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Limitations of the small-scale yielding approach in theoretical investigations of yield and fracture at a crack-tip

E. SMITH (MANCHESTER)

LARSSON and CARLSSON's finite-element results have shown that the small-scale yielding approach to the problem of yield and fracture at a crack tip has an unduly restricted range of validity in certain situations. Furthermore Rice, using an approximate analytical approach based on a simple model for plane strain yielding at a crack tip, has shown that this restricted range is associated with a non-singular and non-vanishing stress which accompanies the classic inverse square-root singularity at a crack tip. The present paper substantiates Rice's conclusions by clearly demonstrating the effect of the additional stress term, using an exact approach for the analogous anti-plane strain model.

Wyniki otrzymane metodą elementów skończonych przez Larrsona i Carlssona wskazują, że zakres stosowalności założeń małego uplastycznienia w analizie obszaru uplastycznienia i propagacji wierzchołka szczeliny jest w pewnych przypadkach niewłaściwie ustalony. Następnie Rice, stosując przybliżone metody analityczne oparte na prostym modelu uplastycznienia w wierzchołku szczeliny wykazał, że ten zakres stosowalności związany jest z nieosobliwym i nieznikającym naprężeniem towarzyszącym klasycznej, pierwiastkowej osobliwości naprężeń, występującej w wierzchołku szczeliny. Praca uzasadnia więc wnioski wyciągnięte przez Rice'a, pokazując jasno wpływ dodatkowego członu rozwinięcia naprężeń w ścisłym rozwiązaniu pokrewnego zagadnienia antypłaskiego stanu odkształcenia.

Результаты полученные методом конечных элементов Ларссоном и Карлссоном показывают, что область применяемости предположений малой пластичности, в анализе области перехода в пластическое состояние и распространения вершины щели, в некоторых случаях неправильно определена. Затем Рейс, применяя приближенные аналитические методы, опирающиеся на простой модели перехода в пластическое состояние в вершино щели, показал, что эта область применяемости связана с несингулярным и с неисчезающим напряжением, сопутствующим классической, типа радикала сингулярности напряжений в вершине щели. Итак, работа обосновывает следствия приведенные Рейсом, показывая ясно влияние дополнительного члена разложения напряжений в точном решении смежной задачи антиплоского деформационного состояния.

1. Introduction

A COMPREHENSIVE understanding of yield and fracture processes at a crack tip is an essential pre-requisite in the framing of criteria for the prevention of catastrophic failures in engineering components, and the development of laboratory tests that correlate with service conditions. In general, the complexity of these processes and their interaction with geometrical parameters preclude an easy attainment of such an understanding. However, if a solid is subject to sufficiently low applied loads that the plastic zone size at a crack tip is small compared with the crack size and any other characteristic dimension of the solid such as its width, a situation referred to hereafter as "small-scale yielding", the problem becomes more amenable to theoretical analysis. Thus, if r, θ are polar coordinates referred to the crack tip as origin, linear elasticity gives the near-tip stress distribution for plane strain (mode I) loading as

$$p_{ij} = K_I r^{-\frac{1}{2}} f_{ij}(\theta) + (\text{non-singular terms}), \tag{1.1}$$

where K_I is the stress-intensity factor and $f_{ij}(\theta)$ are known functions of θ, which are normalized so that the singular contribution to the tensile stress normal to the crack plane, at a point directly ahead of the tip, is $K_I(2\pi r)^{-\frac{1}{2}}$. Similar expressions apply for plane strain shear (mode II) and anti-plane strain (mode III) types of loading with respectively K_{II} and K_{III} replacing K_I, although the functions $f_{ij}(\theta)$ are of course different. Equation (1.1) is inaccurate within and near the small yield zone, but the basis of the small-scale yielding approach [1] is that the dominant singular term governs the deformation within that zone. Thus the elastic-plastic problem reduces to one formulated in the pattern of a boundary layer problem, the reduced problem depending on the analysis of a model consisting of a semi-infinite crack in an infinite solid, with boundary conditions such that there is an asymptotic approach to the elastic singularity stress distribution at large distances:

$$p_{ij} = K_I r^{-\frac{1}{2}} f_{ij}(\theta) \quad \text{as} \quad r \to \infty. \tag{1.2}$$

The small yield zone is surrounded by the dominant elastic singularity, and both the applied loads and body shape influence the deformation within the zone only through the value of K_I as determined by classic elasticity theory.

