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Unsteady shearing flows and plane shear waves in simple fluids

S. ZAHORSKI (WARSZAWA)

IT IS SHOWN that certain unsteady shearing flows belong to the class of motions with superposed proportional stretch histories discussed elsewhere [3, 9]. These flows include two-dimensional harmonic oscillations, plane circular shearing as well as other homothermal motions discussed by CARROLL [4]. The corresponding representations of the constitutive equation of a simple fluid are also considered. Certain solutions of the governing equations in the case of plane harmonic oscillations are obtained either for fluids with linear shear response or for flows with small amounts of shear. Various properties of two mutually perpendicular plane shear waves are discussed in greater detail for very low and very high (ultrasonic) angular frequencies.

Pokazano, że pewne nieustalone przepływy ścinające należą do klasy ruchów z nałożonymi proporcjonalnymi historiami deformacji, wprowadzonej poprzednio [3, 9]. Przepływy te obejmują zarówno dwuwymiarowe harmoniczne oscylacje, płaskie kołowe ścinania, jak i inne ruchy homotermiczne dyskutowane przez CARROLLA [4]. Rozważono również odpowiednie reprezentacje równania konstytutywnego cieczy prostej. Pewne rozwiązania równań ruchu w przypadku płaskich harmonicznych oscylacji uzyskano dla cieczy z liniową reakcją na ścinanie lub dla przepływów z małymi gradientami ścinania. Różne własności dwóch wzajemnie prostopadłych płaskich fal ścinania przedyskutowano bardziej szczegółowo przy bardzo niskich i bardzo wysokich (naddźwiękowych) częstościach kołowych.

Показано, что некоторые неустановившиеся сдвиговые течения принадлежат к классу движений с наложенными пропорциональными историями деформации, который введен раньше [3, 9]. Эти течения охватывают так двумерные гармонические осцилляции, плоские круговые сдвиги, как и другие гомотермические движения, обсуждаемые Карролом [4]. Рассмотрены тоже соответствующие представления определяющего уравнения простой жидкости. Некоторые решения уравнений движения, в случае плоских гармонических осцилляций, получены для жидкости с линейной реакцией на сдвиг или для течений с малыми градиентами сдвига. Разные свойства двух взаимно перпендикулярных плоских волн сдвига обсуждены более подробно при очень низких и очень высоких (сверхзвуковых) круговых частотах.

1. Introduction

IT IS WELL known that numerous unsteady shearing flows belong to such particular classes of motions in which much can be learned without making essential simplifications concerned with a form of the constitutive equations (cf. e.g. [1, 2]). Apart from unsteady viscometric flows and motions with constant or proportional stretch history [3], there exist many other shearing flows, especially those of oscillatory character, which exhibit interesting kinematic or dynamic properties when applied to compressible or incompressible elastic and dissipative media. These are, for example, unsteady homothermal motions [4], various oscillatory plane motions [5, 6, 7], certain structural orientation patterns [4] as well as motions leading to progressive or standing shear waves with linear, circular or even elliptical polarization [8, 9, 10].

In the present paper two types of unsteady plane shearings are considered as belonging to the class of motions with superposed proportional stretch histories [3]. For compressible as well as for incompressible simple fluids (cf. [1]), the corresponding constitutive equations are derived and the Rivlin-Ericksen type of representations developed. The resulting governing differential equations can be solved in a closed form in the case of harmonic dependence on time and linear shear response (generalized viscosity independent of shear rate or small amounts of shear). Certain spatially periodic solutions, leading to linearly polarized two shear waves, are considered in greater detail. Various properties of waves such as damping effects, phase shifts, maximum amounts of shear etc. are discussed for very low as well as for very high (ultrasonic) frequencies.

