frac{r_k}{a_m - a_k}\right)^2 v_k. \tag{6.5}$$

We calculate the average $\langle\cdot\rangle = \frac{1}{|U|}\int\int_D \cdot dxdy$, where U is a "representative cell" in Maxwell's formalism, i.e., U is a domain which has area $|U| = \frac{\pi r_k^2}{v_k}$ for each $k = 1, 2, ..., n$ and which contains only one group of the inclusions. Here v_k is the area fraction of the k−th inclusions. In particular, U can be a rectangular cell, doubly periodically continued onto the whole plane.

Use of the averaged equations

$$\langle \sigma_{xx} - \sigma_{yy} \rangle = 2\mu_e \left\langle \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right\rangle$$

yields the formula for the effective shear modulus

$$\mu_e = \mu \frac{\dfrac{v}{2} - \displaystyle\sum_{k=1}^{n} \mathrm{Re}\,\beta_k v_k + \ \mathrm{Re}\,\Psi_0}{\dfrac{v}{2} - \displaystyle\sum_{k=1}^{n} \frac{\mu}{\mu_k} \mathrm{Re}\,\beta_k v_k + \mathrm{Re}\,\Psi_0}, \tag{6.6}$$

where β_k is calculated by (5.3), Ψ_0 is calculated by (6.5) in which γ_k is calculated according to (5.4).

The effective modulus λ_e is found from the averaged Eq. (6.2)

$$\langle \sigma_{yy} \rangle = \lambda_e \langle \theta \rangle + 2\mu_e \left\langle \frac{\partial v}{\partial y} \right\rangle. \tag{6.7}$$

It follows from Eq. (5.8) that

$$\langle \sigma_{yy} \rangle = \sum_{k=1}^{n} (2\alpha_k + \mathrm{Re}\,\beta_k)\, v_k - \mathrm{Re}\,\Psi_0.$$

Use of Eq. (5.11) yields

$$\left\langle \frac{\partial v}{\partial y} \right\rangle = \sum_{k=1}^{n} (\alpha_k (\kappa_k - 1) + \mathrm{Re}\,\beta_k) \frac{v_k}{2\mu_k} - \frac{\mathrm{Re}\,\Psi_0}{2\mu} + \frac{\kappa - 3}{8\mu} v,$$

$$\langle \theta \rangle = \sum_{k=1}^{n} \frac{\kappa_k - 1}{\mu_k} \alpha_k v_k + \frac{\kappa - 1}{4\mu} v.$$

Then Eq. (6.7) implies

$$\lambda_e = \frac{\displaystyle\sum_{k=1}^{n} (2\alpha_k + \mathrm{Re}\,\beta_k)\, v_k - \mathrm{Re}\,\Psi_0}{\dfrac{\kappa - 1}{4\mu} v + \displaystyle\sum_{k=1}^{n} \frac{\kappa_k - 1}{\mu_k} \alpha_k v_k} - \mu_e \frac{\displaystyle\sum_{k=1}^{n} (\alpha_k (\kappa_k - 1) + \mathrm{Re}\,\beta_k) \frac{v_k}{\mu_k} - \frac{\mathrm{Re}\,\Psi_0}{\mu} + \frac{\kappa - 3}{4\mu} v}{\dfrac{\kappa - 1}{4\mu} v + \displaystyle\sum_{k=1}^{n} \frac{\kappa_k - 1}{\mu_k} \alpha_k v_k}. \tag{6.8}$$

In order to calculate the effective coefficients of the thermal expansion, we come back to Eq. (2.1) and local field derived in Sec. 5. First, we calculate

$$\nu_e = \frac{\lambda_e}{2\left(\lambda_e + \mu_e\right)}, \qquad E_e = \frac{\mu_e\left(3\lambda_e + 2\mu_e\right)}{\lambda_e + \mu_e}. \tag{6.9}$$

The second homogenized Eq. (2.1) yields the formula

$$E_e \alpha_e^T = \left\langle E\alpha^T \right\rangle + \sum_{k=1}^{n} \frac{v_k}{1+\nu_k}\left(\alpha_k\left(\kappa_k - 1\right) + \operatorname{Re}\beta_k\right) - \frac{1}{1+\nu}\operatorname{Re}\Psi, \tag{6.10}$$

where

$$\left\langle E\alpha^T \right\rangle = \sum_{k=1}^{n} E_k \alpha_k^T v_k + vE\alpha^T \tag{6.11}$$

is the mean value of the piece-wise constant function $E\alpha^T$,

$$\Psi = -\sum_{k=1}^{n}\sum_{m\neq k} \gamma_k^{(1)} \left(\frac{r_k}{a_m - a_k}\right)^2 v_k. \tag{6.12}$$

Here we also use the relation $\dfrac{E}{2\mu} = \dfrac{1}{1+\nu}$.

Applying (4.1) – (4.4) we calculate

$$\left\langle \nu\left(\sigma_{xx} + \sigma_{yy}\right) - \sigma_{yy}\right\rangle = \sum_{k=1}^{n} \left(1+\nu_k\right)\left[2\alpha_k\left(2\nu_k - 1\right) + \operatorname{Re}\beta_k\right] v_k + (1+\nu)\operatorname{Re}\Psi.$$

Therefore, Eqs. (6.10) – (6.13) yield the formula

$$E_e \alpha_e^T = \left\langle E\alpha^T \right\rangle + \sum_{k=1}^{n} \frac{v_k}{1+\nu_k}\left[\alpha_k^{(1)}\left(\nu_k^2 - 2\nu_k^3\right) + \left(\nu_k^2 + 2\nu_k + 2\right)\operatorname{Re}\beta_k^{(1)}\right] - \frac{\nu^2}{1+\nu}\operatorname{Re}\Psi, \tag{6.13}$$

where $\alpha_k^{(1)}$ and $\beta_k^{(1)}$ are calculated from the formulae (3.7) and (3.8), Ψ has the form Eq. (6.12), E_e is given by Eq. (6.9).

7. Conclusion

We study two-dimensional thermoelastic composite materials with circular inclusions, when the temperature is constant everywhere in the material. Using the complex potentials of Kolosov-Muskhelishvili, we deduce the problem to a system of functional equations which can be solved by the method of successive approximations. This allows us to construct a simple algorithm to determine the local stress and displacement fields in analytic form with the accuracy of up to the term $O\left(R^2\right)$, where $R = \max_{k,m} r_k d_{km}^{-1}$, r_k is the radius of the k-th inclusion, d_{km} is the distance between centers of the k-th and m-th inclusion ($k \neq m$) (see formulae in Secs. 4 and 5). The effective elastic constants are written also in analytic form (6.6) and (6.8) up to $O(R^4)$. The effective coefficient of thermal expansion is expressed by Eq. (6.13).

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A cracked piezoelectric material under generalized plane electromechanical impact

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SOLUTION OF THE GENERALIZED plane deformation problem of a piezoelectric material strip with a crack is proposed. Laplace and Fourier transforms are used to reduce the problem to the solution of singular integral equations in the Laplace transform plane. Laplace inversion yields the results in the time domain. This analysis yields six independent stress and three electric displacement components. The model is general enough to account for arbitrary polarization direction, under transient or steady state load, for any mechanical or electrical mode of cracking. Numerical solutions for a piezoelectric material strip under electromechanical impact are illustrated. The influences of strip thickness and crack position on time-dependent crack tip fields are investigated. The results show that the transient electric displacement loads can increase or reduce the stress intensity factors at different time, dependent on the applied electric displacement load direction.

Key Words: Piezoelectric materials, fracture mechanics, crack, stress intensity factor, impact load.

1. Introduction

IN DESIGNING THE PIEZOELECTRIC materials, one must take into consideration imperfections, such as crack, that are often preexisting or are generated by external impact forces during the service life. The existence of a crack can significantly change the dynamic response of piezoelectric materials and structures. A significant amount of research has been performed in modeling the crack in piezoelectric materials [1 – 5]. In particular, the dynamic stress intensity factor of a cracked dielectric medium in a uniform electric field was studied by SHINDO and his colleagues [6]. NARITA and SHINDO [7] investigated the scattering of Love waves by a surface-breaking cracks in a piezoelectric layer over an elastic half plane, while the crack is normal to the interface. The anti-plane shear crack growth rate of a piezoelectric ceramic body with finite width [8], and the dynamic bending of a symmetric piezoelectric laminated plate with a through crack [9] were investigated by NARITA and SHINDO. More recently, the electroelastic problem for a piezoelectric layer with an anti-plane shear crack connected with two elastic half-planes [10], and the central crack problem in a finite piezoelectric strip [11] have been systematically studied.

In this paper, the Griffith crack in a piezoelectric material is considered with the polarization direction perpendicular to crack plane or parallel to crack plane. Dynamic loading is analyzed by using Laplace transform. The Fourier transform is used to handle the space variable. Stress and electric displacement intensity factors are determined for different crack length and crack position under different fracture mode.

2. Solutions of the crack problem

Field equations for piezoelectric materials subjected to mechanical and electrical fields can be written as

$$\sigma_{ij} = c_{ijkl}u_{k,l} + e_{lij}\phi_{,l}, \qquad D_i = e_{ikl}u_{k,l} - \in_{i,l} \phi_{,l}, \tag{2.1}$$

$$\sigma_{ij,j} = \rho\partial^2 u_i/\partial t^2, \qquad D_{i,i} = 0, \tag{2.2}$$

where σ_{ij}, u_i, D_i, and ϕ are stresses, displacements, electric displacements, and electric potential, respectively; c_{ijkl}, e_{ijk}, $\in_{il}$, and ρ are elastic constants, piezoelectric constants, dielectric permittivities, and density, respectively; t is time variable, and a comma indicates partial derivation. The electric field, E_i, is related to the electric potential, $\phi : E_i = -\phi_{,i}$.

Inserting (2.2) into (2.1) and applying the Laplace transform with respect to time t, we have

$$c_{ijkl}u^*_{k,lj} + e_{lij}\phi^*_{,lj} = \rho p^2 u^*_i, \quad e_{ikl}u^*_{k,li} - \in_{il} \phi^*_{,li} = 0, \tag{2.3}$$

where "p" is the Laplace transform parameter. The quantities with superscript "*" denote the Laplace transform.

In the following, unless it is declared, the superscript "*" will be omitted for simplicity. Generalized plane deformation means that the field variable u_i and ϕ are functions of x_1 and x_2 only. Assume that the piezoelectric medium is infinite along the x_1 axis. We look for a solution to (2.3) of the form

$$u_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_k(s)F(s)e^{|s|\lambda x_2}e^{-isx_1}ds, \tag{2.4}$$

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_4(s)F(s)e^{|s|\lambda x_2}e^{-isx_1}ds, \tag{2.5}$$

where $i = \sqrt{-1}$, $F(s)$ is an unknown function to be determined. Substituting (2.4) and (2.5) into (2.1) and then into (2.3), we obtain

$$\begin{bmatrix} \Theta_{jk} & \Theta_j \\ \Theta_k & \Theta_0 \end{bmatrix}\begin{Bmatrix} A_j \\ A_4 \end{Bmatrix} = 0, \tag{2.6}$$

where j and k take the values 1, 2, and 3, and

$$\begin{aligned}
\Theta_{jk} &= -c_{j1k1} - \operatorname{sgn}(s)i(c_{j1k2} + c_{j2k1})\lambda + c_{j2k2}\lambda^2 - \delta_{jk}\rho p^2/s^2, \\
\Theta_j &= -e_{1j1} - \operatorname{sgn}(s)i(e_{1j2} + e_{2j1})\lambda + e_{2j2}\lambda^2, \\
\Theta_0 &= \epsilon_{11} + \operatorname{sgn}(s)i(\epsilon_{12} + \epsilon_{21})\lambda - \epsilon_{22}\lambda^2.
\end{aligned}$$

Equation (2.6) is an eigenvalue problem consisting of four equations, a nontrivial set (A_1, A_2, A_3, A_4) exists if λ is a root of the determinant. The eight roots for λ, form four conjugate pairs. In terms of these eigenvalues, a general expression for the displacements and electric potential can be written as

$$\{V(x_1,x_2)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [A(x_2)]\{F(s)\}e^{-isx_1}ds, \tag{2.7}$$

where

$$\{V(x_1,x_2)\} = \begin{Bmatrix} u_k \\ \phi \end{Bmatrix}, \qquad k = 1,2,3,$$

$$[A(x_2)] = \begin{bmatrix} A_{k\alpha}e^{|s|\lambda_\alpha x_2} \\ A_{4\alpha}e^{|s|\lambda_\alpha x_2} \end{bmatrix}, \quad \{F(s)\} = \{F_\alpha\}^T, \quad \alpha = 1,...,8.$$

Define a new vector, $\{t_2\}$, as given by

$$\{t_2(x_1,x_2)\} = (\sigma_{21}\ \sigma_{22}\ \sigma_{23}\ D_2)^T.$$

We can obtain the following expression by using (2.1) and (2.7):

$$\{t_2(x_1,x_2)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [B_2]\{F\}se^{-isx_1}ds, \tag{2.8}$$

where $[B_2(x_2)]$ is a 4×8 matrix.

3. The crack problem

Consider a piezoelectric medium of height h, see Fig. 1. The body contains a crack of length $2a$ lying along x_1 axis. Let $\{t_{20}(x)\}$ represent $\{t_2(x,0)\}$, and the superscripts (1) and (2) refer to the quantities associated with the materials occupying the lower and upper parts, respectively. The initial displacement, velocities and electric potential are zero. The boundary conditions are assumed to have the following forms:

$$\begin{aligned}
\{t_2(x,-h^{(1)})\} &= \{t_{10}(x,t)\},\\
\{t_2(x,h^{(2)})\} &= \{t_{30}(x,t)\},\\
\{t_{20}(x)\} &= \{t_0(x,t)\}, \qquad -a < x < a.
\end{aligned}$$

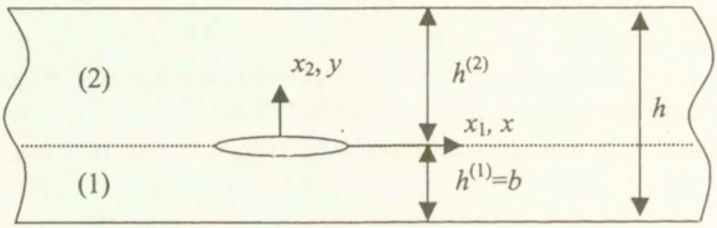

FIG. 1. A crack in a piezoelectric material strip: geometries and coordinates.

Suppose the Laplace transforms of $\{t_{10}\}$, $\{t_{30}\}$, and $\{t_{20}\}$ are $\{t_{10}^*\}$, $\{t_{30}^*\}$, and $\{t_{20}^*\}$, respectively. The unknown vector $\{F\}$ can be expressed in terms of $\{t_{10}^*\}$, $\{t_{20}^*\}$ and $\{t_{30}^*\}$ by applying the inverse Fourier transform to Eq. (2.8), i.e.,

$$\{F^{(2)}(s)\} = \frac{1}{s}\left[B^{(2)}\right]^{-1} \int_{-\infty}^{\infty} \left\{ \begin{array}{c} t_{30}^*(x) \\ t_{20}^*(x) \end{array} \right\} e^{isx} dx, \tag{3.1}$$

$$\{F^{(1)}(s)\} = \frac{1}{s}\left[B^{(1)}\right]^{-1} \int_{-\infty}^{\infty} \left\{ \begin{array}{c} t_{20}^*(x) \\ t_{10}^*(x) \end{array} \right\} e^{isx} dx, \tag{3.2}$$

where

$$[B^{(2)}] = \left[\begin{array}{c} B_2^{(2)}(h^{(2)}) \\ B_2^{(2)}(0) \end{array} \right], \quad [B^{(1)}] = \left[\begin{array}{c} B_2^{(1)}(0) \\ B_2^{(1)}(-h^{(1)}) \end{array} \right].$$

Substituting (3.1) and (3.2) back into (2.7), we have

$$\{V^{(1)}(r,y)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{s}[D_1^{(1)}(y)\ D_2^{(1)}(y)]\int_{-\infty}^{\infty}\left\{\begin{array}{c} t_{20}^*(x) \\ t_{10}^*(x)\end{array}\right\} e^{isx}dxe^{-isr}ds, \tag{3.3}$$

$$\{V^{(2)}(r,y)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{s}[D_1^{(2)}(y)\ D_2^{(2)}(y)]\int_{-\infty}^{\infty}\left\{\begin{array}{c} t_{30}^*(x) \\ t_{20}^*(x)\end{array}\right\} e^{isx}dxe^{-isr}ds. \tag{3.4}$$

where $[D_1^{(1)}]$, $[D_2^{(1)}]$, $[D_1^{(2)}]$, and $[D_2^{(2)}]$ are 4×4 matrices:

$$[D_1^{(1)}(y)\ \ D_2^{(1)}(y)] = [A^{(1)}(y)][B^{(1)}]^{-1},$$

$$[D_1^{(2)}(y)\ \ D_2^{(2)}(y)] = [A^{(2)}(y)][B^{(2)}]^{-1}.$$

Introduce the electromechanical dislocation density function $\{d\}$ along the crack faces as

$$\{d(x)\} = \partial(\{V^{(2)}(x,0)\} - \{V^{(1)}(x,0)\})/\partial x. \tag{3.5}$$

Equations (3.3) and (3.4) can be used to give

$$\{d(r)\} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\left([L]\int_{-\infty}^{\infty}\{t_{10}^*\}e^{isx}dx + [M]\int_{-\infty}^{\infty}\{t_{20}^*\}e^{isx}dx + [N]\int_{-\infty}^{\infty}\{t_{30}^*\}e^{isx}dx\right)e^{-isr}ds, \tag{3.6}$$

where

$$[L(s)] = [D_2^{(1)}(0)],\quad [M(s)] = [D_1^{(1)}(0)] - [D_2^{(2)}(0)],\quad [N(s)] = -[D_1^{(2)}(0)].$$

Solving $\{t_{20}^*(x)\}$ from Eq. (3.6), we have

$$\{t_{20}^*(x)\} = \int_{-a}^{a}[K_1(x,r)]\{d(r)\}dr - \{t_b(x)\}, \tag{3.7}$$

where

$$[K_1(x,r)] = \frac{1}{2\pi i}\int_{-\infty}^{\infty}[M(s)]^{-1}e^{is(r-x)}ds, \tag{3.8}$$

$$\text{(3.9)} \qquad \{t_b(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[M]^{-1}\left([L]\int_{-\infty}^{\infty}\{t_{10}^*\}e^{isr}dr + [N]\int_{-\infty}^{\infty}\{t_{30}^*\}e^{isr}dr\right)e^{-isx}ds.$$

Thus $\{d\}$ is the only unknown vector in the problem which may be determined from the crack faces boundary condition. The singular behavior of the kernel $[K_1(x,r)]$ may be obtained from the asymptotic analysis of the integral in (3.8). Observe that as s tends to infinity, $[M(s)]$ tends to $\operatorname{sgn}(s)[M(\infty)]$. It can be deduced from (3.8) that

$$[K_1(x,r)] = \frac{1}{\pi}\frac{[M(\infty)]^{-1}}{r-x} + [\Lambda(x,r)],$$

$$[\Lambda(x,r)] = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left([M(s)]^{-1} - \operatorname{sgn}(s)[M(\infty)]^{-1}\right)e^{is(r-x)}ds.$$

In view of the uniform convergence, the function defined by

$$\{t_{0b}(x)\} = \int_{-a}^{a}[\Lambda(x,r)]\{d(r)\}dr,$$

is bounded in the closed interval $-a \le x \le a$. Observe that (3.7) gives $\{t_{20}^*(x)\}$ outside as well as inside of the crack. For the latter, (3.7) may be expressed as

$$\text{(3.10)} \qquad \{t_0^*(x)\} = \frac{[M(\infty)]^{-1}}{\pi}\int_{-a}^{a}\frac{\{d(r)\}}{r-x}dr + \int_{-a}^{a}[\Lambda(x,r)]\{d(r)\}dr - \{t_b(x)\},$$

where $\{t_0^*(x)\}$ is the Laplace transform of $\{t_0(x,t)\}$.

A numerical technique to solve singular integral equations of the above form has been developed by ERDOGAN and GUPTA [12], where a weighted residual technique is employed to reduce the singular integral equations to a set of algebraic equations in unknown coefficients of Chebychev polynomials of the first kind. This method was also used by WANG [13] for the fracture analysis of the nonhomogeneous materials. Note that the solution for $\{d\}$ has the form

$$\text{(3.11)} \qquad \{d(r)\} = \sum_{m=1}^{M}\{C^m\}T_m(\bar{r})/\sqrt{1-\bar{r}^2},$$

where $\bar{r} = r/a$, $T_m(\bar{r})$ is the Chebychev polynomials of the first kind. Upon evaluating $\{C^m\}$ from (3.10) and (3.11), the displacements and electric potential differences between the crack faces can be evaluated from Eqs. (3.5) and (3.11) as

$$\{\Delta(r)\} = -a \sum_{m=1}^{\infty} \{C^m\} \frac{\sin(m\, ar\cos\bar{r})}{m}, \qquad \bar{r} < 1.$$

The stress and electric displacement intensity factors, i.e., in-plane normal traction (Mode I), in-plane shear (Mode II), anti-plane shear (Mode III), in-plane electric displacement (Mode IV), $\{K\} = \{K_{\mathrm{II}}\ K_{\mathrm{I}}\ K_{\mathrm{III}}\ K_{\mathrm{IV}}\}^T$ can be calculated by

$$\text{(3.12)} \qquad \{K\} = \left(\sqrt{2[(-a) - x]}\right)_{x\to(a)^-} \{t^*_{20}(x)\}$$

$$= [M(\infty)]^{-1}\sqrt{a}\sum_{m=1}^{M}(-1)^m\{C^m\},$$

for the left-hand side crack-tip, and

$$\text{(3.13)} \qquad \{K\} = \left(\sqrt{2[x - a]}\right)_{x\to(a)^+} \{t^*_{20}(x)\} = -[M(\infty)]^{-1}\sqrt{a}\sum_{m=1}^{M}\{C^m\},$$

for the right-hand side crack-tip.

Once the elastic and electric fields in Laplace transform plane are obtained, the corresponding values in the time domain are given by the Laplace inversion. The numerical Laplace inversion methods used here is due to DURBIN [14], a study of Durbin's method has been made by NARAYANAN and BESKOS [15].

Suppose that $F^*(p)$ is the Laplace transform of the function $F(t)$, then $\lim\limits_{p\to 0} pF^*(p) = \lim\limits_{t\to\infty} F(t)$. Therefore, we can obtain the static solutions of the field intensity factors in the physical space using (3.12) and (3.13).

4. The special cases

The polarization direction of the piezoelectric material is conventionally chosen as the x_3 direction. Constitutive relations for piezoelectric ceramics polarized along x_3 direction exhibiting transversely isotropic behavior (hexagonal symmetry) can be written in the following form:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix},$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{Bmatrix} + \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}.$$

Assume that the field variable u, v, and ϕ are functions of x and y only. The model described above is general enough to treat the fracture problems for piezoelectric materials with polarization axis along arbitrary direction. For the crack configuration and the orthogonal coordinate system shown in Fig. 1, we can obtain some special mechanical and electrical responses in any of the three directions.

CASE 1. Polarization axis perpendicular to the x-y plane

The in-plane displacements are governed by the equations

$$c_{11}\frac{\partial^2 u}{\partial x^2} + (c_{12} + c_{66})\frac{\partial^2 v}{\partial x \partial y} + c_{66}\frac{\partial^2 u}{\partial y^2} = \rho\frac{\partial^2 u}{\partial t^2},$$

$$c_{66}\frac{\partial^2 v}{\partial x^2} + (c_{12} + c_{66})\frac{\partial^2 u}{\partial x \partial y} + c_{22}\frac{\partial^2 v}{\partial y^2} = \rho\frac{\partial^2 v}{\partial t^2}.$$

This is a plane elastic crack problem. The piezoelectric effect has no influence on in-plane displacements and stresses.

The electrical potential and out-of-plane displacement w are governed by

$$c_{44}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+e_{15}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)=\rho\frac{\partial^2 w}{\partial t^2},$$

$$e_{15}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-\in_{11}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)=0.$$

The electrical potential is coupled only with the out-of-plane displacement w. This is the anti-plane electro-elastic crack problem.

CASE 2. Polarization axis along the y direction

The governing equations for in-plane displacements and electric potential are the following

$$c_{11}\frac{\partial^2 u}{\partial x^2}+c_{44}\frac{\partial^2 u}{\partial y^2}+(c_{13}+c_{44})\frac{\partial^2 v}{\partial x\partial y}+(e_{31}+e_{15})\frac{\partial^2 \phi}{\partial x\partial y}=\rho\frac{\partial^2 u}{\partial t^2},$$

$$(c_{13}+c_{44})\frac{\partial^2 u}{\partial x\partial y}+c_{44}\frac{\partial^2 v}{\partial x^2}+c_{33}\frac{\partial^2 v}{\partial y^2}+e_{15}\frac{\partial^2 \phi}{\partial x^2}+e_{33}\frac{\partial^2 \phi}{\partial y^2}=\rho\frac{\partial^2 v}{\partial t^2},$$

$$(e_{31}+e_{15})\frac{\partial^2 u}{\partial x\partial y}+e_{15}\frac{\partial^2 v}{\partial x^2}+e_{33}\frac{\partial^2 v}{\partial y^2}-\in_{11}\frac{\partial^2 \phi}{\partial x^2}-\in_{33}\frac{\partial^2 \phi}{\partial y^2}=0.$$

The electrical potential is coupled with in-plane displacements u and v.