An exact solution for the elastic-plastic field in this new situation is more readily attainable, and this approach has been used for a variety of loading states and material flow characteristics. Formulation of the problem in this manner enables the size r_p of the plastic zone and the crack tip opening displacement δ, when definable, to be given by general formulae of the type

$$\begin{aligned} r_p &= \alpha K^2/\sigma_0^2, \\ \sigma &= \beta K^2/E\sigma_0, \end{aligned} \tag{1.3}$$

where E is Young's modulus, σ_0 is the yield stress, while α and β are dimensionless factors which may depend on, for example, Poisson's ratio v and the material's work-hardening characteristics, but are independent of the applied loads and the body shape. The magnitude of r_p in relation to the solid thickness is important in deciding whether plane strain deformation conditions are indeed possible, and the attainment by δ of some critical value δ_t has frequently been used as a condition for crack extension; consequently, the expressions (1.3) are of considerable interest and importance. It must be emphasized that these expressions, obtained by the small-scale yielding approach, are exact only in the limiting situation where the plastic zone size becomes vanishingly small; it is therefore important to know the extent to which such solutions can be applied before they fail to give a reasonably accurate description of the true state of affairs. Clearly, the ideal way of ascertaining this extent is to consider situations where the yield zone is unrestricted in size, and compare exact results with those determined via the small-scale yielding approach. The approach was found [1, 2] to be valid up to applied stress levels that were relatively large in relation to the general yield stress, although the complete solutions upon which the validity of the conclusions were based applied to macroscopic plasticity anti-plane strain and Dugdale-Bilby-Cottrell-Swinden (DBCS) yield models of a specific type, an important feature being that the loadings were symmetric with respect to the crack plane. On the other hand, LARSSON and CARLSSON [3] employed a finite element method to study plastic-elastic mode I deformation for a limited number of specimen geometries, and noted significant deviations from the small-scale yielding predictions at applied loads that produced plastic zone sizes lower than those allowed by the ASTM limits for fracture toughness tests described in terms of K_I values; for example, α differed by a factor of two between compact tension and centre-cracked specimens at loads corresponding to the ASTM limits.

Rice suggested that this geometrical effect might be due to differences in the non-singular terms in Eq. (1.1) and Larsson and Carlsson verified that a boundary layer formulation, in which Eq. (1.2) is replaced by the requirement of an asymptotic approach to the elastic field given by the stresses in Eq. (1.2) together with those representing the non-singular contribution, gave results in accord with the finite element results for their specific specimen geometries. Subsequently, RICE [4] examined the characteristics of a simple plane strain yielding model in which two slip bands are formed symmetrically at a crack tip. The results he obtained supported his earlier suggestion that the deviations were related to the presence of the non-singular stress terms. However, Rice's analytical approach was only approximate and in view of the practical relevance of the problem in regard to fracture toughness test procedures, it is obviously desirable to confirm Rice's view. Such is the aim of the investigation recorded in this paper.

The doubt concerning Rice's discussion [4] is associated with his approximating procedure (Sect. 2), and it is obviously preferable to use an exact analytical approach; for plane strain deformation this would seem to be impossible with the analytical techniques that are currently available. However, Rice has suggested that there is a similar possibility of non-singular and non-vanishing stress terms producing marked deviations from the small-scale yielding approach predictions in the analogous anti-plane strain situation, but situations where such terms are present have only recently been investigated by KARIHALOO [5]; he also used an approximate procedure, representing the plastic relaxation by appropriate super-dislocations. Recognizing that exact solutions are often readily obtained for antiplane strain deformation, the present paper (Sect. 3) considers an appropriate anti-plane strain model which can be analyzed exactly. It is clearly demonstrated that deviations from the small-scale yielding approach predictions are due to the presence of the non-singular terms and Rice's conclusions are therefore substantiated.