2. Unsteady plane shearing flows

Consider the following two types of plane shearing motions:

$$\begin{aligned} x &= X+\varphi(Z)f_1(\tau), \\ y &= Y+\psi(Z)f_2(\tau), \\ z &= Z, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} x &= X+\varphi(Z)f_1(\tau)+\psi(Z)f_2(\tau), \\ y &= Y+\varphi(Z)f_2(\tau)-\psi(Z)f_1(\tau), \\ z &= Z, \end{aligned} \tag{2.2}$$

where x, y, z, denote the Cartesian coordinates of a particle at an arbitrary time τ, X, Y, Z — the Cartesian coordinates of the same particle in a reference configuration, φ and ψ are certain functions of the variable Z, while f_1 and f_2 are smooth functions of time only. In the first case only translational motions in the material planes xy are possible, while in the second case some rotational motions in these planes are admissible. It is seen, moreover, that either for $\varphi(z) = \psi(z)$ or for $f_1(\tau) = f_2(\tau)$ the more general motions described by Eq. (2.2) take particular forms of the simpler motions (2.1). The corresponding velocity and acceleration fields can easily be calculated from the above equations.

The deformation gradients at time τ with respect to reference configurations can in both cases be expressed as

$$\mathbf{F}(\tau) = 1+\mathbf{M}_1 f_1(\tau)+\mathbf{M}_2 f_2(\tau) = \exp\big(\mathbf{M}_1 f_1(\tau)+\mathbf{M}_2 f_2(\tau)\big), \tag{2.3}$$

where

$$[\mathbf{M}_1] = \begin{bmatrix} 0 & 0 & \varphi' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{M}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \psi' \\ 0 & 0 & 0 \end{bmatrix} \tag{2.4}$$

for the motions described by Eq. (2.1), and

$$[\mathbf{M}_1] = \begin{bmatrix} 0 & 0 & \varphi' \\ 0 & 0 & -\psi' \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{M}_2] = \begin{bmatrix} 0 & 0 & \psi' \\ 0 & 0 & \varphi' \\ 0 & 0 & 0 \end{bmatrix} \tag{2.5}$$

for those described by Eq. (2.2). Primes denote the derivatives with respect to z. It is also seen from Eqs. (2.4) and (2.5) that

$$\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_2\mathbf{M}_1 = \mathbf{M}_1^T\mathbf{M}_2 = \mathbf{M}_2^T\mathbf{M}_1 = \mathbf{0}, \tag{2.6}$$

where the superscript T denotes the transpose.

Introducing the relative deformation gradients with respect to the reference configuration chosen at present time t ($\tau \leqslant t$),

$$\mathbf{F}_t(\tau) = \mathbf{F}(\tau)\mathbf{F}^{-1}(t), \quad \mathbf{F}_t(t) = 1, \tag{2.7}$$

we arrive at the history of the right Cauchy-Green relative deformation tensor (cf. [1]) in the form

$$\mathbf{C}(s) = \mathbf{F}_t^T(t-s)\mathbf{F}_t(t-s) = \exp\big(\bar{g}_1(s)\mathbf{M}_1^T+\bar{g}_2(s)\mathbf{M}_2^T\big)\exp\big(\bar{g}_1(s)\mathbf{M}_1+\bar{g}_2(s)\mathbf{M}_2\big), \tag{2.8}$$

where $s \in [0, \infty)$, and

$$\bar{g}_i(s) = f_i(t-s)-f_i(t), \quad i = 1, 2. \tag{2.9}$$

Bearing in mind the definitions of the velocity gradients $\mathbf{L}(t)$ (cf. [1]), we also have

$$\begin{gathered}\mathbf{L}(t) = \dot{\mathbf{F}}(t)\mathbf{F}^{-1}(t), \quad \mathbf{L}_i(t) = \mathbf{M}_i\dot{f}_i(t), \quad i = 1, 2,\\ \mathbf{L}(t) = \mathbf{L}_1(t)+\mathbf{L}_2(t),\end{gathered} \tag{2.10}$$

where dots denote the material differentiation with respect to time. Thus, instead of Eq. (2.8) we arrive at

$$\mathbf{C}(s) = \exp\big(g_1(s)\mathbf{L}_1^T+g_2(s)\mathbf{L}_2^T\big)\exp\big(g_1(s)\mathbf{L}_1+g_2(s)\mathbf{L}_2\big), \tag{2.11}$$

where $g_i(s) = \bar{g}_i(s)/\dot{f}_i(t)$, $i = 1, 2$.