The anti-plane displacement is governed by the equation

$$c_{66}\frac{\partial^2 w}{\partial x^2}+c_{44}\frac{\partial^2 w}{\partial y^2}=\rho\frac{\partial^2 w}{\partial t^2}.$$

This is an anti-plane elastic crack problem for orthotropic materials.

5. Results and discussion

We consider a PZT-5H piezoelectric ceramics strip as an example. The thickness of the medium is h. The elastic constants are $c_{11}=12.6\times10^{10}$ N/m^2, $c_{13}=8.41\times10^{10}$ N/m^2, $c_{33}=11.7\times10^{10}$ N/m^2, $c_{44}=2.3\times10^{10}$ N/m^2. The piezoelectric constants are $e_{31}=-6.5$ C/m^2, $e_{33}=23.3$ C/m^2, $e_{15}=17.44$ C/m^2. The dielectric permittivities are $\in_{11}=150.3\times10^{-10}$ C/Vm, $\in_{33}=130.0\times10^{-10}$ C/Vm. The mass density is $\rho=7500$ Kg/m^3. We postulate the crack faces are free from mechanical traction and are electrically insulated.

5.1. Polarization axis perpendicular to the x-y plane

This is the case of anti-plane mechanical deformation and in-plane electric fields. Assume a sudden mechanical loading τ_0 applied to the upper and lower surfaces of the strip. It is found that the electric displacement intensity factor is zero at the crack tip. Figure 2 shows the variation of stress intensity factor K_{III} with time. The results are the same as those for homogeneous material without the piezoelectric effect. It follows that for homogeneous materials under mechanical loading, the piezoelectric effect has no influence on the stress intensity factor.

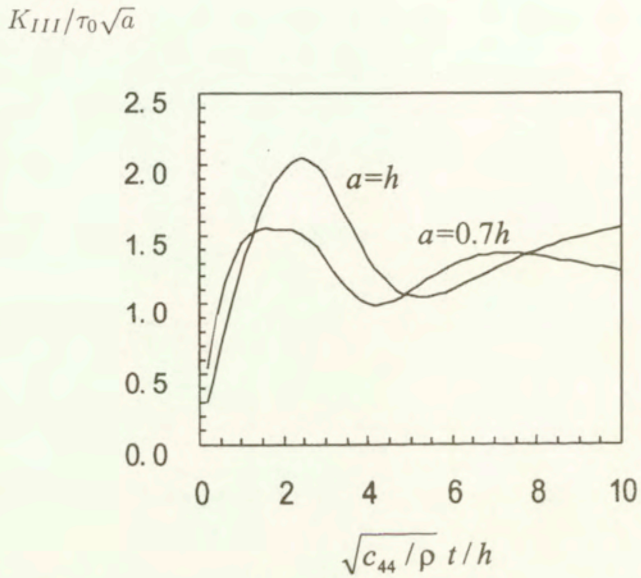

FIG. 2. Anti-plane stress intensity factor for a centrally cracked PZT-5H ceramics strip under transient mechanical load.

Assume a transient electric displacement load D_0 applied to the upper and lower surfaces of the strip. It is found that the dynamic stress intensity factors are not zero and they are plotted in Fig. 3. When time t approaches infinity the steady state stress intensity factors become zero. These results show that, differ from the static case, the transient electric displacement load can produce stress in the crack plane ahead of the crack tip.

5.2. Polarization axis along the y direction

The in-plane electrical field is coupled with in-plane displacements u and v. Assume that the upper and lower surfaces of the strip are loaded by a sudden stress σ_0 and a sudden electric displacement D_0. If the crack is located in the mid-plane of the strip, the shear stress in the plane of $y = 0$ is zero. Figures 4 and 5 show, respectively, the variation of normal stress intensity factor and electric

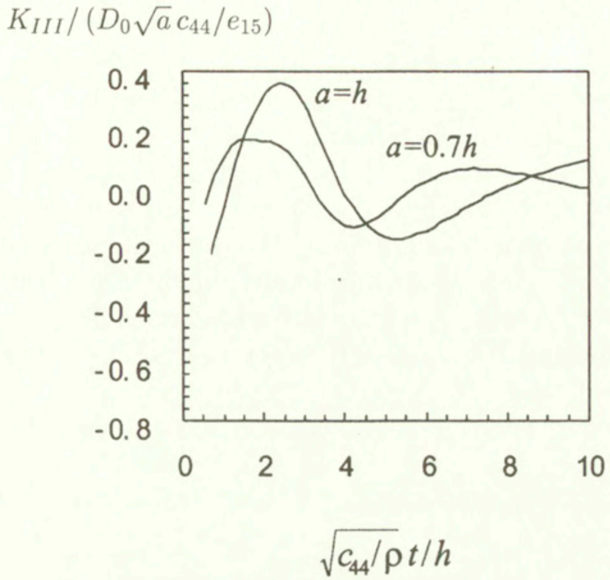

FIG. 3. Anti-plane stress intensity factor for a centrally cracked PZT-5H ceramics strip under transient electrical load.

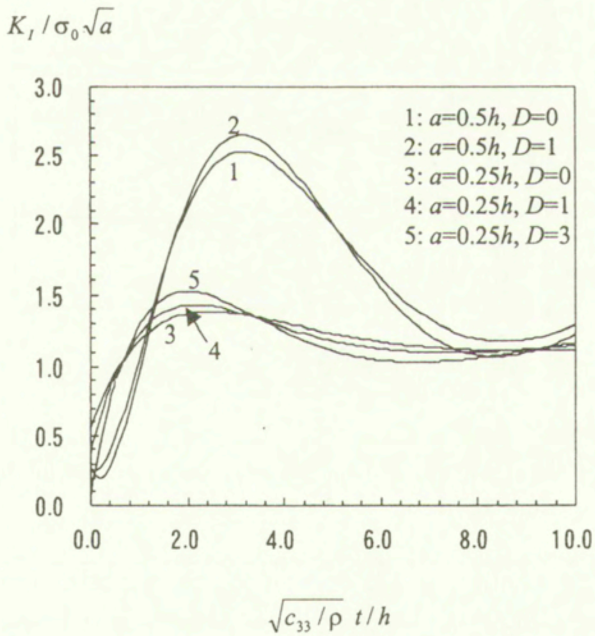

FIG. 4. Variation of normal stress intensity factor with time for different crack lengths and electric loads ($b/h = 0,5$).

displacement intensity factor with time. It is seen that as the relative thickness of the strip decreases, the field intensity factors increase. The stress intensity factors can increase or decrease with applied electric load, depending on the time t. At very early times, the presence of electric displacement load will reduce the stress intensity factors. The oppose trends are found for larger t. The presence of transient electric displacement load will always increase the peak stress intensity factor. Note that the applied electric displacement load is positive. Suppose a negative electric displacement load is applied, the transient electric displacement load will always reduce the peak stress intensity factor. This is different from the available date for piezoelectric materials under static loads. It has been shown that the steady-state stress intensity factor does not depend on the applied electric displacement load.

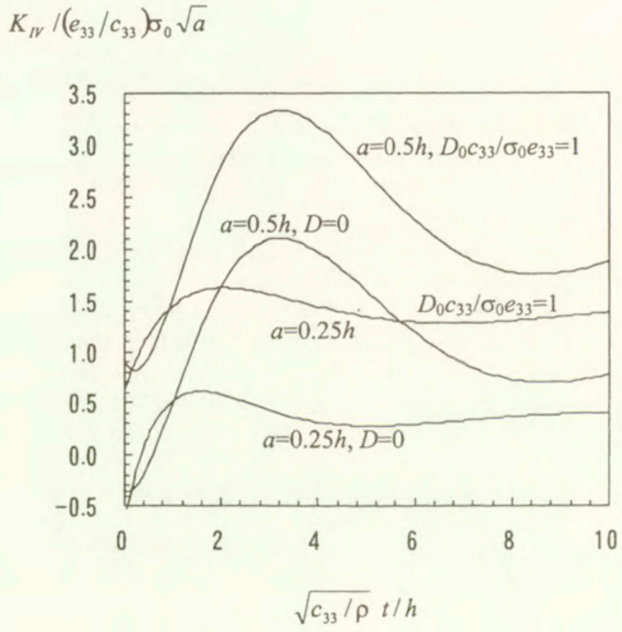

FIG. 5. Variation of electric displacement intensity factor with time for different crack lengths and electric loads ($b/h = 0.5$).

Figures 6, 7 and 8 show the influence of the crack position on the stress and electric displacement intensity factors. When the crack is located in the center of the strip, K_{II} is zero. The Mode II stress intensity factor is not zero when crack is not located in the mid-plane of the strip. As the crack approaches the surface of the strip, the field intensity factors increase quickly and the crack will more likely extend.

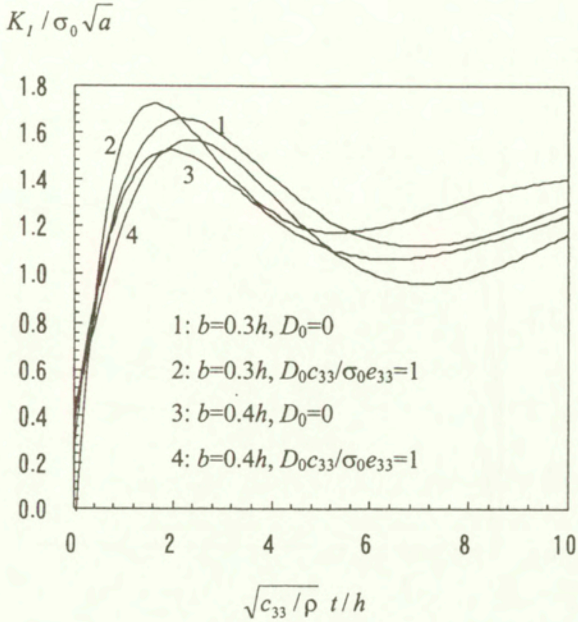

FIG. 6. Variation of normal stress intensity factor with time for different crack positions and electric loads ($a = 0.25h$).

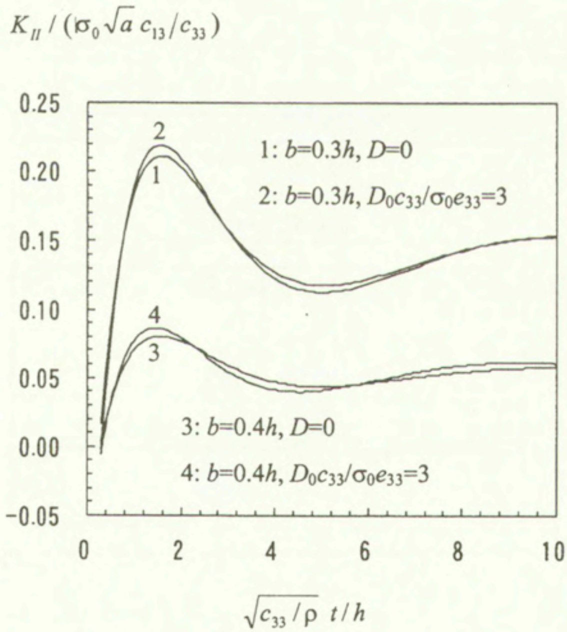

FIG. 7. Variation of shear intensity factor with time for different crack positions and electric loads ($a = 0.25h$).

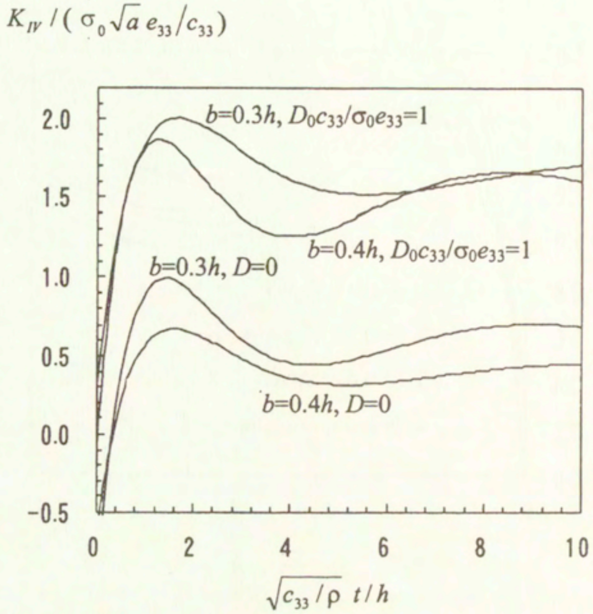

FIG. 8. Variation of electric displacement intensity factor with time for different crack positions and electric loads ($a = 0.25h$).

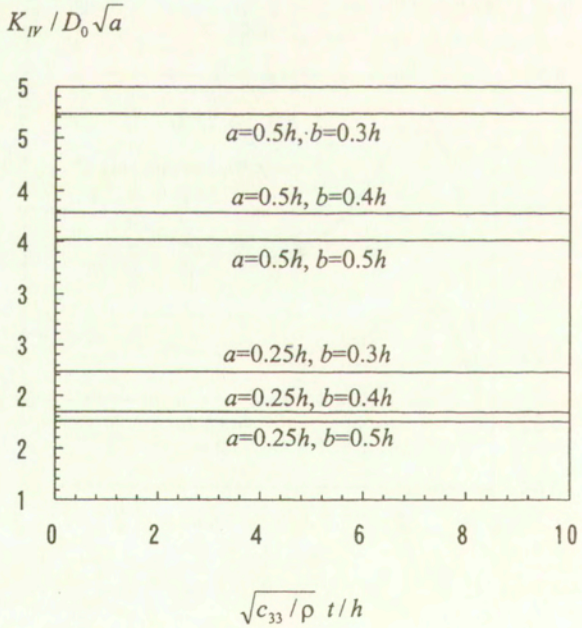

FIG. 9. Variation of electric displacement intensity factor with time (pure electric load).

Consider a pure electric displacement load D_0 applied on the surfaces of the strip. It is found that the electric displacement intensity factors are time independent as shown in Fig. 9. The same fact was also found for anti-plane crack problem.

6. Conclusion

Fracture is one of the properties that limit the use of piezoelectric materials as sensors and actuators in smart materials and structures technology. The present work focuses on the fracture mechanics analysis for finite thickness piezoelectric medium with the polarization axis along arbitrary direction. Particularly, numerical solutions for a piezoelectric material strip under in-plane and anti-plane electromechanical impacts are analyzed. It is found that as the relative length of the crack increases, the field intensities increase quickly. The crack position has a pronounced influence on field intensities ahead of the crack tip. The electric field can retard or enhance the crack tip stress intensity factors at different times, depending on the applied field displacement load direction.

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Prediction of mechanical behaviour of inhomogeneous and anisotropic materials using an incremental scheme

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A NEW MULTISITE homogenisation technique is proposed, suitable for the description of the effective properties of composite materials. After a short presentation of the multi-site approach, the traditional self-consistent approximation is developed. The limitations of such an approach are emphasised and a new alternative technique, based on the differential scheme idea, is proposed. The great versatility of this method in predicting the global elastic properties of various inhomogeneous materials is demonstrated. The present examples correspond to extreme situations of composites with voids and that of quasi-rigid (compared to the matrix) reinforcements. Very good agreement between the predicted and measured (or calculated by FEM) equivalent properties are found.

Notations

$\boldsymbol{\sigma}$, ($\boldsymbol{\Sigma}$)	Cauchy local (global) stress tensor
$\boldsymbol{\varepsilon}$, (**E**)	local (global) strain tensor
A, (**B**)	strain (stress) concentration tensor
c, (**C**)	local (global) elasticity tensor
u	local displacement field
G, ($\boldsymbol{\Gamma}$)	Green's (modified) tensor
$\mathbf{T}^{II}$	interaction tensor
I	fourth range unit tensor
S	number of steps
f, $\mathbf{f}_i$	global volume fraction, current volume fraction at step i
Δ**f**, $\Delta\mathbf{f}_i$	global increment volume fraction, current volume fraction at step i $\left(\Delta f = \frac{f}{S}\right)$

1. Introduction

THE SELF-CONSISTENT SCHEME (SCS) appears to be a powerful tool for prediction of the effective properties of polycrystalline materials, both in elasticity as well as in elasto-plasticity [1, 2, 3]. Some attempts of its direct application to the natural or manufactured multi-phase composites have demonstrated the inadequacy of such an approach [4, 5, 6]. In fact, the aberrant behaviour is predicted when the reinforcement volume or mass fraction and/or the degree of inhomogeneity between this reinforcement and matrix becomes considerable [5, 6]. This is due to the inaccurate description of interactions between the reinforcement and the surrounding material introduced by the self-consistent approximation. In fact, in this approach to the inclusion modelling, an inhomogeneity is embedded in the effective material, the properties of which can be very different from those of the surrounding medium.

At least three approaches have been proposed to solve this problem. In the MORI-TANAKA model (MTS) [7], the inclusion interacts directly with a uniform matrix, which is subjected to an overall loading equal to the external loading applied to the heterogeneous material. To describe more accurately the interactions of the heterogeneity with the surrounding material, the idea of a "composite" inclusion has been developed. In this approach, the reinforcement is coated with a shell of the matrix material and the resulting composite is embedded in an infinite body with unknown properties of the equivalent homogeneous medium. This model was introduced by CHRISTENSEN and LO [8] for elastic materials and then extended to the nonlinear behaviour by HÉRVÉ and ZAOUI [9]. Recently, CHERKAOUI *et al.* [10] have proposed an alternative but approximate solution of the multi-coated inclusion.

Another approximate method, called the Differential Scheme (DS), was proposed by BRUGGEMAN [11] to predict the effective conductivity of inhomogeneous media. The DS is based on a progressive construction of the composite material. The effective properties of the composite are obtained by gradual addition of infinitesimal quantities of reinforcements. The actual effective properties are determined using the small concentration relations. To find the global properties of the material, a differential tensorial equation has to be solved. The method has been used and improved by many authors. ROSCOE [12] proposed an interpretation of the method based on the assumption that there exist the reinforcements of different size. The construction of the composite begins with insertion of the smallest inclusions and ends with the greatest ones. In this way, at each step, the application of the small concentration procedure is justified. BOUCHER [13] has used the DS to predict the behaviour of porous materials. He obtained a good agreement of the predicted properties with experimental results[1]. Works by MCLAUGHLIN [14], SALGANIK [15], LAWS and DVORAK [16], HASHIN [17] can be cited to illustrate the application of the DS in the field of viscous and cracked media. NORRIS [18] has emphasised that the incremental construction of

[1]It is well known that the SCS does not work at all for such materials when the volume fraction of voids exceeds 25%.

the multi-phase composite is not unique. Consequently, the predicted material properties are not exact. All the applications we have found in the literature are limited to the case of isotropic behaviour of the constituents. The form of these constituents is frequently limited to spherical, cylindrical or penny-shaped inclusions.

The purpose of this paper is to establish a new scheme for prediction of the effective elastic properties of composite materials. The general framework is that of the kinematic integral equation that appears as the formal solution for inhomogeneous materials. This equation will be briefly recalled in the next section where we will show that all existing models can be derived from this expression by adopting appropriate simplifications. In the next section we introduce, following FASSI-FEHRI [19], two tensors to describe the interactions between an inclusion and the surrounding material, and between two inclusions interacting through the matrix. These applications are used to construct one-site and multi-site approximations of the integral equation. The same approach has been applied by FASSI-FEHRI and BERVEILLER [20] for the self-consistent modelling of anisotropic materials with cubic symmetries. El MOUDEN [21] has used similar tensors to generalise the Mori-Tanaka model to the multi-site situation. In his approach the matrix has been supposed to be isotropic. ZATTARIN and LIPINSKI [22] have developed a general multi-site self-consistent scheme (MSCS) for a cluster or periodic arrangement of inclusions. Their results show relatively rapid deviation of the predicted properties from the experimental values for strongly inhomogeneous cases.

The approach developed in this paper is based on the DS idea and was first proposed by VIÉVILLE [23] and VIÉVILLE and LIPINSKI [6] for one-site approximations. We call it Multi-Site Incremental Scheme (MIS). It consists in the finite increment construction of the composite. In each increment, for which the small concentration solution is not valid, the self-consistent procedure is used to calculate the actual properties of the resulting material. The model takes into account the anisotropy of the constituents, the morphological texture of the composite as well as the spatial repartition of the inhomogeneities (topological texture).

The model is used to predict the effective properties of reinforced composites and porous materials. In both cases the calculated properties are close to the experimental data.

2. Preliminary remarks

Consider a representative volume (RV) of a macro-homogeneous and micro-inhomogeneous body in the sense introduced by HILL [1] and MANDEL [30]. We restrict our considerations to the case of linear elastic behaviour of the constitutive phases and the small transformation approximation. The usual Hooke's law is supposed to be valid on the global and local levels. Let $\mathbf{E}$ be the global strain tensor and $\boldsymbol{\Sigma}$ be the overall stress tensor. Our goal is to determine the overall moduli of the body, such that

$$\Sigma_{ij} = C^{eff}_{ijkl} E_{kl}, \tag{2.1}$$

knowing the local Hooke's law at each point r of the RV

$$\sigma_{ij}(r) = c_{ijkl}(r)\varepsilon_{kl}(r), \tag{2.2}$$

where $\boldsymbol{\sigma}(r)$, $\boldsymbol{\varepsilon}(r)$ and $\mathbf{c}(r)$ are respectively the local stress, strain and elastic stiffness tensors. All these second-order tensors are symmetric and the above fourth-order tensors are characterised be the usual elasticity symmetries.

We suppose that all constituents are perfectly bonded together. Under this condition, we can link the macroscopic variables with microscopic ones by the volume averaging procedure over RV:

$$\Sigma_{ij} = \frac{1}{V}\int_V \sigma_{ij}(r)dV, \tag{2.3}$$

$$E_{ij} = \frac{1}{V}\int_V \varepsilon_{ij}(r)dV. \tag{2.4}$$

Following MANDEL [30] and HILL [1], we introduce now a fourth-order concentration tensor $\mathbf{A}$ such that

$$\varepsilon_{ij}(r) = A_{ijkl}(r)E_{kl}. \tag{2.5}$$

Combining (2.2), (2.3), and (2.5) and comparing the resulting expression with (2.1), one can determine the effective properties of the RV

$$C_{ijkl} = \frac{1}{V}\int_V c_{ijmn}(r)A_{mnkl}(r)dV. \tag{2.6}$$

In this paper we use a general kinematic integral equation proposed by DEDERICHS and ZELLER [24]. This equation, what will be shown in the next section, enables to deduce an approximation of $\mathbf{A}$ and to develop new models. In the following we propose a multi-site approximation of this tensor.

3. Multi-Site Self-Consistent Scheme (MSCS)

3.1. Integral equation

Consider a heterogeneous body in equilibrium under the uniform boundary conditions of the type

$$u_i(r) = E_{ij}x_j, \qquad \forall r \in S,$$

where $r = x_i e_i$ and S is the external surface of the body. When the body forces are neglected, the local equilibrium equation is written as follows:

$$\sigma_{ij,j} = 0,$$

and the displacement vector of any point of the medium can be expressed from the following integral equation proposed first by DEDERICHS and ZELLER [24]

$$u_m(r) = U^{\circ}_m(r) - \int\limits_V G_{mj,i'}(r,r')\delta c_{ijkl}(r')u_{k,l'}(r')dV, \tag{3.1}$$

where $G_{mj}(r,r')$ is a Green tensor corresponding to the m-th component of the displacement vector at point r due to the j-th component of a unit force applied at r' of a fictitious homogeneous medium with elastic properties defined by $\mathbf{C}^0$. $U^{\circ}_m(r)$ describes the displacement field, solution of a corresponding auxiliary problem with homogeneous properties, and

$$\delta c_{ijkl}(r) = c_{ijkl}(r) - C^{\circ}_{ijkl}$$

defines the deviation of the local properties from $\mathbf{C}^0$.

