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A note on the existence and global stability of deformations of compressible nonlinearly elastic solids

M. ARON and C. CHRISTOPHER

School of Mathematics and Statistics University of Plymouth
Drake Circus Plymouth,
PL4 8AA United Kingdom

The existence and global stability of certain deformations of isotropic compressible hyperelastic solids is investigated for the case when the body forces are zero and boundary conditions of place are satisfied. Certain global stability results are obtained and, using a phase-plane analysis, the existence of deformations describing the bending of rectangular blocks into annular cylindrical sectors is established.

1. Introduction

The main problem of nonlinear elastostatics is the determination of solutions to equilibrium equations which satisfy appropriate boundary conditions and which are stable, in the sense that they minimise the total energy relative to an appropriate class of variations. Exact solutions to relevant boundary value problems are, however, rarely available and when approximate solutions are sought, it is essential that the existence and stability of solutions are established beforehand. As discussed recently in [1], there are two ways of answering questions of this nature. One way is to establish the existence of minimisers for the total energy within an appropriate function space which are subsequently shown to satisfy the corresponding Euler-Lagrange equation; another way is to establish the existence of solutions to the boundary value problem for the Euler-Lagrange equation, which are subsequently shown to be minimizers to the total energy.

In this paper we are concerned with the existence and global stability (i.e. stability relative to variations of class C^1 and of arbitrary magnitude) of equilibrium solutions for isotropic compressible elastic solids and, in this context, we draw attention to the existence and uniqueness result established in [2] for deformations that describe the straightening of annular cylindrical sectors and to the global stability results obtained in [3] – [6]. Here, assuming (as in [2] – [6]) that the body forces are zero and that the deformations satisfy boundary conditions of place, we obtain certain generalisations of the stability results in [5, 6] (which, in turn, are generalisations of the global stability results obtained in [2, 3] respectively) and using an approach similar to that adopted in [2], find con-

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ditions which ensure the existence and uniqueness of deformations that describe the bending of rectangular blocks into annular cylindrical sectors. We show that in certain instances it is possible to combine these results to conclude that certain deformations exist and are globally stable.

2. Preliminaries

We consider deformations of isotropic compressible hyperelastic solids which, in a reference configuration, occupy a domain D of the Euclidean space R^3. Such deformations are smooth and invertible transformations $\mathbf{f} : D \to \hat{D}$, where $\hat{D}$ denotes another domain of R^3. The deformations $\mathbf{f}$ are required to satisfy

$$\det\mathbf{F} > 0, \qquad \mathbf{F} \equiv \nabla\mathbf{f}, \tag{2.1}$$

where det and ∇ stand for determinant and gradient, respectively. The principal stretches, denoted here by λ_i, $i = 1,2,3$, are the eigenvalues of $\left(\mathbf{F}\mathbf{F}^T\right)^{1/2}$ where $\mathbf{F}^T$ denotes the transpose of $\mathbf{F}$.

The materials under consideration are characterised by strain-energy functions

$$W = W(\mathbf{F}) = \hat{W}(\lambda_1, \lambda_2, \lambda_3) \tag{2.2}$$

whose domains are restricted to deformations that obey (2.1), which are symmetric in λ_1, λ_2, λ_3, and which satisfy

$$\hat{W}(\lambda_1, \lambda_2, \lambda_3) \geq 0, \qquad \hat{W}_i(1,1,1) = 0, \qquad \hat{W}_i \equiv \frac{\partial \hat{W}}{\partial \lambda_i}, \qquad i = 1,2,3, \tag{2.3}$$

equality in $(2.3)_1$ being possible if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 1$. If $W = W(\mathbf{F})$ is of class C^2, then it is convex if and only if [7]

$$\begin{gathered} \sum_{i,j=1}^{3} \hat{W}_{ij}\xi_i\xi_j \geq 0, \quad \forall(\xi_1,\xi_2,\xi_3) \in R^3, \quad \hat{W}_{ij} \equiv \frac{\partial^2 \hat{W}}{\partial\lambda_i \partial\lambda_j}, \quad i,j,= 1,2,3, \\ \hat{W}_m + \hat{W}_n \geq 0, \qquad m,n = 1,2,3, \quad m \neq n. \end{gathered} \tag{2.4}$$

As pointed out by many authors (see [8], Sec. 52, for instance), the condition (2.4) is not acceptable in nonlinear elasticity.

The stress-deformation relation is

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \tag{2.5}$$

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where $\mathbf{S}$ denotes the first Piola-Kirchhoff stress tensor [8, Sec. 43A] and the equilibrium condition reads

$$\text{Div}\,\mathbf{S}(\mathbf{F}) = \mathbf{0} \quad \text{on } D, \tag{2.6}$$

when the body forces are zero. According to the principle of virtual work ([9], Sec. 2.6), an equilibrium solution $\mathbf{f}$ satisfies

$$\int_D \operatorname{tr}\left[\mathbf{S}(\nabla\mathbf{v})^T\right] dD = 0 \tag{2.7}$$

for all $\mathbf{v} \in V \equiv \{\mathbf{v} \in C^1(D) : \mathbf{v}|_{\partial D} = 0,\ \det(\mathbf{F} + \nabla\mathbf{v}) > 0\}$, where tr is the trace operator, $C^1(D)$ is the set of continuously differentiable functions on D and ∂D is the boundary of D.

In the absence of body forces, an equilibrium solution $\mathbf{f}$ (that satisfies boundary conditions of place) is said to be globally stable if

$$\int_D W(\mathbf{F} + \nabla\mathbf{v})dD \geq \int_D W(\mathbf{F})dD, \qquad \mathbf{F} = \nabla\mathbf{f}, \tag{2.8}$$

for every variation $\mathbf{v} \in V$. When D is a bounded open set, a strain-energy function W which satisfies the condition (2.8) at a particular deformation gradient $\mathbf{F}$ is said to be quasiconvex at $\mathbf{F}$ (see [9], Sec. 5, for instance).

3. Stability results

In this section we confine our attention to a sub-class of the class of materials characterised by (2.2) and (2.3), namely the materials whose strain-energy function can be written in the form

$$\hat{W} = H\left[\hat{P}(\lambda_1, \lambda_2, \lambda_3)\right] + h(J), \tag{3.1}$$

where $P(\mathbf{F}) \equiv \hat{P}(\lambda_1, \lambda_2, \lambda_3)$ is a scalar-valued function (assumed to be symmetric in $\lambda_1, \lambda_2, \lambda_3$) and where, for convenience, we employ the notation $J \equiv \lambda_1\lambda_2\lambda_3$. Our first result is similar to that established in [3] and [5].

PROPOSITION 1. Let $h(J) = aJ + b$, where a and b are constants, let $P = P(\mathbf{F})$ be a convex function, and let $H'' \geq 0$. Then, for boundary conditions of place and zero body forces, an equilibrium solution $\mathbf{f}^*$ is globally stable provided that

$$H'\left[P(\mathbf{F}^*)\right] \geq 0, \qquad \mathbf{F}^* \equiv \nabla\mathbf{f}^*. \tag{3.2}$$

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P r o o f. By the well-known identities $\partial J/\partial \mathbf{F} \equiv J\mathbf{F}^{-T}$ and $\mathrm{Div}(J\mathbf{F}^{-T}) \equiv \mathbf{0}$ (see [5] and [10]), it is clear that we have $\mathrm{Div}(\partial h/\partial \mathbf{F}) = \mathbf{0}$ and thus, for the materials under consideration, the equilibrium condition (2.6) reduces to

$$\mathrm{Div}\left\{ H'\left[P(\mathbf{F})\right] \frac{\partial P}{\partial \mathbf{F}} \right\} = \mathbf{0}. \tag{3.3}$$

By assumption, $\mathbf{f}^*$ is an equilibrium solution and therefore, in view of (2.7), (3.3) leads to

$$\int_D H'\left[P(\mathbf{F}^*)\right] \mathrm{tr}\left[\frac{\partial P}{\partial \mathbf{F}}(\mathbf{F}^*)(\nabla \mathbf{v})^T\right] dD = 0, \qquad \forall \mathbf{v} \in V. \tag{3.4}$$

Our convexity assumptions also imply

$$P(\mathbf{F}^* + \nabla \mathbf{v}) - P(\mathbf{F}^*) \geq \mathrm{tr}\left[\frac{\partial P}{\partial \mathbf{F}}(\mathbf{F}^*)(\nabla \mathbf{v})^T\right], \qquad \forall \mathbf{v} \in V, \tag{3.5}$$

and

$$H[P(\mathbf{F}^* + \nabla \mathbf{v})] - H[P(\mathbf{F}^*)] \geq H'[P(\mathbf{F}^*)][P(\mathbf{F}^* + \nabla \mathbf{v}) - P(\mathbf{F}^*)], \qquad \forall \mathbf{v} \in V, \tag{3.6}$$

which, together with (3.2) and (3.4), yield

$$\int_D \{H[P(\mathbf{F}^* + \nabla \mathbf{v})] - H[P(\mathbf{F}^*)]\} dD \geq 0. \tag{3.7}$$

Since

$$\int_D \{h\left[J(\mathbf{F}^* + \nabla \mathbf{v})\right] - h\left[J(\mathbf{F}^*)\right]\} \, dD = 0, \qquad \forall \mathbf{v} \in V, \tag{3.8}$$

we find (see (3.1)) that

$$\int_D \left[W(\mathbf{F}^* + \nabla \mathbf{v}) - W(\mathbf{F}^*)\right] dD \geq 0, \qquad \forall \mathbf{v} \in V, \tag{3.9}$$

which shows that $\mathbf{f}^*$ is globally stable.

For illustration we restrict ourselves to the case of plane strain and consider a class of materials characterised by a strain-energy function of the form

$$\hat{W} = \frac{2\mu}{\gamma}\left[H(\lambda_1^\gamma + \lambda_2^\gamma) - \lambda_1\lambda_2\right] \tag{3.10}$$

where $\mu > 0$ and $\gamma > 1$ are constants and where H is a function that satisfies $H(2) = 1$, $H'(2) = 1/\gamma$, $H'' > 0$. We note that if $\gamma = 1$, (3.10) represents

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the class of harmonic materials introduced in [11]. An example of material that belongs to the class (3.10) is provided by

$$\hat{W} = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2\right) + \frac{\lambda+\mu}{8}\left(\lambda_1^2 + \lambda_2^2 - 2\right)^2 - \mu\lambda_1\lambda_2, \tag{3.11}$$

where μ and λ are the infinitesimal Lamé moduli that satisfy $\mu > 0$, $\mu + \lambda > 0$.

Using the specialisation of conditions (2.4) to plane strain (see [7]), we find that

$$P(\mathbf{F}) = \hat{P}(\lambda_1, \lambda_2) = \lambda_1^\gamma + \lambda_2^\gamma, \qquad \gamma > 1, \tag{3.12}$$

is convex. Also, according to the existence result established in [2], equilibrium solutions of the form

$$x = f(R), \qquad y = \alpha\Theta \qquad \alpha = \text{const}, \qquad \alpha > 0, \tag{3.13}$$

where (x, y) are Cartesian coordinates in $\hat{D}$ and (R, Θ) are polar coordinates in D, exist for materials (3.10), boundary conditions of place, and zero body forces, provided that

$$(\gamma - 1)H' + \gamma\lambda_1^\gamma H'' > 0. \tag{3.14}$$

We note that the condition (3.14) is equivalent (for material (3.10)) to the tension-extension condition ([8], Sec. 51)

$$\hat{W}_{11} > 0 \tag{3.15}$$

and that the deformation (3.13) represents the straightening of annular cylindrical sectors. Clearly, if the condition $H' > 0$ is satisfied (for materials (3.11) this is equivalent to $\lambda < 0$), by the above proposition (3.13) are globally stable solutions (relative to the class of materials (3.10)) for any of the boundary conditions of place. If the latter condition is violated however, the stability condition (3.2) may be satisfied only if the boundary conditions are restricted in a suitable manner (see [5], for instance).

Our next Proposition 2 generalises (in a certain sense) the global stability results obtained in [4] and [6].

PROPOSITION 2. *Let $h'' \geq 0$ ($h'' \not\equiv 0$), $H'' \geq 0$ and let $P = P(\mathbf{F})$ be a convex function. Then, for boundary conditions of place and zero body forces, an equilibrium solution $\mathbf{f}^*$ that obeys the condition*

$$\det(\nabla \mathbf{f}^*) = \text{const}, \tag{3.16}$$

is globally stable provided that the condition (3.2) is satisfied.

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P r o o f. Since, by (3.16) and the identities quoted at the beginning of the proof of Proposition 1, we have

$$\text{Div}\left[\frac{\partial h}{\partial \mathbf{F}}(\det \mathbf{F}^*)\right] = \mathbf{0} \tag{3.17}$$

we find, as before, that (3.2) implies (3.7) and, by the convexity of h, it also follows that

$$\int_D \{h[J(\mathbf{F}^* + \nabla \mathbf{v})] - h[J(\mathbf{F}^*)]\}\, dD \geq h'[J(\mathbf{F}^*)] \cdot \int_D [J(\mathbf{F}^* + \nabla \mathbf{v}) - J(\mathbf{F}^*)]\, dD = 0, \qquad \forall \mathbf{v} \in V. \tag{3.18}$$

The condition (3.9) therefore follows from (3.7) and (3.18).

4. An existence result

In this section we consider the plane-strain bending of a rectangular block into an annular cylindrical sector. This deformation may be given in the form

$$r = f(X), \qquad \theta = \alpha Y, \qquad \alpha = \text{const}, \qquad \alpha > 0, \tag{4.1}$$

where (r, θ) are polar coordinates in $\hat{D}$ and (X, Y) are Cartesian coordinates in D, and the equilibrium condition is (see [12], Sec. 5.2)

$$\begin{gathered} \frac{d}{dX}\hat{W}_1[f'(X), \alpha f(X)] = \alpha \hat{W}_2[f'(X), \alpha f(X)], \\ X \in (-A, A), \qquad A = \text{ const}, \quad A > 0. \end{gathered} \tag{4.2}$$

In order to satisfy the boundary conditions of place we prescribe α and require

$$f(-A) = f_1, \qquad f(A) = f_2, \qquad f_2 > f_1 > 0. \tag{4.3}$$

We seek solutions to the boundary-value problem described above, which satisfy

$$f, f' > 0 \qquad \text{on} \qquad X \in [-A, A], \tag{4.4}$$

and to this end we assume that $\hat{W}$ is of class C^3 and satisfies the tension-extension condition (3.15).

The region of physical interest (see (4.3) and (4.4)) is given by

$$\Omega \equiv \{(f, g) : f_1 \leq f \leq f_2, g > 0\}, \tag{4.5}$$

where g represents the possible values of f', and we define

$$\varphi(f, g) \equiv g\hat{W}_1(g, \alpha f) - \hat{W}(g, \alpha f), \qquad (f, g) \in \Omega. \tag{4.6}$$

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Clearly any C^2 solution of (4.2) satisfies

$$\varphi(f,g) = \varphi(f(X), f'(X)) = c, \tag{4.7}$$

for some constant c, and conversely, by the condition (see (3.15))

$$\frac{\partial \varphi}{\partial g} > 0, \qquad (f,g) \in \Omega, \tag{4.8}$$

and the implicit function theorem, any solution of (4.7) which is in Ω must be of class C^2 and satisfy (4.2).

Solutions of (4.2) also correspond to solutions of the two-dimensional system

$$\frac{d}{dX} f = g, \qquad \frac{d}{dX} g = \frac{\alpha\left(\hat{W}_2(g,\alpha f) - g\hat{W}_{12}(g,\alpha f)\right)}{\hat{W}_{11}(g,\alpha f)} \tag{4.9}$$

and thus we may regard X as a time parameter of an autonomous planar system defined in Ω. The level curves $\varphi = c$ therefore give the trajectories of (4.9). Since the vector field (4.9) is C^1, it is locally Lipshitz and, given any point (f_0, g_0) in Ω and some value X_0 of X, by standard results (see e.g. [13]) there exists a unique maximal solution $(f(X), g(X))$ of (4.9) with $f(X_0) = f_0$, $g(X_0) = g_0$, which must be at least C^2. We can choose such a solution to start at $f = f_1$, at time $X = -A$, as required.

The condition (4.8) implies that each level curve intersects each line $f =$ constant at most once. Thus a trajectory can pass from $f = f_1$ to $f = f_2$ if and only if there is a value of c for which the level curve $\varphi = c$ intersects each line $f =$ constant between $f = f_1$ and $f = f_2$. Introducing the definitions

$$\begin{aligned} q_{\min}(f) &\equiv \lim_{g \to 0} \varphi(f,g) \in [-\infty, \infty), \\ q_{\max}(f) &\equiv \lim_{g \to \infty} \varphi(f,g) \in (-\infty, \infty], \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} q_0 &\equiv \max_{f \in [f_1, f_2]} q_{\min}(f) \in [-\infty, \infty], \\ q_1 &\equiv \min_{f \in [f_1, f_2]} q_{\max}(f) \in [-\infty, \infty], \end{aligned} \tag{4.11}$$

we find that c must belong to the interval (q_0, q_1) and thus we must certainly have $q_0 < q_1$ for a solution of (4.2) that satisfies (4.3).

Let $c \in (q_0, q_1)$ and denote the solution curve $(f, g_c(f))$, where $g_c > 0$, the curve which satisfies $\varphi = c$. As discussed above, each curve defines a unique

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trajectory $(f(X), g(X))$ such that $f(-A) = f_1$. It will also satisfy $f(A) = f_2$ if and only if the time taken to cross from $f = f_1$ to $f = f_2$ is $2A$, that is

$$\int_{f_1}^{f_2} \frac{1}{f'(X)} df = \int_{f_1}^{f_2} \frac{1}{g_c} df = 2A. \tag{4.12}$$

Since $\varphi(f, g_c(f)) = c$ we have (by (4.8)) that, for fixed f,

$$\frac{d}{dc} \int_{f_1}^{f_2} \frac{1}{g_c} df = -\int_{f_1}^{f_2} \frac{g_c'}{g_c^2} df = -\int_{f_1}^{f_2} \frac{1}{\varphi_g g_c^2} df < 0, \tag{4.13}$$

which shows that there is at most one solution which satisfies the boundary conditions.

The maximum and minimum values of the integral in (4.12) are given when c approaches q_0 and q_1 respectively. Defining

$$I_k \equiv \lim_{c \to q_k} \int_{f_1}^{f_2} \frac{1}{g_c} df, \qquad k = 0, 1, \tag{4.14}$$

we have demonstrated the following proposition.

PROPOSITION 3. If the $T - E$ condition (3.15) is satisfied, then there is a unique solution of class C^2 to Eq. (4.2), which satisfies Eqs. (4.3) and (4.4), if and only if $q_0 < q_1$ and

$$I_1 < 2A < I_0. \tag{4.15}$$

REMARK. Interpreting the following integrals in the obvious sense when the integrand is infinite (see [2]), we have

$$I_k = \int_{f_1}^{f_2} \frac{1}{g_{q_k}} df, \qquad k = 0, 1. \tag{4.16}$$

Thus if $q_{\max}(f)$ is constant, then the level curves tend uniformly to infinity as c increases, and so $I_1 = 0$. Similarly, if $q_{\min}(f)$ is constant, then $I_0 = \infty$. It then follows that if condition (3.15) is satisfied, and if $q_{\max}(f)$ and $q_{\min}(f)$ are both constant, for each given value A, there is a unique solution to the bending problem for all possible boundary conditions of place.

An example of material which satisfies the condition (3.15) and for which $q_{\max}(f)$ and $q_{\min}(f)$ are both constant is provided by

$$\hat{W} = \mu \left[\frac{1}{2} \lambda_1^3 \lambda_2 + \frac{1}{2} \lambda_1 \lambda_2^3 - 2\lambda_1 \lambda_2 + 1 \right], \qquad \mu = \text{const}, \quad \mu > 0. \tag{4.17}$$

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From (4.7) we find in this case that

$$f(X) = (C_1 X + C_2)^{3/4} \tag{4.18}$$

where C_1, C_2 are constants of integration. It is easily seen in this instance that, from the boundary conditions (4.3), the arbitrary constants can be determined uniquely for any choice of f_1 and f_2 (with $f_2 > f_1 > 0$).

For materials (3.10) (where we now allow $\gamma \geq 1$ and assume that (3.14) holds) however, we find that (see (4.10))

$$q_{\min}(f) = -\frac{2\mu}{\gamma} H(\alpha^\gamma f^\gamma), \tag{4.19}$$

which shows that, for a prescribed value of A, the condition (4.15) restricts the choice of the boundary conditions available. As discussed in the previous section, this choice may be further restricted by the stability requirement (3.2). For a harmonic material of a special kind both these types of restrictions are considered in [14].

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Suction through point-like opening and stability of boundary layer

I. Yu. POPOV

Department of Higher Mathematics,
Leningrad Institute of Fine Mechanics and Optics,
14 Sablinskaya, St.-Petersburg, 197101, Russia, USSR

A MODEL OF BOUNDARY layer suction through small opening is suggested. A procedure of neutral curve construction is described. The dependence of the shift of neutral curves on the intensity of suction and on the position of the opening is studied.

Key words: boundary layer, hydrodynamic stability, laminar flow, Orr-Sommerfeld equation, Tollmien-Schlichting wave.

1. Introduction

THE LAMINAR-TURBULENT transition in wall boundary layers is important for many practical applications. To study the problem it is necessary to treat the downstream development of the disturbances. The transition is strongly influenced by nonlinear effects, which become important when, by growth, the unstable disturbance has reached a certain level [20]. But at the first stage of the process when disturbance is sufficiently small, one can consider the linearized problem [3, 8]. Linear boundary layer stability problem reduces to investigation of Tollmien-Schlichting wave evolution [19]. In the framework of this approach stability corresponds to the downstream decrease of the Tollmien-Schlichting wave amplitude, and instability to its increase.

Suction of boundary layer is a widely used method of stream laminarization [7, 19]. We shall study the linearized problem of boundary layer. But even linear problem faces great difficulties [13, 17]. That is why it is interesting to construct rough and sufficiently simple models allowing one to estimate the influence of this small perturbation on the stability of boundary layer. It is conventional to use an approximation of uniformly distributed suction [7], but in the framework of this approach we have no possibility to investigate the influence of aperture position. There is another way- to replace a small opening (strip) by a point-like one [5, 14]. A goal of the present paper is further development of this idea. Namely, we analyze a shift of neutral curves and, correspondingly, the critical value of the Reynolds number under the influence of additional perturbation of the velocity of

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main flow by a point source (sink) at the wall. The result is compared with [11, 17]. The model allows one to vary the choice of the model function - disturbance of the main flow due to aperture in the boundary (in the model it is replaced by a point-like window). In [10 – 12] the excitation of the Tollmien-Schlichting waves and the receptivity of boundary layer to the Dirac line source (sink) at the wall was studied. One can use this function which corresponds to the point-like source as a disturbance in our model. But comparison of our model with the results [13, 17] shows that, in order to obtain appropriate result (outside some neighbourhood of the aperture), it is sufficient to use more rough and simpler model functions, for example, a point source for the Stokes flow [6, 16]. It should be mentioned that Stokes-flow solutions are widely used in multi-layer analysis of boundary layer for the flow in the inner sublayer [2, 18].

There is a so-called zero-width slit model which is rather effective in diffraction theory and creeping flow investigation [6, 15, 16]. It is based on the theory of self-adjoint extensions of symmetric operators. The model is similar to the zero-range potential method in quantum mechanics. The suggested approach allows one to apply the operator extension theory methods in a hydrodynamic problem.

Consider a two-dimensional flow of viscous incompressible fluid over the semi-axis $\Gamma, \Gamma = \{(x,y) : x \geq 0, y = 0\}, x, y$ are Cartesian coordinates on the plane. It is convenient to use a stream function Ψ instead of the velocity (u_x, u_y) of the flow: $u_x = \Psi'_y, u_y = -\Psi'_x$. Then the Navier-Stokes equations transform to the following boundary-initial value problem for the stream function Ψ:

$$\frac{\partial \Delta\Psi}{\partial t} + \frac{\partial\Psi}{\partial y}\frac{\partial\Delta\Psi}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\Delta\Psi}{\partial y} - v\Delta^2\Psi = 0,$$

(1.1)

$$\Psi\mid_\Gamma = \frac{\partial\Psi}{\partial n}\mid_\Gamma = 0, \Psi\mid_{t=0} = \Psi_0,$$

where v is the kinematic viscosity.