2. Rice's approximate approach

Figure 1 illustrates Rice's model [4] for plane strain yielding at a crack tip; plastic relaxation occurs by slip on two discrete planes inclined at angles $\pm\phi$ with the crack plane, the resistance to slip being τ_0, the shear yield stress. Rice determined the lengths of the plastic zones and the crack tip opening displacement by means of the following approximate argument. Consider in the first instance a mode II shear-type crack, for

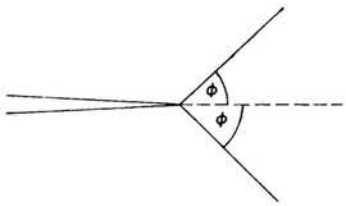

FIG. 1. Rice's model for plane strain yielding at a crack tip; plastic relaxation occurs by slip on two discrete planes, the resistance to slip being τ_0, the shear yield stress.

which there is an associated stress intensity factor $K^{(s)}$ such that the shear stress immediately ahead of the crack, in its own plane, is

$$p_{yx}^{(s)} = K^{(s)}(2\pi r)^{-1/2}. \tag{2.1}$$

If the intensified stress is relaxed by slip in the crack plane, the yield stress being τ_0, small-scale yielding results for the plastic zone size and the crack tip displacement are, respectively,

$$\begin{aligned} r_p^{(s)} &= \frac{\pi}{8}\left[\frac{K^{(s)}}{\tau_0}\right]^2, \\ \delta^{(s)} &= \frac{(1-v^2)[K^{(s)}]^2}{E\tau_0}. \end{aligned} \tag{2.2}$$

For the mode I tensile-type crack in the elastic situation, the shear stress along planes at angles $\pm\phi$ to the crack plane is

$$p_{\phi r} = \frac{\sin\phi\cos(\phi/2)K}{2(2\pi r)^{\frac{1}{2}}}. \tag{2.3}$$

K being the mode I elastic stress intensity factor. By comparing the relations (2.1) and (2.3), $K^{(s)}$ is identified as

$$K^{(s)} = \frac{\sin\phi\cos(\phi/2)K}{2} \tag{2.4}$$

and the plastic zone size and the crack tip displacement are then given by the relations (2.2) as

$$\begin{aligned} r_p &= \frac{\pi\sin^2\phi(1+\cos\phi)K^2}{64\tau_0^2}, \\ \delta &= 2\delta^{(s)}\sin\phi = \frac{(1-v^2)\sin^3\phi(1+\cos\phi)K^2}{4E\tau_0}, \end{aligned} \tag{2.5}$$

δ being the total relative displacement of the upper and lower crack faces at the tip. The approximate nature of this analytical approach arises from the identification of $K^{(s)}$ by the expression (2.4), and the neglect of any interaction between the two slip regions. Indeed with only one slip region, its length is still given by the relation (2.5) although the crack opening displacement is then only a half of the value given by Eqs. (2.5).

To account for a non-singular and non-vanishing stress $p_{xx} = T$ which produces a shear stress $p_{\phi r} = -T\sin\phi\cos\phi$ along a plane at an angle ϕ to the crack plane, Rice argues that this acts in addition to the shear yield stress τ_0, and therefore replaces τ_0 in the relations (2.5) by $(\tau_0 + T\sin\phi\cos\phi)$, arriving at the following solutions:

$$r_p = \frac{\pi\sin^2\phi(1+\cos\phi)K^2}{64(\tau_0+T\sin\phi\cos\phi)^2}, \tag{2.6}$$
$$\delta = \frac{(1-v^2)\sin^3\phi(1+\cos\phi)K^2}{4E(\tau_0+T\sin\phi\cos\phi)}.$$