The strain tensor field may now be calculated by differentiation of Eq. (3.1)

$$\varepsilon_{mn}(r) = E^{\circ}_{mn} - \int\limits_V \Gamma_{mnij}(r,r')\delta c_{ijkl}(r')\varepsilon_{kl}(r')dV \tag{3.2}$$

where a new quantity, called a modified Green tensor, is introduced:

$$\Gamma_{mnij}(r,r') = \frac{1}{2}\left(G_{mj,i'n}(r,r') + G_{nj,i'm}(r,r')\right). \tag{3.3}$$

Expressions (3.1) and (3.2) constitute the formal solution of the heterogeneous body behaviour, and enable to deduce a form of the concentration tensor $\mathbf{A}(r)$. This tensor is obtained by a recurrent procedure (see for instance [3, 19]) which begins with the Voigt approximation of the strain field characterised by

$$\mathbf{A}(r) = \mathbf{A}^0(r) = \mathbf{I}, \tag{3.4}$$

$$\boldsymbol{\varepsilon}(r) = \mathbf{E} = \mathbf{E}^0, \tag{3.5}$$

where $\mathbf{I}$ is a fourth range unit tensor and $\mathbf{A}^0$ satisfies the equation

$$\boldsymbol{\varepsilon}(r) = \mathbf{A}^0(r):\mathbf{E}^0. \tag{3.6}$$

Introducing (3.5) into (3.2) one obtains a higher-order approximation. This procedure is repeated, leading to the following expression for the concentration tensor $\mathbf{A}^0(r)$:

$$A(r) = I + \int_V \mathbf{\Gamma}(r,r'):\delta\mathbf{c}(r')dV - \int_V \mathbf{\Gamma}(r,r'):\delta\mathbf{c}(r'):(\mathbf{\Gamma}(r',r''):\delta\mathbf{c}(r'')dV)dV + \dots .$$

The averaging operation applied to the expression (3.6) gives the link between $\mathbf{E}^0$ and $\mathbf{E}$, and consequently, the final result for the strain concentration tensor $\mathbf{A}^0(r)$ can be written as

$$\mathbf{A}(r) = \mathbf{A}^0(r): \left(\frac{1}{V}\int_V \mathbf{A}^0(r)dV\right)^{-1} . \tag{3.7}$$

In what follows we analyse the problem of a set of N ellipsoidal inclusions interacting through the homogeneous matrix. The solution of this problem constitutes a basis of the multi-site modelling of heterogeneous media, taking into account the effects of the spatial distribution of reinforcements upon the effective macroscopic behaviour of the material.

3.2. Multi-site modelling

Consider an infinite medium with elastic constants $\mathbf{C}^0$, containing N inclusions. Each inclusion I is characterised by its volume V^{I} and the elastic constants $\mathbf{c}^{\mathrm{I}}$. In this case, the field of the elastic properties of the material can be expressed as shown below:

$$\mathbf{c}(r) = \mathbf{C}^0 + (\mathbf{c}^1 - \mathbf{C}^0)\theta^1(r) + \dots + (\mathbf{c}^I - \mathbf{C}^0)\theta^I(r) + \dots + (\mathbf{c}^N - \mathbf{C}^0)\theta^N(r), \tag{3.8}$$

where θ^{I} is the characteristic function such that:

$$\theta^I(r) = \begin{cases} 1 & \forall r \in V^I, \\ 0 & \forall r \notin V^I. \end{cases} \tag{3.9}$$

Similar expressions are available for all N inclusions.

Let us introduce new notations

$$\Delta\mathbf{c}^I = \mathbf{c}^I - \mathbf{C}^0.$$

Integral equation (3.2) can now be presented as

$$\varepsilon(r) = \mathbf{E}^0 - \int \Gamma(r,r'):\big(\Delta\mathbf{c}^1\theta^1(r')$$

$$+ ... + \Delta\mathbf{c}^I\theta^I(r') + ... + \Delta\mathbf{c}^N\theta^N(r')\big):\varepsilon(r')dV$$

which, using the properties of θ (3.9), can be rewritten as

$$\varepsilon(r) = \mathbf{E}^0 - \int\limits_{V_1} \Gamma(r,r'):\Delta\mathbf{c}^1:\varepsilon(r')dV - ... - \int\limits_{V_1} \Gamma(r,r'):\Delta\mathbf{c}^1:\varepsilon(r')dV$$

$$- ... - \int\limits_{V_N} \Gamma(r,r'):\Delta\mathbf{c}^N:\varepsilon(r')dV.$$

In spite of the important simplification, the above equation still remains very difficult to solve. To simplify the considerations, let us calculate the average strain inside each inclusion

$$\varepsilon^1 = \frac{1}{V_I}\int\limits_{V_I} \varepsilon(r)dV, \tag{3.10}$$

and approximate the real strain field by a function

$$\varepsilon(r) = \varepsilon^1\theta^1(r) + ...\varepsilon^I\theta^I(r) + ...\varepsilon^N\theta^N(r).$$

Thus, the integral equation becomes:

$$\varepsilon(r) = \mathbf{E}^0 - \left(\int\limits_{V_1} \Gamma(r,r')dV\right):\Delta\mathbf{c}^1:\varepsilon^1 - ... - \left(\int\limits_{V_I} \Gamma(r,r')dV\right):\Delta\mathbf{c}^I:\varepsilon^I$$

$$- ... - \left(\int\limits_{V_N} \Gamma(r,r')dV\right):\Delta\mathbf{c}^N:\varepsilon^N.$$

Now, one can calculate the average strain tensor inside each inclusion using relation (3.10). This leads to the system of N tensorial equations for N unknown strain tensors ε^I:

$$
(3.11)\qquad
\begin{aligned}
&\boldsymbol{\varepsilon}^1 = \mathbf{E}^0 - \mathbf{T}^{11}{:}\Delta\mathbf{c}^1 : \boldsymbol{\varepsilon}^1 - ... - \mathbf{T}^{11}{:}\Delta\mathbf{c}^I : \boldsymbol{\varepsilon}^I - ... - \mathbf{T}^{1N}{:}\Delta\mathbf{c}^N : \boldsymbol{\varepsilon}^N\\
&\vdots\\
&\boldsymbol{\varepsilon}^I = \mathbf{E}^0 - \mathbf{T}^{I1}{:}\Delta\mathbf{c}^1 : \boldsymbol{\varepsilon}^1 - ... - \mathbf{T}^{II}{:}\Delta\mathbf{c}^I : \boldsymbol{\varepsilon}^I - ... - \mathbf{T}^{IN}{:}\Delta\mathbf{c}^N : \boldsymbol{\varepsilon}^N\\
&\vdots\\
&\boldsymbol{\varepsilon}^N = \mathbf{E}^0 - \mathbf{T}^{N1}{:}\Delta\mathbf{c}^1 : \boldsymbol{\varepsilon}^1 - ... - \mathbf{T}^{NI}{:}\Delta\mathbf{c}^I : \boldsymbol{\varepsilon}^I - ... - \mathbf{T}^{NN}{:}\Delta\mathbf{c}^N : \boldsymbol{\varepsilon}^N
\end{aligned}
$$

$\mathbf{T}^{IJ}$ are called interaction tensors. They are defined by the expression

$$
(3.12)\qquad \mathbf{T}^{IJ} = \frac{1}{V_I}\int\limits_{V_I}\int\limits_{V_J}\Gamma(r,r')\,dV\,dV
$$

and have been studied, for case of two ellipsoidal inclusions embedded in an anisotropic matrix, by BERVEILLER and FASSI-FEHRI [20]. A numerical calculation of these tensors has been performed by LIPINSKI [3], and recently improved by ZATTARIN *et al.* [25].

When the reference homogeneous material $\mathbf{C}^0$ is chosen to be an unknown effective medium $\mathbf{C}^{\text{eff}}$, the self-consistent multi-site approximation is obtained. In this case; $\mathbf{E}^0 = \mathbf{E}$ and the concentration tensors for all inclusions are expressed by:

$$
(3.13)\qquad \mathbf{A}^{\mathrm{I}}_{\mathrm{sc}} = \left(\mathbf{I} + \mathbf{T}^{\mathrm{II}}{:}\Delta\mathbf{c}^{\mathrm{I}}\right)^{-1} : \left(\mathbf{I} - \Sigma_{j\neq I}\mathbf{T}^{IJ}{:}\Delta\mathbf{c}^{J}{:}\mathbf{A}^{J}_{\mathrm{sc}}\right).
$$

The usual one-site self-consistent method is obtained when all tensors $\mathbf{T}^{IJ}$ with $I \neq J$ are neglected. Of course, the determination of the concentration tensors has to be performed by iterations because the $\mathbf{A}^I_{\mathrm{sc}}$ tensor depends on all unknown $\mathbf{A}^J_{\mathrm{sc}}$ operators. The one-site self-consistent approximation constitutes a very good starting point for this iterative procedure.

3.3. Confrontation with experimental data

The self-consistent, one- or multi-site, models predict the accurate results when they are applied to the multi-phase polycrystalline materials whose elastic or elastic-plastic properties change slowly from one constituent to another [5, 19]. Its direct application to the artificial composites we want to illustrate below what leads to a considerable deviation from the experimental measurements when the relative difference between the matrix and reinforcement characteristics becomes significant. Consider for instance a composite with isotropic epoxy matrix reinforced by the boron fibres, supposed also to be isotropic. Such a composite has been experimentally studied by SABODH [26]. The elastic properties of the constituents are respectively:

– for epoxy : Young's modulus $E = 4.14$ GPa and Poisson's ratio $v = 0.35$;
– for boron : Young's modulus $E = 414$ GPa and Poisson's ratio $v = 0.20$.

The cylindrical fibers have been reproduced numerically by ellipsoidal inclusions with the aspect ratio of 1000. The regular spatial distribution of reinforcements has been introduced corresponding to the periodic square arrangement of parallel reinforcements. The MSCS has been used to predict the elastic properties of the composite. Table 1 shows the evolution of the transversal Young's modulus and the axial shear modulus as functions of the volume fraction of fibres, and a comparison between the experimental and predicted values is given. A reasonably good agreement is obtained for the volume fraction less than 25%. When the concentration of fibres is greater than 40%, the interactions between the inclusion and the surrounding material are strongly overestimated and the predicted moduli become much too high compared to the experimental measurements.

Table 1. Comparison between experimental and predicted moduli for epoxy-boron composite. Experimental data from Sabodh [26].

Volume fraction	Young's modulus [GPa]		Shear modulus [GPa]	
	Experimental	Predicted	Experimental	Predicted
0.25	7.67	7.67	2.93	3.42
0.3	9.15	9.15	3.06	4.03
0.35	11.18	11.18	3.18	4.63
0.40	12.74	13.82	3.79	7.54
0.50	15.36	24.87	5.72	17.89
0.60	20	53.3	8.91	41.1
0.70	33.25	119	14.4	72.36

4. Incremental Scheme (IS)

In order to avoid the divergence of the self-consistent model, VIÉVILLE [23] and VIÉVILLE *et al.* [6] proposed to modify the SCS using the idea of progressive construction of the material developed in the DS approach. Contrary to the DS, construction of the material is made using finite increments of the volume fractions, and for each increment the self-consistent approximation of the homogenisation process is performed instead of a small concentration solution.

4.1. Two-phase material

Let us first analyse a simple case of two-phase material considered by VIÉVILLE [23]. The resulting composite will be characterised by a volume fraction of the second phase $0 < f < 1$. Generally, the phase with the greatest volume fraction is considered as a matrix, and the second phase as a reinforcement. The composite is built by the S step procedure in such a manner that at step i, the volume fraction of the second phase is

$$f_i = \frac{f}{S} \cdot i = \Delta f \cdot i. \tag{4.1}$$

When the step number S tends to infinity, the usual DS is found. Suppose, to clarify the idea, that at the beginning we have some volume of the matrix V^m of the reinforcement V^r, and that the resulting composite should have a volume V^c such that

$$V^m + V^r = V^c = V.$$

After $(i-1)$ steps the composite corresponds to

$$(i-1)\Delta f \cdot V^r + (1-(i-1)\Delta f) \cdot V^m = V^c_{i-1}, \tag{4.2}$$

where $V^c_{i-1} = V$ is the composite volume for step $i-1$. This volume becomes the matrix for the next step of the building procedure (N.B. this new matrix contains reinforcements).

At step i, to preserve the total volume, we have to cut off some volume of the new matrix to introduce an unknown volume fraction of reinforcement Δf_i such that the following equation will be verified

$$\Delta f_i \cdot V^r + (1-\Delta f_i) \cdot V^c_{i-1} = i\Delta f \cdot V^r + (1-i\Delta f) \cdot V^m. \tag{4.3}$$

Now, introducing (4.2) into (4.3) and comparing factors of V^m or V^r, one can determine the value of Δf_i

$$\Delta f_i = \frac{\Delta f}{1-(i-1)\Delta f}. \tag{4.4}$$

Expression (4.4) shows that the volume fraction of reinforcements continuously increases as a function of the step number i. This corresponds to the ROSCOE'S [29] interpretation of the DS where the filling process begins with the smallest inclusions to end up with the biggest ones. However, as it has been emphasised by HASHIN [17], this interpretation is contradictory to the requirement of infinitesimality of the volume fraction increment.

It is important to point out that the overall properties of the equivalent homogeneous material depend upon the number of steps S. Figure 1 illustrates the evolution of Young's modulus of an equivalent homogeneous material composed of 50% of an isotropic matrix, with shear modulus $G_M = 1.4$ MPa and Poisson's ratio $v_M = 0.499$, and 50% of isotropic randomly distributed spheres, with shear modulus $G_S = 30.2$ GPa and Poisson's ration $v_S = 0.160$. It is easy to determine, for this case, the value of Young's modulus using the DS solution given by BOUCHER [13] in case of spherical inclusions.

One can observe that the degree of heterogeneity of the studied material is very important. Indeed we have:

$$\frac{G_S}{G_M} \cong 21571.$$

On the other hand, Fig. 1 shows that the SCS predicts the value of Young's modulus, given for $S = 1$, which is much higher than that obtained by the DS. Moreover, the One-Site Incremental Scheme (OIS) converges very rapidly to the DS solution.

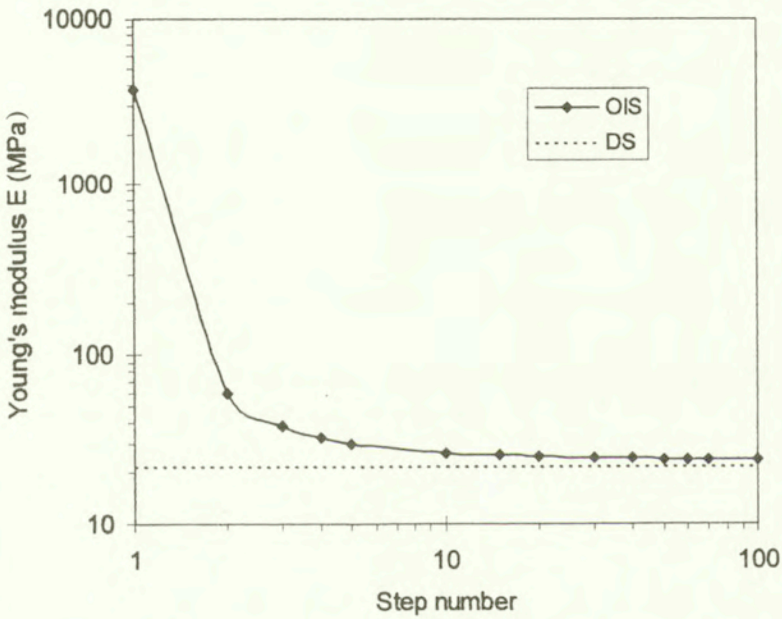

FIG. 1. Influence of the number of steps on the value of Young's modulus. Results obtained using One-Site Incremental Scheme.

4.2. Multi-phase material

Consider now the case of multiphase material (more than two phases) composed of a matrix with N families of reinforcements. To build this material we start with some volume of the matrix V^m, of the reinforcements V^{rJ} (J takes the values from 1 to N) and suppose that the resulting composite should have a volume V^c such that

$$V^m + V^{rJ} = V^c = V.$$

If f^J is the global volume fraction for the family J and S the total number of steps used to build the composite, the incremental fraction for the family J is

$$\Delta f^J = \frac{f^J}{S}. \tag{4.5}$$

After $(i-1)$ steps, the material corresponds to

$$\sum_{J=1}^{N}(i-1)\Delta f^J \cdot V^{rJ} + \left[1 - \sum_{J=1}^{N}(i-1)\Delta f^J \cdot V^{rJ}\right] \cdot V^m = V_{i-1}^c \tag{4.6}$$

where $V_{i-1}^c = V$ is the composite volume after step $i-1$. This volume becomes a matrix for the next step of the building procedure.

At step i, to preserve the total volume, we have to cut off some volume of the new matrix to introduce an unknown volume fraction of reinforcements Δf_i^J such that the following equation should be satisfied:

$$\sum_{J=1}^{N}\Delta f_i^J \cdot V^{rJ} + \left[1 - \sum_{J=1}^{N}\Delta f_i^J\right] \cdot V_{i-1}^c = i\sum_{J=1}^{N}\Delta f^J \cdot V^{rJ} + \left[1 - i\sum_{J=1}^{N}\Delta f^J\right] \cdot V^m. \tag{4.7}$$

Now, introducing (4.6) into (4.7) and comparing factors of V^m or V^r, one can determine the following system of equations for the unknown fractions Δf_i^J:

$$\begin{bmatrix} \dfrac{1}{(i-1)\Delta f^1} & .. & -1 & .. & -1 \\ . & .. & . & .. & . \\ -1 & .. & \dfrac{1}{(i-1)\Delta f^J} & .. & -1 \\ . & .. & . & .. & . \\ -1 & .. & -1 & .. & \dfrac{1}{(i-1)\Delta f^N} \end{bmatrix} \begin{Bmatrix} \Delta f_i^1 \\ . \\ . \\ \Delta f_i^J \\ . \\ . \\ \Delta f_i^N \end{Bmatrix} = \frac{1}{(i-1)} \begin{Bmatrix} 1 \\ . \\ . \\ 1 \\ . \\ . \\ 1 \end{Bmatrix}.$$

The solution of this system takes a form

$$\Delta f_i^J = \frac{\Delta f^J}{1-(i-1)\Delta f^r} \tag{4.8}$$

where

$$\Delta f^r = \sum_{J=1}^{N} \Delta f^J = \frac{\sum_{J=1}^{N} f^J}{S} = \frac{f^r}{S}, \tag{4.9}$$

and f^r is the total volume fraction of reinforcements.

It is important to emphasise that the above solution corresponds to the radial or proportional path in the volume fraction N-dimensional space. As it has been observed by NORRIS [18], this is not a unique possibility to fill the composite. Figure 2 illustrates this idea in 2D fraction space for a three-phase material. The path I corresponds to the Roscoe-Boucher scheme and to the actual proposition, and path II illustrates any arbitrary filling process. It is easy to show, see for instance NORRIS [18], that the resulting equivalent material properties are path-dependent. All the results presented in this paper have been obtained by increasing the volume fractions along path I, expression (4.8). The influence of the filling process corresponding to any other path is not discussed in this paper.

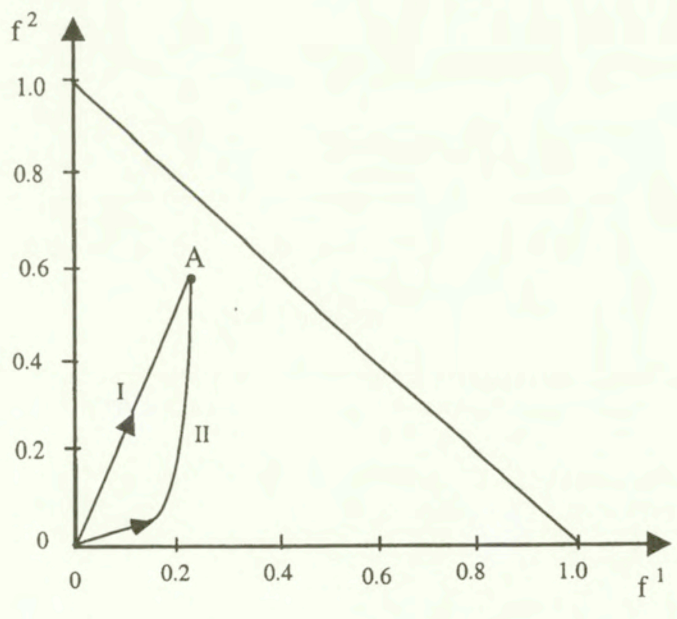

FIG. 2. Path definition in fraction space. I – actual method, II – arbitrary path.

5. Applications

In this section we present some numerical results in order to show the results of the above-described method.

The first application concerns again the epoxy-boron composite of the Sec. 3.3. We compare the transverse Young's modulus predicted by the Multi-Site Self-Consistent Scheme (MSCS) and by the Multi-Site Incremental Scheme (MIS), with experimental values from SABODH [26].

In Fig. 3 we can observe that a relatively good agreement is obtained with MIS even for concentration of fibres. MSCS model, according to Table 1, overestimates (when $f = 0.7$) the Young's modulus by a factor of almost 4.

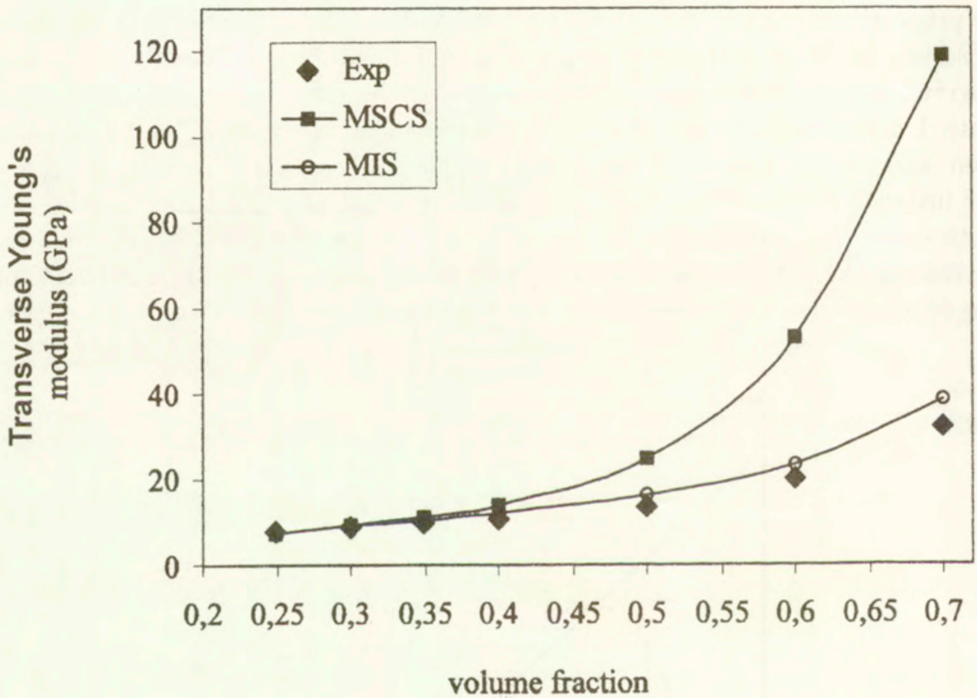

FIG. 3. Comparison of the predicted transverse Young's modulus, using MSCS and MIS with experimental data from SABODH [26].

In the second application we consider a two-phase material consisting of a rigid sphere embedded in an isotropic matrix. The elastic properties are

$$v^M = v^S = 0.3$$

and

$$\frac{E^S}{E^M} = 50,$$

where v^M and v^S are, respectively, Poisson's ratio of the matrix and sphere, E^M and E^S are their Young's moduli. The regular spatial distribution of spheres is introduced corresponding to a periodic cubic arrangement. We used the Multi-Site Self-Consistent Scheme (MSCS) and the Multi-Site Incremental Scheme (MIS) to predict the elastic properties of this composite. The calculated properties are compared with the results obtained by Finite Element Method. The commercial code ANSYS V 5.6 has been used to perform all these calculations. Figure 4 illustrates the mesh of an elementary volume from which the full composite can be constructed by symmetries and periodicity. This mesh is composed on 4600 hexahedral eight-node elements and corresponds to the volume fraction of reinforcements $f = 0.4$. The same figure shows also the obtained axial stress distribution. One can observe that this stress component distribution inside the inclusion is very complex.

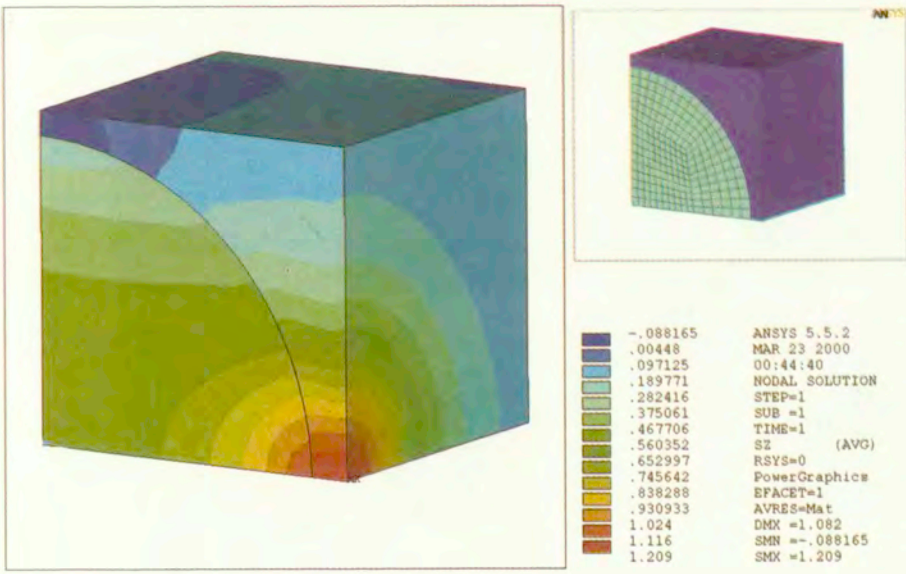

FIG. 4. Finite Element mesh and axial stress distribution.