Let ψ be the stream function of the main stationary flow, $u = \psi'_y, v = -\psi'_x$. A solution of the problem with slightly disturbed initial condition we denote by $\psi + \varphi$. Assuming that the disturbance is small, we linearize Eq. (1.1). Consider a region in which the boundary layer has been formed. Assuming that the velocity field is parallel and its components depend only on the transversal variable y, one can seek φ in the form of a Tollmien-Schlichting wave: $\varphi(x,y) = f(y)\exp(i\alpha(x - ct))$. It is possible to take into account also the transversal component v of the velocity keeping in mind that u and its derivatives are much larger than v and its derivatives [1, 7]. Taking into account that $u''_{xx} \ll u''_{yy}, v''_{xx} \ll v''_{yy}$ in the boundary layer and considering v and v_{yy} as parameters, one obtains the following modified

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Orr-Sommerfeld problem (in dimensionless form) [1, 7]:

$$(1.2) \qquad f'''' - 2\alpha^2 f'' + \alpha^4 f = R(i\alpha(u-c)(f'' - \alpha^2 f) - i\alpha u''_{yy} f + v(f''' - \alpha^2 f') - v''_{yy} f'), \qquad f(0) = f'(0) = \alpha f(1) + f'(1) = 0,$$
$$f(y) \to \text{const} \quad \text{for} \quad y \to \infty,$$

where R is the Reynolds number for the boundary layer. The system of units is such that the width of the boundary layer is equal to 1. The first two conditions mean that the disturbance of the velocity field must vanish at the boundary. As for the remaining two conditions, they show that the perturbation of the velocity is concentrated in the boundary layer. It is possible to use other conditions, for example, $f(y) \to 0$ and $f'(y) \to 0$ for $y \to \infty$, but for our purposes it is more convenient to use the above mentioned condition in which the scale is fixed: the width of the boundary layer is 1.

Let α be real. It is convenient to use a neutral curve, i.e. a set of points on the plane α, R, for which $\Im c = 0$, to describe the stability. The domain of instability consists of points for which $\Im c > 0$ (because the amplitude of the corresponding Tollmien-Schlichting wave increases). Minimal value R_* of R on the neutral curve is named the critical value of the Reynolds number. For $R < R_*$ the flow is stable for any value of α. To determine R_* it is an important problem of linear hydrodynamic stability. It is essential for various applications to clarify how the critical Reynolds number is influenced by different variations of the system.

For the main flow we have $u = U(y), v = 0$ in the boundary layer. We shall estimate the influence of small aperture on the stability by means of replacement of coefficients u, v in Eq. (1.2) by the components of the flow velocity in the boundary layer in the case when the aperture is present [8]. Unfortunately, explicit construction of the velocity field in this case is very complicated. That is why it if useful to find simpler model functions εg for the perturbation of equation coefficients: $u = U(y) + \varepsilon g'_x, v = -\varepsilon g'_y$, where ε is a small parameter. Outside some neighbourhood of the opening, stationary velocity field for small aperture differs slightly from that for the model with point source at the boundary. The simplest example of such source is a potential source: $g = \arctan(y(x-a)^{-1})$, where $(a, 0)$ is a point of the opening. It should be mentioned that one can use the operator extension theory method [6, 15, 16] to construct a stream function for the potential or creeping flow with point source.

2. Construction of asymptotic expansion

Let us construct main terms of an asymptotic expansion of the solution in small parameter $(\sqrt{\alpha R})^{-1}$. We shall follow the scheme of [1], which is a modifica-

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tion of that from [9]. Let f_1, f_2 be the solutions of the Orr-Sommerfeld equation without viscosity (for details, see the Appendix):

$$f_1(y) = (u-c)\sum_{n=0}^{\infty} q_n(y)\alpha^{2n},$$

$$q_0 = 1, q_n(y) = \int_0^y (u-c)^{-2}dy \int_0^y (u-c)^2 q_{n-1}(y)dy,$$

$$f_2(y) = (u-c)\sum_{n=0}^{\infty} t_n(y)\alpha^{2n},$$

$$t_0 = \int_0^y (u-c)^{-2}dy, t_n(y) = \int_0^y (u-c)^{-2}dy \int_0^y (u-c)^2 t_{n-1}(y)dy.$$

Let us make a substitution

$$f(y) = \exp(\int_0^y p(y)dy)$$

in Eq. (1.2) to determine two additional solutions. Then one obtains for $p(y)$:

$$(3.1) \qquad (u-c)(p' + p^2 - \alpha^2) - u'' + (iR\alpha)^{-1}vR(p' + 3pp' + p^3 - \alpha^2 p) - v''_{yy}Rp(i\alpha R)^{-1} = (i\alpha R)^{-1}(p^4 + 6p^2p' + 3(p')^2 + 4pp'' + p''' - 2\alpha^2(p' + p^2) + \alpha^4).$$

We search a solution of (2.1) in the form of a series in powers of $(\sqrt{\alpha R})^{-1}$:

$$(3.2) \qquad p(y) = \sum_{n=0}^{\infty}(\alpha R)^{\frac{1-n}{2}} p_n(y).$$

Substitute (2.2) into (2.1) and select the terms of order αR and $\sqrt{\alpha R}$:

$$(u-c)p_0^2 = -ip_0^4,$$

$$(u-c)p_0' + 6ip_0^2p_0' - ivRp_0^3 = (-4ip_0^3 - 2(u-c)p_0)p_1.$$

Hence,

$$p_0 = \pm\sqrt{i(u-c)}, p_1 = -5p_0'(2p_0)^{-1} + vR/2.$$

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Consequently, two "viscous" solutions of (1.2) have the form:

$$f_3(y) = (u-c)^{-5/4}\exp\left(\int\limits_0^y \left(-\sqrt{i\alpha R(u-c)} + vR/2\right) dy\right),$$

(3.3)

$$f_4(y) = (u-c)^{-5/4}\exp\left(\int\limits_0^y \left(\sqrt{i\alpha R(u-c)} + vR/2\right) dy\right).$$

The solution should satisfy the boundary conditions

$$f(0) = f'(0) = \alpha f(1) + f'(1) = 0, f(y) \to \text{const} \quad \text{for} \quad y \to \infty.$$

But f_4 does not satisfy the last one. That is why we take the solution in the form

$$f(y) = b_1 f_1(y) + b_2 f_2(y) + b_3 f_3(y).$$

Coefficients are determined from the system of boundary conditions. It is a linear algebraic system for the coefficients. It has non-trivial solution in the case when the system determinant is equal to zero. After some calculations, this condition takes the form:

$$-\frac{f_3(0)}{f_3'(0)} = \frac{cz}{u'(0)(1+z)}, \tag{3.4}$$

where

$$z = z_r + iz_i = u'(0)c(f_2'(1) + \alpha f_2(1))(f_1'(1) + \alpha f_1(1))^{-1},$$

$$z_r = -\frac{u'(0)}{u_*'} + \frac{u'(0)cu_*''}{(u_*')^3}\ln c\frac{u'(0)c}{\alpha(1-c)^2}, \tag{3.5}$$

$$u(y_*) = c,\ u_*' = u'(y_*),\ u_*'' = u''(y_*),$$

$$z_i = -\pi u'(0)cu_*''(u_*')^3. \tag{3.6}$$

Under the assumption that $c \simeq y_* u'(0)$ [9], one obtains from (2.4):

$$z + 1 = \left(1 + \frac{f_3(0)}{f_3'(0)y_*}\right)^{-1}.$$

Let $\Phi(y)$ be the conventional solution of the Orr-Sommerfeld equation

$$\Phi'''' - 2\alpha^2\Phi'' + \alpha^4\Phi = Ri\alpha((u-c)(\Phi'' - \alpha^2\Phi) - U_{yy}''\Phi).$$

Then taking into account (1.5), one gets

$$f_3(y) = \Phi(y)\exp\left(\int\limits_0^y (vR/2)dy\right).$$

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Hence, (2.4) gives us

$$(3.7) \qquad z+1=(1+\Phi(0)(y_*(\Phi'(0)+Rv(0)\Phi(0)/2))^{-1})^{-1}$$

$$(1-F(w)(1-v(0)Ry_*F(w)/2)^{-1})^{-1}=G_1(w),$$

where $F(w)=F_r(w)+F_i(w)$ is the Thitiens function [9], $w=y_*(\alpha R u'_*)^{1/3}$. We use Eq. (2.7) for the determination of α, R on the neutral curve. If the model function is such that $v(0)=0$ in a cross-section of the boundary layer which is considered, then it is necessary to take into account next terms of the expansion (1.4). One obtains p_2 by collecting the terms of order $(\alpha R)^0$ in (2.1):

$$p_2=(2p_0^3)^{-1}(vR3(p_0p_0'+p_0^2p_1)-5p_0^2p_1'-iu''$$

$$-5p_0^2p_1^2-12p_0p_1p_1'-3(p_0')^2-4p_0p_0''+\alpha^2p_0^2).$$

Substituting this expression in (2.1), one obtains the following relation for the determination of α, R on the neutral curve in the case when $v(0)=0$ instead of (2.7), by extracting the main term:

$$(3.8) \qquad z+1=(1+\Phi(0)(y_*(\Phi'(0)-5\sqrt{R/\alpha}(4p_0(0))^{-1}v'(0)\Phi(0)))^{-1})^{-1}$$

$$=\ (1-F(w)(1-5\sqrt{R/\alpha}(4p_0(0))^{-1}v'(0)y_*F(w))^{-1})^{-1}=G_2(w),$$

If the model function satisfies both boundary conditions, we should take into account the next term (p_3) of the series (2.2). In this case, one must collect all terms of order $(\sqrt{\alpha R})^{-1}$ in (2.1):

$$p_3=-(2p_0^3)^{-1}(-vR(p_0''+3p_0p_1'+3p_0^2p_2+3p_1p_0'+3p_0p_2^2-\alpha^2p_0)$$

$$+5p_0^2p_2'-2p_1p_2p_0^2+Rp_0v''+4p_0p_1^3+6p_0p_1^2+12p_0'p_0p_2$$

$$+12p_1'p_1p_0+6p_0'p_1'+4p_1p_0''+4p_0p_1''+p_0'''-2\alpha^2p_0'+2p_0p_1).$$

Then in the case $v(0)=v'(0)=0$ one obtains, by extracting the main term in p_3, the following condition for the determination of the neutral curve:

$$(3.9) \qquad z+1=(1+\Phi(0)(y_*(\Phi'(0)-21i(8\alpha(u(0)-c))^{-1}v''(0)\Phi(0)))^{-1})^{-1}$$

$$=\ (1-F(w)(1+21i(8\alpha(u(0)-c))^{-1}v''(0)y_*F(w))^{-1})^{-1}=G_3(w).$$

Expressions (2.7) – (2.9) for $G_j(w)$ have similar structures:

$$(1-F(w)(1+mw^3F(w))^{-1})^{-1}=G(w,m),$$

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if we introduce parameter m which has the value m_j for the case G_j:

$$\begin{aligned}
m_1 &= -v(0)(2y_*^2\alpha u_*')^{-1},\\
m_2 &= -5\sqrt{2}v'(0)(1+i)(8\sqrt{\alpha^3 R u_*}y_*^2 u_*')^{-1},\\
m_3 &= -21v''(0)(8u_* u_*'\alpha^2 R y_*^2)^{-1}.
\end{aligned}$$

3. Procedure of neutral curve construction

Let us describe a procedure of constructing the neutral curve.

Let $G(w,m) = H(w,m)+iQ(w,m), m = r+is, \Im H = \Im Q = \Im r = \Im s$. Then

$$\begin{aligned}
H(w,m) = (1 + 2rw^3F_r - 2sw^3F_i + r^2w^6F_i^2 + \ s^2w^6(F_r^2+F_i^2)\\
+ r^2w^6F_i^2 - rw^3F_r^2 - F_r - rw^3F_i^2)((1+rw^3F_r - sw^3F_i - F_r)^2\\
+ (rw^3F_i + sw^3F_r - F_i)^2)^{-1},\\
Q(w,m) = (F_i - sw^3(F_i^2+F_r^2))((1+w^3(rF_r - sF_i - F_r)^2\\
+ (w^3(rF_i+sF_r) - F_i)^2)^{-1}.
\end{aligned}$$

It is possible to assume a different initial approximation for the unperturbed main flow. We use the following profile: $U(y) = 2y - 5y^4 + 6y^5 - 2y^6$. Coefficients u, v in (1.2) have the following form: $u = U(y) + \varepsilon g_y', v = -\varepsilon g_x'$, where ε is a parameter, which characterizes the intensity of suction.

Let us fix ε. Choose a set of values of function $F(w)$. For each value one solves the following system of equations obtained above:

$$\begin{aligned}
Q(w,m) &= z_i(y_*),\\
H(w,m) &= 1 + f_r(y_*,\alpha),\\
R &= w^3y_*^{-3}(\alpha u_*')^{-1},\\
m &= m(y_*,\alpha,R).
\end{aligned}$$

The solution α, R of the system gives us the point of the neutral curve. Testing a sufficient set of values of the function $F(w)$, one constructs the neutral curve.

4. Suction and critical Reynolds number

To estimate the influence of suction on the stability near the critical point of the neutral curve we can use another technique. Namely, let ε be small, and we

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search a solution of (1.2) in the form of a power series in this small parameter:

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + ..., \qquad c = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \dots . \tag{4.1}$$

Substitute now these expressions into (1.2). Note that the function g depends on x, but x is treated as a parameter, i.e. the stability analysis is carried out under the assumption that the boundary layer is treated locally as a parallel flow. Terms of order ε^0 gives us the conventional Orr-Sommerfeld equation for f_0, c_0 :

$$f_0^{''''} - 2\alpha^2 f_0^{''} + \alpha^4 f_0 = Ri\alpha((u - c_0)(f_0^{''} - \alpha^2 f_0) - u_{yy}'' f_0). \tag{4.2}$$

For terms of order ε one obtains the inhomogeneous Orr-Sommerfeld equation for f_0, c_0 :

$$\begin{aligned} & f_0^{''''} - 2\alpha^2 f_0^{''} + \alpha^4 f_0 - Ri\alpha((u - c_0)(f_0^{''} - \alpha^2 f_0) - u_{yy}'' f_0) \\ & = R(i\alpha(g_y' - c_1)(f_0'' - \alpha^2 f_0) - i\alpha g_{yyy}''' f_0 - g_x'(f_0''' - \alpha^2 f_0') + g_{yyx}''' f_0'). \end{aligned} \tag{4.3}$$

The condition of solvability of Eq. (4.1) is the orthogonality of the right-hand part to a solution θ of the associated Orr-Sommerfeld equation:

$$\theta^{''''} - 2\alpha^2\theta^{''} + \alpha^4\theta = Ri\alpha((u - c_0)(\theta^{''} - \alpha^2\theta) + 2u_y'\theta').$$

The orthogonality condition gives us the value of c_1:

$$\begin{aligned} c_1 = & \left(i\alpha \int_0^\infty (f_0^{''} - \alpha^2 f_0)\overline{\theta} dy \right)^{-1} \int_0^\infty (i\alpha g_y'(f_0'' - \alpha^2 f_0) \\ & - i\alpha g_{yyy}''' f_0 - g_x'(f_0''' - \alpha^2 f_0') + g_{yyx}''' f_0')\overline{\theta} dy. \end{aligned} \tag{4.4}$$

The sign of $\varepsilon\Im c_1$ shows how the stability changes. Inequality $\varepsilon\Im c_1 > 0$ corresponds to instability, and condition $\varepsilon\Im c_1 < 0$ – to stability.

5. Discussion

It is possible to choose different functions g in the model. Of course, the best is the stream function for the case of small aperture in the boundary. But it is very difficult to construct such solution [13]. That is why it is useful to choose a simpler model functions. Formula (2.7) may be used for the description of uniform suction through the surface, for example. The simplest choice of the model function for the case of suction through small opening is a potential source

$$g = \arctan(y/(x - a)).$$

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Here $(a, 0)$ is a position of the opening. This function satisfies only one boundary condition in (1.1). For this case the function G should be chosen in accordance with formula (2.8). It is possible to choose a model function which satisfies both the boundary conditions. The simplest example is the corresponding solution of biharmonic equation (Stokes flow):

$$g = y^2/((x-a)^2 + y^2).$$

Note that the Stokes-flow solutions are often used as an instrument of boundary layer investigation in multi-layer analysis for inner sublayer (see, for example, [2, 18]). For such a choice of the model function, formula (2.9) for G works. But for a creeping flow, the value of flux across a line is equal to the difference between the values of stream function on the ends of the line. That is why this source is not a source (or sink) of mass, but a source of vorticity. From this point of view it is more appropriate to use a function

$$g = y^2/((x-a)^2 + y^2) + \pi/2 - \arccos(y/\sqrt{(x-a)^2 + y^2}),$$

which gives us a flux π through the opening.

For the last type of the model function we construct a neutral curve. The dependence of its shift on ε (intensity of suction) is shown in Fig. 1. Here $\Delta x = x - a$, where $(a, 0)$ is the position of the opening, x is the coordinate of the cross-

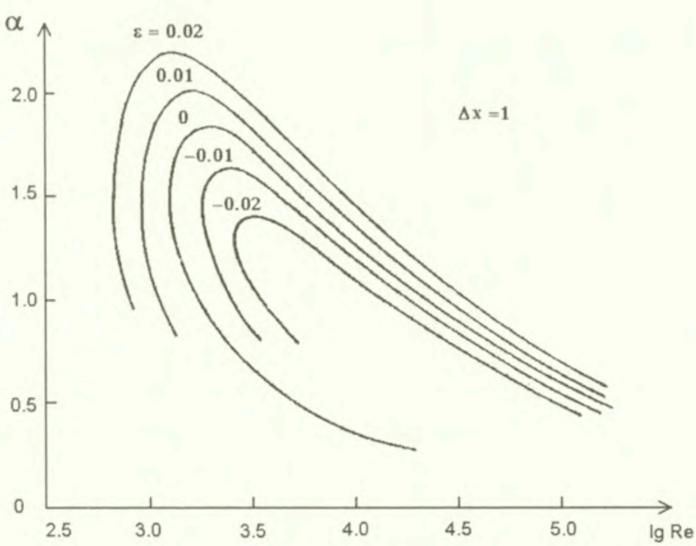

FIG. 1. Dependence of neutral curve position on the intensity of suction. Δx is the distance between the cross-section under consideration and the point of the aperture; the unit of length is the thickness of the boundary layer at the point of the opening. Curve with $\varepsilon = 0$ corresponds to absence of suction.

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section for which the stability is studied. Positive ε corresponds to source. For these values of the parameter we obtain a displacement of the neutral curve to the left and extension of instability domain. For $\varepsilon < 0$ (suction) we obtain a shift to the right of the curve and, correspondingly, reduction of instability domain. The neutral curves for different Δx are shown in Fig. 2. Here the case $\Delta x = \infty$ gives us the neutral curve for the boundary layer without suction. Of course, it is possible to use more realistic model functions (for example, that obtained by approximate calculations for point-like source [12]; in this situation we have not an explicit expression for g , but it is not essential). Nevertheless, one can see that even for such a rough choice of the model function one obtains results which are in good agreement with that for more realistic, but considerably more complicated models (see, for example, [17]).

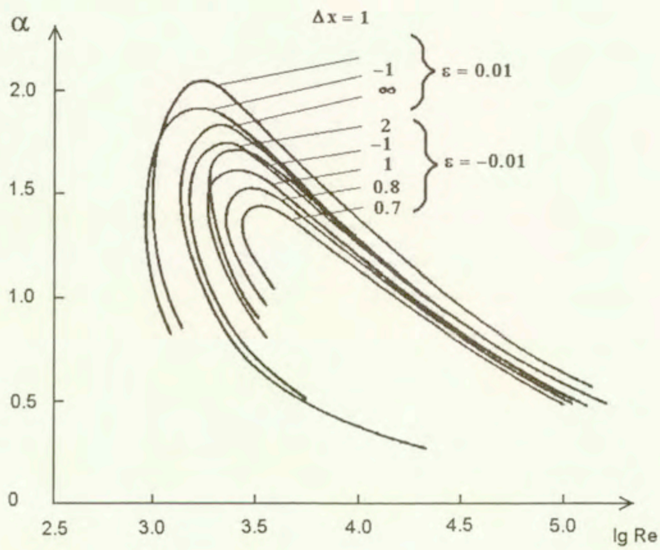

FIG. 2. Dependence of neutral curve position of the position of the aperture. Negative ε corresponds to suction (sink), positive ε- to source.

Obviously, that solution for point-like aperture is not appropriate as a model solution in the neighbourhood of the aperture of finite width (at the point of the opening the model solution has a singularity). One can estimate the size of neighbourhood outside which our model is correct. Comparison of the model function g with the stream function for the flow near aperture [13] shows that the ratio of the radius of the neighbourhood to the width of the boundary layer (at the point of the window) is of the order $10^2\varepsilon$.

Change of stability at critical point due to suction is studied by means of (4.4). The solutions f_0, θ are taken from [4] for critical values of parameters:

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$\alpha = 0.304, R = 519, c_0 = 0.3967$. The dependence of $\Im c_1$ on the distance from the point of suction is shown in Fig. 3 (for the model function $g = \arctan(y/(x-a))$) and in Fig. 4 (for the model function $g = y^2/((x-a)^2 + y^2)$). Note that the second function corresponds to the situation when the total flux through the aperture is zero, for example, there is a cavity coupled with the boundary layer

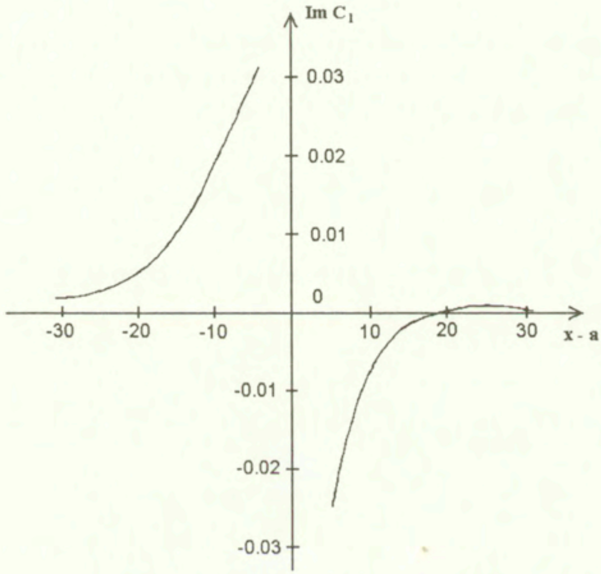

FIG. 3. The dependence of $\Im c_1$ on a distance from the point of suction for the model function $g = \arctan(y/(x-a))$.

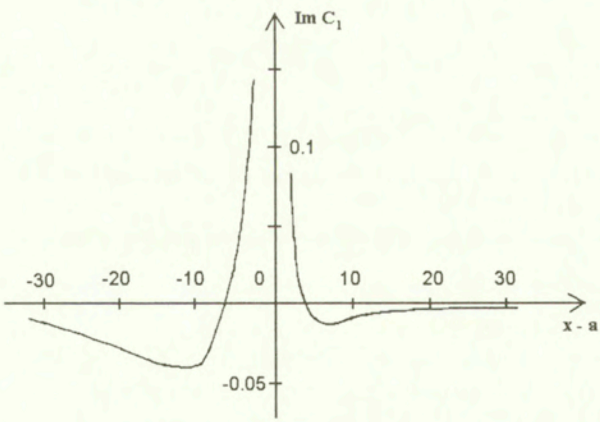

FIG. 4. The dependence of $\Im c_1$ on a distance from the point of suction for the model function $g = y^2/((x-a)^2 + y^2)$.

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through small opening. For the first function, the total flux is not zero – there is a suction. It explains the qualitative difference of the pictures. We must stress that the model is correct outside a neighbourhood of the opening, i.e. the results are not valid for small $x - a$ (the unit of length is the thickness of the boundary layer).