Thus, for the plane strain situation where a crack of length $2c$ lies within an infinite body which is subject to biaxial loading such that $p_{yy} = p_{yy}^{\infty}$ and $p_{xx} = p_{xx}^{\infty}$ remote from the crack,

$$K = p_{yy}^{\infty}(\pi c)^{\frac{1}{2}}, \tag{2.7}$$
$$T = p_{xx}^{\infty} - p_{yy}^{\infty},$$

whereupon, if $T \ll \tau_0$, the relations (2.6) and (2.7) show that

$$r_p = \frac{\pi^2\sin^2\phi(1+\cos\phi)c(p_{yy}^{\infty})^2}{64\tau_0^2}\left[1 - \frac{(p_{xx}^{\infty}-p_{yy}^{\infty})\sin 2\phi}{\tau_0}\right], \tag{2.8}$$
$$\delta = \frac{\pi(1-v^2)\sin^3\phi(1+\cos\phi)c(p_{yy}^{\infty})^2}{4E\tau_0}\left[1 - \frac{(p_{xx}^{\infty}-p_{yy}^{\infty})\sin 2\phi}{2\tau_0}\right].$$

As emphasized by Rice, there is a fundamental difference between these expressions and similar expressions for the available complete solutions for macroscopic plasticity anti-plane strain and DBCS models when non-singular and non-vanishing stresses are absent; the corresponding series for r_p and δ are both of the form

$$r_p = \alpha\left(\frac{K}{\sigma_0}\right)^2\left[1 + \lambda\left(\frac{p_{\text{app}}}{\sigma_0}\right)^2 + --\right], \tag{2.9}$$

where α and λ are constants and p_{app} is a nominal applied stress. With such solutions, the deviation from the small-scale yielding prediction is quadratic in the applied load, whereas for the inclined slip band model (and, by implication, for the macroscopic plasticity plane strain model), the deviation is linear. Rice concludes that this difference explains why LARSSON and CARLSSON [3] observed a substantially more limited range of validity to the small-scale yielding approximation than had been evident from the earlier solutions.

3. An exact approach for a particular anti-plane strain model

The particular anti-plane strain model to be considered is that illustrated in Fig. 2. A crack of length $2c$ lying along the plane $y = 0$ and infinitely long in the z direction is contained within an infinite solid which is subject to applied shear stresses $p_{yz} = p_{yz}^{\infty}$ and $p_{xz} = p_{xz}^{\infty}$. Plastic relaxation occurs at one of the crack tips by slip on a single plane inclined at an angle ϕ to the crack plane, the resistance to slip being τ_0. It is necessary

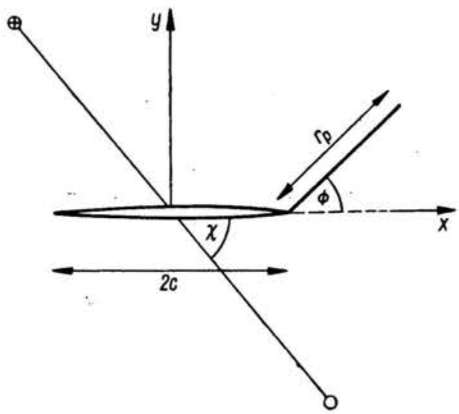

FIG. 2. The anti-plane strain model analyzed in Sect. 3; see the text for a detailed description.

to determine only the size of the plastic zone, for which it is possible to obtain an exact solution, since the crack tip displacement is likely to behave in a similar manner.

This model, for the case where p_{xz}^{∞} is equal to zero, may be analyzed by using results obtained by SIH [6]. He considered the model [6, 7] in which a composite crack consisting of two segments of lengths $2c$ and r_p inclined at an angle $\phi = m\pi$ $(0 \leqslant m \leqslant 1)$ to each other, lies within an infinite body which is subject to a uniform external shear stress τ^* whose direction of application makes an angle χ with the x axis (Fig. 2). The stress intensification at the tip of the segment of length r_p is