It can easily be observed that Eq. (2.8) as well as Eq. (2.11) define particular cases of the motions with superposed proportional stretch histories discussed elsewhere [2, 3]. In other words, the motions described by Eqs. (2.1) and (2.2) belong to the class of the above motions.

It is worth-while to note that the above examples of flows can be considered as certain unsteady homothermal motions defined by CARROLL [4]. On taking functions $f_1(\tau)$ and $f_2(\tau)$ or $\varphi(z)$ and $\psi(z)$ as cosine and sine, respectively, we obtain various examples of motions discussed by Carroll in a series of papers (cf. [4–8]). For sinusoidal dependence on time, Eqs. (2.1) and (2.2) describe plane oscillations; for sinusoidal dependence on z, these equations may characterize certain orientation patterns observed in liquid crystals at rest (cf. [4]).

The constitutive equation of a simple fluid (cf. [1, 2])

$$\mathbf{T}(t) = \mathop{\mathsf{F}}_{s=0}^{\infty}\big(\mathbf{C}(s)\big), \tag{2.12}$$

where $\mathbf{T}(t)$ is the stress tensor at time t, and F denotes an isotropic constitutive functional, after substituting from Eqs. (2.8) or (2.11), can be written in the following forms (cf. [3]):

$$\mathbf{T}(t) = \mathop{\mathsf{G}}_{s=0}^{\infty}\big(\bar{g}_1(s), \bar{g}_2(s); \mathbf{M}_1, \mathbf{M}_2\big) = \mathop{\mathsf{H}}_{s=0}^{\infty}\big(g_1(s), g_2(s); \mathbf{L}_1, \mathbf{L}_2\big). \tag{2.13}$$

Here, G as well as H denote functionals of the first scalar arguments, and are simultaneously isotropic functions of $\mathbf{M}_1$ and $\mathbf{M}_2$ or $\mathbf{L}_1$ and $\mathbf{L}_2$, respectively.

Taking into account the relations (2.6) and either

$$\text{(2.14)} \qquad \mathrm{tr}\mathbf{M}_1\mathbf{M}_1^T = \varphi'^2, \quad \mathrm{tr}\mathbf{M}_1\mathbf{M}_2^T = \psi'^2$$

for $\mathbf{M}_1$ and $\mathbf{M}_2$ given in the form of Eq. (2.4) or

$$\text{(2.15)} \qquad \mathrm{tr}\mathbf{M}_1\mathbf{M}_1^T = \mathrm{tr}\mathbf{M}_2\mathbf{M}_2^T = \varphi'^2+\psi'^2$$

for $\mathbf{M}_1$ and $\mathbf{M}_2$ expressed by Eq. (2.5), we arrive at the following representations:

$$\text{(2.16)} \qquad \begin{aligned} \mathbf{T}(t) = {} & \mathsf{s}_0(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\mathbf{1} \\ & + \mathsf{t}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)(\mathbf{M}_1^T+\mathbf{M}_1) + \mathsf{t}(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)(\mathbf{M}_2^T+\mathbf{M}_2) \\ & + \mathsf{s}_1(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\mathbf{M}_1\mathbf{M}_1^T + \mathsf{s}_1(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\mathbf{M}_2\mathbf{M}_2^T \\ & + \mathsf{s}_2(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\mathbf{M}_1^T\mathbf{M}_1 + \mathsf{s}_2(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\mathbf{M}_2^T\mathbf{M}_2 \\ & + \mathsf{s}_{12}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)(\mathbf{M}_1\mathbf{M}_2^T + \mathbf{M}_2\mathbf{M}_1^T), \end{aligned}$$

where $\varkappa^2$ stands for the arguments φ'^2, ψ'^2 or $\varphi'^2+\psi'^2$, respectively. The material functionals s_0, t, s_1, s_2 and s_{12}, five in number, are scalar functionals of $\bar{g}_i(s)$, $i = 1, 2$, and can be expressed as certain functions of t and $\varkappa^2$. Only s_0 and s_{12} are symmetric with respect to interchange of functions $\bar{g}_1$ and $\bar{g}_2$ or indices 1 and 2. In the case of incompressible simple fluids the term $\mathsf{s}_0\mathbf{1}$ should be replaced by a hydrostatic pressure $p\mathbf{1}$.