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Appendix

Consider the form of series for $f(1)$ (and $f(2)$) in powers of α. It is a solution of the main equation without viscosity:

$$(u-c)(f'' - \alpha^2 f) - i\alpha u''_{yy} f = 0.$$

Substituting f in the form

$$f(y) = (u-c)\sum_{n=0}^{\infty} q_n(y)\alpha^{2n},$$

one gets

$$(u-c)\sum_{n=0}^{\infty}(u'' q_n + 2u'q'_n + (u-c)q''_n)\alpha^{2n}$$

$$= (u-c)^2\sum_{n=1}^{\infty} q_{n-1}\alpha^{2n} + (u-c)u''\sum_{n=0}^{\infty} q_n\alpha^{2n}.$$

Comparing the terms with identical powers of α, one obtains for $n = 0$

$$2(u-c)u'q'_0 + (u-c)^2 q''_0 = 0,$$

i.e.

$$((u-c)^2 q'_0)' = 0.$$

Hence,

$$q_0 = C_1\int_0^y (u-c)^{-2}dy + C_2.$$

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For $n > 0$ one gets

$$((u-c)^2 q_n')' = (u-c)^2 q_{n-1}.$$

Hence, one obtains the following recurrent relation:

$$q_n(y) = \int\limits_0^y (u-c)^{-2} dy \int\limits_0^y (u-c)^2 q_{n-1}(y) dy.$$

Taking two linearly independent solutions q_0, we obtain the form of series for f_1 (f_2):

$$f_1(y) = (u-c) \sum_{n=0}^{\infty} q_n(y) \alpha^{2n},$$

$$q_0 = 1, q_n(y) = \int\limits_0^y (u-c)^{-2} dy \int\limits_0^y (u-c)^2 q_{n-1}(y) dy,$$

$$f_2(y) = (u-c) \sum_{n=0}^{\infty} t_n(y) \alpha^{2n},$$

$$t_0 = \int\limits_0^y (u-c)^{-2} dy, t_n(y) = \int\limits_0^y (u-c)^{-2} dy \int\limits_0^y (u-c)^2 t_{n-1}(y) dy.$$

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Onset of convection in a sparsely packed porous layer with throughflow

I. S. SHIVAKUMARA (1) and C. E. NANJUNDAPPA (2)

(1) *UGC-DSA Centre in Fluid Mechanics, Department of Mathematics, Bangalore University, Bangalore-560001, India*

(2) *Dr. Ambedkar Institute of Technology, Department of Mathematics, Bangalore-560056, India*

THE ONSET of Rayleigh-*Bénard* convection in a sparsely packed porous layer with vertical throughflow is investigated using Brinkman's modification of the Darcy flow model with fluid viscosity different from effective viscosity. The critical Rayleigh numbers are obtained for free-free, rigid-rigid and rigid-free boundaries which are insulated to temperature perturbations. It is noted that an increase in the value of viscosity ratio is to delay the onset of convection. Further, it is observed that the throughflow can be used either to suppress or augment convection, depending on the nature of boundaries and also on the values of physical parameters.

1. Introduction

THE POROUS-*BÉNARD* problem has been widely studied in recent decades owing to its numerous fundamental and industrial applications. Some examples of interest can be found in the thermal insulation engineering, water movements in geothermal reservoirs, underground spreading of chemical waste, geothermal engineering and enhanced recovery of petroleum reservoirs. The growing volume of works devoted to this topic is amply documented in the reviews by NIELD [1], TIEN and VAFAI [2], KAKAC *et al.* [3], KAVIANY [4] and NIELD and BEJAN [5].

Several applications, such as fixed-bed catalytic reactors and in situ coal gasification, involve the non-isothermal flow of fluids through porous media, which is called throughflow. The effect of throughflow in a porous medium was first studied by WOODING [6], who treated the case in which the basic-state temperature field is dominated by the convective effects on the throughflow. Later SUTTON [7] presented a linear stability analysis for small throughflow with rigid and conducting boundaries at both top and bottom and insulating walls at the sides. HOMSY and SHERWOOD [8] extended the analysis of [7] to larger throughflow by

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considering an infinite horizontal domain. All these investigators concluded that the throughflow always stabilizes the convection. JONES and PERSICHETTI [9] investigated the convective instability in a porous medium with throughflow and assumed that the boundaries are conducting and either permeable or impermeable. They found that a small amount of throughflow can provide a destabilizing effect in at least one situation, when the throughflow is from a more restrictive boundary (the dynamically rigid boundary) towards a less restrictive boundary (the dynamically free boundary). They thought that this behaviour may be due to numerical inaccuracy and did not offer any reason for the same. NIELD [10] and SHIVAKUMARA [11] indicated that the destabilization did occur for the above type of situation and a physical explanation for the same was offered. All these studies assumed Darcy flow behaviour in which inertia and viscous effects are neglected.

However, it is well known that in many cases involving porous media with high permeability, the viscous effects due to frictional drag at the boundary and the inertia effects are significant, particularly at high Peclet numbers. These effects are studied recently by SHIVAKUMARA [12] through the use of the Brinkman extended Darcy model. The analysis is limited to the case of the effective viscosity equal to fluid viscosity. But the recent work of GIVLER and ALTOBELLI [13] suggests that this assumption does not result in a good agreement with experiments for high porosity porous media. Therefore, a theoretical solution that is general enough to yield accurate results for porous media having high permeability is of fundamental and practical interest. This is the object of the present paper.

In the present study, the linear stability characteristics of a sparsely packed porous layer with simultaneous temperature gradient and a vertical throughflow is examined using Brinkman's modification of the Darcy model. The effective viscosity is taken to be different from fluid viscosity. The boundaries are assumed to be either rigid/free and insulated to temperature perturbations. The resulting eigenvalue problem is solved in Sec. 3 using regular perturbation technique with wavenumber **a** as a perturbation parameter. The results obtained are discussed in Sec. 4.

2. Mathematical formulation

We consider a fluid-saturated horizontal porous layer of depth d with a constant throughflow of magnitude w_0 in the vertical direction. The physical configuration is shown in Fig. 1. Cartesian axes are chosen with the z-axis directed vertically upwards in the gravitational field. The lower boundary $z = 0$ and the upper boundary $z = d$ are maintained at uniform temperatures T_0 and T_1 ($T_1 < T_0$), respectively.

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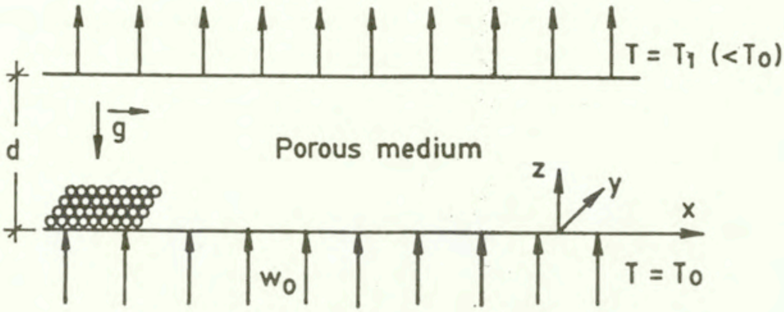

FIG. 1. Physical configuration.

The governing equations are:

$$\nabla \cdot \mathbf{q} = 0, \tag{2.1}$$

$$\rho_0 \left[\frac{1}{\phi} \frac{\partial \mathbf{q}}{\partial t} + \frac{1}{\phi^2} (\mathbf{q} \cdot \nabla) \mathbf{q} \right] = -\nabla p + \rho \mathbf{g} - \frac{\mu}{k} \mathbf{q} + \tilde{\mu} \nabla^2 \mathbf{q}, \tag{2.2}$$

$$A \frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla) T = \kappa \nabla^2 T, \tag{2.3}$$

$$\rho = \rho_0 \left[1 - \alpha (T - T_0) \right]. \tag{2.4}$$

Here $\mathbf{q} = (u, v, w)$ is the velocity vector, T the temperature, p the pressure, $A = (\rho c_p)_m / (\rho c_p)_f$ the ratio of specific heats, κ the effective thermal diffusivity, μ the fluid viscosity, $\tilde{\mu}$ the effective viscosity, ρ the density, α the volumetric expansion coefficient, $\mathbf{g}$ the gravitational acceleration, k the permeability, ϕ the porosity and c_p the heat capacity at constant pressure. The subscripts m and f refer to the fluid-solid mixture and the fluid respectively.

The basic steady state solution is given by

$$\mathbf{q}_b = w_0 \hat{k}, \ \theta_b(z) = T_b - T_0 = (T_1 - T_0) \frac{(1 - e^{w_0 z/\kappa})}{(1 - e^{w_0 d/\kappa})}, \tag{2.5}$$

where $\hat{k}$ is the unit vector in the increasing z – direction and the subscript b denotes the basic state. Note that the basic temperature distribution deviates from linear to nonlinear in z due to throughflow which has a significant influence on the stability of the system.

We now perturb the steady state basic solution as follows:

$$\mathbf{q} = w_0 \hat{k} + \mathbf{q}', \quad T - T_0 = \theta_b(z) + \theta', \quad p = p_b(z) + p', \tag{2.6}$$

where $\mathbf{q}'$, θ' and p' are infinitesimal perturbations.

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Substituting Eq. (2.6) into Eqs. (2.1) to (2.4) and neglecting nonlinear terms, we get the following equations (after dropping the primes):

$$\nabla \cdot \mathbf{q} = 0, \tag{2.7}$$

$$\rho_0 \left[\frac{1}{\phi}\frac{\partial \mathbf{q}}{\partial t} + \frac{w_0}{\phi^2}\frac{\partial \mathbf{q}}{\partial z} \right] = -\nabla p + \alpha\rho_0 g\theta\,\hat{k} - \frac{\mu}{k}\mathbf{q} + \tilde{\mu}\,\nabla^2\mathbf{q}, \tag{2.8}$$

$$A\frac{\partial \theta}{\partial t} + w_0\frac{\partial \theta}{\partial z} + \frac{\partial \theta_b}{\partial z}w = \kappa\nabla^2\theta. \tag{2.9}$$

We eliminate the pressure and the horizontal velocity components from the governing Eqs. (2.7) to (2.8) by standard manipulations. Then we non-dimensionalize the equations using the notations

$$z^* = z/d,\ t^* = (\kappa/d^2\phi)t, \quad \mathbf{q}^* = (d/\kappa)\mathbf{q}, \quad \theta^* = \theta/(T_0 - T_1)$$

and drop the asterisks for simplicity to obtain the following equations:

$$\frac{1}{\mathrm{Pr}}\left[\frac{\partial}{\partial t} + Q\frac{\partial}{\partial z}\right]\nabla^2 w - (\Lambda\nabla^2 - \sigma^2)\nabla^2 w = R\nabla_1^2\theta, \tag{2.10}$$

$$\frac{A}{\phi}\frac{\partial \theta}{\partial t} + Q\frac{\partial \theta}{\partial z} + f(z)w = \nabla^2\theta. \tag{2.11}$$

Here $R = \alpha g(T_0 - T_1)d^3/\nu\kappa$ is the Rayleigh number, $\mathrm{Pr} = \nu/\kappa\phi^2$ is the modified Prandtl number, $Q = w_0 d/\kappa$ is the throughflow-dependent Peclet number, $\sigma^2 = d^2/k$ is the inverse of Darcy number, $\Lambda = \tilde{\mu}/\mu$ is the ratio of viscosities, $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian operator, $\nabla^2 = \nabla_1^2 + \partial^2/\partial z^2$ is the Laplacian operator and $\partial\theta_b(z)/\partial z = f(z) = -Qe^{Qz}/(e^Q - 1)$ is the nonlinear steady state basic temperature gradient. Since the instability appears in non-oscillatory form (see [8]), we drop the time derivatives in Eqs. (2.10) and (2.11) and seek a steady cellular solution in the form $(w, \theta) = [W(z), \Theta(z)]\exp\{i(lx + my)\}$, to get

$$(D^2 - a^2)\left[\Lambda(D^2 - a^2) - \sigma^2 - MD\right]W = Ra^2\Theta, \tag{2.12}$$

$$(D^2 - a^2)\Theta - QD\Theta = f(z)W. \tag{2.13}$$

Here $D = d/dz$, $a^2 = l^2 + m^2$ is the square of the overall horizontal wavenumber and $M = Q/\mathrm{Pr}$. These equations are to be solved using appropriate boundary conditions depending on the nature of boundaries. We take the boundaries to be either rigid (however permeable) or free from tangential stress and insulating to temperature perturbations (see CHANDRASEKHAR [14] and NIELD [15]). Accordingly, the boundary conditions take the following form:

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On the rigid boundary

$$W = DW = 0 = D\Theta \tag{2.14}$$

and on the stress-free boundary

$$W = D^2W = 0 = D\Theta. \tag{2.15}$$

Equations (2.12) and (2.13) together with the boundary conditions (2.14) and / or (2.15) constitutes a two-point boundary value problem with R as an eigenvalue.

3. Method of solution

Since the boundaries are insulated to temperature perturbations, the solution for the eigenvalue problem can be obtained in a closed form for arbitrary Q because, in this case, R attains its minimum value (R=Rc) at an arbitrarily small wavenumber [15]. Accordingly, we look for solutions of Eqs. (2.12) and (2.13) in the form

$$(W, \Theta) = (W_0, \Theta_0) + a^2(W_1, \Theta_1) + \cdots . \tag{3.1}$$

Substituting Eq. (3.1) into Eqs. (2.12) and (2.13) and equating the coefficients of the same powers in a^2, we get

$$\Lambda D^4W_0 - MD^3W_0 - \sigma^2D^2W_0 = 0\,, \tag{3.2}$$

$$D^2\Theta_0 - QD\Theta_0 = f(z)W_0\,, \tag{3.3}$$

$$\Lambda D^4W_1 + (-2\Lambda D^2 + \sigma^2 + MD)W_0 - MD^3W_1 - \sigma^2D^2W_1 = R\Theta_0\,, \tag{3.4}$$

$$D^2\Theta_1 - QD\Theta_1 = \Theta_0 + f(z)W_1. \tag{3.5}$$

These equations have to be solved subject to the boundary conditions

$$W_i = DW_i = D\Theta_i = 0 \quad (i = 0, 1) \qquad \text{at} \quad z = 0, 1 \tag{3.6}$$

in the case of rigid-rigid boundaries, and

$$W_i = D^2W_i = D\Theta_i = 0 \qquad (i = 0, 1) \qquad \text{at} \quad z = 0, 1 \tag{3.7}$$

in the case of stress-free boundaries.

For any combination of boundaries (i.e. rigid-rigid, free-free and rigid-free), the solution of Eqs. (3.2) and (3.3) is $W_0 = 0$ and $\Theta_0 = 1$. Then Eqs. (3.4) and (3.5) become

$$\Lambda D^4W_1 - MD^3W_1 - \sigma^2D^2W_1 = R\,, \tag{3.8}$$

$$D^2\Theta_1 - QD\Theta_1 = 1 + f(z)W_1. \tag{3.9}$$

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The general solution of Eq. (3.8) is found to be

$$W_1 = C_0 + C_1 z + C_2 e^{\alpha z} + C_3 e^{\beta z} - Rz^2/2\sigma^2\,, \tag{3.10}$$

where $\alpha, \beta = \left[M \pm \sqrt{M^2 + 4\Lambda\sigma^2}\right]/2\Lambda$ and C_i $(i = 0$ to $3)$ are constants of integration. Applying the solvability condition [15] to Eqs. (3.8) and (3.9) [15, 16] (i.e. the zeroth order solutions are orthogonal to the first order solutions), it follows that

$$< RW_0 >= 0 \text{ and } < [1 + f(z)W_1]\Theta_0 >= 0, \text{ where } < \cdots >= \int_0^1 (\cdots)dz\,.$$

In these, the first condition is satisfied trivially and the second requires that

$$< 1 + f(z)W_1 >= 0\,. \tag{3.11}$$

To determine R, which in fact is a critical Rayleigh number Rc, we should find the constants C_i $(i = 0$ to $3)$ in Eq. (3.10) which depend on the nature of boundaries. The critical Rayleigh numbers obtained for different boundary combinations are detailed below.

3.1. Both boundaries free

In this case the required boundary conditions are

$$W_1 = D^2 W_1 = D\Theta_1 = 0 \qquad \text{at} \quad z = 0, 1. \tag{3.12}$$

The constants appearing in Eq. (3.10) are now determined using the boundary conditions, and they are found to be

$$\begin{aligned}
C_0 &= R\left[\alpha^2(e^\alpha - 1) - \beta^2(e^\beta - 1)\right]/\Delta_1\sigma^2\,,\\
C_1 &= R\left[\alpha^2\beta^2(e^\beta - e^\alpha) + 2(\alpha^2 - \beta^2)(1 - e^\alpha)(1 - e^\beta)\right]/2\Delta_1\sigma^2\,,\\
C_2 &= R\beta^2(e^\beta - 1)/\Delta_1\sigma^2\,,\\
C_3 &= -R\alpha^2(e^\alpha - 1)/\Delta_1\sigma^2\,,
\end{aligned}$$

where $\Delta_1 = \alpha^2\beta^2(e^\beta - e^\alpha)$.

Making use of these values of C_i $(i = 0$ to $3)$ in Eq. (3.11), we get an expression for Rc, which can be written, after some algebraic simplifications, in the form

$$Rc = \frac{4Q^2\sigma^2(\gamma^2 - \delta^2)^2\,(\cosh q_2 - \cosh q_1)}{[b_1 \cosh q_1 + b_2 \cosh q_2 + b_3 \cosh q_3 + b_4 \sinh q_1 + b_5 \sinh q_2 + g_1]}\,, \tag{3.13}$$

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where

$$b_1 = \frac{4Q^2(\gamma^2-\delta^2)(\gamma-\delta)}{Q+\gamma+\delta} + 4(\gamma^2-\delta^2)^2 ,$$

$$b_2 = -\frac{4Q^2(\gamma^2-\delta^2)(\gamma+\delta)}{Q+\gamma-\delta} - 4(\gamma^2-\delta^2)^2 ,$$

$$b_3 = 16\gamma\delta Q^2 - 8\delta Q^3 \frac{\{\gamma^2+\delta^2+2\gamma(Q+\gamma)\}}{(Q+\gamma)^2-\delta^2} ,$$

$$b_4 = -2Q\{(\gamma^2-\delta^2)^2 - 8\gamma\delta\} ,$$

$$b_5 = 2Q\{(\gamma^2-\delta^2)^2 + 8\gamma\delta\} ,$$

$$g_1 = -16Q\gamma\delta\ (\sinh q_3 + \sinh q_4) ,$$

with $\delta = \sqrt{(M^2+4\Lambda\sigma^2)}/2\Lambda$, $\gamma = M/2\Lambda$, $q_{1,2} = Q/2 \pm \delta$ and $q_{3,4} = Q/2 \pm \gamma$.

It is observed that Rc is an even function of Q and the direction of throughflow (i.e. Q>0 or Q<0) does not change the value of Rc.

As $\sigma \to \infty$ (Darcy case), Eq. (3.13) gives

$$Rc \sim \frac{2Q^2\sigma^2}{[Q\coth(Q/2) - 2]} . \tag{3.14}$$

Equation (3.14) is an even function of Q but independent of Pr and it coincides with that of NIELD [10]. Further as $Q \to 0$ (i.e. in the absence of throughflow), Eq. (3.13) gives

$$Rc \sim \frac{12\Lambda\tilde{\sigma}^5}{[\tilde{\sigma}^3 - 12\tilde{\sigma} + 24\tanh(\tilde{\sigma}/2)]} , \tag{3.15}$$

where $\tilde{\sigma} = \sigma/\sqrt{\Lambda}$. From this equation it follows that, with $\Lambda = 1$

$$Rc \sim 120 \quad \text{as } \sigma \to 0 \tag{3.16}$$

and

$$Rc \sim 12\sigma^2 \quad \text{as } \sigma \to \infty \tag{3.17}$$

which are the known exact values.

3.2. Both boundaries rigid

The boundary conditions are

$$W_1 = DW_1 = D\Theta_1 = 0 \qquad \text{at} \quad z = 0, 1. \tag{3.18}$$

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Then from Eq. (3.10), on using Eq. (3.18), we get

$$\begin{aligned}
C_0 &= R\left[\alpha(1+e^{\alpha}) - \beta(1+e^{\beta}) + 2(e^{\beta}-e^{\alpha})\right]/\Delta_2\sigma^2\,,\\
C_1 &= R\left[2\alpha(e^{\beta}-1) - 2\beta(e^{\alpha}-1) - \alpha\beta(e^{\beta}-e^{\alpha})\right]/\Delta_2\sigma^2\,,\\
C_2 &= R\left[\beta(1+e^{\beta}) - 2(e^{\beta}-1)\right]/\Delta_2\sigma^2\,,\\
C_3 &= R\left[2(e^{\alpha}-1) - \alpha(1+e^{\alpha})\right]/\Delta_2\sigma^2\,,
\end{aligned}$$

where $\Delta_2 = 2(\beta-\alpha)\left(1-e^{\alpha}-e^{\beta}+e^{\alpha\beta}\right) + 2\alpha\beta(e^{\alpha}-e^{\beta})$.

Substituting these values of C_i ($i = 0$ to 3) in Eq. (3.11) and performing some integration, we obtain an expression for Rc, which can be written in the form

$$Rc = \frac{2Q^2\sigma^2[-2\delta(\sinh q_4 + \sinh q_3 - \sinh q_1 - \sinh q_2) + f_1]}{(d_1\cosh q_1 + d_2\cosh q_2 + d_3\cosh q_3 + d_4\sinh q_1 + f_2)}, \tag{3.19}$$

where

$$\begin{aligned}
d_1 &= \frac{2Q(\gamma+\delta)^2}{Q+\gamma+\delta} - (\gamma^2-\delta^2)\,,\\
d_2 &= -\frac{2Q(\gamma-\delta)^2}{Q+\gamma-\delta} + (\gamma^2-\delta^2)\,,\\
d_3 &= \frac{4Q^3\delta}{(Q+\gamma)^2-\delta^2} - 4Q\delta\,,\\
d_4 &= \frac{Q(\gamma+\delta)^2(\gamma-\delta)}{Q+\gamma+\delta} - 4\delta\,,\\
d_5 &= -\frac{Q(\gamma-\delta)^2(\gamma+\delta)}{Q+\gamma-\delta} - 4\delta\,,\\
d_6 &= \frac{Q^2(\gamma+\delta)(Q+2\delta)}{Q+\gamma+\delta} + 4\delta\,,\\
d_7 &= -\frac{Q^3(\gamma+\delta)}{Q+\gamma-\delta} + 4\delta\,,\\
f_1 &= (\gamma^2-\delta^2)(\cosh q_1 - \cosh q_2)\,,\\
f_2 &= d_5\sinh q_2 + d_6\sinh q_3 + d_7\sinh q_4.
\end{aligned}$$

We note that Rc is an even function of Q, as in the previous case, because of symmetric boundary conditions and hence the direction of throughflow does not have any influence on the stability of the system.

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As $\sigma \to \infty$, Eq. (3.19) gives

$$Rc \sim \frac{2Q^2\sigma^2}{[Q\coth(Q/2) - 2]}, \tag{3.20}$$

which is the same as Eq. (3.14). Further as $Q \to 0$, Eq. (3.19) gives

$$Rc \sim \frac{12\Lambda\tilde{\sigma}^4(2 + \tilde{\sigma}\sinh\tilde{\sigma} - 2\cosh\tilde{\sigma})}{[4(6-\tilde{\sigma}^2) + \tilde{\sigma}(24+\tilde{\sigma}^2)\sinh\tilde{\sigma} - 8(\tilde{\sigma}^2+3)\cosh\tilde{\sigma}]}. \tag{3.21}$$

Hence (with $\Lambda = 1$)

$$Rc \sim 720 \quad \text{as } \sigma \to 0 \tag{3.22}$$

and

$$Rc \sim 12\sigma^2 + 72\sigma \quad \text{as} \quad \sigma \to \infty. \tag{3.23}$$

The first term on the right-hand side of Eq. (3.23) is the value given by the Darcy equation and the second term is the Brinkman boundary layer correction.

3.3. Lower boundary rigid and upper boundary free

In this case the boundary conditions are given by

$$W_1 = DW_1 = 0 = D\Theta_1 \qquad \text{at} \quad z = 0, \tag{3.24}$$

$$W_1 = D^2W_1 = 0 = D\Theta_1 \qquad \text{at} \quad z = 1. \tag{3.25}$$

Hence

$$\begin{aligned}
C_0 &= R\left[\alpha^2 e^\alpha - \beta^2 e^\beta + 2(e^\beta - \beta) - 2(e^\alpha - \alpha)\right]/2\Delta_3\sigma^2,\\
C_1 &= R\left[\alpha^2\beta e^\alpha - \alpha\beta^2 e^\beta + 2\alpha(e^\beta - 1) - 2\beta(e^\alpha - 1)\right]/2\Delta_3\sigma^2,\\
C_2 &= R\left[\beta^2 e^\beta - 2(e^\beta - \beta - 1)\right]/2\Delta_3\sigma^2,\\
C_3 &= R\left[2(e^\alpha - \alpha - 1) - \alpha^2 e^\alpha\right]/2\Delta_3\sigma^2,
\end{aligned}$$

where $\Delta_3 = \beta^2 e^\beta(e^\alpha - \alpha - 1) - \alpha^2 e^\alpha(e^\beta - \beta - 1)$.