Figure 5 shows the evolution of the Young modulus as a function of the volume fraction of spheres. A good agreement is obtained between the FEM and the MIS approach, while the MSCS strongly overestimates the value of the Young modulus for the volume fractions greater than 20%. One can observe that, despite the complex stress field inside the inclusion (Fig. 4), the MIS approach predicts very accurately the effective properties of the composite.

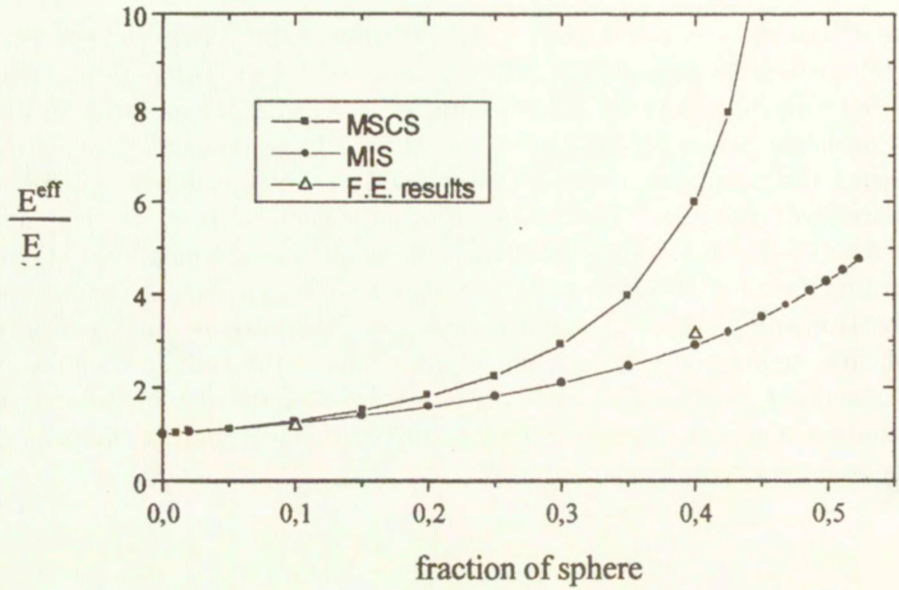

FIG. 5. Evolution of Young's modulus, using MSCS, MIS and FEM approaches, for a composite constituted of rigid spheres embedded in a matrix.

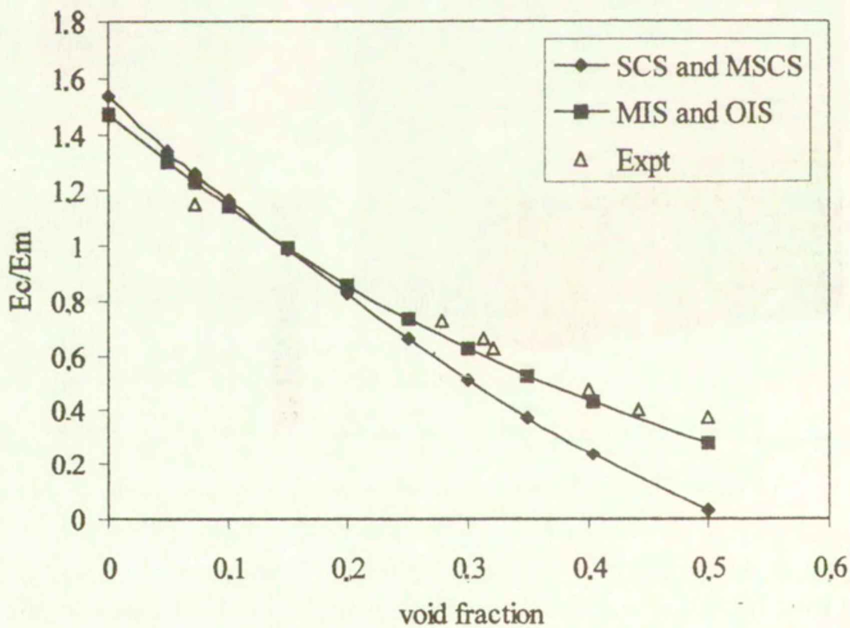

FIG. 6. Evolution (due to the SCS and IS approach) of the Young's modulus for a three-phase composite, and comparison with experimental results.

In the last example, we compare the experimental results obtained by ISHAI and COHEN [27] with the Self-Consistent Scheme (SCS) and the Incremental Scheme (IS) predictions. The material is a three-phase composite consisting of sand and voids in an epoxy matrix. We suppose that sand and voids are in a random arrangement. In this case of perfect disorder, Multi-Site (MSCS or MIS) and One-Site (SCS or OIS) give rise to similar results. HUANG *et al.* [28] showed, using the Cohen data, that the volume fractions of sand f_S and void f_V are interdependent and vary according to:

$$f_S = 0.173(1 - f_V).$$

As Fig. 6 shows, the IS predicts correctly the experimental data over the entire range of void fraction, while the SCS underestimates the results for void fractions greater than $f_V = 0.2$.

6. Conclusions

A new transition scale based model has been proposed for the prediction of effective properties of composite materials. This Multi-Site Incremental Scheme takes into account the anisotropy of the constituents as well as the morphological and the topological textures, what enables the prediction of the effective properties for general cases.

The model has been applied to characterise the elastic behaviour of two- and three-phase composites. The incremental procedure used in this approach permits to avoid the problems of Self-Consistent Scheme distortions for large volume fraction of reinforcement or void.

The comparison of the numerical results with the experimental data known from literature shows a very good agreement between the calculations and measurements. The same conclusion concerns the comparison between the predicted values and Finite Elements results.

The MIS scheme appears as a powerful alternative to the complex and time-consuming Finite Element applications for prediction of the effective properties of elastic heterogeneous materials.

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On material and geometrical instabilities in finite elasticity and elastoplasticity

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MATERIAL INSTABILITY phenomena arise in homogeneous stress states if nonlinear stress-strain relations are considered. The stability behaviour is investigated by looking at the Gateaux derivative of the first Piola-Kirchhoff stress tensor in the direction of the deformation gradient. This requires to solve a nine-dimensional matrix eigenvalue problem. In the present contribution, it is shown that material instabilities can be clearly differentiated from instabilities of geometrical character. The latter aspect is especially important for the design of new materials, since unstable solution paths under common loading conditions are not desirable. Geometrical instabilities, however, can usually be avoided by choosing appropriate boundary conditions. The derivation in this work leads to a simple stability criterion which allows to describe the stability behaviour of many materials in a very general context.

1. Introduction

MANY CLASSICAL stability investigations in elasticity concern the buckling of rods, plates and shells. A common example is an Euler column exposed to an axial force. At a certain load value, the homogeneous stress state is no longer stable which leads in the presence of imperfections to buckling of the structure into a stable inhomogeneous stress state. The treatment of such stability problems requires to consider large deformations or in other words, the geometrical nonlinearity in the system. For such so-called structural stability behaviour, the nonlinearity of the material law (physical nonlinearity) plays usually a subordinate role and is neglected in most cases. Structural instabilities occur mainly in compressive stress states.

For stability problems in finite elasticity, this is no longer the case. Here, large deformations and *nonlinear* stress-strain relations are considered. It can be expected that this increases the variety of instability phenomena noticeably. In particular, a new class of stability problems comes into play, even if the investigation is restricted completely to homogeneous stress states. Instabilities of such kind are usually due to the physical nonlinearity in the system. The question arises as to whether these so-called material instabilities are of physical origin, or only a result of the material model. Actually, one finds both cases. A physical material stability phenomenon was observed in an experiment documented by TRELOAR [41], where the symmetric deformation state of a biaxially equally loaded sheet became unstable and therefore inaccessible after a certain load level. Instead, the originally square sheet developed a rectangular form.

It is important to emphasize that the appearance of such material instabilities in finite elasticity does not mean that fundamental constitutive requirements such like polyconvexity (see BALL [2], CIARLET [10]) are violated. A good overview of constitutive inequalities is given in MARSDEN and HUGHES [22]. See for more details BAKER and ERICKSEN [1], HILL [14, 17, 18], OGDEN [26], BALL [2] and SIMPSON and SPECTOR [37, 38, 39]. One can show experimentally that these constitutive restrictions are physically reasonable for elastic materials.

This is different in elastoplasticity, where the bifurcation into shear bands is observed experimentally. The phenomenon can be explained mathematically by the loss of ellipticity of the underlying differential equation system. Thus, if one wishes to describe the behaviour of such materials realistically, the material model must be sophisticated enough to include shear banding (localization). Note that polyconvexity includes strong ellipticity such that the latter effect is excluded in finite elasticity. Such phenomena belong to another class of material instabilities which have to be additionally investigated in the context of elastoplasticity.

In many publications dealing with stability problems in finite elasticity, some special examples and material models are considered but only a few general conclusions are derived. Very common examples are for instance a biaxially loaded sheet (see e.g. SHIELD [35], OGDEN [26], KEARSLEY [20], CHEN [8], MÜLLER [24, 25], REESE [28], REESE and WRIGGERS [29]) or a triaxially loaded cube (see e.g. RIVLIN [31], SAWYERS [34], BALL and SCHAEFFER [3], REESE and WRIGGERS [30]). One main goal of the present work is to develop a stability criterion which would be general enough to reproduce the results of previous investigations in a very simple way. This criterion is based on a common aspect of the stability behaviour of elastic materials which has not been fully explored in earlier works. This is partially due to the fact that most investigations are based on the assumption of fixed principal axes which represents a major restriction and makes the complete understanding of the material stability behaviour more difficult. Exceptions are the quite formal derivations of HILL [16, 19], OGDEN [26], Sec. 6.2 and CHEN [9], who also developed general criteria. The results of HILL and OGDEN are similar to those derived in the present paper but based on a different derivation. The emphasis of the present work lies rather on the physical description of non-unique solutions. A further goal is to predict unstable solution paths and to be able to avoid them in practice. Since the present approach is not restricted to purely elastic material behaviour, it is more general than the one of OGDEN [26].

In contrast to other publications, including the ones of HILL [16, 19], OGDEN [26] and CHEN [9], in the present contribution, the classical split of the tangent stiffness into material and geometrical parts is exploited to obtain a useful classification of the instability phenomena into two groups. The second group is again subdivided into three different cases. It will be shown further, that the two main groups represent instabilities of geometrical and material character, respectively.

Although we restrict ourselves completely to homogeneous stress and deformation states, we still observe instabilities of purely geometrical character. Note that these are different from the typical structural instability phenomena (buckling), since they are associated with rigid body rotations as eigenmodes and occur usually in the natural state (undeformed configuration).

Concerning the group of material instabilities, we have to differentiate between purely elastic and elastoplastic material behaviour. In elasticity, material instabilities are characterized by the material tensor losing its positive definiteness. It is important to emphasize that these singularities represent *multiple* bifurcations. Such deformation states become unstable under arbitrary linear combinations of stretch and shear modes.

In elastoplasticity, material instabilities might arise in tension as well as in compression which is due to the fact that they usually appear in the form of shear bands. This aspect will be discussed in detail.

The paper is organized as follows. Based on the balance of linear momentum, the eigenvalue problem for the Gateaux derivative of the first Piola-Kirchhoff stress tensor in the direction of the deformation gradient is formulated. Using frame indifference, the latter fourth order tensor can be split into material and geometrical parts. In Sec. 3, this result is used to classify the instabilities into two main groups. In certain cases, in particular for isotropic elasticity and associated elastoplasticity, we can reformulate the eigenvalue problem for the tangent tensor in such a way that a decoupled structure is obtained. The originally nine-dimensional eigenvalue problem then reduces to one three-dimensional and three two-dimensional sub-problems (Sec. 4). Finally (Sec. 5), the solution of these eigenvalue problems is discussed in a very general context, i.e. neither a special material model nor certain deformation states are specified. In this way, generally applicable stability criteria are derived. In Sec. 6, the use of these criteria is validated by means of several examples.

2. Preliminary remarks

The derivation starts from the balance of linear momentum

$$\operatorname{Div} \mathbf{P} = \mathbf{0}, \tag{2.1}$$

where the absence of volumetric forces (e.g. due to gravitation or inertia) has been exploited and $\mathbf{P}$ denotes the first Piola-Kirchhoff stress tensor. Using the assumption that the strain and stress states are homogeneous, the integration of (2.1) leads to

$$\tilde{\mathbf{g}}\,(\mathbf{F}) = \tilde{\mathbf{P}}\,(\mathbf{F}) - \mathbf{P}_L = \mathbf{0}. \tag{2.2}$$

Here, $\mathbf{P}_L$ denotes the first Piola-Kirchhoff stress tensor known either from the tractions $\mathbf{T}_L = \mathbf{P}_L \cdot \mathbf{N}$ prescribed on the boundary $\partial \mathcal{B}_T$ or from the deformation given on $\partial \mathcal{B}_u$. The whole boundary of the reference volume $\mathcal{B}_0$ is given by $\partial \mathcal{B}_0 = \partial \mathcal{B}_T \cup \partial \mathcal{B}_u$. The tensor $\mathbf{F}$ represents the material deformation gradient. In the following, a colon will stand for the scalar product of two tensors, whereas one dot between two tensors characterizes the usual tensor multiplication. For simplicity, we start here with the special case of finite elasticity. Thus, we need to consider here only the dependence on $\mathbf{F}$. In order to account for the inelastic material behaviour, internal variables will be introduced. The extension of the derivation will be discussed in Sec. 4.2.

Due to the restriction to hyperelastic material behaviour, the stress tensor

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} \tag{2.3}$$

can be derived from a potential W, the so-called strain energy function per unit reference volume.

From the implicit function theorem (see HILDEBRANDT and GRAVES [13]) we obtain the result that the solution of (2.2) is regular, if the tensor $\dfrac{\partial \mathbf{g}}{\partial \mathbf{F}}$ is invertible, and singular otherwise. In the latter case, one can find a non-zero tensor $\Delta\mathbf{F}$ which fulfills the relation

$$\Delta \mathbf{g} := D\tilde{\mathbf{g}}\,(\mathbf{F}) : \Delta \mathbf{F} = \frac{d}{d\alpha}\left(\hat{\mathbf{g}}(\mathbf{F} + \alpha\,\Delta\mathbf{F})\right)\Big|_{\alpha=0} = \frac{\partial \mathbf{g}}{\partial \mathbf{F}} : \Delta\mathbf{F} = \mathbf{0}. \tag{2.4}$$

The tensor $D\tilde{\mathbf{g}}\,(\mathbf{F}) : \Delta\mathbf{F}$ represents the so-called Gateaux derivative of $\tilde{\mathbf{g}}\,(\mathbf{F})$ in the direction $\Delta\mathbf{F}$, where $\Delta\mathbf{F}$ is identical with the eigentensor $\tilde{\mathbf{\Phi}}$ of the eigenvalue problem

$$\text{EIG:} \qquad (\mathcal{A} - \omega\,\mathbf{1}^4) : \tilde{\mathbf{\Phi}} = \mathbf{0},$$

if the eigenvalue ω vanishes. See in this context also BEATTY [6, 7] and the text-book of MARSDEN and HUGHES [22]. Note that the fourth order tensor

$$\mathcal{A} := \frac{\partial \mathbf{g}}{\partial \mathbf{F}} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}} = \frac{\partial^2 \tilde{W}(\mathbf{F})}{\partial \mathbf{F}^2} \tag{2.5}$$

denotes the derivative of the first Piola-Kirchhoff stress tensor $\mathbf{P}$ with respect to the deformation gradient $\mathbf{F}$. It can be stated that for asymmetrical bifurcations, symmetrical bifurcations and limit points, a singular solution always indicates the beginning of an unstable solution path. Thus, to check the stability behaviour on the "primary" solution path, the detection of singularities is sufficient.

Since the constitutive equations are required to be frame-indifferent (see e.g. TRUESDELL and NOLL [42]), the function $\tilde{\mathbf{P}}(\mathbf{F})$ reduces further to

$$\mathbf{P} = \mathbf{F} \cdot \breve{\mathbf{S}}(\mathbf{E}) = \mathbf{F} \cdot \frac{\partial \breve{W}(\mathbf{E})}{\partial \mathbf{E}}, \tag{2.6}$$

where $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1})$ represents the Green-Lagrange strain tensor. This fact has not been used to derive $\mathcal{A}$ and is therefore not included in (EIG). Using (2.6), we may rewrite the expression $\Delta\mathbf{g}$ as

$$\Delta\mathbf{g} = \mathcal{A} : \Delta\mathbf{F} = \Delta\mathbf{F} \cdot \breve{\mathbf{S}}(\mathbf{E}) + \mathbf{F} \cdot (D\breve{\mathbf{S}}(\mathbf{E}) : \Delta\mathbf{E}) = \Delta\mathbf{F} \cdot \breve{\mathbf{S}}(\mathbf{E}) + \mathbf{F} \cdot \left(\frac{\partial \mathbf{S}}{\partial \mathbf{E}} : \Delta\mathbf{E}\right), \tag{2.7}$$

where $\Delta\mathbf{E}$ is defined by the Gateaux derivative

$$\Delta\mathbf{E} := D\tilde{\mathbf{E}}(\mathbf{F}) : \Delta\mathbf{F} = \frac{\partial \mathbf{E}}{\partial \mathbf{F}} : \Delta\mathbf{F} = \operatorname{sym}(\mathbf{F}^T \cdot \Delta\mathbf{F}). \tag{2.8}$$

The fourth order tensor

$$\mathcal{L} := \frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \frac{\partial^2 \breve{W}(\mathbf{E})}{\partial \mathbf{E}^2} \tag{2.9}$$

is termed a material tensor. If we replace $\Delta\mathbf{F}$ by the eigentensor $\tilde{\mathbf{\Phi}}$ and carry out the scalar multiplication of (2.7) with $\tilde{\mathbf{\Phi}}$ we obtain in case of a singular solution

$$\boxed{\tilde{\mathbf{\Phi}} : \mathcal{A} : \tilde{\mathbf{\Phi}} = \operatorname{sym}(\mathbf{F}^T \cdot \tilde{\mathbf{\Phi}}) : \mathcal{L} : \operatorname{sym}(\mathbf{F}^T \cdot \tilde{\mathbf{\Phi}}) + \mathbf{S} : (\tilde{\mathbf{\Phi}}^T \cdot \tilde{\mathbf{\Phi}}) = 0.} \tag{2.10}$$

One could derive the result (2.10) also on the basis of the relations (6.2.76) and (6.1.22) stated in OGDEN [26]. A similar statement is found in HILL [19] (Eq. (3.38 a)), where one has to replace the spatial velocity gradient $\frac{\partial v_i}{\partial x_j}$ by $\tilde{\mathbf{\Phi}} \cdot \mathbf{F}^{-1}$. Neither OGDEN nor HILL, however, exploited the information contained in (2.10) any further. This is now done in the following section.

3. Classification of singular solutions

Using index notation, it can be easily shown that the term $\mathbf{S} : (\tilde{\mathbf{\Phi}}^T \cdot \tilde{\mathbf{\Phi}}) = S_{ij}\,\tilde{\Phi}_{ki}\,\tilde{\Phi}_{kj}$ is alternatively represented by means of

$$S_{ij}\,\tilde{\Phi}_{ki}\,\tilde{\Phi}_{kj} = \tilde{\Phi}_{ij}\,\delta_{ik}\,S_{jl}\,\tilde{\Phi}_{kl} := \tilde{\mathbf{\Phi}} : \mathcal{M} : \tilde{\mathbf{\Phi}}, \tag{3.1}$$

where the fourth order tensor $\mathcal{M}$ is computed from

$$\mathcal{M} = \delta_{ik}\,S_{jl}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l. \tag{3.2}$$

The vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$ represent Cartesian basis vectors and the Einstein summation convention is assumed to hold for lower case indices. The Eq. (2.10) then reads

$$\boxed{\tilde{\mathbf{\Phi}} : \mathcal{A} : \tilde{\mathbf{\Phi}} = \underbrace{\mathrm{sym}\,(\mathbf{F}^T \cdot \tilde{\mathbf{\Phi}}) : \mathcal{L} : \mathrm{sym}\,(\mathbf{F}^T \cdot \tilde{\mathbf{\Phi}})}_{:=\,\omega_{\mathcal{L}}} + \underbrace{\tilde{\mathbf{\Phi}} : \mathcal{M} : \tilde{\mathbf{\Phi}}}_{:=\,\omega_{\mathcal{M}}} = 0.} \tag{3.3}$$

The scalar factors $\omega_{\mathcal{L}}$ and $\omega_{\mathcal{M}}$ represent the eigenvalues in the eigenvalue problems

$$(\mathcal{L} - \omega_{\mathcal{L}}\,\mathbf{1}^4_{\mathrm{sym}}) : \underbrace{\mathrm{sym}\,(\mathbf{F}^T \cdot \tilde{\mathbf{\Phi}})}_{:=\,\breve{\mathbf{\Phi}}} = \mathbf{0} \quad \text{and} \quad (\mathcal{M} - \omega_{\mathcal{M}}\,\mathbf{1}^4) : \tilde{\mathbf{\Phi}} = \mathbf{0}, \tag{3.4}$$

respectively.

Consider now a given deformation state $\mathbf{F} = \bar{\mathbf{F}}$ and assume that the tensor $\tilde{\mathcal{A}}\,(\bar{\mathbf{F}})$ has at least one vanishing eigenvalue. Let us further introduce the index $\star$ for all eigentensors of $\tilde{\mathcal{A}}\,(\bar{\mathbf{F}})$ which fulfill the equation $\tilde{\mathcal{A}}\,(\bar{\mathbf{F}}) : \tilde{\mathbf{\Phi}} = \mathbf{0}$. Then, for each eigentensor $\tilde{\mathbf{\Phi}}^{\star}$, (only) one of the following two statements is true.

(1) The eigentensor $\tilde{\mathbf{\Phi}}^{\star}$ satisfies the equation

$$\breve{\mathbf{\Phi}}^{\star} := \mathrm{sym}\,(\bar{\mathbf{F}}^T \cdot \tilde{\mathbf{\Phi}}^{\star}) = \mathbf{0}. \tag{3.5}$$

From (3.3), we obtain the relation

$$\tilde{\mathbf{\Phi}}^{\star} : \tilde{\mathcal{A}}\,(\bar{\mathbf{F}}) : \tilde{\mathbf{\Phi}}^{\star} = \tilde{\mathbf{\Phi}}^{\star} : \tilde{\mathcal{M}}\,(\bar{\mathbf{F}}) : \tilde{\mathbf{\Phi}}^{\star} = \omega_{\mathcal{M}} = 0. \tag{3.6}$$

Thus, $\tilde{\mathcal{M}}\,(\bar{\mathbf{F}})$ has one zero eigenvalue associated with the eigentensor $\tilde{\mathbf{\Phi}}^{\star}$.

(2) The eigentensor $\tilde{\mathbf{\Phi}}^{\star}$ does *not* satisfy (3.5). Let $\bar{\mathbf{E}}$ be defined by $\bar{\mathbf{E}} := \frac{1}{2}\,(\bar{\mathbf{F}}^T \cdot \bar{\mathbf{F}} - \mathbf{1})$.

(a) $\omega_{\mathcal{M}} = 0 \;\Rightarrow\; \omega_{\mathcal{L}} = 0$
The eigenvalue of $\breve{\mathcal{L}}\,(\bar{\mathbf{E}})$ associated with the eigentensor $\breve{\mathbf{\Phi}}^{\star}$ vanishes.

(b) $\omega_{\mathcal{M}} < 0 \;\Rightarrow\; \omega_{\mathcal{L}} > 0$

The eigenvalue of $\breve{\mathcal{L}}(\bar{\mathbf{E}})$ associated with the eigentensor $\breve{\mathbf{\Phi}}^{\star}$ is positive.

(c) $\omega_{\mathcal{M}} > 0 \Rightarrow \omega_{\mathcal{L}} < 0$
The eigenvalue of $\breve{\mathcal{L}}(\bar{\mathbf{E}})$ associated with the eigentensor $\breve{\mathbf{\Phi}}^{\star}$ is negative.

It remains to discuss what these cases mean physically. It is well-known that the split of the tangent matrix

$$A_{ijkl} = F_{im}\, L_{mjnl}\, F_{kn} + M_{ijkl}. \tag{3.7}$$

$(M_{ijkl} = \delta_{ik}\, S_{jl})$ represents a split into a material and a geometrical part. The latter equation is also given by HILL [19] and OGDEN [26] (Sec. 6.1.2).

In *Case 1*, the term $\breve{\mathbf{\Phi}} : \mathcal{L} : \breve{\mathbf{\Phi}}$, i.e. the material part of (3.3), vanishes. A singularity of this type is not related to the choice of the material model and can be considered therefore to be of purely geometrical character. The reversed argument allows the following statement. If $\mathcal{M}$ has a zero eigenvalue and the eigentensor accociated with the vanishing eigenvalue fulfills (3.5), the tensor $\mathcal{A}$ must be singular, too. It will become clear later that this case becomes relevant only in the natural state $(\mathbf{S} = \mathbf{0})$.

Usually, however, *Case 2* is detected, where (apart from 2a) either the first or the second term in (3.3) has a negative sign. The first class is characterized by $\mathcal{L}$ losing its positive definiteness (*Case 2c*). The second type arises, if $\mathcal{M}$ has at least one negative eigenvalue with respect to the eigenform $\tilde{\mathbf{\Phi}}$ (*Case 2b*).