It is easy to notice that the last term involving functional s_{12} is responsible for mutual coupling of two component motions; when only one-dimensional flow is considered this term vanishes altogether (cf. [4]).

On the basis of Eqs. (2.16) the corresponding stress components can be written as

$$\text{(2.17)} \qquad \begin{aligned} T^{13} &= \mathsf{t}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi', \\ T^{23} &= \mathsf{t}(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\psi', \\ T^{12} &= 2\mathsf{s}_{12}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi'\psi', \\ T^{11} &= \mathsf{s}_0(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2) + \mathsf{s}_1(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi'^2, \\ T^{22} &= \mathsf{s}_0(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2) + \mathsf{s}_1(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\psi'^2, \\ T^{33} &= \mathsf{s}_0(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2) + \mathsf{s}_2(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi'^2 + \mathsf{s}_2(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\psi'^2 \end{aligned}$$

for the flow described by Eqs. (2.1), and

$$\text{(2.18)} \qquad \begin{aligned} T^{13} &= \mathsf{t}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi' + \mathsf{t}(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\psi', \\ T^{23} &= -\mathsf{t}(\bar{g}_1(s), \bar{g}_2(s), \varkappa^2)\psi' + \mathsf{t}(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\varphi', \\ T^{12} &= 2\mathsf{s}_{12}(\bar{g}_1(s), \bar{g}_2(s), \varkappa^2)(\varphi'^2-\psi'^2), \\ T^{11} &= \mathsf{s}_0(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2) + \mathsf{s}_1(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi'^2 \\ &\quad + \mathsf{s}_1(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\psi'^2 + 2\mathsf{s}_{12}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi'\psi', \\ T^{22} &= \mathsf{s}_0(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2) + \mathsf{s}_1(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\psi'^2 \\ &\quad + \mathsf{s}_1(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\varphi'^2 - 2\mathsf{s}_{12}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi'\psi', \\ T^{33} &= \mathsf{s}_0(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2) + \mathsf{s}_2(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)(\varphi'^2+\psi'^2) \\ &\quad + \mathsf{s}_2(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)(\varphi'^2+\psi'^2) \end{aligned}$$

for the more general flow (2.2).

When either $\varphi(z) = \psi(z)$ or $f_1(\tau) = f_2(\tau)$, the above formulae simplify considerably. Similar simplifications can also be achieved for a particular choice of time dependence (cf. Sect. 5).

3. Representations of the Rivlin-Ericksen type

Although the constitutive equations (2.16) contain a relatively small number of material functionals, it is sometimes more useful to operate with other explicit forms of representations involving kinematic tensors of the Rivlin-Ericksen type (cf. [1]).

To this end we can introduce the following definitions of the partial Rivlin-Ericksen (R-E) tensors (cf. [11]):

$$\text{(3.1)} \qquad \mathbf{A}_{1(i)} = \mathbf{L}_i^T + \mathbf{L}_i, \qquad \mathbf{A}_{n+1(i)} = \dot{\mathbf{A}}_{n(i)} + \mathbf{A}_{n(i)}\mathbf{L}_i + \mathbf{L}_i^T\mathbf{A}_{n(i)}, \qquad \begin{array}{l} i = 1, 2 \\ n = 1, 2, 3, \ldots, \end{array}$$

where superposed dots denote the material time-derivatives, and where the partial velocity gradients $\mathbf{L}_i$ have been defined in Eq. (2.10). It can be checked that the R-E tensors $\mathbf{A}_n$ and partial R-E tensors $\mathbf{A}_{n(i)}$ are related as below:

$$\text{(3.2)} \qquad \mathbf{A}_n = (-1)^n \frac{d^n\mathbf{C}(s)}{ds^n}\bigg|_{s=0}, \qquad \mathbf{A}_n = \mathbf{A}_{n(1)} + \mathbf{A}_{n(2)}, \qquad n = 1, 2, 3, \ldots.$$

It can be proved in a way similar to that used for the motions with proportional stretch histories (cf. [11, 2, 3]) that the first and second partial R-E tensors $\mathbf{A}_{1(1)}$, $\mathbf{A}_{1(2)}$ and $\mathbf{A}_{2(1)}$, $\mathbf{A}_{2(2)}$ determine $\mathbf{L}_1$ and $\mathbf{L}_2$ uniquely, if the tensors $\mathbf{A}_{1(1)}$ and $\mathbf{A}_{1(2)}$ have three eigen-values distinct. This is the case for the flows considered.