Making use of these values in Eq. (3.11), Rc is found in the form

$$Rc = \frac{2Q^2\sigma^2(e^Q - 1)\Delta_3}{\left[e_1 e^{Q+\gamma+\delta} + e_2 e^{Q+\gamma} + e_3 e^{Q+\delta} + e_4 e^Q + e_5 e^\gamma + e_6 e^\delta + e_7 e^{(\gamma+\delta)} + e_8\right]}, \tag{3.26}$$

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where

$$e_1 = \frac{Q^3(\delta^2-2)}{Q+\gamma} + (\gamma^2-\delta^2)(Q^2+2Q+2)\,,$$

$$e_2 = Q\gamma^2(\delta+2) + 2Q\delta - 2(1+\delta)\gamma^2 - \frac{2Q^2\gamma(1+\delta)}{Q+\gamma}\,,$$

$$e_3 = \frac{2Q^2\delta(\gamma+1)}{Q+\delta} - Q\delta^2(\gamma+2) + 2Q\gamma + 2(1+\gamma)\delta^2\,,$$

$$e_4 = -2Q(\gamma+\delta)\,,$$

$$e_5 = \frac{Q\delta^2(\gamma^2-2)}{Q+\delta} + 2(1+\delta)\gamma^2\,,$$

$$e_6 = -\frac{Q\gamma^2(\delta^2-2)}{Q+\gamma} - 2(1+\gamma)\delta^2\,,$$

$$e_7 = 2(\delta^2-\gamma^2)\,,$$

$$e_8 = \frac{2Q^3(\gamma-\delta)(\gamma+\delta+Q+1)}{(Q+\gamma)(Q+\delta)} - 2Q(Q+1)(\gamma-\delta).$$

A glance at Eq. (3.26) reveals that Rc is not an even function of Q. Hence the direction of throughflow alters the value of Rc.

As $\sigma \to \infty$, Eq. (3.26) gives

$$Rc \sim \frac{2Q^2\sigma^2}{[Q\coth(Q/2)-2]}, \tag{3.27}$$

which is the same as Eq. (3.14). Thus we observe that for a densely packed porous medium, Rc is independent of the types of boundaries. In other words, the Darcy model fails to take care of the boundary and inertia effects.

Further as $Q \to 0$, Eq. (3.26) gives

$$Rc \sim \frac{12\Lambda\tilde{\sigma}^5(\sinh\tilde{\sigma} - \tilde{\sigma}\cosh\tilde{\sigma})}{[4\tilde{\sigma}(\tilde{\sigma}^2-6)\sinh\tilde{\sigma} + (24-\tilde{\sigma}^4)\cosh\tilde{\sigma} + 12(\tilde{\sigma}^2-2)]}. \tag{3.28}$$

From this equation it follows that (with $\Lambda = 1$)

$$Rc \sim 320 \quad \text{as } \sigma \to 0 \tag{3.29}$$

and

$$Rc \sim 12\sigma^2 + 36\sigma \quad \text{as } \sigma \to \infty. \tag{3.30}$$

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The second term on the right-hand side of Eq. (3.30) is the boundary layer correction as seen in the previous case (see Eq. (3.23)).

4. Results and discussion

Onset of convection in a sparsely packed porous layer with throughflow is investigated in understanding the control of convective instability by the adjustment of vertical throughflow. The Brinkman-extended Darcy model with fluid viscosity different from effective viscosity is used to describe the flow in the porous media. The boundaries are assumed to be either rigid-permeable or free of tangential stresses and insulated to temperature perturbations. Analytical expression for critical Rayleigh number Rc, for free-free, rigid-rigid and rigid-free boundaries have been obtained and are evaluated numerically for different values of Q, Λ and Pr. The results are presented graphically in Figs. 2 – 9.

Figure 2 shows the variations of Rc as a function of $|Q|$ for both free-free and rigid-rigid (i.e. symmetric) boundary conditions. These plots are for $\sigma^2 = 10, 100$, $\Lambda = .1, 1, 10$ and $\mathrm{Pr} = 7$. Note that increase in the values of σ^2 and Λ increases Rc as expected and makes the system more stable. The direction of throughflow is not altering the stability of the system and increase in the value of $|Q|$ is to increase Rc. This is because the effect of throughflow is to confine significant thermal gradients to a thermal boundary layer at the boundary toward which the throughflow is directed. The effective length scale is thus smaller than the thickness of the porous layer d and so the Rayleigh number, which is proportional to the cube of the porous layer thickness, will be much smaller than the actual value of Rc. Therefore higher values of Rc are needed for the onset of convection with an increase in $|Q|$.

The presence of constant vertical throughflow in the basic state brings in the effect of inertia through the appearance of Prandtl number, which complicates the situation. The effect of Prandtl number on the stability of the system is shown distinctly in Figs. 3 and 4 for free-free and rigid-rigid boundaries, respectively. These figures are for $|Q| = 2$, $\Lambda = 0.1, 1, 10$ and $\sigma^2 = 10, 100$. For $\sigma^2 = 10$, the values of Rc are shown on the left-hand side of the figure while for $\sigma^2 = 100$ they are shown on the right-hand side. From these figures it is evident that Rc decreases initially with an increase in the value of Pr and passes through a minimum, depending on the value of Λ, before attaining an asymptotic value with further increase of Pr. Also, an increase in the value of Λ is to decrease the range of values of Pr upto which the system becomes unstable (i.e. Rc decreases). The variation in Rc with Pr for free-free boundaries is found to be not so significant as compared to the rigid-rigid boundaries.

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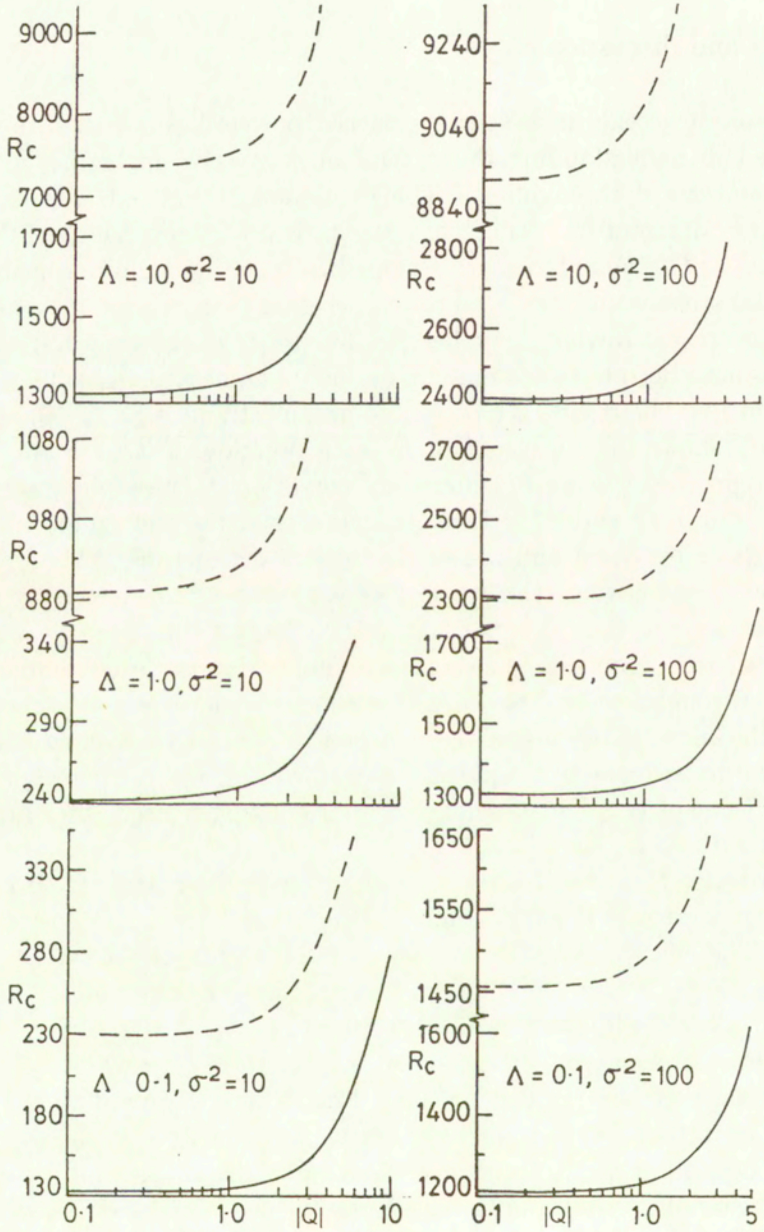

FIG. 2. Critical rayleigh number R_c vs. $|Q|$ for different values of σ^2 and Λ for free-free (——) and rigid-rigid (- - - -) boundaries when $\Pr = 7 \cdot 0$.

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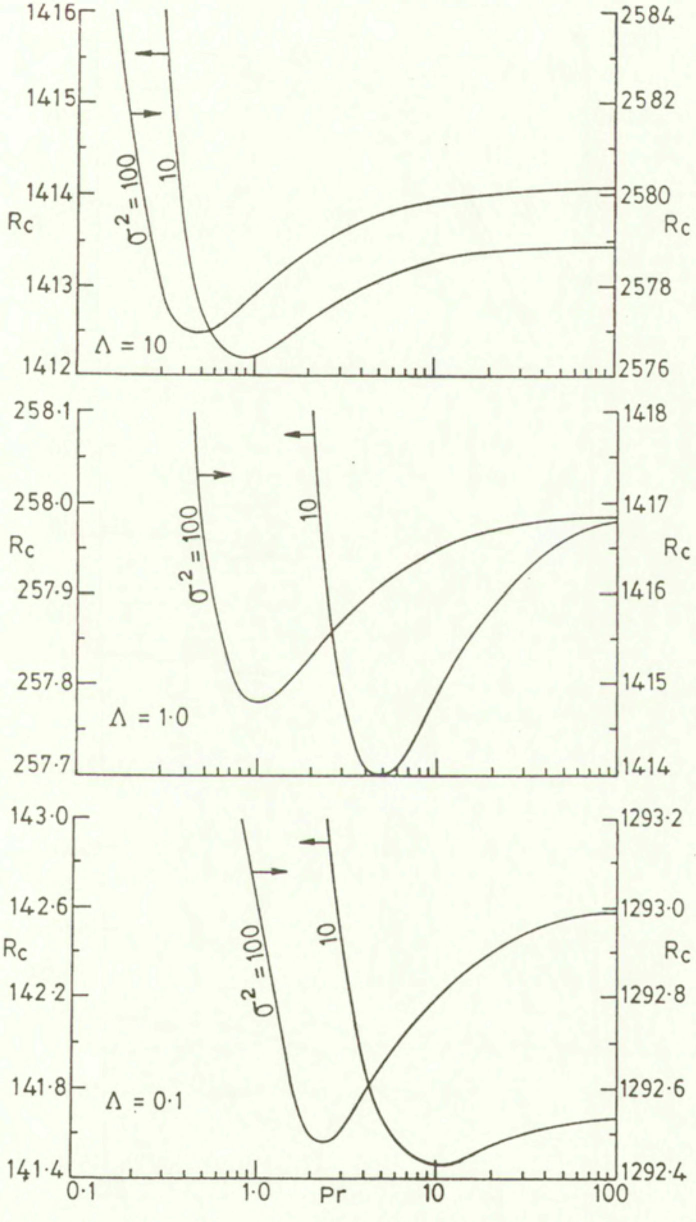

FIG. 3. Critical rayleigh number vs. Pr for different values of Λ and σ^2 for free-free boundaries when $|Q| = 2 \cdot 0$.

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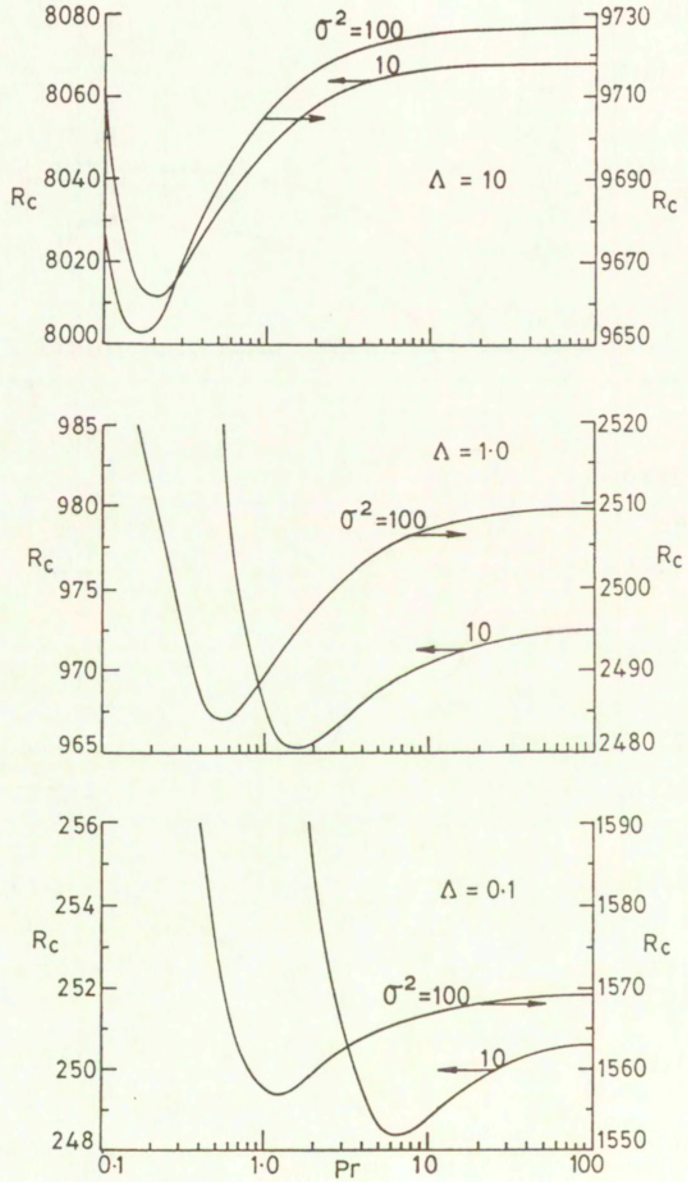

FIG. 4. Critical rayleigh number vs. Pr for different values of Λ and σ^2 for rigid-rigid boundaries when $|Q| = 2 \cdot 0$.

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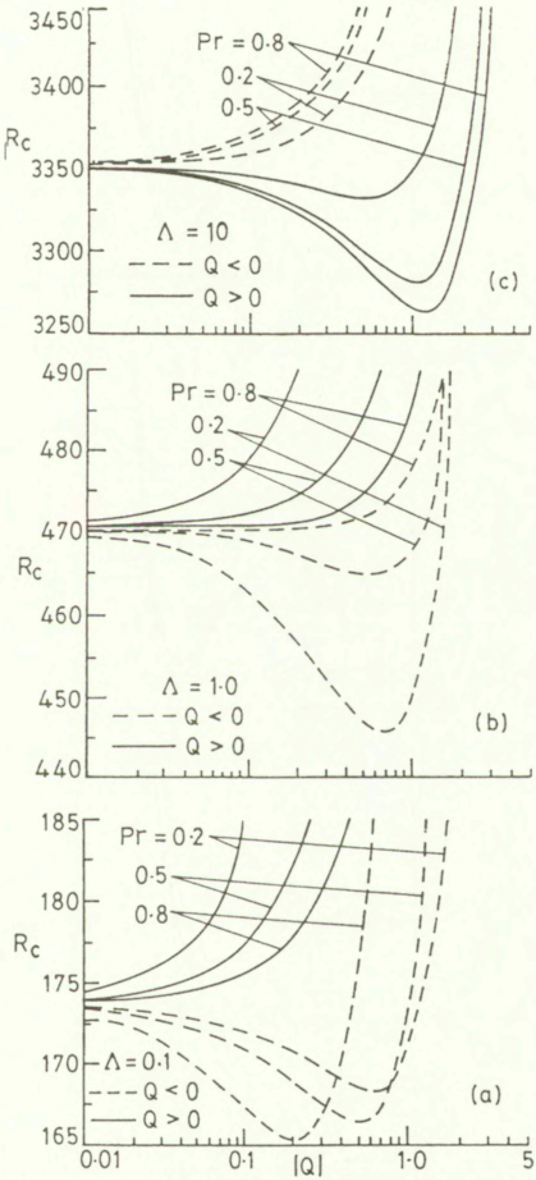

FIG. 5. Critical rayleigh number vs. $|Q|$ for different values of Pr and Λ for rigid-free boundaries when $\sigma^2 = 10$.

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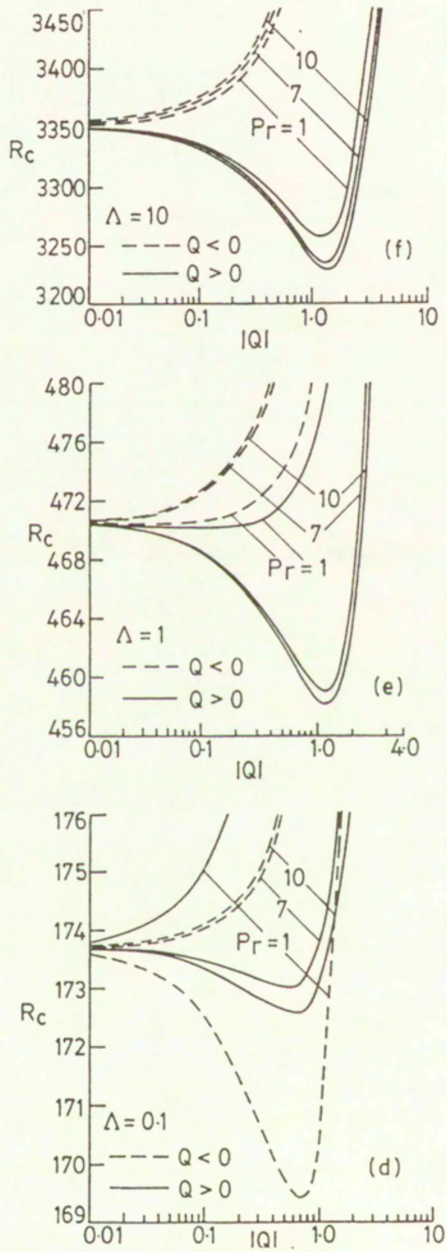

FIG. 5 (continued)

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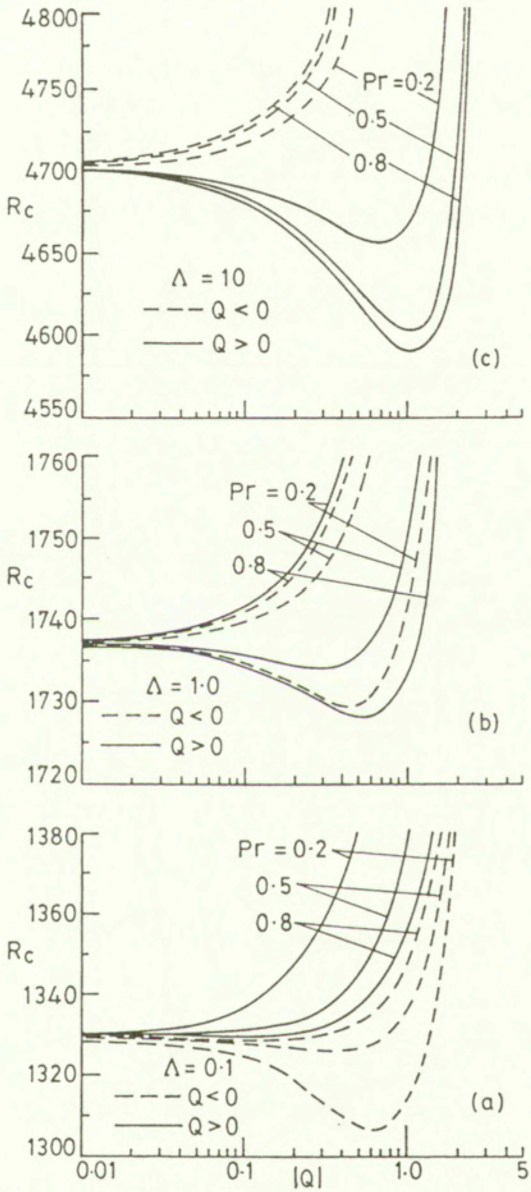

FIG. 6. Critical rayleigh number vs. $|Q|$ for different values of Λ and Pr for rigid-free boundaries when $\sigma^2 = 100$.

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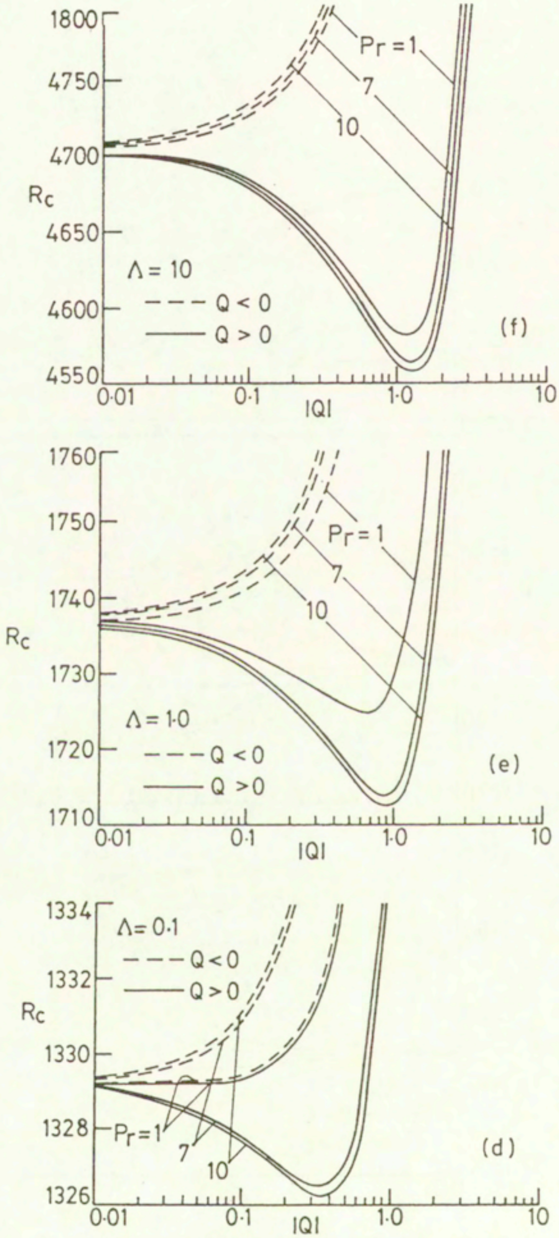

FIG. 6 (continued)

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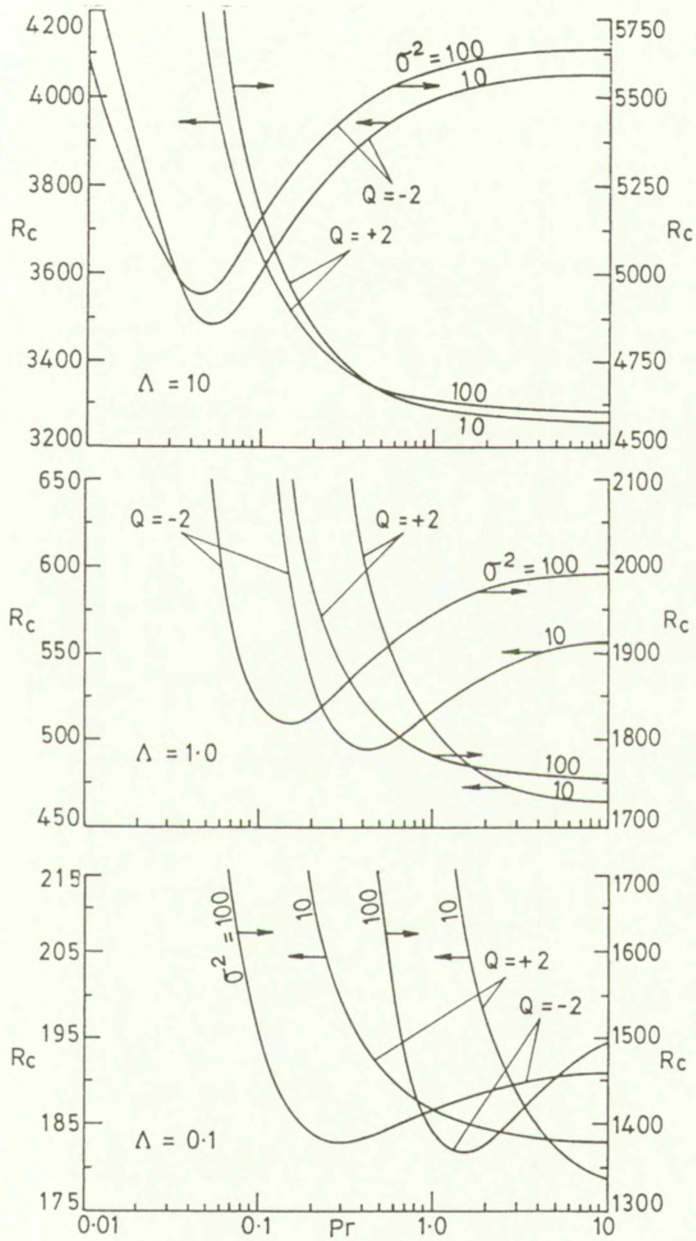

FIG. 7. Critical rayleigh number vs. Pr for different values of Λ and σ^2 for rigid-free boundaries when $Q = \pm 2$.