To conclude, *Case 1* represents a geometrical instability, whereas *Case 2c* characterizes in any case a material instability. *Case 2a* would be both, but it does not appear in the context of physically reasonable models. *Case 2b* becomes relevant, if negative stresses dominate. For later use, we exploit the polar decomposition of $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ to rewrite (3.7) as

$$A_{ijkl} = R_{ix}\, \underbrace{(U_{xm}\, L_{mjnl}\, U_{yn} + \delta_{xy}\, S_{jl})}_{E^{1}_{xjyl}}\, R_{ky} \tag{3.8}$$

leading to

$$\tilde{\mathbf{\Phi}} : \mathcal{A} : \tilde{\mathbf{\Phi}} = (\mathbf{R}^{T} \cdot \tilde{\mathbf{\Phi}}) : \mathcal{E}^{1} : (\mathbf{R}^{T} \cdot \tilde{\mathbf{\Phi}}). \tag{3.9}$$

In a similar way, the push-forwards $C_{ixky} = F_{im}\, F_{xj}\, F_{km}\, F_{yl}\, L_{mjnl}$ and $\tau_{xy} = F_{xj}\, S_{jl}\, F_{yl}$ ($\boldsymbol{\tau}$ Kirchhoff stress tensor) are used to reformulate (3.7) as

$$A_{ijkl} = (F^{-1})_{jx}\, \underbrace{(C_{ixky} + \delta_{ik}\, \tau_{xy})}_{E^{2}_{ixky}}\, (F^{-1})_{ly} \tag{3.10}$$

which yields

$$\tilde{\mathbf{\Phi}} : \mathcal{A} : \tilde{\mathbf{\Phi}} = (\tilde{\mathbf{\Phi}} \cdot \mathbf{F}^{-1}) : \mathcal{E}^2 : (\tilde{\mathbf{\Phi}} \cdot \mathbf{F}^{-1}). \tag{3.11}$$

If one of the three tensors, $\mathcal{A}$, $\mathcal{E}^1$ or $\mathcal{E}^2$, is singular, the same holds for the other two. The eigentensors of $\mathcal{E}^1$ and $\mathcal{E}^2$ associated with these singular solutions are, however, different. The eigentensors $\mathbf{\Phi}^1$ of $\mathcal{E}^1$ are obtained from the ones of $\mathcal{A}$ by multiplying $\tilde{\mathbf{\Phi}}$ by $\mathbf{R}^T$ from the left. Analogously, we compute the eigentensors $\mathbf{\Phi}^2$ of $\mathcal{E}^2$ by multiplying $\tilde{\mathbf{\Phi}}$ by $\mathbf{F}^{-1}$ from the right. The L_2-norm $||\mathbf{\Phi}^1||$ is set equal to $||\tilde{\mathbf{\Phi}}||$. On the other hand, we require the relation

$$||\mathbf{\Phi}^2|| = \sqrt{\mathbf{\Phi}^2 : \mathbf{\Phi}^2} = \sqrt{\mathrm{tr}\,(\tilde{\mathbf{\Phi}} \cdot \mathbf{C}^{-1} \cdot \tilde{\mathbf{\Phi}})} \tag{3.12}$$

to hold for $\tilde{\mathbf{\Phi}}^2$. Both tensors, $\mathcal{E}^1$ and $\mathbf{\Phi}^1$ live completely in the reference configuration, whereas $\mathcal{A}$ and $\tilde{\mathbf{\Phi}}$ are two-field tensors. $\mathcal{E}^2$ and $\mathbf{\Phi}^2$ live in the current configuration.

It will be shown later that it is convenient to work with $\mathcal{E}^1$ or $\mathcal{E}^2$ in some cases. The stability investigation can then be carried out in the following way:

- Solution of the eigenvalue problem

$$\mathsf{EIG}^1: \qquad (\mathcal{E}^1 - \omega\, \mathbf{1}^4) : \mathbf{\Phi}^1 = \mathbf{0}$$

  or

$$\mathsf{EIG}^2: \qquad (\mathcal{E}^2 - \omega\, \mathbf{1}^4) : \mathbf{\Phi}^2 = \mathbf{0},$$

  respectively.

- In case of a singular solution → case differentiation:

Case 1: The symmetric part of $\mathbf{U}^T \cdot \mathbf{\Phi}^1$ or $\mathbf{\Phi}^2$, respectively, is equal to the zero tensor.

Case 2: The symmetric part of $\mathbf{U}^T \cdot \mathbf{\Phi}^1$ or $\mathbf{\Phi}^2$, respectively, is **not** equal to the zero tensor. The sign of

$$\omega_{\mathcal{M}} = \Phi^1_{xj}\, \delta_{xy}\, S_{jl}\, \Phi^1_{yl} = \Phi^2_{ix}\, \delta_{ik}\, \tau_{xy}\, \Phi^2_{ky} \tag{3.13}$$

determines whether we deal with *Case 2a*, *Case 2b* or *Case 2c*.

4. Stability investigation

4.1. Finite elasticity

Up to this point, the derivation is general enough to include anisotropic material behaviour. The goal of the following derivation is to show the use of the stability criterion derived in Sec. 2. For simplicity, we restrict ourselves here to transverse isotropy. Transversely isotropic material behaviour is for instance observed in fiber-reinforced materials, where the material properties in the fiber direction are different from those in the plane perpendicular to the fiber. In this plane the material behaviour is assumed to be isotropic. The considerations carried out in the context of transverse isotropy would hold analogously for general anisotropy.

According to the theoretical works of BOEHLER [4, 5], the strain energy function $\breve{W}(\mathbf{E})$ reduces to an isotropic function of $\mathbf{E}$ and the so-called structural tensor $\mathbf{M} = \mathbf{n} \otimes \mathbf{n}$. The normed vector $\mathbf{n}$ is oriented parallel to the fibers. It is then straightforward to show that the potential W is a function of the three invariants

$$I_1 := \operatorname{tr} \mathbf{E}, \quad I_2 := \frac{1}{2} \left(I_1^2 - \operatorname{tr}(\mathbf{E}^2) \right), \quad I_3 := \det \mathbf{E} \tag{4.1}$$

of $\mathbf{E}$ and the first invariants of $\mathbf{E} \cdot \mathbf{M}$ and $\mathbf{E}^2 \cdot \mathbf{M}$, respectively:

$$I_4 := \operatorname{tr}(\mathbf{E} \cdot \mathbf{M}) = \mathbf{E} : \mathbf{M}, \qquad I_5 := \operatorname{tr}(\mathbf{E}^2 \cdot \mathbf{M}) = \mathbf{E}^2 : \mathbf{M}. \tag{4.2}$$

In general, $\mathbf{E}$ and $\mathbf{M}$ are not coaxial such that the 6 x 6-matrix obtained from writing $\mathcal{L}$ in the Voigt notation would be a full matrix. If, however, coaxiality of $\mathbf{S}$ with $\mathbf{E}$ or $\mathbf{U}$ is assumed, we may write the nine-dimensional matrix representation of $\mathcal{E}^1$ as

$$[\mathcal{E}^1] = \begin{bmatrix} [\mathcal{E}^1]^{\text{stretch}} & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathcal{E}^1]^{\text{shear(12)}} & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathcal{E}^1]^{\text{shear(23)}} & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathcal{E}^1]^{\text{shear(31)}} \end{bmatrix}, \tag{4.3}$$

where the sub-matrices $[\mathcal{E}^1]^{\text{stretch}}$ and $[\mathcal{E}^1]^{\text{shear(12)}}$ take the form

$$[\mathcal{E}^1]^{\text{stretch}} = \begin{bmatrix} \lambda_1^2 L_{1111} & \lambda_1 \lambda_2 L_{1122} & \lambda_1 \lambda_3 L_{1133} \\ \lambda_2 \lambda_1 L_{1122} & \lambda_2^2 L_{2222} & \lambda_2 \lambda_3 L_{2233} \\ \lambda_3 \lambda_1 L_{1133} & \lambda_3 \lambda_2 L_{2233} & \lambda_3^2 L_{3333} \end{bmatrix} + \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & S_{22} & 0 \\ 0 & 0 & S_{33} \end{bmatrix} \tag{4.4}$$

and

$$[\mathcal{E}^1]^{\text{shear(12)}} = \begin{bmatrix} \lambda_1^2 L_{1212} & \lambda_1 \lambda_2 L_{1212} \\ \lambda_2 \lambda_1 L_{1212} & \lambda_2^2 L_{1212} \end{bmatrix} + \begin{bmatrix} S_{22} & 0 \\ 0 & S_{11} \end{bmatrix}, \tag{4.5}$$

respectively. The matrices $[\mathcal{E}^1]^{\text{shear}(23)}$ and $[\mathcal{E}^1]^{\text{shear}(31)}$ are constructed analogously. The coefficients refer to the axes defined by $\mathbf{n}_I$ ($I = 1,2,3$, $\mathbf{n}_I \cdot \mathbf{n}_J = \delta_{IJ}$) with $\mathbf{n}_1 = \mathbf{n}$. The quantities λ_I ($I = 1,2,3$) are given by

$$\lambda_I := U_{II} = \sqrt{2\,E_{II} + 1}, \quad U_{IJ} = 0 \text{ if } I \neq J, \tag{4.6}$$

where U_{II} represent the principal values of the right stretch tensor $\mathbf{U}$.

Due to the special structure of $[\mathcal{E}^1]$, the nine-dimensional eigenvalue problem (EIG^1) can be subdivided into four smaller ones: one for the "stretch matrix" $[\mathcal{E}^1]^{\text{stretch}}$ and three others for the "shear matrices" $[\mathcal{E}^1]^{\text{shear}(12)}$, $[\mathcal{E}^1]^{\text{shear}(23)}$ and $[\mathcal{E}^1]^{\text{shear}(31)}$. This decoupled structure makes an analytical approach possible. Moreover, due to the fact that the latter two matrices are derived from $[\mathcal{E}^1]^{\text{shear}(12)}$ by merely exchanging the indices, we have to solve the two-dimensional eigenvalue problem only once. In isotropic elasticity, the coaxiality of $\mathbf{S}$ and $\mathbf{E}$ is always fulfilled such that the decoupled form of $[\mathcal{E}^1]$ is possible in general.

Many previous works (e.g. BALL and SCHAEFFER [3]) use the assumption that the principal axes do not rotate and can be chosen equal to the fixed Cartesian coordinate axes. In this special case, the left and the right stretch tensor, $\mathbf{V}$ and $\mathbf{U}$, have the same principal axes, and $\mathbf{R}$ is equal to the identity tensor. Then, also the matrix $[\mathcal{A}]$ would be sparse. Such an assumption, however, is here not necessary, since the coaxiality of $\mathbf{S}$ and $\mathbf{E}$ is already sufficient to obtain an analytically tractable form of $\mathcal{E}^1$.

An alternative approach has been given by OGDEN [26], who wrote the expression $\Delta\mathbf{g} = \Delta\tilde{\mathbf{P}}(\mathbf{F}) = \mathbf{0}$ (see the relation (2.4)) in the form

$$\Delta\tilde{\mathbf{P}}(\mathbf{F}) = \Delta(\mathbf{R}\cdot\mathbf{U}\cdot\check{\mathbf{S}}(\mathbf{E})) = \Delta(\mathbf{R}\cdot\check{\mathbf{T}}(\mathbf{U})) = \Delta\mathbf{R}\cdot\mathbf{T} + \mathbf{R}\cdot\Delta\mathbf{T} = \mathbf{0}. \tag{4.7}$$

The quantity $\mathbf{T}$ represents the so-called Biot stress tensor. From the latter equation one obtains the statement

$$\Delta\mathbf{T} = -\mathbf{R}^T\cdot\Delta\mathbf{R}\cdot\mathbf{T}. \tag{4.8}$$

The Biot stress tensor $\mathbf{T}$ is symmetric, if the coaxiality of $\mathbf{S}$ and $\mathbf{U}$ holds.

Since the coaxiality of $\mathbf{S}$ and $\mathbf{U}$ implies the coaxiality of $\boldsymbol{\tau}$ and $\mathbf{V}$, $[\mathcal{E}^2]$ has the same decoupled form. The sub-matrices then read

$$[\mathcal{E}^2]^{\text{stretch}} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} \\ C_{1122} & C_{2222} & C_{2233} \\ C_{1133} & C_{2233} & C_{3333} \end{bmatrix} + \begin{bmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{bmatrix} \tag{4.9}$$

and

$$[\mathcal{E}^2]^{\text{shear}(12)} = \begin{bmatrix} C_{1212} & C_{1212} \\ C_{1212} & C_{1212} \end{bmatrix} + \begin{bmatrix} \tau_{22} & 0 \\ 0 & \tau_{11} \end{bmatrix}. \tag{4.10}$$

4.2. Finite isotropic elastoplasticity

In the case of finite isotropic elastoplasticity with isotropic hardening, we start from the Helmholtz free energy

$$\Psi = W(\mathbf{C}_e) + f(\xi) = W(I_1^{\mathbf{C}_e}, I_2^{\mathbf{C}_e}, I_3^{\mathbf{C}_e}) + f(\xi) \tag{4.11}$$

where $\mathbf{C}_e$ is defined by $\mathbf{C}_e = \mathbf{F}_p^{-T} \cdot \mathbf{C} \cdot \mathbf{F}_p^{-1} = \mathbf{F}_e^T \cdot \mathbf{F}_e$ and $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = 2\,\mathbf{E} + \mathbf{1}$ denotes the right Cauchy-Green tensor. Note that here the multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p$ into elastic and plastic parts has been exploited. Since the present model is restricted to isotropy, the function W depends only on the invariants of $\mathbf{C}_e$ which are identical to the invariants of $\mathbf{C} \cdot \mathbf{C}_p^{-1}$ ($\mathbf{C}_p = \mathbf{F}_p^T \cdot \mathbf{F}_p$ "plastic" right Cauchy-Green tensor). Then, besides the accumulated plastic strain ξ, the tensor $\mathbf{C}_p$ plays the role of an internal variable. We further define the yield function $\Phi(\boldsymbol{\tau}, q)$ which is formulated in terms of the invariants of the Kirchhoff stress tensor $\boldsymbol{\tau} = 2\,\dfrac{\partial W}{\partial \mathbf{b}_e} \cdot \mathbf{b}_e$ and the stress-like quantity $q = \dfrac{\partial f(\xi)}{\partial \xi}$. Postulating the evolution equations ($\mathbf{b}_e = \mathbf{F} \cdot \mathbf{C}_p^{-1} \cdot \mathbf{F}^T$)

$$\mathbf{d}_p := \frac{1}{2}\,\mathbf{F}^{-T} \cdot \dot{\mathbf{C}}_p \cdot \mathbf{F}^{-1} = \dot{\gamma}\,\mathbf{b}_e^{-1} \cdot \frac{\partial \Phi}{\partial \boldsymbol{\tau}} \quad \text{and} \quad \dot{\xi} = \dot{\gamma}\,\frac{\partial \Phi}{\partial q} \tag{4.12}$$

and using the Kuhn-Tucker conditions $\dot{\gamma} \leq 0$, $\Phi \leq 0$ and $\dot{\gamma}\,\Phi = 0$, together with the consistency condition $\dot{\Phi}$ yields the plastic multiplier $\dot{\gamma}$ as

$$\dot{\gamma} = k\,(2\,\frac{\partial \Phi}{\partial \mathbf{b}_e} \cdot \mathbf{b}_e) : \mathbf{d}, \tag{4.13}$$

where $k = k(\mathbf{b}_e, \xi)$ represents an isotropic function of its two arguments, and the partial derivative of Φ with respect to $\mathbf{b}_e$ is given via

$$\frac{\partial \Phi}{\partial \mathbf{b}_e} = \frac{\partial \Phi}{\partial \boldsymbol{\tau}} : \frac{\partial \boldsymbol{\tau}}{\partial \mathbf{b}_e}. \tag{4.14}$$

The deformation rate tensor $\mathbf{d}$ is computed from $\mathbf{d} = \dfrac{1}{2}\,\mathbf{F}^{-T} \cdot \dot{\mathbf{C}} \cdot \mathbf{F}^{-1}$. If we use the fact that the material behaviour is rate-independent, the time derivative $(...)^{\cdot}$ can be replaced by the derivative with respect to some arbitrary parameter s. In the following, we choose the notation

$$\frac{d\,(...)}{ds} := \Delta\,(...) \quad \text{and} \quad \mathbf{d}_\Delta := \frac{1}{2}\,\mathbf{F}^{-T} \cdot \Delta\mathbf{C} \cdot \mathbf{F}^{-1}, \tag{4.15}$$

$$\mathbf{d}_{p\,\Delta} := \frac{1}{2}\,\mathbf{F}^{-T} \cdot \Delta\mathbf{C}_p \cdot \mathbf{F}^{-1}.$$

The evolution Eq. $(4.12)_1$ can finally be reformulated in the form

$$\mathbf{d}_{p\,\Delta} = k\,\left(\mathbf{b}_e^{-1} \cdot \frac{\partial \Phi}{\partial \boldsymbol{\tau}}\right) \otimes \left(2\,\frac{\partial \Phi}{\partial \mathbf{b}_e} \cdot \mathbf{b}_e\right) : \mathbf{d}_\Delta = \mathcal{G} : \mathbf{d}_\Delta. \tag{4.16}$$

Note that the present model is based on a *hyperelastic* stress relation (see also LUBLINER [21] and SIMO and HUGHES [36]), whereas the model discussed by HILL [16, 19] is formulated in rate form (*hypoelastic* stress relation).

The purpose of the present section is to show that the considerations made in Sec. 2 can be easily extended to a model of isotropic elastoplasticity as given in the form discussed above. In this context, (2.6) is rewritten as

$$\mathbf{P} = \mathbf{F} \cdot \frac{\partial W(I_1^{\mathbf{C}_e}, I_2^{\mathbf{C}_e}, I_3^{\mathbf{C}_e})}{\partial \mathbf{C}} = \mathbf{F} \cdot \mathbf{S}\,(\mathbf{C}, \mathbf{C}_p), \tag{4.17}$$

where $\mathbf{S}$ is represented by means of the function

$$\mathbf{S}\,(\mathbf{C}, \mathbf{C}_p) = \alpha_1\, \mathbf{C}^{-1} + \alpha_2\, \mathbf{C}_p^{-1} + \alpha_3\, \mathbf{C}_p^{-1} \cdot \mathbf{C} \cdot \mathbf{C}_p^{-1}. \tag{4.18}$$

The scalar factors α_i $(i = 1, 2, 3)$ are given as functions of the invariants of $\mathbf{C} \cdot \mathbf{C}_p^{-1}$. It is not difficult to see that the statements of the Secs. 2 and 3 concern the more general case of elastoplasticity, if the increment $\Delta\mathbf{S}$ is computed via

$$\Delta\mathbf{S} = \Bigg(\underbrace{\frac{\partial \mathbf{S}}{\partial \mathbf{E}}}_{\mathcal{L}^{\mathrm{el}}} + 2\,\frac{\partial \mathbf{S}}{\partial \mathbf{C}_p} : \frac{\partial \mathbf{C}_p}{\partial \mathbf{C}}\Bigg) : \Delta\mathbf{E} = \mathcal{L}\,(\mathbf{C}, \mathbf{C}_p, \xi) : \Delta\mathbf{E} \tag{4.19}$$

and the material tensor in (2.9) is replaced by the tensor $\mathcal{L}$ of the latter relation. Obviously, unlike in finite elasticity, $\mathcal{L}$ does not necessarily possess the symmetry property $L_{ijkl} = L_{klij}$. We will derive the fourth-order tensor $\mathcal{C}$ which has the same symmetry properties as $\mathcal{L}$ and is given by

$$\mathbf{F} \cdot \Delta\mathbf{S} \cdot \mathbf{F}^T = \mathcal{C} : \mathbf{d}_\Delta. \tag{4.20}$$

Using (4.18), the increment $\mathbf{F} \cdot \Delta\mathbf{S} \cdot \mathbf{F}^T$ is determined by

$$\begin{aligned} \mathbf{F} \cdot \Delta\mathbf{S} \cdot \mathbf{F}^T = {} & \Delta\alpha_1\, \mathbf{1} + \Delta\alpha_2\, \mathbf{b}_e + \Delta\alpha_3\, \mathbf{b}_e^2 \\ & -2\,\alpha_1\, \mathbf{d}_\Delta + 2\,\alpha_3\, \mathbf{b}_e \cdot \mathbf{d}_\Delta \cdot \mathbf{b}_e - 2\,\alpha_2\, \mathbf{b}_e \cdot \mathbf{d}_{p\,\Delta} \cdot \mathbf{b}_e \\ & -2\,\alpha_3\,(\mathbf{b}_e \cdot \mathbf{d}_{p\,\Delta} \cdot \mathbf{b}_e^2 + \mathbf{b}_e^2 \cdot \mathbf{d}_{p\,\Delta} \cdot \mathbf{b}_e). \end{aligned} \tag{4.21}$$

After a longer derivation, we obtain $\mathcal{C}$ as

$$\begin{aligned} \mathcal{C} = {} & \mathbf{1} \otimes \mathbf{B}_1 - \mathbf{1} \otimes (\mathbf{b}_e \cdot \mathbf{B}_1) : \mathcal{G} + \mathbf{b}_e \otimes \mathbf{B}_2 - \mathbf{b}_e \otimes (\mathbf{b}_e \cdot \mathbf{B}_2) : \mathcal{G} \\ & + \mathbf{b}_e^2 \otimes \mathbf{B}_3 - \mathbf{b}_e^2 \otimes (\mathbf{b}_e \cdot \mathbf{B}_3) : \mathcal{G} - 2\,\alpha_1\, \mathbf{1}^4_{\mathrm{sym}} + 2\,\alpha_3\, \mathcal{X}\,(\mathbf{b}_e, \mathbf{b}_e) \\ & -2\,\alpha_2\, \mathcal{X}\,(\mathbf{b}_e, \mathbf{b}_e) : \mathcal{G} - 2\,\alpha_3\,(\mathcal{X}\,(\mathbf{b}_e, \mathbf{b}_e^2) + \mathcal{X}\,(\mathbf{b}_e^2, \mathbf{b}_e)) : \mathcal{G}, \end{aligned} \tag{4.22}$$

where the equation

$$\Delta\alpha_i = \underbrace{2\,\frac{\partial \alpha_i}{\partial \mathbf{b}_e} \cdot \mathbf{b}_e}_{:=\mathbf{B}_i} : (\mathbf{d}_\Delta - \mathbf{b}_e \cdot \mathbf{d}_{p\,\Delta}), \tag{4.23}$$

has been exploited. The coefficients of $\mathcal{X}(\mathbf{Y}, \mathbf{Z})$ with respect to Cartesian coordinates read

$$X_{ijkl} = \frac{1}{2}(Y_{ik} Z_{lj} + Y_{il} Z_{kj}). \tag{4.24}$$

Since the tensors $\mathbf{B}_i$ represent isotropic functions of $\mathbf{b}_e$, the Voigt notation of $\mathcal{C}$ with respect to the principal axes of $\mathbf{b}_e$ takes the sparse form referred to in Sec. 4.1. Clearly, this is not true for $\mathcal{L}$, so that it suggests to carry out the stability investigation by means of the solution of (EIG^2).