$$K = \frac{\tau_* \pi^{\frac{1}{2}} r_p^{\frac{1}{2}} \sin(\beta+\chi+m\alpha)}{(2\cos\beta)^{\frac{1}{2}} \left[\sin\frac{(\alpha+\beta)}{2}\sin\frac{(\alpha-\beta)}{2}\right]^{\frac{1}{2}}} \cdot \left[\frac{\sin\frac{(\alpha-\beta)}{2}}{\sin\frac{(\alpha+\beta)}{2}}\right]^m, \tag{3.1}$$

with

$$\frac{r_p}{2c} = \tan\frac{(\alpha+\beta)}{2}\tan\frac{(\alpha-\beta)}{2}\left[\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}\right]^m, \tag{3.2}$$

$$\sin\beta = m\sin\alpha.$$

These results are immediately applicable to the original relaxed crack model, for the case where $p_{xz}^{\infty} = 0$, by substituting $p_{yz}^{\infty} = \tau = \tau_*\cos\chi_*$ and $\tau_0 = \tau_*\sin\chi_*\sin\phi$ with $\chi = (\pi/2)+\chi_*$; in this case K in the relation (3.1) is zero since the stress must be bounded at the tip of the plastic region and, consequently,

$$\beta+\chi_*+m\alpha = \pi/2. \tag{3.3}$$

The relations (3.2) and (3.3), which are exact, enable the length r_p of the plastic zone to be determined. Substituting $\psi = (\pi/2)-\chi_*$, it is immediately seen that α, β and ψ are small when the applied stress and therefore the plastic zone size are small; a straightforward but detailed consideration of the relations (3.2) and (3.3) then gives

$$\frac{r_p}{2c} = \frac{(1-m^2)}{4}\left(\frac{1+m}{1-m}\right)^m \frac{\sin^2 m\pi}{4m^2}\left(\frac{p_{yz}^{\infty}}{\tau_0}\right)^2 [1+0(p_{yz}^{\infty}/\tau_0)^2] \tag{3.4}$$

the important feature of this result being that the second term in the square bracket is of the form $0(p_{yz}^{\infty}/\tau_0)^2$ and not $0(p_{yz}^{\infty}/\tau_0)$. For the case where there is an additional applied stress $p_{xz} = p_{xz}^{\infty}$, it is only necessary to replace τ_0 by $(\tau_0+p_{yz}^{\infty}\sin m\pi)$ in the expression (3.4) which simplifies for small applied stresses to

$$\frac{r_p}{2c} = \frac{(1-m)^2}{4}\left(\frac{1+m}{1-m}\right)^m \frac{\sin^2 m\pi}{4m^2}\left(\frac{p_{yz}^{\infty}}{\tau_0}\right)^2\left[1-\frac{2p_{xz}^{\infty}\sin m\pi}{\tau_0}+0(p_{\text{app}}/\tau_0)^2\right]. \tag{3.5}$$

It is instructive to compare this result with that obtained using Rice's approximate approach outlined in the previous section. In the elastic situation the shear stress along the plane at an angle ϕ to the crack plane is

$$p_{\phi z} = \frac{\cos(\phi/2)K}{(2\pi r)^{\frac{1}{2}}}, \tag{3.6}$$

where K refers to the mode III loading situation. By comparing the relation (3.6) with the anti-plane strain analogue of the relation (2.1) the new $K^{(a)}$ is identified as

$$K^{(a)} = \cos(\phi/2)K \tag{3.7}$$

and it immediately follows that the length of the plastic zone and the crack tip displacement, given by the anti-plane strain analogue of Eqs. (2.2), i.e.

$$\begin{aligned} r_p^{(a)} &= \frac{\pi}{8}\left[\frac{K^{(a)}}{\tau_0}\right]^2, \\ \delta^{(a)} &= \frac{[K^{(a)}]^2}{2\mu\tau_0} \end{aligned} \tag{3.8}$$

are, for the case where the applied stress is $p_{yz} = p_{yz}^{\infty}$ and therefore $K = p_{yz}^{\infty}(\pi c)^{\frac{1}{2}}$:

$$\begin{aligned} r_p &= \frac{\pi\cos^2(\phi/2)K^2}{8\tau_0^2}, \\ \delta &= \frac{\cos^2(\phi/2)K^2}{2\mu\tau_0}. \end{aligned} \tag{3.9}$$