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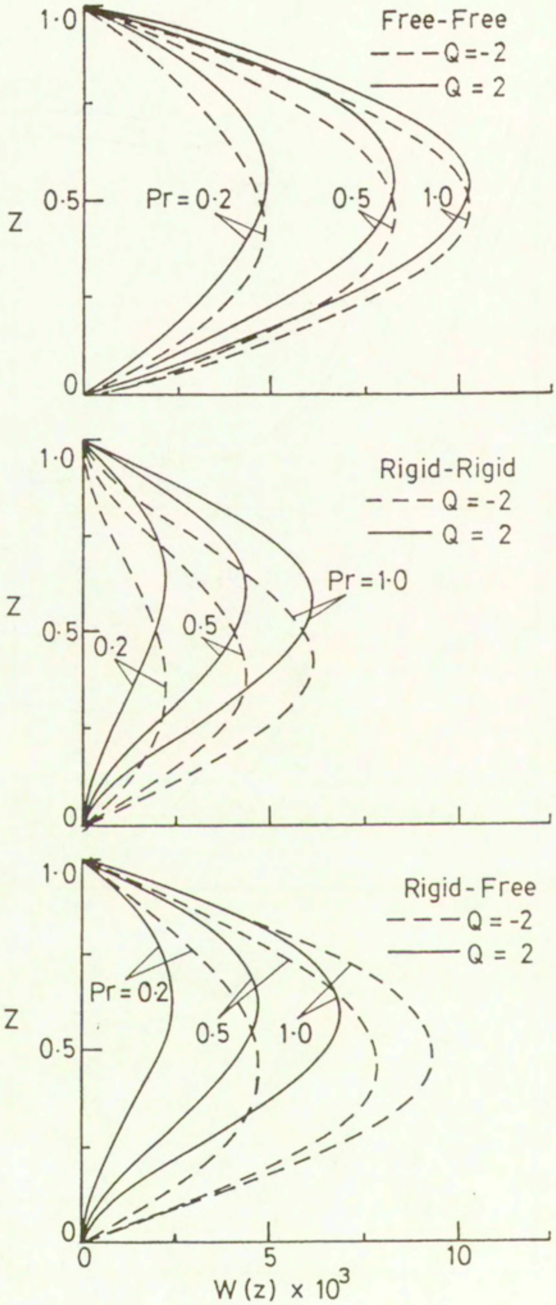

FIG. 8. Velocity eigenfunction for different values of Pr for different boundaries when $\Lambda = 0 \cdot 1$ and $\sigma^2 = 10$.

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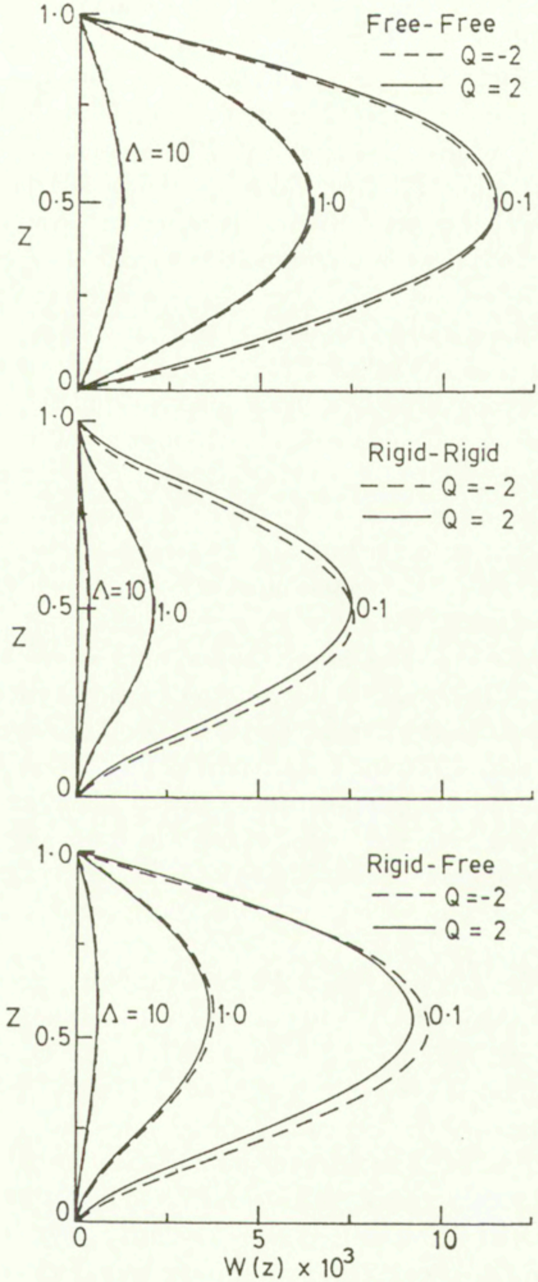

FIG. 9. Velocity eigenfunction for different values of Λ for different boundaries when $\mathrm{Pr} = 7 \cdot 0$ and $\sigma^2 = 10$.

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When the boundaries are of asymmetric type (i.e. rigid-free) the situation observed is totally different, in contrast to the symmetric boundaries, and the same is depicted in Figs. 5 and 6 for $\sigma^2 = 10$ and 100 respectively, for different values of Pr and Λ. Figures 5(a, b, c) and 6(a, b, c) show the results for $\mathrm{Pr} < 1$, while Figs. 5(d, e, f) and 6(d, e, f) are for $\mathrm{Pr} > 1$. These figures show that the direction of throughflow alters the stability of the system and also the existence of a critical Prandtl number below which the downward flow destabilizes the system and above which upward flow destabilizes the system. Note that the destabilization manifests itself as a minimum in the $Rc - |Q|$ curve up to certain values of Q depending on the values of σ^2, Λ , Pr and as well as the direction of throughflow. For antigravity throughflow ($Q > 0$), it is found that an increase in the value of Pr decreases the value of Rc and thus makes the system unstable. For gravity aligned throughflow ($Q < 0$) an opposite kind of behaviour is noticed, in general, with some exceptions when $\sigma^2 = 10$ and $\Lambda = 0.1$ and 1 (see Figs. 5a and 5b). Increase in the value of Λ tends to make the system more stable and also to increase the range of values of Q up to which the system gets destabilized. For $\Lambda = 0.1$ and 10, it is found that the destabilization is greater for $\mathrm{Pr} < 1$ and $\mathrm{Pr} > 1$ respectively for the values of σ^2 considered, where as for $\Lambda = 1$ the destabilization is greater for $\mathrm{Pr} < 1$ when $\sigma^2 = 10$ and for $\sigma^2 = 100$ the same is found to be true for $\mathrm{Pr} > 1$. The destabilization may be due to the distortion of the steady state basic temperature distribution by the vertical throughflow. A measure of this is given by the term $< f(z)W\Theta >$ and this can be interpreted as a rate of transfer of energy into the disturbance by interaction of the perturbation convective motion with basic temperature gradient. The maximum temperature occurs at a place where the perturbed vertical velocity is large and this leads to an increase in energy supply for destabilization. The destabilization may also be due to other mechanisms, i.e. the momentum transport and the thermal energy transport.

Figure 7 shows Rc as a function of Pr for rigid-free boundaries. The results exhibited are for two values of $\sigma^2 = 10$ (shown on the left-hand side of the figure) and 100 (shown on the right-hand side of the figure) for $Q = \pm 2$ and $\Lambda = 0.1$, 1, 10. It may be noted that the Prandtl number plays a dual role in deciding on the stability of the system, depending on the direction of throughflow. For antigravity throughflow Rc decreases monotonically with Pr, while for gravity aligned throughflow it decreases initially with Pr and increases again with further increase of Pr before attaining an asymptotic value.

The velocity eigenfunction $W(z)$ for different boundary combinations are illustrated in Figs. 8 and 9 for different values of Pr and Λ in the case $\sigma^2 = 10$ and $Q = \pm 2$. As can be seen, increase in the value of Pr (see Fig. 8) and decrease in the value of Λ (see Fig. 9) increases the convection in the porous layer.

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Modeling of elastic slab with periodic breaks

E. DANICKI

Institute of Fundamental Technological Research
Polish Academy of Sciences
ul. Świętokrzyska 21, 00–049 Warsaw, Poland
e-mail: edanicki@ippt.gov.pl

An elastic slab having periodic breaks across its cross-section is loaded by space-harmonic forces applied to its surfaces. Due to the breaks, the slab displacement response to the load includes a series of space-harmonics. A method is proposed for evaluation of the slab harmonic response within a presumed spectral domain.

1. Introduction

Periodic structures are frequently encountered in mechanical constructions: as a modern example recall a space-shuttle frame with periodically attached ceramic tiles of comparable stiffness. Other examples are: 1) an elastic body (a halfspace, for instance) with surface-breaking or subsurface cracks, 2) similarly looking is the composite ultrasonic transducer with deep periodic cuts that lower its acoustic impedance, 3) a plate with ribs, composite structure with periodic fillings between bonds, etc. In all cases the period of the structure (the period of elastic slabs that can be distinguished as attached to a solid frame) can be comparable to a wavelength of the applied traction, so that any simplifications like homogenization or equivalent discrete loading is not appropriate for serious analysis. Also note that in all the above cases, there is a uniform frame that keeps the slabs in order.

The solid frames in the above examples, an elastic halfspace or an elastic plate, can be conveniently characterized by a planar harmonic Green's function that is the relation between the surface displacement and the surface traction for any spatial frequency of these surface wave-fields, say $u = GT$ in standard notation. If we know the similar description for the layer of periodic slabs mentioned above, say $u_s = HT_s$, then the boundary-value problem for static deformation or vibration of the structure can be formulated like $u = u_s$, $T = T_s$, on the contact plane. Evaluation of H is the ultimate goal of the analysis; simple one-dimensional structure is considered in this paper (Fig. 1) but there are no substantial obstacles in generalization of the presented method.

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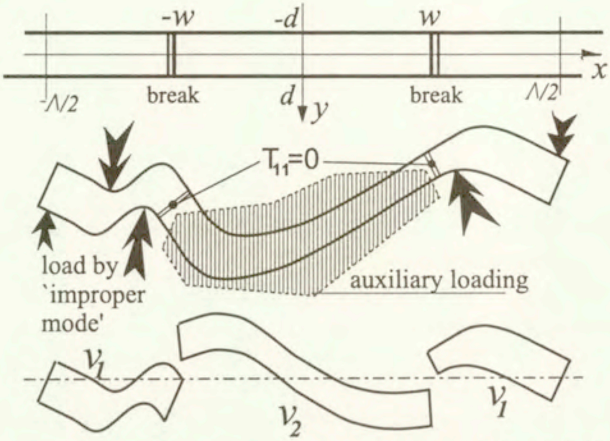

FIG. 1. The analyzed elastic slab with periodic breaks, and the slab deformed under the load of 'improper mode' in the domain $(-\Lambda/2, -w) + (w, \Lambda/2)$ (with auxiliary load in the shadowed area that allows the first part of the slab to relax at breaks). This, and the similarly evaluated deformation in the other domain, yield together what is shown in the bottom figure. Note the displacement discontinuity at breaks.

The vibration (or static deformation) of such structures poses a rather difficult problem that, naturally, can be solved using purely numerical methods. Analytical approach, being valuable due to its ability to bring better understanding of the analyzed structure, requires models of its components; here – a model of a broken slab. It is evident that the broken slab works as an entity only when attached to a certain solid frame (otherwise its periodic components would spill in chaos). This must be taken into account by allowing suitable functional space for deformation and stress fields in the system.

A natural choice is a domain of harmonic functions; moreover, the harmonic analysis is a primary tool in the theory of periodic structures [1]. Here, the mechanical field is expanded in a Bloch series like this

$$\Psi(x,y) = \sum_n \Psi^{(n)}(y) e^{-j(r+nK)x}, \tag{1.1}$$

where $K = 2\pi/\Lambda, \Lambda$ is the period of breaks distributed along the x-axis of the $2d$-thick elastic slab (Fig. 1, here the field is considered to be independent of z, and z-component of displacement is neglected for simplicity; also note that there are two breaks per period); r is a spectral variable, and $0 < r < K$ to avoid ambiguity (the inverse Fourier transform becomes an integral over r, any aperiodic field can be represented by the above series [2]).

The planar harmonic Green's function (actually a matrix $\mathbf{H}$ dependent on r) characterizing the analyzed slab is sufficient for most applications. It is involved

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in the relation

$$[\mathbf{u}^{(n)}] = [\mathbf{H}^{(nm)}][\mathbf{T}^{(m)}], \tag{1.2}$$

where $\mathbf{u}$ is the particle displacement vector on either of the slab surfaces ($y = \mp d$); $\mathbf{T}$ is the corresponding traction (surface load) on these surfaces, and n, m are the harmonic numbers in the Bloch series (1.1). It is convenient to define $\mathbf{u}^{(n)}$ and $\mathbf{T}^{(m)}$ as 4-dimensional vectors describing fields on both slab surfaces, $y = -d$ d, explicitly: $\mathbf{u} = [u_1^-, u_2^-, u_1^+, u_2^+]^T$, and similarly $\mathbf{T} = [T_{21}^-, T_{22}^-, T_{21}^+, T_{22}^+]^T$, with superscripts $-, +$ referring to $y = \mp d$.

For a homogeneous slab without breaks, all matrices $\mathbf{H}^{(nm)} = 0$ except for $n = m$, and $\mathbf{H}^{(nn)}$ dependent on spectral variable $p = r + nK \in (-\infty, \infty)$, can be easily derived analytically (Appendix) from the known equations of motion of isotropic homogeneous elastic layer [3 – 5]; this is an ordinary boundary-value problem of mechanics [6] discussed briefly in the next section below. The slab with breaks however, includes all matrices $\mathbf{H}^{(mn)}$ because of its periodic inhomogeneity. This is the aim of this paper to propose a method for their evaluation. The method is described on a certain simple example.

2. Fields in homogeneous slab

Deformation of isotropic homogeneous elastic slab is governed by a system of second order differential equations

$$\mathcal{L}\mathbf{u} = 0, \tag{2.1}$$

which, assuming x-dependence in harmonic form $\exp(-jpx)$ (and also $\exp j\omega t$ in the case of time-dependent vibrations), leads to the system

$$\mu \frac{d^2}{dy^2} u_x - p^2(\lambda + 2\mu) u_x - jp(\lambda + \mu) \frac{d}{dy} u_y = 0,$$
$$(\lambda + 2\mu) \frac{d^2}{dy^2} u_y - p^2 \mu u_y - jp(\lambda + \mu) \frac{d}{dy} u_x = 0,$$

in the example for isotropic elastic body characterized by Lamé constants μ, λ, under a possible static surface load that is to be accounted for in boundary conditions, but without any internal forces [6].

The system (2.1) has four partial solutions $\phi_k(y)$; conditions $\phi_k(y = 0) = 1$ normalize these solutions. The field inside the slab is

$$\begin{aligned} u_i(x, y) &= \textstyle\sum_k \tilde{U}_{i(k)} \phi_k(y) F_k e^{-jpx}, \\ T_{ij}(x, y) &= \textstyle\sum_k \tilde{T}_{ij(k)} \phi_k(y) F_k e^{-jpx}, \end{aligned} \tag{2.2}$$

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where coefficients F_k can be evaluated from the boundary conditions of the given boundary-value problem,

$$T_{2i}(y=\mp d) = \sum_k \tilde{T}_{2i(k)} \phi_k(y=\mp d) F_k, \tag{2.3}$$

for instance (neglecting harmonic dependence on x), which substituted into Eqs. (2.2) yield Eq. (1.2) presented earlier, with substitution $p = r + nK$.

The field at $y = 0$ is

$$\begin{aligned} u_i(x) &= \textstyle\sum_k \bar{U}_{i(k)} F_k e^{-jpx}, \\ T_{11}(x) &= \textstyle\sum_k \bar{T}_{(k)} F_k e^{-jpx}, \end{aligned} \tag{2.4}$$

for instance, due to $\phi_k(0) = 1$.

3. Slab with breaks – deformation modes

For simplicity of the presented example, only the stress component T_{11} is required to vanish at breaks and only at the center of the slab cross-section (at $y = 0$). There are two required conditions, at $x = -w$ and $x = w$ ($w < \Lambda/2$, Fig. 1):

$$T_{11}|_{x=-w} = 0, \text{ and } T_{11}|_{x=w} = 0. \tag{3.1}$$

The involved stress depends on the applied Bloch expansion components,

$$T_{11}(x=-w, y=0) = \sum_n \bar{\mathbf{T}}^{(n)} \mathbf{F}^{(n)} e^{j(r+nK)w}, \tag{3.2}$$

$$T_{11}(x=-w, y=0) = \sum_n \bar{\mathbf{T}}^{(n)} \mathbf{F}^{(n)} e^{-j(r+nK)w}, \tag{3.3}$$

where each $\mathbf{F}^{(n)}$ is a 4-dimensional column vector $[F_k]$, and similarly $\bar{\mathbf{T}}^{(n)}$. The summation over harmonics is limited to finite numbers, say $n \in (-N, N]$; this defines the representation space for the mechanical field in the slab. Applying large N, one gets better representation at the cost of computation time.

Now, it is necessary to evaluate a vector $\mathbf{F}$ (comprising all $\mathbf{F}^{(n)}$) that minimizes the divergence of the solution from the conditions (3.2); explicitly, a minimum is sought of

$$\mathcal{E} = \mathbf{F}'\mathbf{A}'\mathbf{A}\mathbf{F}, \quad \mathbf{A}^{(n)} = \begin{bmatrix} \bar{\mathbf{T}}^{(n)} e^{j(r+nK)w} \\ \bar{\mathbf{T}}^{(n)} e^{-j(r+nK)w} \end{bmatrix}, \tag{3.4}$$

where $\mathbf{F}'$ and $\mathbf{A}'$ are Hermitian conjugate to column vectors $[\mathbf{F}^{(n)}]$ and $[\mathbf{A}^{(n)}]$, correspondingly, and $\bar{\mathbf{T}}^{(n)}$ depends on all $\bar{T}_{(k)}$ of Eqs. (2.4).

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It is evident that the null-space $\mathbf{O}$ of $\mathbf{E} = \mathbf{A}'\mathbf{A}$ solves the problem. In this particular case, the size of the null-space is $2N - 2$, that is shorter by 2 (2 is the number of conditions at breaks) from the whole space dimension $2N$. The *MATLAB* [7] is a convenient tool for computation of $\mathbf{O}$.

Each vector $\mathbf{F}$ from the null-space $\mathbf{O}$ satisfies conditions (3.1) as accurately as possible within the assumed representation space, and thus they should be accounted for in the displacement and stress fields of both the homogeneous (for which all the above equations were derived), and the broken slabs as well. This is very important because they constitute the majority of possible vectors in the considered space. We call these solution the 'proper modes' of deformation of homogeneous slab because they simultaneously satisfy the conditions for the broken slab.

Using Eqs. (2.2), it is possible to evaluate all harmonic components of displacement and traction on the slab surfaces $y = \mp d$

$$[\mathbf{u}^{(n)}] = \tilde{\mathbf{U}}\mathbf{O}\boldsymbol{\alpha}, \ [\mathbf{T}^{(n)}] = \tilde{\mathbf{T}}\mathbf{O}\boldsymbol{\alpha}, \tag{3.5}$$

where $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{T}}$ are somewhat rearranged matrices $[\tilde{U}_{i,(k)}(r + nK)\phi(\mp d)]$ and $[\tilde{T}_{i,(k)}(r + nK)\phi(\mp d)]$ that appeared earlier in Eqs. (2.2), and $\boldsymbol{\alpha}$ is a column vector of 'modal amplitudes' (note that the same amplitudes appear in both the above equations).

In fact, the amplitudes $\boldsymbol{\alpha}$ are the least important as we seek the slab characterization in the form of Eq. (1.2), where only the Bloch harmonics of $\mathbf{u}$ and $\mathbf{T}$ are involved. There is no way to eliminate $\boldsymbol{\alpha}$ from Eqs. (3.4) however, because this system of equations is incomplete. There are too few unknowns $\boldsymbol{\alpha}$, because the dimension of $\mathbf{O}$ is smaller by 2 (in this example) from the number of involved harmonics ($2N$).

4. Improper modes

The null-space is only by 2, in our example, smaller than the whole space defined by the matrix $\mathbf{E}$, thus there are only 2 'improper modes' $\mathbf{Z}$ which are eigenvectors of $\mathbf{E}$ not belonging to $\mathbf{O}$. In general, these 'modes' do not satisfy conditions (3.1). The surface traction associated with these modes can be evaluated from the second of Eqs. (3.4), replacing $\mathbf{O}$ by $\mathbf{Z}$ and $\boldsymbol{\alpha}$ by $\boldsymbol{\beta}$,

$$[\mathbf{T}^{(n)}] = \sum_n \tilde{\mathbf{T}}\mathbf{Z}\boldsymbol{\beta} = \mathbf{T}_z, \tag{4.1}$$

in the spectral representation. The first equation of (3.4) is useless here because it concerns the unbroken homogeneous slab where T_{11} does not vanish at $x = \mp w$. It is evident that the broken slab will deform differently under the load $\mathbf{T}_z$. Due

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to the breaks, the displacement may suffer a jump at the breaking points. It is the task of this section to evaluate the broken slab response (in its surface displacements) to the 'modal' loading represented by Eq. (4.1).

The following trick is applied to evaluate this response. There are two breaks per period, at $x = \mp w$. Let us define a periodic 'window function' first that will be used to discriminate the domains between and outside breaks:

$$f(x) = \begin{cases} 1, -\Lambda/2 < x < -1, \\ 0, -w < x < w, \\ 1, w < x < \Lambda/2, \end{cases} \tag{4.2}$$

it is different from zero outside the breaks ($1 - f$ has a support between breaks), and periodic in the remaining domain of x. Due to the breaks, the slab displacements under the load

$$\mathbf{t}_1^{(k)}(x) = f(x)\mathcal{F}^{-1}\{\mathbf{T}_z^{(k)}\}, \ \mathbf{T}_z^{(k)} = \tilde{\mathbf{T}}\mathbf{Z}_k \tag{4.3}$$

is rather constrained to the domain of $\mathbf{t}^{(k)}$, that is to the support of f (neglecting certain residual internal tractions at breaks that can build up in the applied approximation), and independent of the load $\mathcal{F}^{-1}\{\mathbf{T}_z\}(1-f)$ in the other domain. In the above equation, k is the number that counts the considered mode $\mathbf{Z}_k$, Eq. (4.1), and $\mathcal{F}^{-1}$ means the inverse Fourier transform (actually the fast Fourier transform, FFT, is used in computations).

Let us evaluate the slab response in the support of f first. Applying the equations for a homogeneous unbroken slab, the spectral representation to $\mathbf{t}_1^{(k)}$ is evaluated first. The resulting $T_{11}(x = \mp d, y = 0)$ from Eqs. (2.4) indicates how much the evaluated solution (2.1 – 4) differs from the conditions at breaks. Considering still the homogeneous slab, an auxiliary load is applied in the other domain

$$\mathbf{t} = \sum_k \gamma_k \mathcal{F}^{-1}\{\mathbf{T}_z^{(k)}\}(1 - f) \tag{4.4}$$

being a combination of 'improper modes' ($k = 1, 2$ in our example) to make the conditions at breaks (7) satisfied. There are two conditions, and two constants γ_k, which can be evaluated to satisfy these conditions. The physical interpretation of this step relies on helping, by means of $\mathbf{t}$, the slab in the other domain to deform properly in order to allow the slab in the first domain to deform freely at breaks $x = \mp w$, that is to help realizing the conditions $T_{11} = 0$ there. In other words, we apply (by means of $\mathbf{t}$) additional T_{11} at breaks to compensate the force exerted by the other part of the slab that would not allow it to deform freely there.

Under both combined loads, $\mathbf{t}_1 + \mathbf{t}$, the slab deformation $\mathbf{v}_1^{(k)}$ can be evaluated from Eqs. (2.2) (this requires evaluation of the corresponding F_k first, from the

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second equation). We are interested only in the displacements in the domain of the applied load $\mathbf{t}_1^{(k)}$. This is $\mathbf{v}_1 f$: the *broken* slab response to the load $\mathbf{t}_1$, with vanishing stress at breaks (within the applied approximation); both the analyzed stress and displacements belonging to the same domain of x. Here, the displacement is evaluated with accuracy to the additive 'proper' modes which yield zero stress at breaks; the ambiguity to be removed by means of the energy conservation law. Some computed results are shown in Fig. 2.