The derivation in Sec. 2 was based on the assumption that $\mathcal{E}^2$ has the major symmetry $E^2_{ijkl} = E^2_{klij}$. Obviously, this holds for the second term of E^2_{ijkl}, $\delta_{ik}\,\tau_{jl}$, independently of the material model chosen. The material tensor $\mathcal{C}$, however, as well as $\mathcal{L}$, possesses the major symmetry only in special cases. One of these cases is obtained, if we use the von Mises yield function (σ_Y yield stress)

$$\Phi = ||\operatorname{dev}\boldsymbol{\tau}|| - (\sigma_Y - q) \tag{4.25}$$

and the neo-Hookean elasticity relation (μ shear modulus, Λ Lamé constant, $J_e^2 = \det \mathbf{b}_e$)

$$\boldsymbol{\tau} = \mu(\mathbf{b}_e - \mathbf{1}) + \frac{\Lambda}{2}(J_e^2 - 1)\,\mathbf{1}. \tag{4.26}$$

The scalar factors α_i and the tensors $\mathbf{B}_i$ $(i = 1, 2, 3)$ take here the forms

$$\alpha_1 = -\mu + \frac{\Lambda}{2}(J_e^2 - 1), \quad \alpha_2 = \mu, \quad \alpha_3 = 0 \tag{4.27}$$

and

$$\mathbf{B}_1 = \Lambda\, J_e^2\, \mathbf{1}, \quad \mathbf{B}_2 = \mathbf{B}_3 = \mathbf{0}. \tag{4.28}$$

Using $\operatorname{dev}\boldsymbol{\tau} = \mu \operatorname{dev}\mathbf{b}_e$, the tensor $\mathcal{G}$ reads

$$\mathcal{G} = \tilde{k}\;(\mathbf{b}_e^{-1} \cdot \operatorname{dev}\mathbf{b}_e) \otimes (\operatorname{dev}\mathbf{b}_e \cdot \mathbf{b}_e), \tag{4.29}$$

where

$$\tilde{k} = k\,\frac{4\,\mu}{\operatorname{dev}\mathbf{b}_e : \operatorname{dev}\mathbf{b}_e} \tag{4.30}$$

represents another isotropic function of $\mathbf{b}_e$ and ξ. We finally obtain for $\mathcal{C}$ the expression

$$\begin{aligned}\mathcal{C} = \Lambda\, J_e^2\, \mathbf{1} \otimes \mathbf{1} - \tilde{k}\,\Lambda\, J_e^2\,(\mathbf{1} \otimes \mathbf{b}_e) : (\mathbf{b}_e^{-1} \cdot \operatorname{dev}\mathbf{b}_e) \otimes (\operatorname{dev}\mathbf{b}_e \cdot \mathbf{b}_e)\\ + (2\,\mu - \Lambda\,(J_e^2 - 1))\,\mathbf{1}^4_{\mathrm{sym}} - 2\,\mu\,\mathcal{X}(\mathbf{b}_e, \mathbf{b}_e) : \mathcal{G}.\end{aligned} \tag{4.31}$$

Due to $(\operatorname{dev}\mathbf{b}_e) : \mathbf{1} = 0$, the right-hand term of the first line vanishes. To check the symmetry of the right-hand term of the second line, we use index notation. By means of (4.24) one obtains

$$X_{ijkl}\, G_{klmn} = \frac{\tilde{k}}{2}\left((b_e)_{ik}\,(b_e)_{jl} + (b_e)_{il}\,(b_e)_{jk}\right) \tag{4.32}$$

$$\left((b_e^{-1})_{kx}\,(\operatorname{dev} b_e)_{xl}\,(\operatorname{dev} b_e)_{my}\,(b_e)_{yn}\right) = \tilde{k}\,(b_e)_{ix}\,(\operatorname{dev} b_e)_{xj}\,(\operatorname{dev} b_e)_{my}\,(b_e)_{yn}.$$

In tensor notation, this yields with $\mathbf{b}_e \cdot \operatorname{dev}\mathbf{b}_e = \operatorname{dev}\mathbf{b}_e \cdot \mathbf{b}_e$ the equation

$$\mathcal{X}(\mathbf{b}_e, \mathbf{b}_e) : \mathcal{G} = \tilde{k}\,(\mathbf{b}_e \cdot \operatorname{dev}\mathbf{b}_e) \otimes (\mathbf{b}_e \cdot \operatorname{dev}\mathbf{b}_e). \tag{4.33}$$

Thus, $\mathcal{C}$ has major as well as minor symmetries.

The "elastic" part of $\mathcal{C}$, $\mathcal{C}^{\mathrm{el}}$, is derived by carrying out the push-forward of $\mathcal{L}^{\mathrm{el}}$ by means of $\mathbf{F}$. In (4.31) the terms of $\mathcal{C}^{\mathrm{el}}$ can be easily identified as those terms which do *not* depend on $\mathcal{G}$. Taking into account the result (4.33), it is evident that the tensor $\mathcal{C} - \mathcal{C}^{\mathrm{el}}$ consists of dyadic products of the form $\mathbf{A}(\mathbf{b}_e, \xi) \otimes \mathbf{B}(\mathbf{b}_e, \xi)$. So only $\mathcal{C}^{\mathrm{el}}$ contributes to $[\mathcal{E}^2]^{\mathrm{shear}(12)}$ or, in other words, C_{1212} can be replaced by C^{el}_{1212}. In finite elasticity, L_{1212} is determined by formulating the relation between $\sum_{I=1}^{3} S_{II}(E_{11}, E_{22}, E_{33})\,\Delta(\mathbf{n}_I \otimes \mathbf{n}_I)$ and $\sum_{J=1}^{3} E_{JJ}\,\Delta(\mathbf{n}_J \otimes \mathbf{n}_J)$ (see OGDEN [26]). C_{1212} is given by

$$C_{1212} = \lambda_1^2\,\lambda_2^2\,L_{1212} = \lambda_1^2\,\lambda_2^2\,\frac{S_{11} - S_{22}}{\lambda_1^2 - \lambda_2^2} = \frac{\tau_{11}\,\lambda_2^2 - \tau_{22}\,\lambda_1^2}{\lambda_1^2 - \lambda_2^2}, \tag{4.34}$$

if $\lambda_1^2 \neq \lambda_2^2$. In elastoplasticity, we exploit the fact that $\bar{\mathbf{S}} := \mathbf{F}_p \cdot \mathbf{S} \cdot \mathbf{F}_p^T$ represents an isotropic function of $\mathbf{C}_e$. The relation between $\sum_{I=1}^{3} \bar{S}_{II}(C_{e\,11}, C_{e\,22}, C_{e\,33})\,\Delta(\bar{\mathbf{n}}_I \otimes \bar{\mathbf{n}}_I)$ and $\dfrac{1}{2}\sum_{J=1}^{3} C_{e\,JJ}\,\Delta(\bar{\mathbf{n}}_J \otimes \bar{\mathbf{n}}_J)$ is then determined analogously to L_{1212} in finite elasticity. Let us call the result $\bar{L}^{\mathrm{el}}_{1212}$. Using further the equations

$$\mathbf{F} \cdot \Delta\mathbf{S} \cdot \mathbf{F}^T = \mathbf{F} \cdot \mathbf{F}_p^{-1} \cdot \Delta\bar{\mathbf{S}} \cdot \mathbf{F}_p^{-T} \cdot \mathbf{F}^T = \mathbf{F}_e \cdot \Delta\bar{\mathbf{S}} \cdot \mathbf{F}_e^T \tag{4.35}$$

and

$$\mathbf{F}^{-T} \cdot \Delta\mathbf{C} \cdot \mathbf{F}^{-1} = \mathbf{F}^{-T} \cdot \mathbf{F}_p^{-T} \cdot \Delta\mathbf{C}_e \cdot \mathbf{F}_p^{-1} \cdot \mathbf{F}^{-1} = \mathbf{F}_e^{-T} \cdot \Delta\mathbf{C}_e \cdot \mathbf{F}_e^{-1} \tag{4.36}$$

which hold for *fixed* $\mathbf{F}_p$, it becomes clear that the coefficient C^{el}_{1212} is given by

$$C^{\mathrm{el}}_{1212} = \lambda_{1\,e}^2\,\lambda_{2\,e}^2\,\bar{L}^{\mathrm{el}}_{1212}. \tag{4.37}$$

We then obtain

$$C^{\mathrm{el}}_{1212} = \lambda_{1\,e}^2\,\lambda_{2\,e}^2\,\bar{L}^{\mathrm{el}}_{1212} = \lambda_{1\,e}^2\,\lambda_{2\,e}^2\,\frac{\bar{S}_{11} - \bar{S}_{22}}{\lambda_{1\,e}^2 - \lambda_{2\,e}^2} = \frac{\tau_{11}\,\lambda_{2\,e}^2 - \tau_{22}\,\lambda_{1\,e}^2}{\lambda_{1\,e}^2 - \lambda_{2\,e}^2}, \tag{4.38}$$

if $\lambda_{1\,e}^2 \neq \lambda_{2\,e}^2$.

To conclude this section, it should be emphasized that the stability investigation based on (EIG^2) seems to be the most convenient, since it is analytically tractable in both cases, finite elasticity and finite elastoplasticity. The results of OGDEN [26] are directly recovered by investigating (EIG^1). After all, it is certainly possible to work with either one of the three eigenvalue problems (EIG), (EIG^1) or (EIG^2). The results are equivalent. But the goal is to derive an analytical stability criterion which is suitably achieved with (EIG^2).

5. Solution of the eigenvalue problem (EIG^2)

In the following, we derive the conditions for which $[\mathcal{E}^2]^{\text{shear}(12)}$ and $[\mathcal{E}^2]^{\text{stretch}}$ are singular. Further, we classify these singular solutions according to the cases listed in Sec. 2.

The analysis presented in this paper is based on the assumption $\partial\mathcal{B}_0 = \partial\mathcal{B}_T$ which means that tractions are prescribed on the whole boundary. Displacement boundary conditions reduce the size of the eigenvalue problem, since not all possible eigenmodes are consistent with the boundary conditions. This usually simplifies the stability investigations but does not lead to new results. Therefore, it is not necessary to consider this case in the present work.

5.1. Solution of the two-dimensional sub-problem

With the shorthand notations $C = C_{1212}$, $\tau_I = \tau_{II}$ and $\Phi_{ij} = \Phi^2_{ij}$, the two--dimensional eigenvalue problem is rewritten as

$$\mathsf{EIG\text{-}SH(12)}: \left(\begin{bmatrix} C & C \\ C & C \end{bmatrix} + \begin{bmatrix} \tau_2 & 0 \\ 0 & \tau_1 \end{bmatrix} - \omega \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{Bmatrix} \Phi_{12} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The eigenvalues are determined by

$$\omega_{1,2} = X \pm \sqrt{X^2 - \det\,[\mathcal{E}^2]^{\text{shear}(12)}}, \tag{5.1}$$

where X is a shorthand notation for the expression $X = C + \dfrac{1}{2}(\tau_1 + \tau_2)$. A necessary condition for vanishing of at least one eigenvalue is

$$\det\,[\mathcal{E}^2]^{\text{shear}(12)} = C\,(\tau_1 + \tau_2) + \tau_1\,\tau_2 = 0 \tag{5.2}$$

leading to

$$\omega_1 = X + |X| \quad \text{and} \quad \omega_2 = X - |X|. \tag{5.3}$$

Thus, if X has a positive sign, ω_2 vanishes and ω_1 is equal to $2\,X$, i.e. positive. If X has a negative sign, ω_1 vanishes and ω_2 is equal to $2\,X$, i.e. negative. For $X = 0$, we have a double zero eigenvalue.

In finite elasticity, we have to differentiate between the two situations $\lambda_1 \neq \lambda_2$ (SH-R) and $\lambda_1 = \lambda_2 = \lambda$ (SH-S). In elastoplasticity, we have instead $\lambda_{1\,e} \neq \lambda_{2\,e}$ (SH-R) and $\lambda_{1\,e} = \lambda_{2\,e} = \lambda_e$ (SH-S), where $\lambda_{i\,e}^2$ $(i = 1, 2, 3)$ denote the eigenvalues of $\mathbf{b}_e$.

SH-R:

In the *regular* situation $\lambda_1 \neq \lambda_2$, the coefficient C is determined by

$$C = \frac{\tau_1\,\lambda_2^2 - \tau_2\,\lambda_1^2}{\lambda_1^2 - \lambda_2^2}, \tag{5.4}$$

see e.g. OGDEN [26] for the case of finite elasticity. The same relation holds for isotropic elastoplasticity, if we choose the shorthand notation $\lambda_i = \lambda_{i\,e}$ which is assumed to hold from now on. Inserting the expression for C into (5.2) yields

$$\boxed{\text{C-SH-R} : \lambda_1 \neq \lambda_2 \;\Rightarrow\; \frac{\tau_1^2}{\lambda_1^2} - \frac{\tau_2^2}{\lambda_2^2} = 0.}$$

The condition (C-SH-R) is fulfilled either with $P := \dfrac{\tau_1}{\lambda_1} = \dfrac{\tau_2}{\lambda_2}$ or $P := \dfrac{\tau_1}{\lambda_1} = -\dfrac{\tau_2}{\lambda_2}$. Taking the first possibility (C-SH-R-1) gives

$$\frac{1}{\lambda_1 + \lambda_2}\begin{bmatrix} P\,\lambda_2^2 & -P\,\lambda_1\,\lambda_2 \\ -P\,\lambda_1\,\lambda_2 & P\,\lambda_1^2 \end{bmatrix}\begin{Bmatrix} \Phi_{12} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{5.5}$$

Thus, we obtain $\Phi_{12} = \dfrac{\lambda_1}{\lambda_2} f$, $\Phi_{21} = f$, where f represents an arbitrary factor. Since the symmetric part of $\boldsymbol{\Phi}$ is not zero, we have here *Case 2*.

The second choice (C-SH-R-2) leads to

$$\frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} P\,\lambda_2^2 & P\,\lambda_1\,\lambda_2 \\ P\,\lambda_1\,\lambda_2 & P\,\lambda_1^2 \end{bmatrix}\begin{Bmatrix} \Phi_{12} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{5.6}$$

Here, the eigenform associated with the vanishing eigenvalue is described by $\Phi_{12} = -\dfrac{\lambda_1}{\lambda_2} f$, $\Phi_{21} = f$. This would be *Case 1*, if the principal stretches λ_1 and λ_2 were equal. Since this is not true, we detect again *Case 2*.

SH-S:

In the *special* situation $\lambda_1 = \lambda_2 = \lambda$, the coefficient C is determined by

$$C = \lim_{\lambda_1^2 \to \lambda_2^2} \frac{\tau_1\,\lambda_2^2 - \tau_2\,\lambda_1^2}{\lambda_1^2 - \lambda_2^2} = \frac{\partial(\tau_1\,\lambda_2^2)}{\partial\lambda_1^2} - \frac{\partial(\tau_2\,\lambda_1^2)}{\partial\lambda_1^2} = \frac{1}{2}\,(C_{1111}^{\text{el}} - C_{1122}^{\text{el}}). \tag{5.7}$$

Note that in finite elasticity, $\mathcal{C}^{\mathrm{el}}$ is identical with $\mathcal{C}$. In elastoplasticity, this holds only for the shear part. The difference has an important effect on the stability behaviour as will be pointed out in Sec. 5.2.

From (5.2) we obtain

$$\boxed{\mathsf{C\text{-}SH\text{-}S}: \lambda_1 = \lambda_2 \ \Rightarrow\ \tau\left(2C + \tau\right) = 0,}$$

where the relation $\tau = \tau_1 = \tau_2$ has been used. There are two possibilities to fulfill (C-SH-S). The first (C-SH-S-1) is represented by the criterion

$$\text{(5.8)} \qquad \mathsf{CRIT}: \qquad C = -\frac{\tau}{2} \Rightarrow \quad \frac{1}{2}\begin{bmatrix} \tau & -\tau \\ -\tau & \tau \end{bmatrix} \begin{Bmatrix} \Phi_{12} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The associated eigenform is $\Phi_{12} = \Phi_{21} = f$ (SH-12), a typical shear mode (see Fig. 1), i.e. we have *Case 2*. The condition (CRIT) has an important meaning in the context of the present stability investigation. We will come back to this point later.

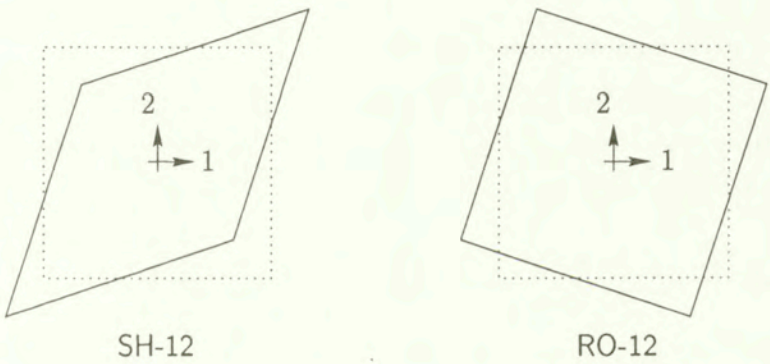

FIG. 1. Shear eigenmode (SH-12) and rotational eigenmode (RO-12).

The second possibility to fulfill (C-SH-S) is $\tau = 0$ (C-SH-S-2) leading to

$$\text{(5.9)} \qquad \begin{bmatrix} C & C \\ C & C \end{bmatrix} \begin{Bmatrix} \Phi_{12} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The associated eigenform is $\Phi_{12} = -\Phi_{21} = f$ (RO-12), a rotational eigenmode (see Fig. 1). The symmetric part of the eigentensor vanishes, we obtain *Case 1*! See Table 1 for a summary of the latter results.

For physically reasonable material models, τ vanishes only if the external loading is chosen in such a way that the boundaries normal to the 1- and the 2-axis are stress-free. This can be achieved for any kind of elastic material model showing the purely geometrical character of singularities belonging to *Case 1*. When we look at *Case 2*, however, it is important to specify a material model, firstly, in order to differentiate between the possibilities a–c and, secondly, to check whether fulfilling the conditions C-SH-R or C-SH-S is consistent with the physical and mathematical assumptions upon which the derivation of these equations is based.

Table 1. Results of the two-dimensional sub-problem.

- $\lambda_1 \neq \lambda_2$:

(1) $\quad \dfrac{\tau_1}{\lambda_1} = \dfrac{\tau_2}{\lambda_2} \quad \rightarrow \quad$ *Case 2*

(2) $\quad \dfrac{\tau_1}{\lambda_1} = -\dfrac{\tau_2}{\lambda_2} \quad \rightarrow \quad$ *Case 2*

- $\lambda_1 = \lambda_2$:

(1) (CRIT) : (SH-12) $\rightarrow$ *Case 2*

(2) $\tau = 0$: (RO-12) $\rightarrow$ *Case 1*

5.2. Solution of the three-dimensional sub-problem

Before discussing the results of Sec. 5.1 in the context of a special material model, let us continue with the investigation of the three-dimensional sub-problem

$$\text{EIG-ST}: \quad \left(\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} \\ C_{1122} & C_{2222} & C_{2233} \\ C_{1133} & C_{2233} & C_{3333} \end{bmatrix} + \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix} - \omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \Phi_{11} \\ \Phi_{22} \\ \Phi_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

At this point, we have to differentiate between the elastic and the elastoplastic case. Consider first the special situation that all stretches are equal (ST-SS). In finite elasticity, this is a sufficient condition for the statements $\tau_1 = \tau_2 = \tau_3$, $C_{1111} = C_{2222} = C_{3333}$ and $C_{1122} = C_{1133} = C_{2233}$. In elastoplasticity, also the loading *history* has to be taken into account. Thus, we have to require in addition, that the stretches are equal at *any* time of the loading process. This is tacitly assumed in the following.

We investigate the two sitations $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ (ST-SS) as well as $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 \neq \lambda$ (ST-S).

ST-SS:

Elasticity. In the special situation $\lambda_1 = \lambda_2 = \lambda_2 = \lambda$ ($\Rightarrow \tau_1 = \tau_2 = \tau_3 = \tau$), the determinant of $[\mathcal{E}^2]^{\text{stretch}}$ reads

$$\det [\mathcal{E}^2]^{\text{stretch}} = H^3 + 2\,N^3 - 3\,H\,N^2 = 0, \tag{5.10}$$

where the shorthand notations

$$H = C_{1111} + \tau \quad \text{and} \quad N = C_{1122} \tag{5.11}$$

have been introduced. One evident solution of the latter equation is

$$N = H \quad \Leftrightarrow \quad C_{1111} + \tau = C_{1122}. \tag{5.12}$$

It is identical with (CRIT). Moreover, it is shown easily, that fulfillment of (CRIT) leads in the context of (EIG-ST) to a *two-fold* zero eigenvalue. The associated eigenforms are linear combinations of the stretch modes $\tilde{\Phi}_{11} = f$, $\tilde{\Phi}_{22} = -f$, $\tilde{\Phi}_{33} = 0$ (ST-12) and $\tilde{\Phi}_{11} = 0$, $\tilde{\Phi}_{22} = f$, $\tilde{\Phi}_{33} = -f$ (ST-23). Thus, also the third stretch mode $\tilde{\Phi}_{11} = -f$, $\tilde{\Phi}_{22} = 0$, $\tilde{\Phi}_{33} = f$ (ST-31) is a relevant eigenform. The third zero eigenvalue is obtained for $H = -2\,N$ and correlated with the eigenform $\tilde{\Phi}_{11} = f$, $\tilde{\Phi}_{22} = f$, $\tilde{\Phi}_{33} = f$ (ST-VOL). We obtain *Case 2*. The modes (ST-12) and (ST-VOL) are plotted in Fig. 2.

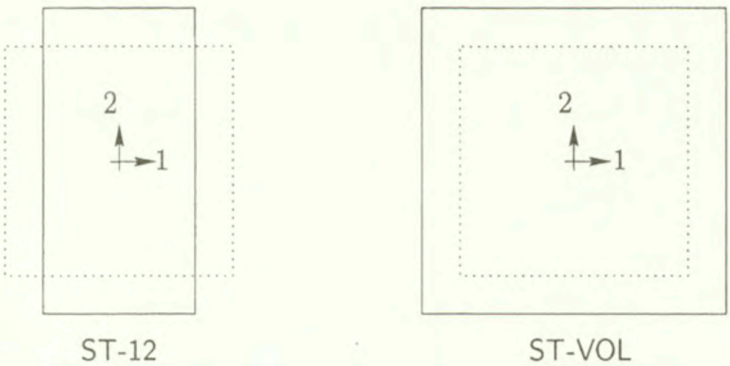

FIG. 2. Stretch modes (ST-12) and (ST-VOL).

Going back to the complete eigenvalue problem (EIG^2), we can make the following statement. If the condition (CRIT) is fulfilled and all three principal stretches are equal, we have a *five-fold* zero eigenvalue corresponding to linear combinations of the three stretch eigenmodes ST-12, ST-23 and ST-31 and the three shear eigenmodes SH-12, SH-23 and SH-31. See Table 1 and 2.

Elastoplasticity. The conditions for $[\mathcal{E}^2]^{\text{stretch}}$ becoming singular are computed in the same way as shown above. But we observe the following important fact. Due to the fact that $C_{iijj} \neq C^{\text{el}}_{iijj}$ $(i, j = 1, 2, 3)$ holds, the condition (5.12) is **not** identical with (CRIT). The "stretch" and the "shear" singularities do not occur simultaneously!

ST-S:

Elasticity. Let us investigate now the case $\lambda_1 = \lambda_2 = \lambda$ in combination with $\lambda_3 \neq \lambda$. The determinant $\det[\mathcal{E}^2]^{\text{stretch}}$ takes the form

$$\det[\mathcal{E}^2]^{\text{stretch}} = H_3\, H^2 + 2\, N\, N_3^2 - 2\, H\, N_3^2 - H_3\, N^2 = 0. \tag{5.13}$$

In the latter equation, the letters H_3 and N_3 stand for the expressions

$$H_3 = C_{3333} + \tau_3 \quad \text{and} \quad N_3 = C_{1133}. \tag{5.14}$$

Again, one possibility to satisfy $\det[\mathcal{E}^2]^{\text{stretch}} = 0$ is to fulfill (CRIT). The associated eigenmode is (ST-12) and as such related to *Case 2*. The other eigenvalues vanish, if the condition

$$H_3\,(H + N) - 2\, N_3^2 = 0 \tag{5.15}$$

is satisfied. Due to the fact that the form of the latter equation is crucially dependent on the material model, the discussion of (5.15) is postponed to Sec. 5.

In analogy to (ST-SS), fulfillment of (CRIT) leads here to a *two-fold* zero eigenvalue of $\mathcal{E}^2$. The relevant eigenforms are linear combinations of the modes (ST-12) and (SH-12) (see Table 1 and 2).

Elastoplasticity. We have the same situation as in the case of three principal stretches. See the summary in Table 3.

Table 2. Results of the three-dimensional sub-problem (elasticity).

- $\lambda_1 = \lambda_2 \neq \lambda_3$:

(CRIT) : (ST-12) → *Case 2*

- $\lambda_1 = \lambda_2 = \lambda_3$:

(CRIT) : (ST-12), (ST-23), (ST-31) → *Case 2*

Table 3. Results of the two- and the three-dimensional sub-problem (elastoplasticity).

- $\lambda_1 = \lambda_2 \neq \lambda_3$:

$C^{\text{el}}_{1111} - C^{\text{el}}_{1122} + \tau_1 = 0$ (CRIT)	:	(SH-12) → Case 2
$C_{1111} - C_{1122} + \tau_1 = 0$	:	(ST-12) → Case 2
further (distinct stretches)	:	see Table 1

- $\lambda_1 = \lambda_2 = \lambda_3$:

$C^{\text{el}}_{iiii} - C^{\text{el}}_{iijj} + \tau_i = 0$ (CRIT)	:	(SH-12), (SH-23), (SH-31) → Case 2
$C_{iiii} - C_{iijj} + \tau_i = 0$	:	(ST-12), (ST-23), (ST-31) → Case 2

The preceding derivation shows that the local stability investigations in finite elasticity in finite elastoplasticity can be carried out in a very general context. In the following, we will consider the conditions derived above in relation to common material models like the neo-Hookean and the Mooney-Rivlin model. Some of the results have already been achieved in earlier works and thus confirm the correctness of the following derivation.