Where there is an additional applied stress $p_{xz} = p_{xz}^{\infty}$, these expressions become

$$\begin{aligned} r_p &= \frac{\pi\cos^2(\phi/2)K^2}{8(\tau_0+p_{xz}^{\infty}\sin\phi)^2}, \\ \delta &= \frac{\cos^2(\phi/2)K^2}{2\mu(\tau_0+p_{xz}^{\infty}\sin\phi)} \end{aligned} \tag{3.10}$$

simplifying, for small applied stresses, to

$$\begin{aligned} r_p &= \frac{\pi^2\cos^2(m\pi/2)c(p_{yz}^{\infty})^2}{8\tau_0^2}\left[1-\frac{2p_{xz}^{\infty}\sin m\pi}{\tau_0}\right], \\ \delta &= \frac{\pi\cos^2(m\pi/2)c(p_{yz}^{\infty})^2}{2\mu\tau_0}\left[1-\frac{p_{xz}^{\infty}\sin m\pi}{\tau_0}\right]. \end{aligned} \tag{3.11}$$

The expressions for r_p using the exact approach (relation (3.5)) and Rice's approximate approach (relation (3.11)) are not identical. However, the results differ only by a multiplicative constant, and the approximate approach clearly gives the correct stress dependency, manifested by the equality of the expressions in the square brackets in the two cases, and also because these expressions are multiplied by $(p_{yz}^{\infty}/\tau_0)^2$.

4. Discussion

The preceding section considered a particular anti-plane strain model that could be analysed exactly, and it was shown that deviations from the small-scale yielding approach predictions for the plastic zone size, and by inference the crack opening displacement also, are associated with a non-singular and non-vanishing term in the elastic stress distribution near a crack tip, which produces a linear and not a quadratic deviation in the applied load as is the case when such a term is absent. Rice's approximate procedure [4] applied to the same model gives the same result, and because the general behaviour patterns predicted by the exact and approximate approaches are the same for the anti-plane strain situation, it is logical to infer that the approximate approach also gives the correct behaviour pattern for the plane strain model discussed in Sect. 2. The main object of the present investigation has therefore been attained, in that the validity of Rice's approximate approach has been demonstrated; consequently, his explanation for the deviation from the small-scale yielding approach predictions in the plane strain situation, as being due to the presence of non-singular and non-vanishing terms in the stress distribution around the crack tip in the elastic situation, has been confirmed.

There are other ways of providing additional confirmation for this viewpoint. For example, it might be worth examining the non work-hardening macroscopic plasticity model of a surface crack in a semi-infinite solid deforming under anti-plane strain conditions. The boundary surface is $x = 0$, while the crack is $0 \leqslant x \leqslant c$, $y = 0$ with the applied stresses being $p_{yz} = p_{yz}^{\infty}$ and $p_{xz} = p_{xz}^{\infty}$; the object should be to extend Rice's existing analysis [8] in which $p_{xz}^{\infty} = 0$, and show that the expressions for the plastic zone size and crack opening displacement deviate from the small-scale yielding expressions because of the presence of terms that are linear in p_{xz}^{∞}. Rice's model of Sect. 2 with applied stresses $p_{yy} = p_{yy}^{\infty}$ and $p_{xx} = p_{xx}^{\infty}$ could also be analyzed using numerical procedures; BILBY and SWINDEN [9] conducted an analysis of this type but with $p_{xx}^{\infty} = 0$, and this proposed study would therefore be an extension of their investigation. The analysis could be facilitated by representing some of the dislocations in the inclined slip planes by super-dislocations, a procedure that has been used by KARIHALOO [5] for the type of anti-plane strain model studied in this paper; indeed the present paper can be viewed as complementing his work, while having the extra attribute that the approach is analytical and exact, thereby facilitating comparison with Rice's approximate analytical procedure [4].

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JOINT MANCHESTER UNIVERSITY,
UMIST METALLURGY DEPARTMENT, MANCHESTER U.K.

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