Next, the other domain is analyzed in the same manner. It suffices to remark here that, replacing f by $1-f$, the analysis repeats the above presented one. The result is $\mathbf{v}_2^{(k)}$ – a response to the load $\mathbf{t}_2^{(k)}$ in the domain $(-w, w)$. Finally

$$\mathbf{v}^{(k)}(x) = \mathbf{v}_1^{(k)} f + \mathbf{v}_2^{(k)}(1-f), \text{ and } \mathbf{T}(x) = \mathbf{t}_1^{(k)} + \mathbf{t}_2^{(k)}, \tag{4.5}$$

define the corresponding functions over the whole domain of x, moreover $\mathbf{T}(x)$ is exactly $\mathbf{T}_z$ from Eq. (4.1) in the spatial domain. The Fourier transform of $\mathbf{v}$ yields the needed spectral representation $\mathbf{u}^{(n)}$. Further on, for loads being a combination of 'improper modes' $\tilde{\mathbf{T}}\mathbf{Z}_k\beta_k$, the n-th Bloch component of the displacement vector is

$$[\mathbf{u}^{(n)}] = [\mathcal{F}\{\mathbf{v}^{(k)}\}]\boldsymbol{\beta}, \quad [\mathbf{T}^{(n)}] = \tilde{\mathbf{T}}\mathbf{Z}\boldsymbol{\beta}, \tag{4.6}$$

the second equation being repeated after (4.1) for convenience.

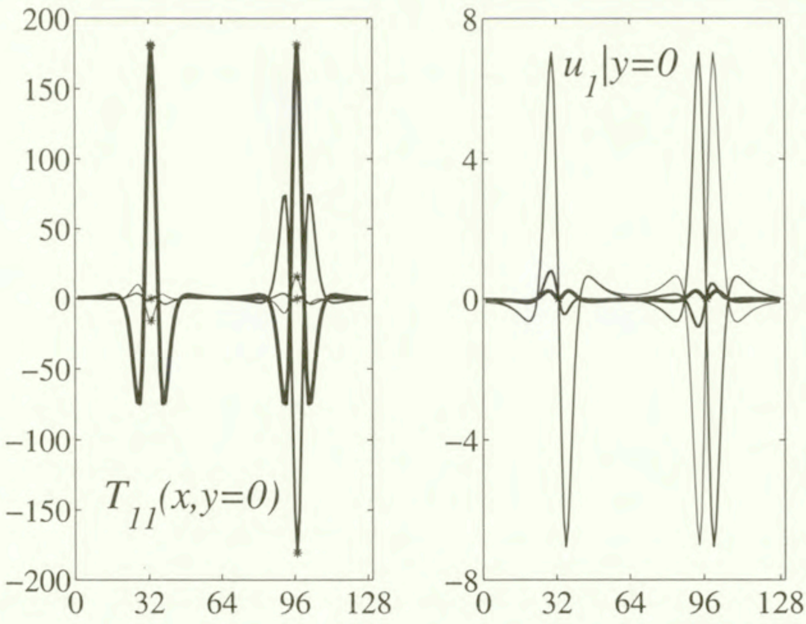

FIG. 2a.

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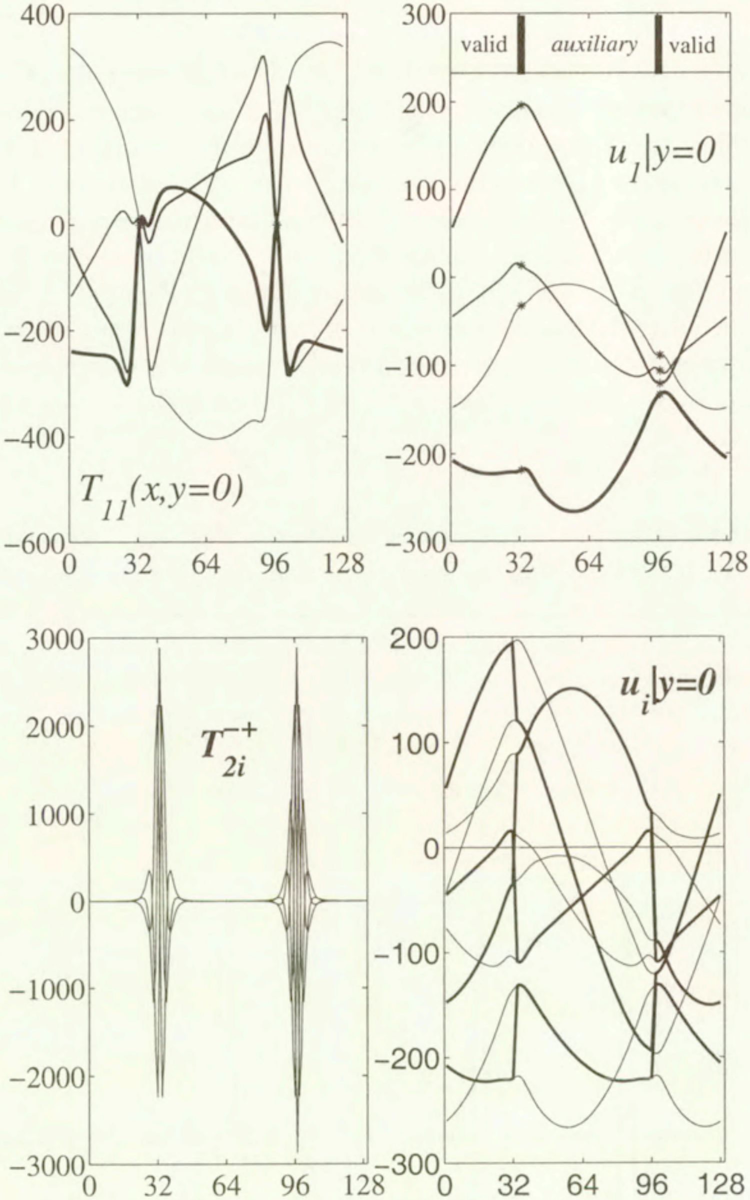

FIG. 2. Real (thick lines) and imaginary parts (thin lines) of $T_{11}(x)$ (left figure) for two 'improper modes.' Note that T_{11} has a maximum rather than zero at breaks; the corresponding homogeneous slab deformation is shown at right. Center: T_{11} and u_1 at $y = 0$ resulting from the loading from (4.3-4) with γ_k evaluated in order to set T_{11} to zero at breaks. Bottom: real and imaginary parts of slab surface traction $T^{\mp}_{21}$ corresponding to 'improper modes' (left figure) and the evaluated broken slab response, u_i at $y = 0$ (both complex parts are shown). Note the displacement discontinuity at breaks (thick lines); thin lines represent $v_{1,2}$ from Eq.(4.5), but evaluated at $y = 0$.

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Now, we can join both Eq.(3.4) and the above Eq.(4.6) to obtain equal numbers of harmonics ($2N$) and unknown coefficients ($\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ combined). They can be eliminated yielding the dependence between $\mathbf{u}^{(n)}$ and $\mathbf{T}^{(m)}$ directly.

5. Discussion

A generalization of the above method in order to satisfy $T_{1i} = 0$, $i = 1, 2$ at many points in the slab cross-section, is straightforward. It requires only the evaluation of the corresponding matrix $\mathbf{A}$ resulting in a new $\mathbf{E}$. It will have smaller null-space $\mathbf{O}$, thus more computations will be needed to evaluate numerous 'improper modes' and the corresponding responses $\mathbf{v}$ of larger number.

Naturally, other conditions can be formulated at breaks, for example using integration of $|T_{1i}|^2$ over the slab cross-section. This results only in a minor modification to $\mathbf{E}$ and the rest of the analysis remains as presented above. In this case however, there may be null-space in no rigorous meaning. But it always exist within the computational accuracy and this suffices for the analysis. Indeed, the solution differs from the conditions at the break by $\|\lambda \mathbf{F}\|$, that is small for small eigenvalues $|\lambda|$ of $\mathbf{E}$.

In the above example, only the traction modal load has been analyzed. Analogously, the modal displacement can be set at the slab surfaces and the traction response sought for. Both results help to find the slab characterization in the form of Eq. (1.2).

Acknowledgements

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Appendix

Planar harmonic Green's function $\mathbf{H}$ for homogeneous elastic slab of a material characterized by Lamé constants λ, μ (in standard notation) and mass density ρ, vibrating with frequency ω (this is zero in statics), possesses the following symmetry:

$$
\text{(A.1)} \qquad \begin{bmatrix} \circ & \triangleleft & \triangleright & \cdot \\ -\triangleleft & \bullet & \cdot & \ast \\ -\triangleright & \cdot & -\circ & \triangleleft \\ \cdot & -\ast & -\triangleleft & -\bullet \end{bmatrix}.
$$

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Its matrix elements are:

$$
\begin{aligned}
H_{11} &= js_tk_t^2[(1-L^2T^2)w+(x^2-z)(T^2-L^2)]/D=-H_{33},\\
H_{12} &= jp[(1+L^2T^2)w(x-2s_ls_t)+(z-x^2)(L^2+T^2)(x+2s_ls_t)\\
&\qquad +4LTx(2xs_ls_t-z)]/D=-H_{21}=-H_{43}=-H_{34},\\
H_{13} &= -j2s_tk_t^2[x^2T(1-L^2)+zL(1-T^2)]/D=-H_{31},\\
H_{14} &= j4ps_ls_tk_t^2x[L(1+T^2)-T(1+L^2)]/D=H_{23}=H_{32}=H_{41},\\
H_{22} &= \;!js_tk_t^2[(1-L^2T^2)w+(x^2-z)(L^2-T^2)]/D=-H_{44},\\
H_{24} &= -j2s_lk_t^2[L(1-T^2)x^2+T(1-L^2)z]/D=-H_{42},\\
k_t^2 &= \rho\omega^2/\mu,\ D=\mu[(1-L^2)(1-T^2)w^2+4x^2z(L-T)^2],\\
k_l^2 &= \rho\omega^2/(\lambda+2\mu),\quad s_{l,t}=\sqrt{k_{l,t}^2-p^2},\quad p<k_{l,t},\\
&\qquad \text{otherwise } =-j\sqrt{p^2-k_{l,t}^2},\\
x &= s_t^2-p^2,\quad z=4p^2s_ls_t,\quad w=x^2+z,\quad T=\exp(-j2s_td),\\
L &= \exp(-j2s_ld).
\end{aligned}
\tag{A.2}
$$

Applying $d\to\infty$ splits the above matrix into independent two of dimensions 2×2, being Green's matrix functions of the upper, $y>0$, or lower, $y<0$, elastic halfspaces. It suffices to take the limits $T\to 0$ and $L\to 0$ simultaneously.

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Thermomechanically consistent formulations of the standard linear solid using fractional derivatives

A. LION

Institute of Mechanics, University of Kassel,
D-34109 Kassel, Mönchebergstraße 7

WE STUDY THE THERMOMECHANICAL properties of a frequently used fractional generalisation of the standard linear solid. Its mathematical structure arises from an ordinary linear differential equation between stress and strain when replacing the first order time rates by fractional derivatives of the order $0 \leq \alpha, \beta < 1$. If the parameters α and β are not further restricted, the model leads to an unphysical behaviour. In the case of harmonic deformations the dissipation modulus can become negative. This corresponds to a negative entropy production and violates the second law of thermodynamics. Then we propose two generalisations of the standard linear solid which are based on a so-called thermodynamically consistent fractional rheological element. It possesses a non-negative free energy and rate of dissipation for arbitrary deformation processes and is compatible with the second law of thermodynamics. The differential equations between stress and strain of the proposed generalisations contain also fractional derivatives of different orders but both the dynamic moduli and the relaxation spectra are non-negative functions of their arguments. No restrictions on the material parameters are required.

Notations

ε, ε_0	strain, strain amplitude
σ, σ_0	stress, stress amplitude
ω	angular frequency
G^*, G', G''	complex modulus, storage modulus and dissipation modulus
δ	specific dissipation
ψ, ψ_{mean}	free energy, temporal mean value of the free energy
ρ	mass density
ν, $H(\nu)$	relaxation frequency, relaxation spectrum
w_d	dissipated energy per loading cycle
$d^\alpha f/dt^\alpha$	fractional derivative of a function $f(t)$
i	imaginary unit
E, G, E_α, E_β, E_γ	elasticity parameters
α, β, γ	parameters of fractional differentiation
τ, τ_R, τ_C	time constants

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1. Introduction

THE FRACTIONAL CALCULUS as a tool to formulate constitutive models is a fairly modern field of scientific research which fits the theory of linear viscoelasticity. Its fundamental idea is based on real-order derivatives of stress and strain which occur in the constitutive equations. If we formulate, for example, a so-called *fractional damping element* of the type $\sigma = E\tau^\alpha d^\alpha \varepsilon / dt^\alpha$ with the order $0 \leq \alpha \leq 1$ of fractional differentiation, we can show that it interpolates between two limit cases: linear Hookean elasticity corresponds to $\alpha = 0$ and linear Newtonian viscosity to $\alpha = 1$ (see e.g. LION [11]). In the intermediate region, i.e. for $0 < \alpha < 1$, it leads to strongly nonlinear stress-strain characteristics (power functions) and to a relaxation or creep behaviour of the power law type; it can be shown that the dynamic moduli and compliances are also of this type. More complicated models containing fractional derivatives lead to creep and relaxation functions, for example of the Mittag Leffler type (cf. KOELLER [10], MAINARDI and BONETTI [15], NONNENMACHER [17] or LION [11]) and are frequently applied to represent the short and long term behaviour of viscoelastic materials (cf. BAGLEY and TORVIK [1], METZELER *et al.* [16], LION [12] or HAUPT *et al.* [7]). In the case of many viscoelastic materials one can observe a very fast rate of relaxation at the very beginning of the process (short term behaviour) and a super-slow rate after some time (long term behaviour). These phenomena can easily be represented using fractional calculus because one needs only a very few number of material parameters. On the basis of relaxation functions of the exponential type one would need a very large number of terms and thus of material constants. The authors BAGLEY and TORVIC [1] have shown that the dynamic behaviour of a corning glass can by represented over seven log cycles in the frequency domain by means of a fractional model containing only four material constants. A model of similar type was applied by HAUPT *et al.* [7] to describe the dynamic properties of polyethylene and by HARTMANN *et al.* [6] to represent the creep behaviour of concrete. Models of the fractional type have also been implemented into finite element codes (cf. SCHMIDT *et al.* [19]).

The fundamental theory of viscoelasticity based on the fractional calculus was originally applied by CAPUTO and MAINARDI [3]. HEYMANS and BAUWENS [9] worked out relations between self similar rheological models and fractional differential equations. In addition to this, they proposed fractional differential equations whose structure is motivated more or less by rheological models. An interesting question concerns the thermodynamical properties of linear viscoelastic models containing fractional derivatives. A first attempt towards this direction was undertaken by BAGLEY and TORVIK [2] who derived restrictions on the material parameters to satisfy the non-negativity of dynamic moduli under sinusoidal loadings. Unfortunately, the free energy function is not specified in their paper.

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For comparison, Friedrich [4] has investigated the temporal decay behaviour of the relaxation function belonging to a fractional differential equation under thermodynamical aspects.

The present work starts with some fundamentals of fractional calculus and the classical theory of linear viscoelasticity. Then we study the thermodynamical properties of a frequently used generalisation of the standard linear solid. We demonstrate that this type of generalisation, named type A, can violate the natural laws of thermodynamics if the material constants are not chosen correctly. Motivated by this result, we propose a systematic approach to formulate fractional generalisations of the standard linear solid, named type B and C. It is based on rheological networks of fractional damping elements. These models lead, for example, to non-negative relaxation spectra and dynamic moduli for any values of their parameters and arbitrary loading processes.

2. Fundamentals

In order to prepare the following studies, let us summarise some basic properties of the dynamic moduli in the sense of the classical theory of linear viscoelasticity. For more details and a deeper understanding concerning the mathematical theory of linear viscoelasticity we refer the interested reader to the textbooks of Tschoegl [20] and Gross [5].

Let us assume that we have a linear viscoelastic material which is loaded by a uniaxial harmonic deformation process $\varepsilon(t)$ of the type

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t). \tag{2.1}$$

As we know, the stationary stress response $\sigma(t)$ is also a sinusoidal function and can be written as

$$\sigma(t) = \varepsilon_0 (G'(\omega) \sin(\omega t) + G''(\omega) \cos(\omega t)). \tag{2.2}$$

The parameter ω is the loading (angular) frequency and ε_0 the deformation amplitude. The frequency-dependent *storage* modulus $G'(\omega)$ describes that part of the stationary stress response which is in phase with the deformation process. The function $G''(\omega)$ is the *dissipation modulus* representing that part of stress which is in phase with the deformation rate.

2.1. Thermodynamic compatibility in terms of the relaxation spectrum

In continuum mechanics it is a common practise to express the second law of thermodynamics in the form of the *Clausius Duhem inequality*. For details we

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refer the reader to HAUPT [8]. In its uniaxial and isothermal form the Clausius Duhem inequality reads

$$\delta = -\rho\dot{\psi} + \sigma\dot{\varepsilon} \geq 0, \tag{2.3}$$

where σ is the stress, ε the strain, δ the rate of dissipation and $\rho\psi$ the specific free energy per unit volume. This inequality expresses the fact that the temporal change in the free energy has to be equal or smaller than the supplied stress power. In the case of linear viscoelastic constitutive models possessing a so-called *relaxation spectrum* $H(\nu)$, it can be motivated that the isothermal free energy, i.e. the mechanical energy which is stored in the material, has the functional form

$$\rho\psi(t) = \frac{1}{2}\int_0^\infty H(\nu)\left(\int_0^t e^{-\nu(t-s)}\dot{\varepsilon}(s)ds\right)^2 d\nu, \tag{2.4}$$

where the variable ν is the relaxation frequency or the reciprocal relaxation time (cf. LION [11]). Now we show that a model of linear viscoelasticity whose free energy is given by (2.4) with

$$H(\nu) \geq 0 \tag{2.5}$$

is compatible with the second law of thermodynamics. To this end we calculate the material time rate of (2.4), insert the result into the Clausius-Duhem inequality (2.3) and rearrange terms:

$$\delta = \left[\sigma - \int_0^\infty H(\nu)\left(\int_0^t e^{-\nu(t-s)}\dot{\varepsilon}(s)ds\right) d\nu\right]\dot{\varepsilon}(t) + \int_0^\infty \nu H(\nu)\left(\int_0^t e^{-\nu(t-s)}\dot{\varepsilon}(s)ds\right)^2 d\nu \geq 0. \tag{2.6}$$

In order to satisfy this inequality for any value of the strain rate at the current time t, the first term in brackets has to vanish. Interchanging the sequence of integration in the first term leads then to the following relations for the stress and the rate of dissipation:

$$\sigma(t) = \int_0^t \left(\int_0^\infty H(\nu)e^{-\nu(t-s)}d\nu\right)\dot{\varepsilon}(s)ds \tag{2.7}$$

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(2.7) [cont.]

$$\text{and} \qquad \delta(t) = \int_0^\infty \nu H(\nu) \left(\int_0^t e^{-\nu(t-s)} \dot{\varepsilon}(s) ds \right)^2 d\nu \geq 0.$$

Taking a look at $(2.7)_2$ we see that the rate of dissipation $\delta(t)$ is non-negative if the relaxation spectrum $H(\nu)$ is non-negative; the same statement is valid for the free energy (2.4). If we have any constitutive equation of linear viscoelasticity and we can show that its relaxation spectrum is non-negative, the model is compatible with the second law of thermodynamics. Thus $H(\nu) \geq 0$ is a sufficient condition for thermodynamic compatibility.

2.2. Thermodynamic compatibility in terms of the dynamic moduli

Let us consider the general stress-strain relation given by $(2.7)_1$ and a harmonic deformation process in the form of (2.1). Based on standard calculations we can demonstrate that the asymptotic stress response has the form of (2.2) where the storage and dissipation moduli read

$$(2.8) \qquad G'(\omega) = \int_0 H(\nu) \frac{\omega^2}{\nu^2+\omega^2} d\nu \quad \text{and} \quad G''(\omega) = \int_0^\infty H(\nu) \frac{\omega\nu}{\nu^2+\omega^2} d\nu.$$

In order to express the free energy $\psi(t)$ and the specific dissipation $\delta(t)$ in terms of the dynamic moduli G' and G'' we take the sinusoidal deformation $\varepsilon(t) = \varepsilon_0 \sin(\omega t)$ into account, evaluate the formulae (2.4) and $(2.7)_2$ and consider large times t, so that the initial transients are vanished. As an intermediate result we obtain the relations

$$(2.9) \qquad \rho\psi(t) = \frac{\varepsilon_0^2}{2} \int_0^\infty H(\nu) \left(\frac{\omega^2}{\nu^2+\omega^2} \sin(\omega t) + \frac{\omega\nu}{\nu^2+\omega^2} \cos(\omega t) \right)^2 d\nu,$$

and

$$(2.10) \qquad \delta(t) = \varepsilon_0^2 \int_0^\infty \nu H(\nu) \left(\frac{\omega^2}{\nu^2+\omega^2} \sin(\omega t) + \frac{\omega\nu}{\nu^2+\omega^2} \cos(\omega t) \right)^2 d\nu$$

leading finally to

$$(2.11) \qquad \rho\psi = \frac{\varepsilon_0^2}{4} \left(\int_0^\infty \frac{H(\nu)\omega^2}{\nu^2+\omega^2} d\nu - \int_0^\infty \frac{H(\nu)(\omega^4 - \omega^2\nu^2)}{(\nu^2+\omega^2)^2} d\nu \cos(2\omega t) \right.$$
$$\left. + 2 \int_0^\infty \frac{H(\nu)\nu\omega^3}{(\nu^2+\omega^2)^2} d\nu \sin(2\omega t) \right)$$

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and

$$(2.12)\qquad \delta = \frac{\varepsilon_0^2\omega}{2}\left(\int\limits_0^\infty \frac{H(\nu)\omega\nu}{\nu^2+\omega^2}d\nu + \omega\int\limits_0^\infty \frac{H(\nu)(\nu^3-\nu\omega^2)}{(\nu^2+\omega^2)^2}d\nu\cos(2\omega t) + 2\omega\int\limits_0^\infty \frac{H(\nu)\omega\nu^2}{(\nu^2+\omega^2)^2}d\nu\sin(2\omega t)\right)$$

Comparing these integrals with the dynamic moduli (2.8), we can show that the final form of the free energy and the specific dissipation under sinusoidal deformations can be written as

$$(2.13)\qquad \rho\psi(t) = \frac{\varepsilon_0^2}{4}\left(G'(\omega) - \left(G'(\omega) - \omega\frac{dG'}{d\omega}\right)\cos(2\omega t) + \left(G''(\omega) - \omega\frac{dG''}{d\omega}\right)\sin(2\omega t)\right)$$

and

$$(2.14)\qquad \delta(t) = \frac{\varepsilon_0^2}{2}\left(G''(\omega) + \omega\frac{dG''}{d\omega}\cos(2\omega t) + \omega\frac{dG'}{d\omega}\sin(2\omega t)\right).$$

These expressions correspond to those which were obtained earlier by Tschoegl [20] but on the basis of different mathematical techniques. As we see, the current values of the free energy $\psi(t)$ and the rate of dissipation $\delta(t)$ depend in a complicated manner on the storage and dissipation moduli as well as on their first and second derivatives with respect to the frequency ω. Both expressions consist of a positive mean value which is superimposed by harmonic oscillations. Since both mechanisms energy storage and dissipation occur in tension and compression as well, it is obvious that ψ and δ depend on the double frequency 2ω.

2.3. Consequences for the storage and dissipation moduli

Let us estimate the amount of energy w_d dissipated per loading cycle. To this end we integrate the specific dissipation (2.14) over one cycle and obtain

$$(2.15)\qquad w_d = \int\limits_t^{t+2\pi/\omega} \delta(s)ds = \pi\varepsilon_0^2 G''(\omega).$$

Since w_d or equivalently, the hysteresis area under cyclic stress strain curves are non-negative, the dissipation modulus $G''(\omega)$ has to be non-negative for any frequency ω:

$$(2.16)\qquad G''(\omega) \geq 0.$$

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Otherwise the model violates the natural laws of thermodynamics. In addition one can show that the expression $w_d = \pi \varepsilon_0^2 G''(\omega)$ is also valid in the case of nonlinear constitutive models whose stationary stress response is given by a complete Fourier series (cf. LION [13]). Calculating the temporal mean value of the free energy (2.13) over one cycle, we obtain

$$\rho\psi_{\text{mean}} = \frac{\varepsilon_0^2}{4} G'(\omega), \tag{2.17}$$

implying $G'(\omega) \geq 0$ and confirming the interpretation of the storage modulus G' assumed by FERRY [21]. Taking an additional look at (2.13) we see that the free energy reduces to

$$\rho\psi(t_k) = \frac{\varepsilon_0^2}{4} \omega \frac{dG'}{d\omega} \tag{2.18}$$

for $\omega t_k = k\pi$ and $k = 0, 1, 2, \ldots$ implying $dG'/d\omega \geq 0$, i.e. the storage modulus is an increasing function. Additional consequences can be derived by analysing (2.13) and (2.14) in more detail but this is not the aim of the present work.