5.3. Loss of strong ellipticity

In the preceding sections, it is assumed that the singularity of $\mathcal{E}^2$ is caused by the loss of positive definiteness of either $[\mathcal{E}^2]^{\text{stretch}}$, $[\mathcal{E}^2]^{\text{shear(12)}}$, $[\mathcal{E}^2]^{\text{shear(23)}}$ or $[\mathcal{E}^2]^{\text{shear(31)}}$. If we term the corresponding eigenmodes $\{\mathbf{\Phi}^{\text{stretch}}\}$, $\{\mathbf{\Phi}^{\text{shear(12)}}\}$, $\{\mathbf{\Phi}^{\text{shear(23)}}\}$ or $\{\mathbf{\Phi}^{\text{shear(31)}}\}$, respectively, the eigenvalues of $\mathcal{E}^2$ can be represented as

$$(5.16)\qquad \omega = \{\mathbf{\Phi}^{\text{stretch}}\}^T [\mathcal{E}^2]^{\text{stretch}} \{\mathbf{\Phi}^{\text{stretch}}\}^T + \{\mathbf{\Phi}^{\text{shear(12)}}\}^T [\mathcal{E}^2]^{\text{shear(12)}} \{\mathbf{\Phi}^{\text{shear(12)}}\}^T + \{\mathbf{\Phi}^{\text{shear(23)}}\}^T [\mathcal{E}^2]^{\text{shear(23)}} \{\mathbf{\Phi}^{\text{shear(23)}}\}^T + \{\mathbf{\Phi}^{\text{shear(31)}}\}^T [\mathcal{E}^2]^{\text{shear(31)}} \{\mathbf{\Phi}^{\text{shear(31)}}\}^T.$$

Evidently, ω might vanish also, if the four terms on the right-hand side of (5.16) cancel each other. For simplicity, let us restrict ourselves to a two-dimensional investigation ($\lambda_3 = 1$), where the stretch matrix reduces to a 2 x 2-matrix and in addition, only the shear matrix $[\mathcal{E}^2]^{\text{shear(12)}}$ has to be considered. Then we may write

$$(5.17)\qquad \omega = \omega^{\text{stretch}} + \omega^{\text{shear(12)}}.$$

The vanishing of ω with $\omega^{\text{stretch}} \neq 0$ is possible only if *either* ω^{stretch} *or* ω^{shear12} becomes negative. In elasticity, this is excluded for $\lambda_1 = \lambda_2$ but theoretically possible for $\lambda_1 \neq \lambda_2$. The change of sign of one of the eigenvalue parts requires that one of the matrices, $[\mathcal{E}^2]^{\text{stretch}}$ or $[\mathcal{E}^2]^{\text{shear(12)}}$, loses its positive definiteness. In other words, if the sub-matrices of $[\mathcal{E}^2]$ can be shown to be positive definite for any arbitrary deformation, singularities of the type $\omega^{\text{stretch}} = -\omega^{\text{shear(12)}}$ with $\omega^{\text{stretch}} \neq 0$ are also excluded.

In elastoplasticity the latter type of instability might occur, even if $\lambda_1 = \lambda_2$ holds. This is due to the fact that in the case of two equal stretches, ω^{stretch} and $\omega^{\text{shear(12)}}$ do not vanish simultaneously. The question arises as to which physical meaning such instabilities could have. It is well known that strong ellipticity of the underlying differential equation system is guaranteed if the so-called acoustic tensor $\mathbf{A} := \mathbf{N} \cdot \mathcal{A} \cdot \mathbf{N}$ ($\mathbf{N}$ arbitrary vector) is positive definite (see e.g. HILL [15], PETRYK [27] for a detailed discussion). According to HADAMARD [12], the singularity of $\mathbf{A}$ indicates a so-called stationary discontinuity. This means that an acceleration wave travels with zero speed through a material continuum. If $\mathbf{A}$ has at least one negative eigenvalue, one speaks of a wave with imaginary speed. One can show further that the loss of ellipticity is sufficient for a bifurcation into a shear band of the orientation $\mathbf{N}$. It is common to use the notion "localization" for this phenomenon, since the deformation is "localized" in some small interior subdomain of the specimen, whereas the boundary remains unperturbed.

The positive definiteness of $\mathbf{A}$ can be expressed also in the form

$$(5.18) \qquad (\boldsymbol{\varphi} \otimes \mathbf{N}) : \mathcal{A} : (\boldsymbol{\varphi} \otimes \mathbf{N}) = (\boldsymbol{\varphi} \otimes \underbrace{\mathbf{N} \cdot \mathbf{F}^{-1}}_{\mathbf{n}}) : \mathcal{E}^2 : (\boldsymbol{\varphi} \otimes \mathbf{N} \cdot \mathbf{F}^{-1}) > 0.$$

Thus, we may state the following. If $\mathcal{E}^2$ has a zero eigenvalue associated with the eigentensor $\mathbf{\Phi}^{\mathrm{loc}} = \boldsymbol{\varphi} \otimes \mathbf{n}$, localization takes place. The vectors $\boldsymbol{\varphi}$ and $\mathbf{n}$ will have components in at least two (plane strain) or all three coordinate directions in general. Since $\mathbf{\Phi}^{\mathrm{stretch}} \neq \mathbf{0}$ and $\mathbf{\Phi}^{\mathrm{shear(12)}} \neq \mathbf{0}$ holds, this is one of the cases, where $\omega = 0$ with $\omega^{\mathrm{stretch}} = -\omega^{\mathrm{shear(12)}} \neq 0$ is valid. Concerning the classification discussed at the end of Sec. 2, the latter type of singularity belongs to *Case 2*, since the symmetric part of $\boldsymbol{\varphi} \otimes \mathbf{n}$ vanishes only in the physically irrelevant situations $\mathbf{n} = \mathbf{0}$ or $\boldsymbol{\varphi} = \mathbf{0}$. For *Case 2b*, the scalar product $\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n}$ must be negative (positive). This means that the stress component orthogonal to the shear band is negative (positive). Both cases are observed experimentally.

6. Examples

6.1. Equitriaxial loading ($\mathbf{P}_L = P_L\, \mathbf{1}$)

Elasticity. In order to remain as simple as possible, we first work with the neo-Hookean model which has been derived by TRELOAR [41] on the basis of micromechanical considerations. Generalization the original model for compressible material behaviour leads to a strain energy function of the form

$$(6.1) \qquad W = \frac{\mu}{2}\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) - \mu \ln J + \frac{\Lambda}{2}\,(J^2 - 1 - 2\ln J)$$

where $J = \det \mathbf{F} = \lambda_1\, \lambda_2\, \lambda_3$ denotes a shorthand notation for the determinant of the deformation gradient. Note that μ and Λ represent elasticity constants, μ being the shear modulus and Λ the Lamé constant.

We consider first the deformation state with equal principal stretches. The eigenvalues of $[\mathcal{E}^2]^{\mathrm{stretch}}$ and $[\mathcal{E}^2]^{\mathrm{shear(12)}}$ can then be evaluated analytically, e.g.

by means of Mathematica. Inserting the material constants $\mu = 1$ and $\Lambda = 10$, one obtains for the "stretch" eigenvalues

$$\omega_1^{\text{stretch}} = 6 + \lambda^2 - 5\,\lambda^6 = \omega_2^{\text{stretch}} \quad \text{and} \quad \omega_3^{\text{stretch}} = 6 + \lambda^2 + 25\,\lambda^6. \tag{6.2}$$

The "shear" eigenvalues take the form

$$\omega_1^{\text{shear}(12)} = \omega_1^{\text{stretch}} \quad \text{and} \quad \omega_2^{\text{shear}(12)} = -6 + \lambda^2 + 5\,\lambda^6. \tag{6.3}$$

The condition (CRIT) is rewritten as

$$6 + \lambda^2 - 5\,\lambda^6 = 0. \tag{6.4}$$

It is now clearly recognizable that fulfilling of (CRIT) leads simultaneously to the vanishing of $\hat{\omega}_1^{\text{stretch}}(\lambda)$, $\hat{\omega}_2^{\text{stretch}}(\lambda)$, $\hat{\omega}_1^{\text{shear}(12)}(\lambda)$, $\hat{\omega}_1^{\text{shear}(23)}(\lambda)$ and $\hat{\omega}_1^{\text{shear}(31)}(\lambda)$. Very interesting is the fact that these functions are even identical. For the present example, we detect the five-fold bifurcation point at $\lambda_{\text{crit}} = 1.061$.

The eigenvalue $\omega_3^{\text{stretch}}$ remains always positive, so that the eigenform (ST-VOL) never becomes relevant. The eigenvalue $\omega_2^{\text{shear}(12)}$ is associated with the rotational eigenmode (RO-12) and consequently changes its sign in the natural state $\tau = 0$ (C-SH-S-2), i.e. for $\lambda = 1$. Analogously, $\omega_2^{\text{shear}(23)}$ and $\omega_2^{\text{shear}(31)}$ change their signs also for $\tau = 0$ but the associated eigenforms are (RO-23) and (RO-31), respectively. To conclude, the deformation state $\lambda_I = \lambda$ is stable only for $1 < \lambda < \lambda_{\text{crit}}$. For all other stretch values, this solution branch is singular ($\lambda = 1$, $\lambda = \lambda_{\text{crit}}$) or unstable ($\lambda < 1$, $\lambda > \lambda_{\text{crit}}$).

It should be emphasized again that the singularity at $\lambda = 1$ occurs independently of the kind of the nonlinearly elastic material model chosen. This is the typical character of singularities of *Case 1*. The singularity for $\lambda = \lambda_{\text{crit}}$, however, can be clearly attributed to *Case 2c*, since $\omega_{\mathcal{M}}$ is positive. Thus, for this stretch value, the material tensors $\mathcal{C}$ or $\mathcal{L}$, respectively, have lost their positive definiteness indicating a material instability in the sense of Sec. 2. Such an instability can be avoided by choosing a different material model, e.g. the one characterized by a constant and positive definite material tensor.

REMARK

The fact that the natural state becomes singular, if rigid body rotations are considered as eigenmodes, has already been discussed by BEATTY [6, 7] and FOSDICK [11]. For the purpose of restricting the class of eigenmodes, the first author introduces the so-called zero moment condition and later uses Korn's inequality in addition. In the context of the present work, additional constraints are not necessary any longer since singularities of *Case 1* can be clearly differentiated from the physically more meaningful *Case 2*. □

In order to compare the results of the present work with the calculations of BALL and SCHAEFFER [3], we derive from (CRIT) the expression

$$\tau = \mu\,(\lambda^2 - 1) + \frac{\Lambda}{2}\,(J^2 - 1) = \Lambda\,(J^2 - 1) - 2\,\mu = -2\,C = C_{1122} - C_{1111} \tag{6.5}$$

leading to $\frac{\Lambda}{2}\,(J^2 - 1) = \mu\,(\lambda^2 - 1) + 2\,\mu$ and

$$\tau_{\text{crit}} = 2\,\mu\,\lambda^2. \tag{6.6}$$

In the limit of incompressiblity ($\Lambda/\mu \to \infty$, $J \to 1 \;\Rightarrow\; \lambda \to 1$), one obtains $\tau_{\text{crit}} \to 2\,\mu$, a result which is in agreement e.g. with RIVLIN [33] and BALL and SCHAEFFER [3].

Due to the fact that infinitely many linear combinations of the three stretch modes and the three shear modes become relevant for the singularity determined by (CRIT), also an infinite number of secondary branches run through this bifurcation point. Among these are six branches, where the deformation state can be either described as being plate-like or rod-like (see BALL and SCHAEFFER [3]). In a plate-like deformation state, two of the principal stretches are equal and larger than the third one. The rod-like deformation state is characterized by the opposite situation, i.e. the two equal principal stretches are smaller than the one in the perpendicular direction. One of the rod-like secondary branches ($\lambda_1 = \lambda_2 = \lambda < \lambda_3$) is depicted in Fig. 3.

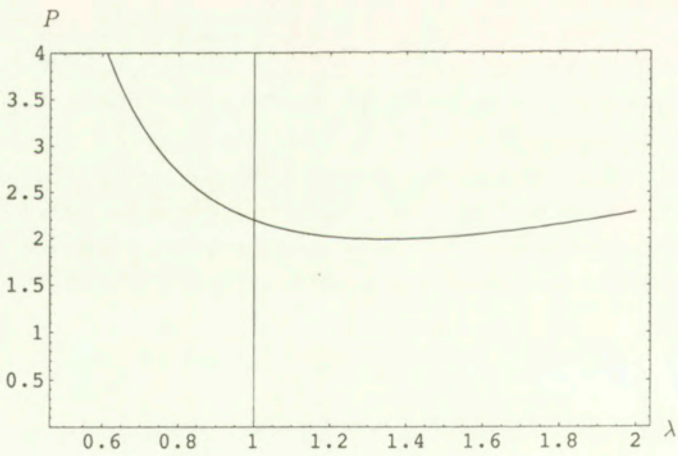

FIG. 3. Stress-stretch curve: $\lambda_1 = \lambda_2 = \lambda < \lambda_3$.

Application of the equality $P_{11} = P_{33}$ yields, with

$$\lambda_3 = \frac{1}{\lambda^3}\,\frac{\mu}{\Lambda}\left(1 \pm \sqrt{1 + (2\,\mu + \Lambda)\,\Lambda\,\lambda^2\,\frac{1}{\mu^2}}\right), \tag{6.7}$$

a relation between λ and λ_3, where the minus sign in (6.7) would apply for a plate-like branch. The determinant of $[\mathcal{E}^2]^{\rm stretch}$ is plotted in Fig. 4.

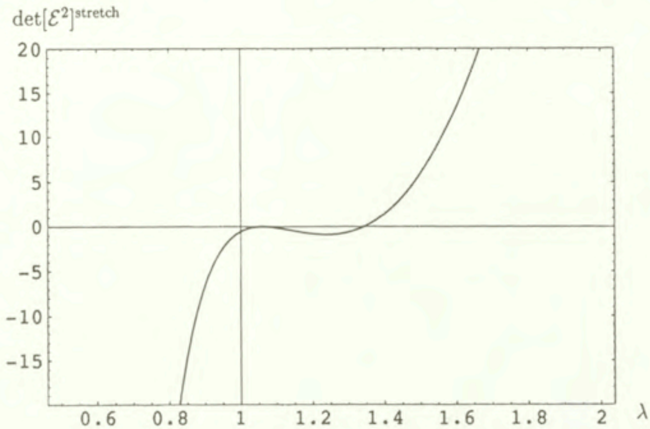

FIG. 4. $\det[\mathcal{E}^2]^{\rm stretch}$ $(\lambda_1 = \lambda_2 = \lambda < \lambda_3)$.

At $\lambda = \lambda_{\rm crit}$, we observe a double zero eigenvalue of $[\mathcal{E}^2]^{\rm stretch}$. This point indicates the bifurcation back to the primary branch or to a plate-like secondary branch. Linear combinations of the stretch eigenforms would lead us to solution branches with three distinct principal stretches. The single zero eigenvalue at $\lambda = 1.343 = \lambda_{\rm lim}$ is associated with the stress minimum in Fig. 3 and represents a singularity of *Case 2c*. So we do not detect any further bifurcation, since the condition (CRIT) cannot be fulfilled on this branch in the context of the neo-Hookean material model. Note that only for $\lambda > \lambda_{\rm lim}$, $[\mathcal{E}^2]^{\rm stretch}$ is positive definite.

The solution of the eigenvalue problem (EIG-SH(12)) yields one positive eigenvalue and one which is only positive for $\lambda > \lambda_{\rm crit}$. The vanishing of the latter eigenvalue indicates again the bifurcation described by (CRIT). The natural state $\tau = 0$ (C-SH-S-2) is never reached on this secondary branch. Thus, the rotational mode (RO-12) does not become relevant and the eigenvalue associated with it remains positive. The eigenvalue problems (EIG-SH(23)) and (EIG-SH(31)) lead to one zero eigenvalue related to the solution $\dfrac{\tau_2}{\lambda_2} = \dfrac{\tau_3}{\lambda_3}$ or $\dfrac{\tau_3}{\lambda_3} = \dfrac{\tau_1}{\lambda_1}$ (C-SH-R-1), respectively. Due to the fact that $P_{ii} = \dfrac{\tau_i}{\lambda_i}$ holds in this example, (C-SH-R-1) is fulfilled on *all* secondary branches (even when all principal stretches are distinct). Since (C-SH-R-2) cannot be satisfied, the second eigenvalue of these eigenvalue problems is positive.

In order to compare the latter results with the literature, let us carry out the same calculation for the incompressible Mooney-Rivlin material model (MOONEY [23], RIVLIN [31, 32])

(6.8) $$W = \sum_{R=1}^{2} \left[\frac{\mu_R}{\alpha_R} (\lambda_1^{\alpha_R} + \lambda_2^{\alpha_R} + \lambda_3^{\alpha_R} - 3) \right] + p\,(J-1) = \bar{W} + p\,(J-1)$$

where the elasticity constants

(6.9) $$\mu_1 = \mu,\; \mu_2 = -\gamma\,\mu,\; \alpha_1 = 2,\; \alpha_2 = -2$$

with the parameter γ have been introduced. Using $P_{11} = P_{33}$, the hydrostatic pressure p is derived from

(6.10) $$p = \frac{1}{\lambda_2\,(\lambda_3 - \lambda_1)} \left(\frac{\partial \bar{W}}{\partial \lambda_3} - \frac{\partial \bar{W}}{\partial \lambda_1} \right).$$

With $\mu = 1$, $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \dfrac{1}{\lambda^2}$, the condition (CRIT) reduces to

(6.11) $$\frac{(1-\lambda^3)\,(2\,\gamma - \lambda + \gamma\,\lambda^3)}{2\,\lambda^2} = 0$$

resulting in the so-called bifurcation condtion

(6.12) $$\hat{\gamma}\,(\lambda) = \frac{\lambda}{\lambda^3 + 2}.$$

The maximum of the function $\hat{\gamma}\,(\lambda)$ is $\gamma_{\max} = \dfrac{1}{3}$ for $\lambda = 1$. Thus, a bifurcation is detected only for $0 < \gamma \leq \dfrac{1}{3}$. The same result can be found in BALL and SCHAEFFER [3]. The bifurcation for $\gamma = \dfrac{1}{3}$ occurs at $\lambda = 1$. It then falls together with the bifurcation from the primary path detected also by the term $1 - \lambda^3$ in (6.12). As already discussed before, fulfilling (CRIT) means that $[\mathcal{E}^2]^{\text{stretch}}$ as well as $[\mathcal{E}^2]^{\text{shear(12)}}$ are singular. The relevant eigenmodes are the stretch mode (ST-12) and the shear mode (SH-12). In contrast to the previous investigation based on the neo-Hookean material, (CRIT) indicates for the Mooney-Rivlin material a second bifurcation from the secondary branch into one with distinct principal stretches (if $0 < \gamma < \dfrac{1}{3}$). Evidently, this is a singularity of *Case 2c*.

Elastoplasticity. For the investigation of the elasto-plastic case, we use the Neo-Hooke model in combination with the von Mises yield function and the evolution equations (4.12) presented in Sec. 4.2.

Employing (4.30) and (4.32), we present the material tensor $\mathcal{C}$ in the form

(6.13) $$\mathcal{C} = 2\,\mu\,\mathbf{1}^4_{\text{sym}} + \Lambda\,J_e^2\,\mathbf{1} \otimes \mathbf{1} - \Lambda\,(J_e^2 - 1)\,\mathbf{1}^4_{\text{sym}} - 2\,\mu\,\tilde{k}\,(\mathbf{b}_e \cdot \operatorname{dev} \mathbf{b}_e) \otimes (\mathbf{b}_e \cdot \operatorname{dev} \mathbf{b}_e).$$

Due to the fact that a purely deviatoric flow rule is assumed, there is no evolution of plastic deformation on the primary branch (all principal stretches equal, $\operatorname{dev} \mathbf{b}_e = \operatorname{dev} \mathbf{b} = \mathbf{0}$), the material behaves elastically.

On the secondary branch with $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$, plastic deformation might evolve, if the yield limit is reached. The evaluation of (CRIT) gives us the same information as above (no further bifurcation for the neo-Hookean case). In addition, we have to investigate the criterion $C_{1111} - C_{1122} + \tau_1 = 0$ which is fulfilled for

$$(6.14) \qquad g\,(\lambda_e, \lambda_{3\,e}) = -\frac{\Lambda}{2}\,(J_e^2 - 1) + \mu\,(\lambda_e^2 + 1) - \frac{2}{9}\,\mu\,\tilde{k}\,\lambda_e^4\,(\lambda_e^2 - \lambda_{e\,3}^2)^2 = 0.$$

If the latter function intersects with $f\,(\lambda_e, \lambda_{e\,3}) = 0$ derived from $P_{11} = P_{33}$, a singular point is detected. Whether such an intersection takes place, depends crucially on the function $\tilde{k}$ which controls the influence of the elastoplastic material behaviour on the stability of some test sample. If such a singular point occurs, it is associated with stretch eigenmodes. One speaks of diffuse failure.

6.2. Equibiaxial loading ($P_{L\,ii} = P$ $(i = 1, 2)$, $P_{L\,33} = 0$, $P_{L\,ij} = 0$ with $i \neq j$)

Elasticity. Here, we start investigating the deformation state $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$. The relationship between λ_3 and λ is derived from the statement $P_{33} = 0$ leading to

$$(6.15) \qquad \lambda_3 = \sqrt{\frac{\Lambda + 2\,\mu}{\Lambda\,\lambda^4 + 2\,\mu}}.$$

The three eigenvalues of $[\mathcal{E}^2]^{\text{stretch}}$ are all positive. The condition (CRIT) cannot be fulfilled with the neo-Hookean material model. As expected, one eigenvalue of $[\mathcal{E}^2]^{\text{shear(12)}}$ is equal to one of $[\mathcal{E}^2]^{\text{stretch}}$. The second eigenvalue is zero for the natural state (C-SH-S-2) indicating a singularity of *Case 1* and negative for $\lambda < 1$. The eigenvalues of $[\mathcal{E}^2]^{\text{shear(23)}}$ and $[\mathcal{E}^2]^{\text{shear(31)}}_{2\,\text{x}\,2}$ are positive which is due to the fact that neither $\dfrac{\tau_1}{\lambda_1} = \dfrac{\tau_3}{\lambda_3}$ nor $\dfrac{\tau_1}{\lambda_1} = -\dfrac{\tau_3}{\lambda_3}$ (C-SH-R) can be fulfilled. To conclude, the solution path investigated above is stable for $\lambda > 1$, singular for $\lambda = 1$ and unstable for $\lambda < 1$. But we do not detect any bifurcation for $\lambda \neq 1$, a result which has already been stated in the literature (see e.g. KEARSLEY [20], CHEN [8] and MÜLLER [24, 25]).

Again, in order to compare with previous results let us examine the same example in the context of the incompressible Mooney-Rivlin material model. The hydrostatic pressure

$$(6.16) \qquad p = -\frac{\partial \bar{W}}{\partial \lambda_3}\,\lambda_3$$

is here derived from $P_{33} = 0$. The condition (CRIT) reduces to the very simple equation

$$\frac{1}{2\,\lambda^4}\,(1 + 3\,\gamma\,\lambda^2 + \lambda^6 - \gamma\,\lambda^8) = 0. \tag{6.17}$$

Evidently, for $\gamma = 0$ this equation can never be fulfilled, whereas for the parameters $\gamma > 0$, always a bifurcation into a path with $\lambda_1 \neq \lambda_2$ takes place (singularity of *Case 2c*). For $\gamma = 0.2$, this is the critical stretch $\lambda_{crit} = 2.27$. This result is in agreement with MÜLLER [24, 25]. The second eigenvalue of $[\mathcal{E}^2]^{shear(12)}$ is zero for $\lambda = 1$ (C-SH-S-2) indicating again a singularity of *Case 1*. Note that the secondary branch is singular due to $P_{11} = \dfrac{\tau_1}{\lambda_1} = \dfrac{\tau_2}{\lambda_2} = P_{22}$ (C-SH-R-1). To conclude, for the Mooney-Rivlin material, the primary deformation state is stable only for $1 < \lambda < \lambda_{crit}$, singular for $\lambda = 1$ or $\lambda = \lambda_{crit}$ and unstable for $\lambda < 1$ or $\lambda > \lambda_{crit}$.

Elastoplasticity. In elastoplasticity, the investigation is carried out analogously to the previous case. We obtain again the condition (6.14) which has to be compared for the present loading with $f\,(\lambda_e, \lambda_{e\,3}) = 0$ derived from $P_{33} = 0$.

7. Conclusions

One purpose of the present paper was to show that the material stability behaviour in finite elasticity and elastoplasticity follows a certain logic which has not been fully investigated in previous works. Important is the fact that the basic aspects can be described without specifying a material model. Similar investigations have been mainly carried out in the context of finite elasticity. The closest one is the approach of OGDEN [26] which exploits the coaxiality of the Biot stress tensor $\mathbf{T}$ and the right stretch tensor $\mathbf{U}$. In the present work, we investigate the eigenvalues of the tensor $\mathcal{E}^2$ with the coefficients $E^2_{ijkl} = C_{ijkl} + \delta_{ik}\,\tau_{jl}$. In this way, the stability investigation leads to the more general case of isotropic elastoplasticity, where $[\mathcal{E}^2]$ can be still written in the suitably decoupled form.

The main results of the present work are repeated in the following.

Elasticity:

1) If all stretches are equal, one finds a five-fold zero eigenvalue associated with arbitrary linear combinations of the three stretch modes and the three shear modes. The condition to obtain the zero eigenvalue is $C = -\dfrac{\tau}{2}$ (CRIT).

2) If the stability investigation is based on the assumption that two principal stretches are equal, one detects a double zero eigenvalue for $C = -\dfrac{\tau}{2}$ (CRIT) associated with one stretch mode and one shear mode.

The present examples were based on the so-called Ogden model (see OGDEN [26]). All singularities detected by means of (CRIT) were identified as material instabilities (*Case 2c*).

3) Independently of the material model chosen, we observe singularities of geometrical character (*Case 1*). Apart from the example 3.3 they occur in the natural state and are associated with rigid body rotations as eigenmodes. Obviously, these instabilities can be easily avoided by hindering the rotation of the system.