2.4. Introduction to fractional calculus

The mathematical theory of the fractional calculus is explained in detail in the textbook of OLDHAM and SPANIER [18] but we also refer the reader to the original paper of CAPUTO and MAINARDI [3]. For our purpose we apply the *Riemann Liouville definition* and define the operator of fractional differentiation of real order $\alpha \geq 0$ as

$$\frac{d^\alpha f}{dt^\alpha} = \frac{d^m}{dt^m} \left(\frac{1}{\Gamma(m-\alpha)} \int_0^t s^{m-\alpha-1} f(t-s) ds \right), \tag{2.19}$$

where the natural number m is chosen so that $m - \alpha > 0$ and $m - \alpha - 1 \leq 0$. The quantity $\Gamma(x)$ is the Eulerian Gamma function satisfying the functional relation $x\Gamma(x) = \Gamma(x+1)$. Carrying out the differentiation d^m/dt^m of (2.19) we find the equivalent representation

$$\frac{d^\alpha f}{dt^\alpha} = \sum_{k=0}^{m-1} \frac{t^{(k-\alpha)} f^{(k)}(0)}{\Gamma(k+1-\alpha)} + \frac{1}{\Gamma(m-\alpha)} \int_0^t s^{m-\alpha-1} \frac{d^m}{dt^m} f(t-s) ds. \tag{2.20}$$

As we see, the sum incorporating the initial conditions of the function $f(t)$ vanishes for large values of t; if $f(t)$ satisfies homogeneous initial conditions, it is even zero. To calculate, for example, the fractional derivative of order $0 \leq \alpha < 1$

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we set $m = 1$ so that the sum is given by only one term. If $f(t)$ satisfies the initial condition $f(0) = 0$, this term is zero and we obtain

$$\frac{d^\alpha f}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} f'(s) ds \quad \text{with} \quad 0 \leq \alpha < 1. \tag{2.21}$$

In the case of $1 \leq \alpha < 2$ we have to set $m = 2$ in (2.20) and to prescribe an additional initial condition. In other cases one has to proceed in a similar way.

Since we shall calculate the dynamic moduli corresponding to fractional differential equations between stresses and strains, we need the fractional derivative of the complex exponential function

$$x(t) = e^{i\omega t} \tag{2.22}$$

in the stationary case, i.e. for large times t where the influence of the initial conditions is vanished. Application of (2.20) then leads to the expression

$$\frac{d^\alpha}{dt^\alpha} e^{i\omega t} = \frac{(i\omega)^m e^{i\omega t}}{\Gamma(m-\alpha)} \int_0^t s^{m-\alpha-1} e^{-i\omega s} ds = \frac{(i\omega)^m e^{i\omega t}}{\Gamma(m-\alpha)} \int_0^\infty s^{m-\alpha-1} e^{-i\omega s} ds \tag{2.23}$$

where the upper limit of integration can be replaced by ∞ under stationary conditions. Introducing the transformation $i\omega s = u$ we obtain the intermediate result

$$\frac{d^\alpha}{dt^\alpha} e^{i\omega t} = \frac{(i\omega)^\alpha e^{i\omega t}}{\Gamma(m-\alpha)} \int_0^\infty u^{(m-\alpha)-1} e^{-u} du, \tag{2.24}$$

where the integral in the limits between $u = 0$ and $u = \infty$ equals the Eulerian Gamma function $\Gamma(m - \alpha)$. Thus we found the simple relation

$$\frac{d^\alpha}{dt^\alpha} e^{i\omega t} = (i\omega)^\alpha e^{i\omega t}, \tag{2.25}$$

which holds under stationary conditions for any real number $\alpha \geq 0$.

3. Standard linear solid

Let us first define the so-called *standard linear solid* which is visualised in Fig. 1.

The model is given by a linear Maxwell element (modulus E and viscosity η) which is in parallel to a Hookean spring with the modulus G. The stress σ splits into σ_1 and σ_2, the strain ε into ε_1 and ε_{in}, and the constitutive relations read

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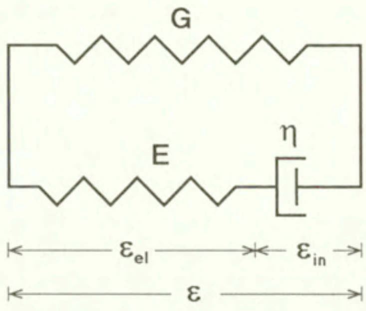

FIG. 1. Standard linear solid.

$$\sigma = \sigma_1 + \sigma_2, \tag{3.1}$$

$$\varepsilon = \varepsilon_{el} + \varepsilon_{in}, \tag{3.2}$$

$$\sigma_1 = G\varepsilon, \tag{3.3}$$

$$\sigma_2 = E\varepsilon_{el}, \tag{3.4}$$

$$\sigma_2 = \eta\dot{\varepsilon}_{in}. \tag{3.5}$$

After some calculations and eliminating the internal variables we obtain the following differential equation of the first order between the stress and the strain:

$$\tau_R\dot{\sigma} + \sigma = G(\tau_C\dot{\varepsilon} + \varepsilon) \quad \text{with} \quad \tau_R = \frac{\eta}{E} \quad \text{and} \quad \tau_C = \tau_R\frac{E+G}{G}. \tag{3.6}$$

For the purpose of calculating the *complex dynamic modulus* $G^* = G' + iG''$ we prescribe a harmonic deformation in the form of a complex exponential function

$$\varepsilon(t) = \varepsilon_0 e^{i\omega t}, \tag{3.7}$$

where ε_0 is the strain amplitude. For the stationary stress response we assume

$$\sigma(t) = \sigma_0(i\omega)e^{i\omega t}, \tag{3.8}$$

where $\sigma_0(i\omega)$ is the frequency-dependent stress amplitude. Inserting (3.7) and (3.8) into (3.6) and application of (2.25) leads to

$$\sigma_0 = G^*(i\omega)\varepsilon_0 \quad \text{with} \quad G^* = G\frac{1+(i\omega\tau_C)}{1+(i\omega\tau_R)}. \tag{3.9}$$

Computing the real and imaginary parts of G^* we obtain the storage and dissipation moduli

$$G' = G\frac{1+\omega^2\tau_R\tau_C}{1+(\omega\tau_R)^2} \quad \text{and} \quad G'' = G\frac{\omega(\tau_C-\tau_R)}{1+(\omega\tau_R)^2}. \tag{3.10}$$

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Taking a look at $(3.10)_2$ we recognise that the dissipation modulus G'' is non-negative for arbitrary frequencies ω. Considering (3.6) we see that the requirement $\tau_C \geq \tau_R$ is automatically satisfied for the standard linear solid.

4. Fractional generalisation (Type A)

To obtain a so-called fractional generalisation of the standard linear solid (3.6) there are different methods. From the point of view of the author, a quite formal generalisation, named as type A, can be obtained if one replaces the first order time derivatives of stress and strain in (3.6) by arbitrary non-integer derivatives $0 \leq \alpha < 1$ and $0 \leq \beta < 1$ (cf. BAGLEY and TORVIK [2]):

$$\tau_R^\alpha \frac{d^\alpha \sigma}{dt^\alpha} + \sigma = G\left(\tau_C^\beta \frac{d^\beta \varepsilon}{dt^\beta} + \varepsilon\right). \tag{4.1}$$

The powers of the time constants τ_R^α and τ_C^β have been introduced for dimensional reasons and in comparison with (3.6), the model now contains 5 non-negative material parameters. For the purpose of computing the dynamic modulus G^* we apply a harmonic deformation in the form of (3.7) and assume stationary conditions so that the stress is given by (3.8). Inserting these assumptions into (4.1) and application of (2.25) leads finally to the modulus

$$G^* = G\frac{1 + (i\omega\tau_C)^\beta}{1 + (i\omega\tau_R)^\alpha}. \tag{4.2}$$

To calculate the real and imaginary parts G' and G'' of G^*, we need the following representation of the α-th power of the imaginary unit $i = \sqrt{-1}$:

$$i^\alpha = e^{i\alpha\pi/2} = \cos(\alpha\pi/2) + i\sin(\alpha\pi/2). \tag{4.3}$$

Taking this formula in combination with the modulus (4.2) into account leads to

$$G' = G \cdot \frac{1 + (\omega\tau_C)^\beta \cos(\beta\pi/2) + (\omega\tau_R)^\alpha \cos(\alpha\pi/2) + (\omega\tau_R)^\alpha (\omega\tau_C)^\beta \cos\left((\beta - \alpha)\frac{\pi}{2}\right)}{1 + (\omega\tau_R)^{2\alpha} + 2(\omega\tau_R)^\alpha \cos(\alpha\pi/2)}, \tag{4.4}$$

$$G'' = G \cdot \frac{(\omega\tau_C)^\beta \sin(\beta\pi/2) - (\omega\tau_R)^\alpha \sin(\alpha\pi/2) + (\omega\tau_R)^\alpha (\omega\tau_C)^\beta \sin\left((\beta - \alpha)\frac{\pi}{2}\right)}{1 + (\omega\tau_R)^{2\alpha} + 2(\omega\tau_R)^\alpha \cos(\alpha\pi/2)} \tag{4.5}$$

for the storage and dissipation moduli. As we see, the storage modulus G' is non-negative for any frequency $\omega \geq 0$ but the numerator of dissipation modulus

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$G''(\omega)$ contains one term with a minus sign. In addition we recognise that the sign of the third term changes in dependence on the difference between the fractional orders α and β of the stress and strain differentiation.

4.1. Investigation of dissipation modulus and relaxation spectrum

To analyse the frequency-dependence of the dissipation modulus (4.5) let us start with the case of $\beta > \alpha$, so that the sign of $\sin((\beta-\alpha)\pi/2)$ is positive. Then we rewrite the modulus G'' as

$$G'' = G\omega^{\alpha}\,\frac{\omega^{\beta-\alpha}\tau_C^{\beta}\sin(\beta\pi/2) - \tau_R^{\alpha}\sin(\alpha\pi/2) + \tau_C^{\beta}\tau_R^{\alpha}\omega^{\beta}\sin(\beta-\alpha)\pi/2)}{1+(\omega\tau_R)^{2\alpha}+2(\omega\tau_R)^{\alpha}\cos(\alpha\pi/2)}. \tag{4.6}$$

Since we assumed both $\beta > 0$ and $\beta - \alpha > 0$,the power functions ω^{β} and $\omega^{\beta-\alpha}$ tend to zero in the limit $\omega \to 0$. Thus for sufficiently small frequencies, the *dissipation modulus becomes negative* provided that the other material parameters are different from zero, i.e. $\alpha > 0$, $G > 0$ and $\tau_R > 0$. In the case of $\alpha = 0$ or $\tau_R = 0$ the dissipation modulus remains non-negative for any value of ω. The same statement holds in the case of $G\tau_C^{\beta} > 0$ and $G = 0$ which can easily be realised by introducing a new parameter for the product $G\tau_C^{\beta}$ in the fractional differential Eq. (4.1). Taking a look at Eq. (4.1), the cases of $\alpha = 0$ or $\tau_R = 0$ are nearly identical but the requirements $G\tau_C^{\beta} > 0$ and $G = 0$ would change the type of the model.

For the purpose of analysing the case of $\alpha > \beta$ we rewrite the dissipation modulus (4.5) as

$$G'' = G\omega^{\beta}\,\frac{\tau_C^{\beta}\sin(\beta\pi/2) - \omega^{\alpha-\beta}\tau_R^{\alpha}\sin(\alpha\pi/2) - \tau_C^{\beta}\tau_R^{\alpha}\omega^{\alpha}\sin((\alpha-\beta)\pi/2)}{1+(\omega\tau_R)^{2\alpha}+2(\omega\tau_R)^{\alpha}\cos(\alpha\pi/2)}. \tag{4.7}$$

Since $\alpha > 0$ and $\alpha - \beta > 0$ is assumed, the terms ω^{α} and $\omega^{\alpha-\beta}$ tend to ∞ in the limit $\omega \to \infty$. As a consequence, the dissipation modulus G'' tends to $-\infty$ for sufficiently large frequencies. Looking at Eq. (2.16) this corresponds to an unphysical effect.

In the third case of $\alpha = \beta$ the dissipation modulus G'' (cf. Eq. (4.5)) can be simplified to the expression

$$G'' = G\,\frac{\omega^{\alpha}(\tau_C^{\alpha} - \tau_R^{\alpha})\sin(\alpha\pi/2)}{1+(\omega\tau_R)^{2\alpha}+2(\omega\tau_R)^{\alpha}\cos(\alpha\pi/2)}, \tag{4.8}$$

which is non-negative for any frequency if the condition $\tau_C^{\alpha} - \tau_R^{\alpha} \geq 0$, or equivalently $\tau_C \geq \tau_R$, is satisfied. If we compare (4.8) with the dissipation modulus of the standard linear solid $(3.10)_2$ we observe similar mathematical forms.

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Practically the same properties can be found if we analyse the relaxation spectrum $H(\nu)$ of the model defined by Eq. (4.1). The common method to compute the spectrum on the basis of the dynamic modulus uses the *inverse Stieltjes transformation* (cf. GROSS [5] or TSCHOEGL [20]):

$$H(\nu) = \frac{1}{2\pi i} \lim_{\gamma \to 0} \left(\frac{G^*(-\nu - i\gamma)}{-\nu - i\gamma} - \frac{G^*(-\nu + i\gamma)}{-\nu + i\gamma} \right). \tag{4.9}$$

To calculate the spectrum on the basis of Eq. (4.9) we rewrite the complex number $-\nu \pm i\gamma$ in terms of the complex exponential function and carry out the limit transition:

$$\lim_{\gamma \to 0} -\nu \pm i\gamma = \lim_{\gamma \to 0} \sqrt{\nu^2 + \gamma^2} e^{\pm i(\pi - \arctan(\gamma/\nu))} = \nu e^{\pm i\pi}. \tag{4.10}$$

Inserting this formula into Eq. (4.9), considering the complex modulus Eq. (4.2) and elementary calculations lead to the following relaxation spectrum:

$$H(\nu) = G \cdot \frac{(\nu\tau_C)^\beta \sin(\beta\pi) - (\nu\tau_R)^\alpha \sin(\alpha\pi) + (\nu\tau_R)^\alpha (\nu\tau_C)^\beta \sin((\beta - \alpha)\pi)}{\pi\nu(1 + (\nu\tau_R)^{2\alpha} + 2(\nu\tau_R)^\alpha \cos(\alpha\pi))}. \tag{4.11}$$

Since the mathematical structure of Eq. (4.11) is similar to that of the dissipation modulus (4.5), the same changes in sign of the numerator occur, but now as a function of the relaxation frequency ν. Since the analysis would run along the same lines as before, we do not repeat it here. Let us take a look at the general expression for the rate of dissipation $(2.7)_2$. Since the spectrum $H(\nu)$ can be negative in certain ranges of the relaxation frequency, there may exist deformation processes which lead to a negative rate of dissipation, excluding the case of $\alpha = \beta$ with $\tau_C^\alpha - \tau_R^\alpha \geq 0$. The result that only the case $\alpha = \beta$ is physically meaningful was proposed by a different line of argumentation by BAGLEY and TORVIK [2].

This analysis has shown that the formal generalisation (type A) of the standard linear solid leads in general to *thermodynamically inconsistent models* which can have a negative dissipation modulus for certain processes. Only in the special case where the parameters of fractional differentiation of stress and strain are equal ($\alpha = \beta$), the model leads to a non-negative dissipation modulus and a non-negative spectrum. This type of a fractional model was originally applied by CAPUTO and MAINARDI [3] but no thermodynamic analysis has been carried out.

5. Fractional generalisation (Type B)

In order to specify a more physically based generalisation of the standard linear solid which is compatible with the natural laws of thermodynamics for any

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values of its parameters, we first consider a so-called *fractional damping element*. It can be understood as an additional rheological element and is compatible with the Clausius Duhem inequality (2.3) (cf. LION [11]). Its free energy can easily be specified and the proof of the thermodynamic compatibility is based on the non-negativity of the relaxation spectrum. A fractional damping element is defined by the linear functional

$$\sigma(t) = E_\alpha \tau^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} \quad \text{or} \quad \sigma(t) = \frac{E_\alpha \tau^\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \dot{\varepsilon}(s) ds \qquad \text{with } 0 \le \alpha < 1, \tag{5.1}$$

where the constant $\tau = 1s$ is introduced for dimensional reasons. Its relaxation spectrum $H(\nu)$ can easily be calculated (cf. TSCHOEGL [20] or LION [14]),

$$H(\nu) = \frac{E_\alpha \tau^\alpha}{\Gamma(1-\alpha)\Gamma(\alpha)\nu^{1-\alpha}}, \tag{5.2}$$

and is positive for $E_\alpha > 0$ implying that the rate of dissipation $(2.7)_2$ is non-negative for any deformation process; thus the thermodynamical compatibility is shown. As a consequence of Eqs. (5.2) and $(2.7)_1$, the functional form Eq. $(5.1)_2$ of the fractional differential Eq. $(5.1)_1$ can be rewritten as

$$\sigma(t) = \frac{E_\alpha \tau^\alpha}{\Gamma(1-\alpha)} \int_0^t \left(\int_0^\infty \frac{1}{\Gamma(\alpha)\nu^{1-\alpha}} e^{-\nu(t-s)} d\nu \right) \dot{\varepsilon}(s) ds. \tag{5.3}$$

This equation can physically be interpreted and expresses that a fractional damping element in the form of Eq. (5.1) corresponds to a superposition of an infinite number of continuously distributed Maxwell elements in parallel.

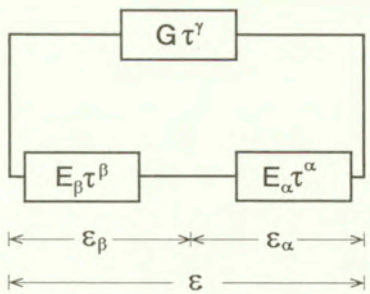

FIG. 2. Fractional linear solid, type B generalisation.

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Motivated by this discussion we replace the rheological elements of the standard linear solid in Fig. 1 by three different fractional damping elements in the form of Eq. (5.1). This idea is visualised in Fig. 2 and named the type B generalisation.

The stress σ is decomposed into the sum of two internal stresses, named σ_1 and σ_2 in Eq. (5.4) but this is not shown in the figure. In the lower branch we have a splitting of the strain ε into the sum of two internal strains ε_α and ε_β corresponding to two fractional elements in series. As we see, the model contains 6 material parameters, namely the non-negative constants E_α, E_β and G as well as three parameters of fractional differentiation $0 \leq \alpha, \beta, \gamma < 1$ and the constitutive equations read

$$\sigma = \sigma_1 + \sigma_2, \tag{5.4}$$

$$\varepsilon = \varepsilon_\alpha + \varepsilon_\beta, \tag{5.5}$$

$$\sigma_1 = G\tau^\gamma \frac{d^\gamma \varepsilon}{dt^\gamma}, \tag{5.6}$$

$$\sigma_2 = E_\alpha \tau^\alpha \frac{d^\alpha \varepsilon_\alpha}{dt^\alpha}, \tag{5.7}$$

$$\sigma_2 = E_\beta \tau^\beta \frac{d^\beta \varepsilon_\beta}{dt^\beta}. \tag{5.8}$$

Comparing these relations with those of the standard linear solid (3.1) – (3.5) we notice a similar structure. The fundamental difference is that the order of differentiation in (3.3) – (3.5) is given by the integers 0, 0 and 1, whereas in (5.6) – (5.8) it is given by the real numbers γ, α and β.

To eliminate the internal strains and stresses ε_α, ε_β, σ_1 and σ_2, we first replace the strain ε_α in Eq. (5.7) using (5.5) and differentiate the result fractionally with the order β:

$$\frac{d^\beta \sigma_2}{dt^\beta} = E_\alpha \tau^\alpha \left(\frac{d^{\alpha+\beta} \varepsilon}{dt^{\alpha+\beta}} - \frac{d^{\alpha+\beta} \varepsilon_\beta}{dt^{\alpha+\beta}} \right). \tag{5.9}$$

Then we differentiate Eq. (5.8) with the order α, rearrange terms and obtain the intermediate result

$$\frac{d^{\alpha+\beta} \varepsilon_\beta}{dt^{\alpha+\beta}} = \frac{1}{E_\beta \tau^\beta} \frac{d^\alpha \sigma_2}{dt^\alpha}, \tag{5.10}$$

which can be used to eliminate the internal strain ε_β in Eq. (5.9). The final result is

$$\tau^\beta \frac{d^\beta \sigma_2}{dt^\beta} = E_\alpha \tau^{\alpha+\beta} \frac{d^{\alpha+\beta} \varepsilon}{dt^{\alpha+\beta}} - \frac{E_\alpha \tau^\alpha}{E_\beta} \frac{d^\alpha \sigma_2}{dt^\alpha}. \tag{5.11}$$

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To eliminate the internal stress in Eq. (5.11) we express σ_2 by means of Eqs. (5.4) and (5.6) and find

$$(5.12)\qquad \tau^{\beta}\frac{d^{\beta}}{dt^{\beta}}\left(\sigma - G\tau^{\gamma}\frac{d^{\gamma}\varepsilon}{dt^{\gamma}}\right) = E_{\alpha}\tau^{\alpha+\beta}\frac{d^{\alpha+\beta}\varepsilon}{dt^{\alpha+\beta}} - \frac{E_{\alpha}\tau^{\alpha}}{E_{\beta}}\frac{d^{\alpha}}{dt^{\alpha}}\left(\sigma - G\tau^{\gamma}\frac{d^{\gamma}\varepsilon}{dt^{\gamma}}\right),$$

leading to the final expression in the form of

$$(5.13)\qquad \tau^{\beta}\frac{d^{\beta}\sigma}{dt^{\beta}} + \frac{E_{\alpha}}{E_{\beta}}\tau^{\alpha}\frac{d^{\alpha}\sigma}{dt^{\alpha}} = G\tau^{\gamma+\beta}\frac{d^{\gamma+\beta}\varepsilon}{dt^{\gamma+\beta}} + E_{\alpha}\tau^{\alpha+\beta}\frac{d^{\alpha+\beta}\varepsilon}{dt^{\alpha+\beta}} + \frac{E_{\alpha}}{E_{\beta}}G\tau^{\gamma+\alpha}\frac{d^{\gamma+\alpha}\varepsilon}{dt^{\gamma+\alpha}},$$

after rearranging terms. In comparison with the type A generalisation specified by Eq. (4.1) we recognise an additional term on the right-hand side of Eq. (5.13); all terms are fractionally differentiated with a different order. If we set, for example, one of the parameters α or β to zero, we recognise that in Eq. (5.13) no strain derivative of the order 0 occurs.

For the purpose of solving the fractional differential equation (5.13) one can prescribe, for example, the deformation process ε. Since the orders α and β of fractional stress differentiation are between 0 and 1, it is sufficient to prescribe only one initial condition, namely $\sigma(0) = 0$. Taking this into account, the fractional differential operator in the form of Eq. (2.21) can be applied to differentiate the stress. The strain differentiation is more complicated because, in dependence on the values of the sums $\gamma+\beta, \alpha+\beta$ and $\gamma+\alpha$, the general form Eq. (2.20) of the differential operator has to be taken into account. If we have, for example, $1 \le \gamma+\beta < 2$, we set $m = 2$ in Eq. (2.20) and have to consider the initial value of the strain rate. Assuming the initial strain $\varepsilon(0)$ to be zero we obtain

$$(5.14)\qquad \frac{d^{\gamma+\beta}\varepsilon}{dt^{\gamma+\beta}} = \frac{t^{1-\gamma-\beta}\dot{\varepsilon}(0)}{\Gamma(2-\gamma-\beta)}\int_0^t s^{1-\beta-\gamma}\ddot{\varepsilon}(t-s)ds.$$

Since the exponent $1-\gamma-\beta$ is negative, the first term incorporating the initial strain rate vanishes for large times or under stationary conditions.