   It is interesting that *Case 2a* and *Case 2b* do not arise in the context of such common material models like the neo-Hookean and the Mooney-Rivlin material model. This fact confirms the introductory remark that geometrical instabilities (with the exception of *Case 1*) have usually a global character. As such, they do not arise, if the investigation is completely restricted to homogeneous deformation states.

Elastoplasticity:

1) If all stretches are equal, no plastic deformation evolves in the case of a deviatoric flow rule. We obtain elastic material behaviour.
2) In the case of two equal principal stretches, one detects a zero eigenvalue for $C = -\frac{\tau}{2}$ (CRIT) which is associated with a shear eigenmode and one for $C_{iiii} - C_{iijj} + \tau_i = 0$ (stretch eigenode). In contrast to elasticitiy, these singularities do not occur simultaneously.
3) As in elasticity, one observes singularities of purely geometrical character (*Case 1*).
4) Localization phenomena are characterized by eigenmodes of the form $\mathbf{\Phi}^{\mathrm{loc}} = \boldsymbol{\varphi} \otimes \mathbf{n}$.

These results enable a very general investigation of the material stability behaviour. Investigations of this kind could become necessary for two reasons. The first case is that instabilities are observed in an experiment. Then, the developed criteria such as e.g. (CRIT) serve to verify whether the developed material model is realistic enough to exhibit these physical effects. The second and usual case is that one wishes to avoid material instabilities. In such a case, the stability conditions derived in this paper are extremely useful to design a material and could be quite easily implemented in any optimization code.

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Brief Notes

On the order of singularity at V-shaped notches in anisotropic bodies

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The self-similar problem of stress singularity at the notch in infinite two-dimensional elastic orthotropic body was considered. The considerations were restricted to the notches symmetrically oriented with respect to the axes of orthotropy. Both, the extension and shear modes were studied. It was confirmed that in the limiting case of zero opening angle (semi-infinite crack), the order of singularity is the same as in the case of isotropic material - $r^{-1/2}$. This is not true in the case of finite opening angles. If the orientation of the notch axis is parallel to the axis of maximal stiffness, the order of singularity is lower than that for the case of perpendicular orientation. In the last case, if the ratio E_T/E_L is small enough, then the order of singularity in tension does not practically decrease with growing opening angle 2α up to $\alpha \approx \pi/4$.

1. Preliminary remarks

The problem of formulation of the fracture criteria at the tips of V-shaped notches in anisotropic materials needs some knowledge on the order of singularity involved [1] (see also [2, 3]). One may expect that values of this parameter essentially depend not only on the opening angle, like in the case of isotropic material, but on the material anisotropy as well. Another important practical problem of computational mechanics of strongly anisotropic materials consists in a proper choice of the parameters of "singular" finite elements for the calculation of the stress distribution in the vicinity of the notch tip. To this end one also needs exact knowledge on the order of singularity. In the foregoing sections we shall briefly sketch the way leading to the family of analytic self-similar solutions in polar co-ordinates as well as to the numeric procedure of choice of these solutions which fulfill the imposed homogenous boundary value conditions at free edges. In the present paper we shall confine our attention to the cases, when the axis of symmetry of the infinite notch coincides with the axis of orthotropy. The solutions for arbitrarily oriented notches can be readily obtained, however their interpretation is not trivial and it will be postponed to the separate paper.

2. Basic relations

For the description of the plane orthotropic problem we shall follow the way proposed in [4]. We assume that the co-ordinate axes $\{x_1, x_2\}$ are chosen along the axes of symmetry of the orthotropic plane elastic medium. In such a case two-dimensional constitutive relations of linear elasticity can be expressed as follows:

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{E_1}\left(\sigma_{11} + \frac{\gamma_1^2+\gamma_2^2-2\gamma_3^2}{2}\sigma_{22}\right), \\ \varepsilon_{22} &= \frac{1}{E_1}\left(\gamma_1^2\gamma_2^2\,\sigma_{22} + \frac{\gamma_1^2+\gamma_2^2-2\gamma_3^2}{2}\sigma_{11}\right), \\ \varepsilon_{12} &= \frac{\gamma_3^2}{E_1}\sigma_{12}, \end{aligned} \tag{2.1}$$

where E_1 denotes Young's modulus corresponding to the uniaxial tension in x_1 direction[1], $\gamma_1, \gamma_2, \gamma_3$ are dimensionless constants, fulfilling the following relations:

$$\gamma_1^2\gamma_2^2 = \frac{E_1}{E_2}, \qquad \gamma_1^2+\gamma_2^2 = 2\left(\frac{E_1}{2\mu} - \nu_{12}\right), \qquad \gamma_3^2 = \frac{E_1}{2\mu}.$$

The meaning of E_2, μ and ν_{12} is obvious. Without any loss of generality $\gamma_1, \gamma_2, \gamma_3$ can be assumed to be positive. Imposing the conditions of the Poisson ratios and elastic energy positiveness, one can obtain the following constraints which should be imposed on γ_1, γ_2, and γ_3:

$$\gamma_1^2+\gamma_2^2 < 2\gamma_3^2 < (\gamma_1+\gamma_2)^2. \tag{2.2}$$

This means that two of them, e.g. γ_1 and γ_2, can, in principle, assume any positive values.

Let us introduce Airy stress function $\Phi(x_1, x_2)$, such that

[1] We assume plane stress conditions, where all material constants under consideration are the same as in the three-dimensional case, corresponding values for the plane strain can be readily obtained by assuming $\varepsilon_{33} = 0$.

$$\begin{aligned} \sigma_{11} &= \Phi_{,22}, \\ \sigma_{22} &= \Phi_{,11}, \\ \sigma_{12} &= -\Phi_{,12}, \end{aligned} \tag{2.3}$$

where comma denotes partial derivative in Cartesian co-ordinates. Combining relations (2.3) and (2.1) and substituting the result into the strain compatibility condition

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12}, \tag{2.4}$$

one can obtain in a standard way the following orthotropic counterpart of the biharmonic equation describing the isotropic material:

$$\Phi_{,2222} + \left(\gamma_1^2 + \gamma_2^2\right)\Phi_{,1122} + \gamma_1^2\gamma_2^2\Phi_{,1111} = 0 \tag{2.5}$$

(compare (2.9) in [4]). The last equation can be rewritten in the following form:

$$\begin{aligned} &\left(\frac{\partial^2}{\partial x_2^2} + \gamma_1^2\frac{\partial^2}{\partial x_1^2}\right)\left(\frac{\partial^2}{\partial x_2^2} + \gamma_2^2\frac{\partial^2}{\partial x_1^2}\right)\Phi\left(x_1, x_2\right) \\ &\qquad = \left(\frac{\partial}{\partial x_2} + i\gamma_1\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} - i\gamma_1\frac{\partial}{\partial x_1}\right) \\ &\qquad \times \left(\frac{\partial}{\partial x_2} + i\gamma_2\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} - i\gamma_2\frac{\partial}{\partial x_1}\right)\Phi\left(x_1, x_2\right) = 0, \end{aligned} \tag{2.6}$$

where the symbol $\times$ (used here for typographic reasons only) denotes superposition of operators, $i = \sqrt{-1}$. Thus, any differentiable function of any of the following complex variables:

$$\eta = x_1 + i\gamma_1 x_2, \quad \overline{\eta} = x_1 - i\gamma_1 x_2, \quad \xi = x_1 + i\gamma_2 x_2, \quad \overline{\xi} = x_1 - i\gamma_2 x_2, \tag{2.7}$$

satisfies Eq. (2.5).

Looking for singular solution it is reasonable to take into consideration power functions of these variables. Adopting polar co-ordinates one can express e.g. η as follows:

$$\eta = r\left(\cos\varphi + i\gamma_1\sin\varphi\right). \tag{2.8}$$

Note that here variables r and φ do not denote the modulus and argument of η, instead we have

$$|\eta| = r\left(\cos^2\varphi + \gamma_1^2\sin^2\varphi\right)^{\frac{1}{2}}, \qquad \operatorname{Arg}(\eta) = \operatorname{Arctan}\left(\gamma_1\tan\varphi\right), \tag{2.9}$$

thus, one can write:

$$\eta^{2-\lambda} = r^{2-\lambda}\left(\cos^2\varphi + \gamma_1^2\sin^2\varphi\right)^{1-\lambda/2} \times \{\cos\left[(2-\lambda)\operatorname{Arctan}(\gamma_1\tan\varphi)\right] + i\sin\left[(2-\lambda)\operatorname{Arctan}(\gamma_1\tan\varphi)\right]\}. \tag{2.10}$$

For practical calculations certain care must be taken to keep the values

$$\beta_1(\varphi) \equiv \operatorname{Arctan}(\gamma_1\tan\varphi), \qquad \beta_2(\varphi) \equiv \operatorname{Arctan}(\gamma_2\tan\varphi) \tag{2.11}$$

in the same quarter-plane as φ: $\operatorname{Sgn}(\cos\beta_i) = \operatorname{Sgn}(\cos\varphi)$, $\operatorname{Sgn}(\sin\beta_i) = \operatorname{Sgn}(\sin\varphi)$ for $i = 1, 2$.

For further considerations we shall restrict our attention to real values of λ parameter[2]. For this case we can look for the following Airy stress function:

$$F(r,\varphi) = r^{2-\lambda}\left[\Phi(\varphi) + \Psi(\varphi)\right], \tag{2.12}$$

where

$$\begin{aligned} \Phi(\varphi) &= A\Phi_1(\varphi) + B\Phi_2(\varphi), \\ \Psi(\varphi) &= C\Psi_1(\varphi) + D\Psi_2(\varphi), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \Phi_1(\varphi) &= \left(\cos^2\varphi + \gamma_1^2\sin^2\varphi\right)^{1-\lambda/2}\cos\left[(2-\lambda)\beta_1\right], \\ \Phi_2(\varphi) &= \left(\cos^2\varphi + \gamma_2^2\sin^2\varphi\right)^{1-\lambda/2}\cos\left[(2-\lambda)\beta_2\right], \\ \Psi_1(\varphi) &= \left(\cos^2\varphi + \gamma_1^2\sin^2\varphi\right)^{1-\lambda/2}\sin\left[(2-\lambda)\beta_1\right], \\ \Psi_2(\varphi) &= \left(\cos^2\varphi + \gamma_2^2\sin^2\varphi\right)^{1-\lambda/2}\sin\left[(2-\lambda)\beta_2\right], \end{aligned} \tag{2.14}$$

A, B, C, D are arbitrary constants.

3. Solution of the boundary value problem

In the polar co-ordinate system, the following expressions for the stress field components in terms of Airy function derivatives hold true [5]:

[2]The authors hope to return in the future to the discussion on the physical sense of high-amplitude oscillations of stress at the vicinity of singular point like $r^{-\alpha}\cos(\ln(r))$, which would follow from the imaginary part of λ if an expression of type (2.10) were taken as a stress function.

$$\begin{aligned}\sigma_{rr} &= \frac{1}{r}\frac{\partial F(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F(r,\varphi)}{\partial \varphi^2},\\ \sigma_{\varphi\varphi} &= \frac{\partial^2 F(r,\varphi)}{\partial r^2},\\ \sigma_{r\varphi} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F(r,\varphi)}{\partial \varphi}\right).\end{aligned} \tag{3.1}$$

As it was already mentioned, in the present paper we shall focus our attention on the restricted class of boundary value problems: V-shaped notches of the opening angle 2α, $0 < \alpha < \pi/2$ symmetric with respect to x_1 axis (compare Fig. 1).

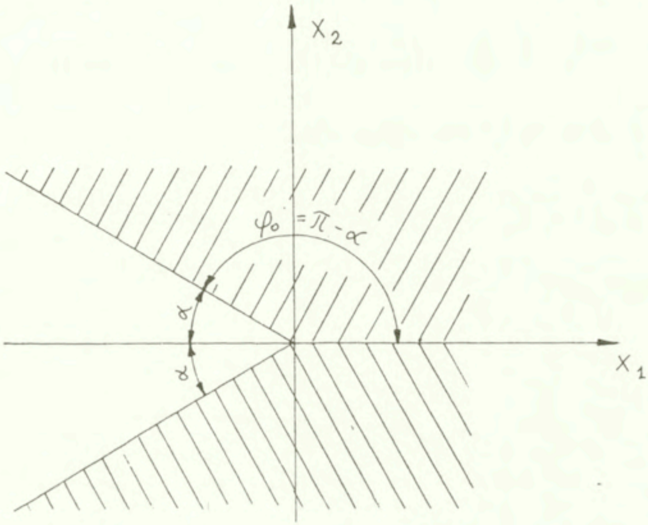

FIG. 1. Orientation of the V-shaped notch with respect to axes of orthotropy.

We assume also the absence of contact forces at the boundaries:

$$\begin{aligned}\sigma_{\varphi\varphi}(r, \pm\varphi_0) &= 0,\\ \sigma_{r\varphi}(r, \pm\varphi_0) &= 0,\end{aligned} \tag{3.2}$$

where $\varphi_0 = \pi - \alpha$. Bearing in mind the symmetry properties of the functions $\Phi(\varphi)$ and $\Psi(\varphi)$ and relations (3.1), one can prove that conditions (3.2) split into two independent problems:

$$\begin{aligned}A\Phi_1(\varphi_0) + B\Phi_2(\varphi_0) &= 0,\\ A\Phi_1'(\varphi_0) + B\Phi_2'(\varphi_0) &= 0,\end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
C\Psi_1(\varphi_0) + D\Psi_2(\varphi_0) &= 0, \\
C\Psi'_1(\varphi_0) + D\Psi'_2(\varphi_0) &= 0,
\end{aligned}
\tag{3.4}
$$

where "prime" stands for the first derivative with respect to φ.

Equations (3.3) describe Mode I (tension) while system (3.4) corresponds to the Mode II (shear). Characteristic equation associated with (3.3) has the following form:

$$
\begin{aligned}
&\left(\gamma_2^2 - \gamma_1^2\right)\cos\left[(2-\lambda)\beta_1(\varphi_0)\right]\cos\left[(2-\lambda)\beta_2(\varphi_0)\right]\tan\varphi_0 \\
&\quad - \left(1+\gamma_1^2\right)\gamma_2\cos\left[(2-\lambda)\beta_1(\varphi_0)\right]\sin\left[(2-\lambda)\beta_2(\varphi_0)\right] \\
&\quad + \left(1+\gamma_2^2\right)\gamma_1\cos\left[(2-\lambda)\beta_2(\varphi_0)\right]\sin\left[(2-\lambda)\beta_1(\varphi_0)\right] = 0,
\end{aligned}
\tag{3.5}
$$

while Eqs. (3.4) yield the following condition

$$
\begin{aligned}
&\left(\gamma_2^2 - \gamma_1^2\right)\sin\left[(2-\lambda)\beta_1(\varphi_0)\right]\sin\left[(2-\lambda)\beta_2(\varphi_0)\right]\tan\varphi_0 \\
&\quad + \left(1+\gamma_1^2\right)\gamma_2\sin\left[(2-\lambda)\beta_1(\varphi_0)\right]\cos\left[(2-\lambda)\beta_2(\varphi_0)\right] \\
&\quad - \left(1+\gamma_2^2\right)\gamma_1\sin\left[(2-\lambda)\beta_2(\varphi_0)\right]\cos\left[(2-\lambda)\beta_1(\varphi_0)\right] = 0.
\end{aligned}
\tag{3.6}
$$

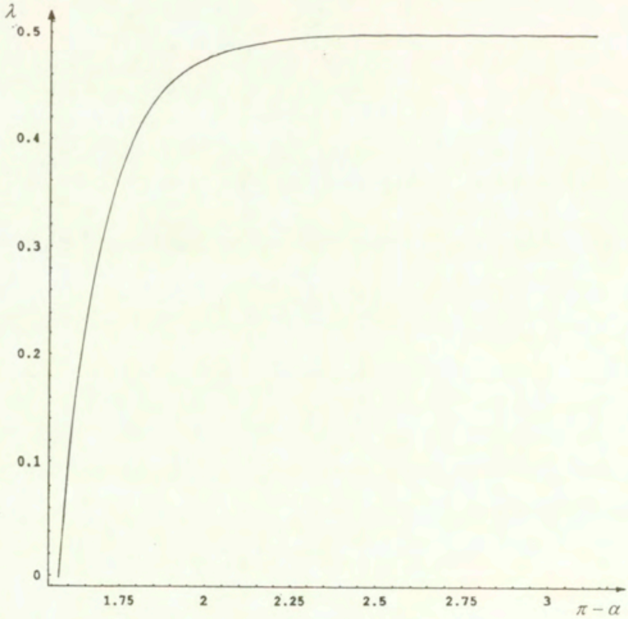

FIG. 2. Order of singularity λ versus decreasing opening angle α, Mode I, $\gamma_1 = 0.4$, $\gamma_2 = 0.3$ ($E_1/E_2 = 0.0144$), notch axis perpendicular to the axis of maximal stiffness.

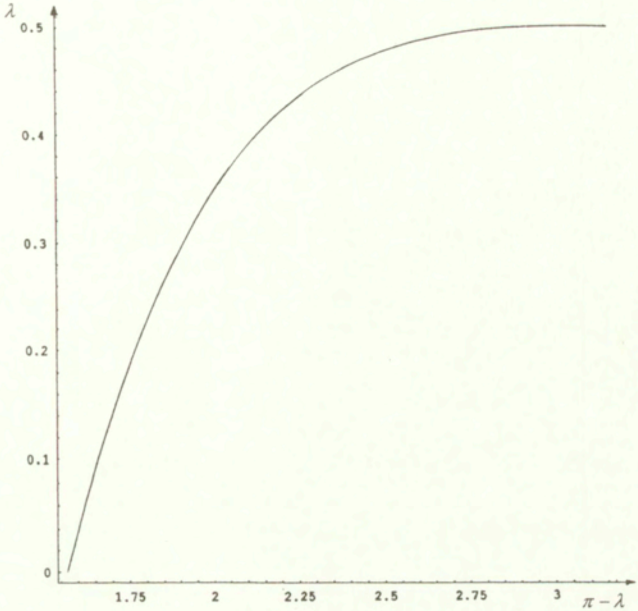

FIG. 3. Order of singularity λ versus decreasing opening angle α, Mode I, almost isotropic material $\gamma_1 = 0.99$, $\gamma_2 = 1.01$.

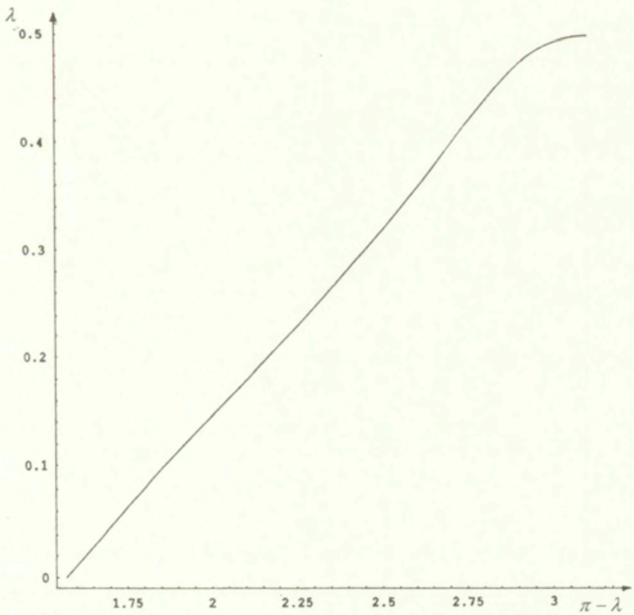

FIG. 4. Order of singularity λ versus decreasing opening angle α, Mode I, $\gamma_1 = 4$, $\gamma_2 = 3$ $(E_1/E_2 = 144)$, notch axis parallel to the axis of maximal stiffness.

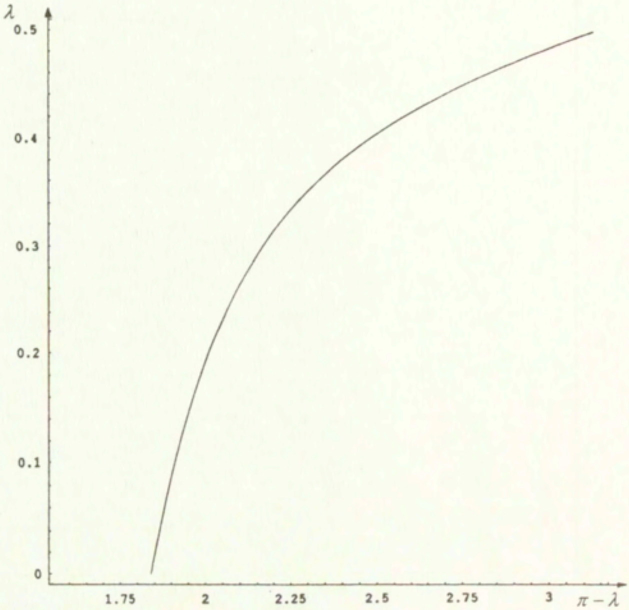

FIG. 5. Order of singularity λ versus decreasing opening angle α, Mode II, $\gamma_1 = 0.4$, $\gamma_2 = 0.3$ ($E_1/E_2 = 0.0144$), notch axis perpendicular to the axis of maximal stiffness.

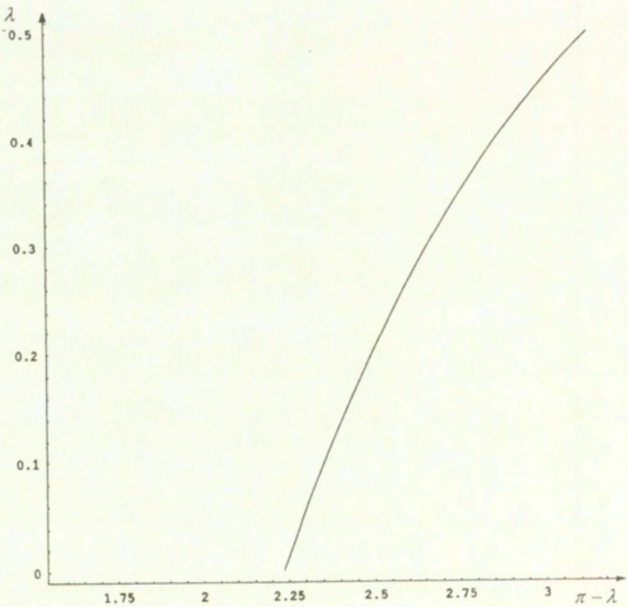

FIG. 6. Order of singularity λ versus decreasing opening angle α, Mode II, almost isotropic material $\gamma_1 = 0.99$, $\gamma_2 = 1.01$.

In Figs. 2, 3, 4 are shown, for different values of γ_2 and γ_2, contour plots (zero level only) of the function defined by the left-hand side of Eq. 3.5. The curves join the points at which Eq. (3.5) is satisfied. Similar plots for Eq. (3.6) are shown in Figs. 5, 6, 7.

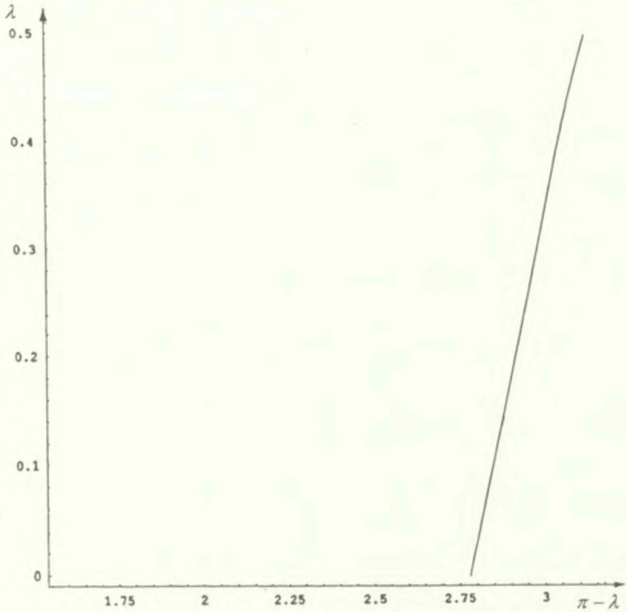

Fig. 7. Order of singularity λ versus decreasing opening angle α, Mode II, $\gamma_1 = 4$, $\gamma_2 = 3$ ($E_1/E_2 = 144$), notch axis parallel to the axis of maximal stiffness.

4. Concluding remarks

It can be readily seen that even strong anisotropy does not change qualitatively the results which had been found earlier for the isotropic case (cf. [1]). It is a proper place here to recall that the ratio of Young moduli E_1/E_2 is equal to the product of the squared gammas $\gamma_2^2\gamma_1^2$, thus, in our examples, longitudinal modulus differs from the transversal one by two orders of magnitude. The following quantitative effects can be observed: the order of singularity λ for both modes of loading is lower for the case of notches having their symmetry axes parallel to the axis of maximal stiffness, and higher for the perpendicular orientation. The isotropic case takes the intermediate position.

As it has been mentioned earlier, the problems of arbitrarily oriented V-notches and, possibly, of the stress distribution at the vicinity of the contact points of three differently oriented wedges made of the same anisotropic material (modelling polycrystalline solids), will be considered in a separate paper.

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