5.2. Investigation of dissipation modulus and relaxation spectrum

Under harmonic loads and the assumption of stationary conditions, the analysis in the frequency domain is much easier. In this case we assume the representations Eqs. (3.7) and (3.8) for the strain and the stress, insert them into (5.13)

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and apply (2.25). Then the complex modulus reads

$$G^* = G(i\omega\tau)^\gamma + \frac{E_\alpha E_\beta (i\omega\tau)^{\alpha+\beta}}{E_\beta (i\omega\tau)^\beta + E_\alpha (i\omega\tau)^\alpha}. \tag{5.15}$$

Splitting (5.15) into real and imaginary parts leads to the formulae

$$G' = G(\omega\tau)^\gamma \cos(\gamma\pi/2) + \frac{E_\alpha^2 E_\beta (\omega\tau)^{2\alpha+\beta} \cos(\beta\pi/2) + E_\alpha E_\beta^2 (\omega\tau)^{\alpha+2\beta} \cos(\alpha\pi/2)}{E_\alpha^2 (\omega\tau)^{2\alpha} + E_\beta^2 (\omega\tau)^{2\beta} + 2E_\alpha E_\beta (\omega\tau)^{\alpha+\beta} \cos((\alpha-\beta)\pi/2)}, \tag{5.16}$$

$$G'' = G(\omega\tau)^\gamma \sin(\gamma\tau/2) + \frac{E_\alpha^2 E_\beta (\omega\tau)^{2\alpha+\beta} \sin(\beta\pi/2) + E_\alpha E_\beta^2 (\omega\tau)^{\alpha+2\beta} \sin(\alpha\pi/2)}{E_\alpha^2 (\omega\tau)^{2\alpha} + E_\beta^2 (\omega\tau)^{2\beta} + 2E_\alpha E_\beta (\omega\tau)^{\alpha+\beta} \cos((\alpha-\beta)\pi/2)}, \tag{5.17}$$

for the storage and dissipation moduli. Taking a look at Eqs. (5.16) or (5.17) we recognise that both functions are non-negative for any value of the frequency ω and any value of the material constants compatible with E_α, E_β, $G \geq 0$ and $0 \leq \alpha, \beta, \gamma < 1$.

Calculating the relaxation spectrum $H(\nu)$ on the basis of the inverse Stieltjes transformation Eqs. (4.9) and (4.10), a series of calculations leads to the final expression

$$H(\nu) = \frac{1}{\pi\nu} \left(G(\nu\tau)^\gamma \sin(\gamma\pi) + \frac{E_\alpha^2 E_\beta (\nu\tau)^{2\alpha+\beta} \sin(\beta\pi) + E_\alpha E_\beta^2 (\nu\tau)^{\alpha+2\beta} \sin(\alpha\pi)}{E_\alpha^2 (\nu\tau)^{2\alpha} + E_\beta^2 (\nu\tau)^{2\beta} + 2E_\alpha E_\beta (\nu\tau)^{\alpha+\beta} \cos((\alpha-\beta)\pi)} \right), \tag{5.18}$$

which is non-negative for any value of material constants $0 \leq E_\alpha, E_\beta, G$ and $0 \leq \alpha, \beta, \gamma < 1$. As a fundamental result we recognise that the fractional model of type B is compatible with the second law of thermodynamics (cf. Sec. 2.1).

5.3. Correlation between the thermodynamical consistent form of type A and the model of type B

The thermomechanical consistent form of the first generalisation, named type A, corresponds to the case of $\alpha = \beta$ in the fractional differential equation (4.1). This model can very easily and without any further investigation be derived on the basis of the type B generalisation Eq. (5.13). The only thing to do is to set $\beta = 0$ and $\gamma = 0$. The result can be interpreted as a spring with modulus G in

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parallel with a fractional Maxwell element consisting of a spring with modulus E_β in series with a fractional damping element. Let us confront both equations:

$$\tau_R^\alpha \frac{d^\alpha \sigma}{dt^\alpha} + \sigma = G\left(\tau_C^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} + \varepsilon\right), \qquad \frac{E_\alpha}{E_\beta}\tau^\alpha \frac{d^\alpha \sigma}{dt^\alpha} + \sigma = G\left(\frac{G+E_\beta}{G}\,\frac{E_\alpha}{E_\beta}\tau^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} + \varepsilon\right). \tag{5.19}$$

As we have shown, the requirement on the material constants for the thermodynamical consistency of the type A generalisation Eq. $(5.19)_1$ is $\tau_C^\alpha - \tau_R^\alpha \geq 0$. If wee take a look at Eq. $(5.19)_2$, we see that this requirement is automatically satisfied if the generalisation is based on fractional damping elements.

6. Fractional generalisation (Type C)

To discuss a further generalisation based on the fractional damping element, let us take a look at Fig. 3, where the type C generalisation is shown. The rheological model corresponds to the Kelvin-Voigt form of the three-parameter solid: a linear spring is in series with a Kelvin element.

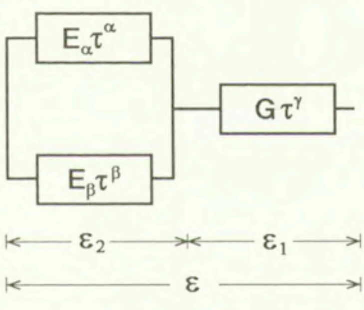

FIG. 3. Fractional linear solid, type C generalisation.

The corresponding constitutive equations read

$$\varepsilon = \varepsilon_1 + \varepsilon_2, \tag{6.1}$$

$$\sigma = \sigma_\alpha + \sigma_\beta, \tag{6.2}$$

$$\sigma = G\tau^\gamma \frac{d^\gamma \varepsilon_1}{dt^\gamma}, \tag{6.3}$$

$$\sigma_\alpha = E_\alpha \tau^\alpha \frac{d^\alpha \varepsilon_2}{dt^\alpha}, \tag{6.4}$$

$$\sigma_\beta = E_\beta \tau^\beta \frac{d^\beta \varepsilon_2}{dt^\beta}, \tag{6.5}$$

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and the fractional differential equation in the form of (6.6), where the internal variables are eliminated, can be obtained in a similar way as described above,

$$\tau^{\gamma}\frac{d^{\gamma}\sigma}{dt^{\gamma}}+\frac{E_{\beta}}{G}\tau^{\beta}\frac{d^{\beta}\sigma}{dt^{\beta}}+\frac{E_{\alpha}}{G}\tau^{\alpha}\frac{d^{\alpha}\sigma}{dt^{\alpha}}=E_{\alpha}\tau^{\alpha+\gamma}\frac{d^{\alpha+\gamma}\varepsilon}{dt^{\alpha+\gamma}}+E_{\beta}\tau^{\beta+\gamma}\frac{d^{\beta+\gamma}\varepsilon}{dt^{\beta+\gamma}}. \tag{6.6}$$

In comparison with (5.13), this equation contains three terms depending on the stress σ and two depending on the strain ε. The dynamic modulus G^{*} is calculated under the assumption of stationary conditions and harmonic stress and strain processes and reads

$$G^{*}=\frac{GE_{\alpha}(i\omega\tau)^{\alpha+\gamma}+GE_{\beta}(i\omega\tau)^{\beta+\gamma}}{G(i\omega\tau)^{\gamma}+E_{\beta}(i\omega\tau)^{\beta}+E_{\alpha}(i\omega\tau)^{\alpha}}. \tag{6.7}$$

Splitting the modulus G^{*} into its real and imaginary parts G' and G'' and calculating the relaxation spectrum on the basis of the inverse Stieltjes transformation specified by Eqs. (4.9) and (4.10), leads to

$$G'(\omega)=\frac{g'(\omega)}{N(\omega)},\qquad G''(\omega)=\frac{g''(\omega)}{B(\omega)},\qquad H(\nu)=\frac{h(\nu)}{M(\nu)}, \tag{6.8}$$

with

$$\begin{aligned} g' = G^{2}(\omega\tau)^{2\gamma}&\left(E_{\alpha}(\omega\tau)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+E_{\beta}(\omega\tau)^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right)\\ &+G(\omega\tau)^{\gamma}\cos\left(\frac{\gamma\pi}{2}\right)\Bigg(E_{\alpha}^{2}(\omega\tau)^{2\alpha}+E_{\beta}^{2}(\omega\tau)^{2\beta}\\ &+2E_{\alpha}E_{\beta}(\omega\tau)^{\alpha+\beta}\cos\left(\frac{(\alpha-\beta)\pi}{2}\right)\Bigg), \end{aligned} \tag{6.9}$$

$$\begin{aligned} g'' = G^{2}(\omega\tau)^{2\gamma}&\left(E_{\alpha}(\omega\tau)^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)+E_{\beta}(\omega\tau)^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\right)\\ &+G(\omega\tau)^{\gamma}\sin\left(\frac{\gamma\pi}{2}\right)\Bigg(E_{\alpha}^{2}(\omega\tau)^{2\alpha}+E_{\beta}^{2}(\omega\tau)^{2\beta}\\ &+2E_{\alpha}E_{\beta}(\omega\tau)^{\alpha+\beta}\cos\left(\frac{(\alpha-\beta)\pi}{2}\right)\Bigg), \end{aligned} \tag{6.10}$$

$$\begin{aligned} N = E_{\alpha}^{2}(\omega\tau)^{2\alpha}+E_{\beta}^{2}(\omega\tau)^{2\beta}+G^{2}(\omega\tau)^{2\gamma}+2E_{\alpha}G(\omega\tau)^{\alpha+\gamma}\cos\left(\frac{(\alpha-\gamma)\pi}{2}\right)\\ +2E_{\beta}G(\omega\tau)^{\beta+\gamma}\cos\left(\frac{(\beta-\gamma)\pi}{2}\right)+2E_{\alpha}E_{\beta}(\omega\tau)^{\alpha+\gamma}\cos\left(\frac{(\alpha-\gamma)\pi}{2}\right), \end{aligned} \tag{6.11}$$

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$$(6.12)\qquad h = G^2(\nu\tau)^{2\gamma}(E_\alpha(\nu\tau)^\alpha \sin(\alpha\pi) + E_\beta(\nu\tau)^\beta \sin(\beta\pi)) + G(\nu\tau)^\gamma \sin(\gamma\pi)(E_\alpha^2(\nu\tau)^{2\alpha} + E_\beta^2(\nu\tau)^{2\beta} + 2E_\alpha E_\beta(\nu\tau)^{\alpha+\beta}\cos((\alpha-\beta)\pi)),$$

$$(6.13)\qquad M = E_\alpha^2(\nu\tau)^{2\alpha} + E_\beta^2(\nu\tau)^{2\beta} + G^2(\nu\tau)^{2\gamma} + 2E_\alpha G(\nu\tau)^{\alpha+\gamma}\cos((\alpha-\gamma)\pi) + 2E_\beta G(\nu\tau)^{\beta+\gamma}\cos((\beta-\gamma)\pi) + 2E_\alpha E_\beta(\nu\tau)^{\alpha+\gamma}\cos((\alpha-\gamma)\pi).$$

Taking a look at the relations (6.8) - (6.13) we see that both the dynamic moduli $G'(\omega)$ and $G''(\omega)$ and the relaxation spectrum $H(\nu)$ are non-negative functions of their arguments for any values of material parameters satisfying $E_\alpha, E_\beta, G \geq 0$ and $0 \leq \alpha, \beta, \gamma < 1$.

6.1. Correlation between the thermodynamical consistent form of type A and the model of type C

The thermomechanical consistent formulation of the type A generalisation, i.e. the case $\alpha = \beta$, can also be derived on the basis of the type C generalisation (6.6). If we set $\beta = 0$ and $\gamma = 0$ we obtain the fractional differential equation in the form of $(6.14)_2$:

$$(6.14)\qquad \tau_R^\alpha \frac{d^\alpha \sigma}{dt^\alpha} + \sigma = G\left(\tau_C^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} + \varepsilon\right), \qquad \frac{E_\alpha}{G+E_\beta}\tau^\alpha \frac{d^\alpha \sigma}{dt^\alpha} + \sigma = \frac{GE_\beta}{G+E_\beta}\left(\frac{E_\alpha}{E_\beta}\tau^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} + \varepsilon\right).$$

As shown above, the requirement on the material parameters to satisfy the thermodynamical consistency of $(6.14)_1$ is $\tau_C^\alpha - \tau_R^\alpha \geq 0$. Comparing $(6.14)_1$ and $(6.14)_2$ we notice that this condition is automatically satisfied if the fractional generalisation of the standard linear solid is based on the rheological model visualised in Fig. 3.

7. Discussion

In this paper we demonstrate that the quite formal generalisation of linear differential equations by replacing the first order time derivatives by fractional derivatives leads in general to thermodynamically inconsistent constitutive models. We show that the relaxation spectrum and the dissipation modulus can become negative, so that the natural laws of thermodynamics are violated. In order to avoid these problems we propose a more physically-based method to

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formulate constitutive relations of the fractional type: to this end an additional rheological element, a so-called *fractional damping element*, is introduced which is compatible with the natural laws of thermodynamics. The idea of the proposed method is to replace the Hookean springs and Newtonian dashpots in a given rheological network by rheological elements of the fractional type. For two special examples we show that the relaxation spectra and the dynamic moduli are non-negative for arbitrary values of the material parameters and the independent process variables. It is obvious that the proposed method of generalisation leads not to the most general form of a fractional constitutive model but it leads to a thermodynamically consistent model. We are sure that this method can also be applied to formulate more complicated models of the fractional type and to formulate three-dimensional stress/strain relations. In the isotropic case one has only to replace the uniaxial stress and strain variables by the stress and strain deviators and to formulate corresponding relations for the hydrostatic pressure and the volume deformation.

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Brief Notes

On the application of a work postulate to frictional contact

R. E. JONES [1] and P. PAPADOPOULOS

Department of Mechanical Engineering, University of California, Berkeley, CA 94720-1740, USA

The work postulate of Naghdi and Trapp is applied to a frictional contact interface to derive an inequality restricting the relation between slip traction and slip direction.

Key words: work inequality, contact mechanics, friction.

1. Introduction

The quasi-thermodynamic postulate of Naghdi and Trapp [1] has been employed extensively in deriving restrictions to the constitutive laws of elastic-plastic materials [2, 3, 4]. The postulate is an extension to finite deformations of an earlier hypothesis by Ilyushin [5] concerning the work done in a closed cycle of homogeneous deformation. In this short paper, it is shown that the work postulate is applicable and relevant to frictional contact (when formulated in a plasticity-like setting) and gives rise to a physically meaningful restriction of the constitutive law for the frictional tractions. This finding serves to further demonstrate the wide-ranging significance of the postulate.

2. Background

Consider two bodies which occupy open regions Ω^α, $\alpha = 1, 2$. Under quasi-static conditions, the motion χ^α of each body is governed by the equilibrium equation

$$\mathrm{div}\mathbf{T}^\alpha + \rho^\alpha \mathbf{b}^\alpha = \mathbf{0} \qquad (\text{no sum on}\,\alpha)\ ,$$

where $\mathbf{T}^\alpha$ denotes the Cauchy stress, ρ^α the mass density, and $\mathbf{b}^\alpha$ the body force. The traction vector $\mathbf{t}^\alpha$ on the smooth boundary surface $\partial\Omega^\alpha$ with outward unit normal $\mathbf{n}^\alpha$ is related to the Cauchy stress $\mathbf{T}^\alpha$ by $\mathbf{t}^\alpha = \mathbf{T}^\alpha \mathbf{n}^\alpha$. The vector $\mathbf{t}^\alpha$ can

[1] Currently at Sandia National Laboratories, Livermore, CA

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be uniquely decomposed as $\mathbf{t}^\alpha = -p^\alpha \mathbf{n}^\alpha + \boldsymbol{\tau}^\alpha$, where $p^\alpha \geq 0$ is the pressure and $\boldsymbol{\tau}^\alpha$ is the tangential traction.

The principle of impenetrability stipulates that $\Omega^\alpha \cap \Omega^\beta = \varnothing$, where $\beta = \mathrm{mod}(\alpha, 2) + 1$. On the contact surface $C = \partial\Omega^\alpha \cap \partial\Omega^\beta$, impenetrability is enforced by p^α, interpreted here as a Lagrange multiplier field. Additionally, the smoothness of $\partial\Omega^\alpha$ implies that $\mathbf{n}^\alpha = -\mathbf{n}^\beta$ on C, so that the traction fields on the two bodies must satisfy the linear momentum balance in the form

$$\mathbf{t}^\beta = -\mathbf{t}^\alpha \; . \tag{2.1}$$

A yield-like function Υ, dependent on $\{p^\alpha, \boldsymbol{\tau}^\alpha\}$, determines the regions of stick and slip as

$$C^\alpha_{\text{stick}} = \{\mathbf{x}^\alpha \in C \mid \Upsilon < 0\}, \qquad C^\alpha_{\text{slip}} = \{\mathbf{x}^\alpha \in C \mid \Upsilon = 0\} \; .$$

The equation $\Upsilon = 0$ defines a surface with closed projections on the $\boldsymbol{\tau}^\alpha$-plane for all $p^\alpha \geq 0$. On C^α_{stick}, the jump in velocity $[\![\mathbf{v}]\!]^\alpha$, defined as

$$[\![\mathbf{v}]\!]^\alpha = \mathbf{v}^\beta - \mathbf{v}^\alpha \; , \tag{2.2}$$

vanishes and $\boldsymbol{\tau}^\alpha$ acts as a Lagrange multiplier to enforce stick. On C^α_{slip}, the tangential traction is constitutively determined by a function $\boldsymbol{\tau}$ which is assumed to depend on p^α and the relative slip direction $\mathbf{d}^\alpha = \dfrac{[\![\mathbf{v}]\!]^\alpha}{\|[\![\mathbf{v}]\!]^\alpha\|}$. Invariance under superposed rigid body motions implies that

$$\mathbf{Q}\boldsymbol{\tau}(p^\alpha, \mathbf{d}^\alpha) = \boldsymbol{\tau}(p^\alpha, \mathbf{Q}\mathbf{d}^\alpha) \; , \tag{2.3}$$

for all proper orthogonal $\mathbf{Q}$.

3. Application of a work postulate

The work postulate of NAGHDI and TRAPP [1] states that the external work done on a body undergoing a smooth and closed cycle of spatially homogeneous deformation is non-negative. For a cycle over the time interval $[t_1, t_2]$ the postulate implies that

$$\int_{t_1}^{t_2} \left[\int_{\partial\Omega^\alpha} \mathbf{t}^\alpha \cdot \mathbf{v}^\alpha \, da + \int_{\Omega^\alpha} \rho^\alpha \mathbf{b}^\alpha \cdot \mathbf{v}^\alpha \, dv \right] dt \; \geq 0 \; . \tag{3.1}$$

Recall that homogeneous deformation maps material points $\mathbf{X}^\alpha$ to $\mathbf{x}^\alpha$, according to

$$\mathbf{x}^\alpha = \mathbf{F}^\alpha \mathbf{X}^\alpha + \mathbf{c}^\alpha \; , \tag{3.2}$$

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where $\mathbf{F}^\alpha$ denotes the deformation gradient. Since the cycle of deformation is assumed to be closed, it follows that $\mathbf{F}^\alpha(t_1) = \mathbf{F}^\alpha(t_2)$ and $\mathbf{c}^\alpha(t_1) = \mathbf{c}^\alpha(t_2)$.

With regard to the work postulate, note that forces on the frictional interface C are external to both Ω^α and Ω^β but internal to the union $\Omega^\alpha \cup \Omega^\beta$. Consequently, if the postulate in the form (3.1) is applied to $\Omega^\alpha \cup \Omega^\beta$ and the corresponding inequalities for Ω^α and Ω^β are subtracted, it follows that

$$\int\limits_{t_1}^{t_2} \left[\int\limits_C (\mathbf{t}^\alpha \cdot \mathbf{v}^\alpha + \mathbf{t}^\beta \cdot \mathbf{v}^\beta)\, da \right] dt \geq 0 .$$

Taking into account (2.1), (2.2), and that impenetrability and stick are workless constraints, the preceding inequality can be also written as

$$\int\limits_{t_1}^{t_2} \left[\int\limits_{C^\alpha_{\text{slip}}} \boldsymbol{\tau}^\alpha \cdot [\![\mathbf{v}]\!]^\alpha \, da \right] dt \leq 0 . \tag{3.3}$$

For the purpose of obtaining constitutive restrictions on $\boldsymbol{\tau}$, consider the contact between a homogeneous deformable body and a flat, rigid and stationary foundation. In particular, assume that in its stress-free, undeformed state ($t = t_1$) the body is a rectangular parallelepiped. For convenience, take a fixed Cartesian basis $\{\mathbf{e}_i\}$ on the surface of the rigid foundation and let $\mathbf{e}_3$ be the outward normal to this surface. Consequently, the contact surface C^α at $t = t_1$ is defined by $X_3^\alpha = 0$. Also, taking into account the homogeneity of the motion, it is clear that the deformable body will remain a parallelepiped. For notational brevity, the superscripts α and β are omitted in the remainder of this note and all quantities are implicitly referred to the deformable body.

In order for the contact to persist, it is sufficient that the normal component of the relative velocity on $X_3 = 0$ vanish. Recalling (2.2) and (3.2), it follows that

$$[\![\mathbf{v}]\!] = -(\dot{\mathbf{F}}\mathbf{X} + \dot{\mathbf{c}}) , \tag{3.4}$$

hence $[\![\mathbf{v}]\!] \cdot \mathbf{e}_3 = 0$ leads to $\dot{F}_{3\gamma} = 0$ ($\gamma = 1, 2$), and $\dot{c}_3 = 0$. The inner integrand in (3.3) is independent of position if the effected motion is such that:

(a) The velocity jump $[\![\mathbf{v}]\!]$ on the interface is uniform;

(b) The surface traction $\mathbf{t}$ on the interface is uniform.

Condition (a) immediately implies a state of uniform stick or slip on C. In either case, Eq. (3.4) yields $\dot{F}_{i\gamma} = 0$, thus $F_{i\gamma}$ are constant throughout the

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homogeneous cycle. It follows that the deformation gradient, relative to the configuration at $t = t_1$, must be of the form

$$\mathbf{F} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + F_{i3}\, \mathbf{e}_i \otimes \mathbf{e}_3 \ , \tag{3.5}$$

where $F_{33}(t) > 0$ for all t. Condition (b) is satisfied if the deformation gives rise to homogeneous stress, thus resulting in uniform traction on any flat surface such as C. This is the case when the homogeneous body is also assumed to be Cauchy-elastic, i.e., $\mathbf{T} = \mathbf{T}(\mathbf{F})$.(2) Existence of a non-empty intersection of the regions

$$\{p = -T_{33}(F_{i3}),\ \boldsymbol{\tau} = T_{\gamma 3}(F_{i3})\mathbf{e}_\gamma, \quad \forall\, F_{i3} \mid F_{33} > 0\}$$

and

$$\{p,\ \boldsymbol{\tau} \quad | \quad \Upsilon(p, \boldsymbol{\tau}) \le 0\}$$

in the neighborhood of $p = 0$, $\boldsymbol{\tau} = \mathbf{0}$ is tacitly assumed, as is the controllability of motions of the type (3.5).

Now, examine a homogeneous cycle of deformation of the form (3.5) starting at $t = t_1$, in which p and $\boldsymbol{\tau}$ increase until $\Upsilon = 0$ at a time $t = t_a$. At that instant, slip is initiated on C and, by fixing $\mathbf{F}$, the body begins to translate rigidly with homogeneous relative velocity $[\![\mathbf{v}]\!] = -\dot{\mathbf{c}}$, constant slipping direction $\bar{\mathbf{d}} = -\dfrac{\dot{\mathbf{c}}}{\|\dot{\mathbf{c}}\|}$, and constant pressure $\bar{p}$. At time t_b, after the body has slipped a distance $|L|$, unloading is effected smoothly so that the body instantaneously returns to stick. Subsequently, through a reverse process, the body is returned to its initial configuration, with slip in the opposite direction occurring during the interval $[t_c, t_d]$. For the given cycle, with the aid of (2.3), inequality (3.3) reduces to

$$\int_{t_a}^{t_b} \boldsymbol{\tau}(\bar{p}, \bar{\mathbf{d}}) \cdot \|\dot{\mathbf{c}}\|\, \bar{\mathbf{d}}\, dt \ + \int_{t_c}^{t_d} \boldsymbol{\tau}(\bar{p}, -\bar{\mathbf{d}}) \cdot \|\dot{\mathbf{c}}\|\, (-\bar{\mathbf{d}}) dt = \boldsymbol{\tau}(\bar{p}, \bar{\mathbf{d}}) \cdot \bar{\mathbf{d}} 2|L| \le 0,$$

which requires

$$\boldsymbol{\tau} \cdot \mathbf{d} \ \le 0\ .$$

Therefore, the Naghdi-Trapp postulate implies that the tangential traction $\boldsymbol{\tau}$ must oppose the slip direction $\mathbf{d}$, as is commonly assumed, and places a corresponding restriction on the constitutive function $\boldsymbol{\tau}$.

(2) This constitutive choice is made in order to render friction the sole source of dissipation. Since the friction law and the bulk material response are uncoupled, no loss in generality results from this assumption.

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