The above result enables to express Eq. $(2.13)_2$ in the following form:

$$\text{(3.3)} \qquad \mathbf{T}(t) = \mathop{\mathrm{K}}_{s=0}^{\infty} \big(g_1(s), g_2(s); \varkappa^2; \mathbf{A}_{1(1)}(t), \mathbf{A}_{1(2)}(t), \mathbf{A}_{2(1)}(t), \mathbf{A}_{2(2)}(t)\big),$$

where as previously $g_i(s) = \bar{g}_i(s)/\dot{f}_i(t)$, $i = 1, 2$.

In the case of the motion described by Eq. (2.1), an expanded form of the constitutive Eq. (3.3) is

$$\begin{aligned} \text{(3.4)} \qquad \mathbf{T} = {} & \alpha_0\mathbf{1} + \bar{\alpha}_1\mathbf{A}_{1(1)} + \bar{\bar{\alpha}}_1\mathbf{A}_{1(2)} + \bar{\alpha}_2\mathbf{A}_{2(1)} + \bar{\bar{\alpha}}_2\mathbf{A}_{2(2)} + \bar{\alpha}_3\mathbf{A}_{1(1)}^2 + \bar{\bar{\alpha}}_3\mathbf{A}_{1(2)}^2 + \bar{\alpha}_4\mathbf{A}_{2(1)}^2 \\ & + \bar{\bar{\alpha}}_4\mathbf{A}_{2(2)}^2 + \bar{\alpha}_5(\mathbf{A}_{1(1)}\mathbf{A}_{2(1)} + \mathbf{A}_{2(1)}\mathbf{A}_{1(1)}) + \bar{\bar{\alpha}}_5(\mathbf{A}_{1(2)}\mathbf{A}_{2(2)} + \mathbf{A}_{2(2)}\mathbf{A}_{1(2)}) \\ & + \alpha_6(\mathbf{A}_{1(1)}\mathbf{A}_{1(2)} + \mathbf{A}_{1(2)}\mathbf{A}_{1(1)}) + \alpha_7(\mathbf{A}_{2(1)}\mathbf{A}_{2(2)} + \mathbf{A}_{2(2)}\mathbf{A}_{2(1)}) \\ & + \bar{\alpha}_8(\mathbf{A}_{1(2)}\mathbf{A}_{2(1)} + \mathbf{A}_{2(1)}\mathbf{A}_{1(2)}) + \bar{\bar{\alpha}}_8(\mathbf{A}_{1(1)}\mathbf{A}_{2(2)} + \mathbf{A}_{2(2)}\mathbf{A}_{1(1)}) \\ & + \bar{\alpha}_9(\mathbf{A}_{1(1)}^2\mathbf{A}_{2(1)}^2 + \mathbf{A}_{2(1)}^2\mathbf{A}_{1(1)}^2) + \bar{\bar{\alpha}}_9(\mathbf{A}_{1(2)}^2\mathbf{A}_{2(2)}^2 + \mathbf{A}_{2(2)}^2\mathbf{A}_{1(2)}^2), \end{aligned}$$

where, for simplicity, we have denoted

$$\text{(3.5)} \qquad \bar{\alpha}_i = \bar{\alpha}_i\big(g_1(s), g_2(s); \varkappa^2\big), \qquad \bar{\bar{\alpha}}_i = \bar{\bar{\alpha}}_i\big(g_2(s), g_1(s); \varkappa^2\big), \qquad i = 0, 1, \ldots, 9.$$

Only the material functionals α_0, α_6 and α_7 are symmetric with respect to the interchange of the functions g_1 and g_2; the remaining functionals may essentially depend on their sequence. For incompressible simple fluids the number of ten functionals reduces by one since the term $\alpha_0\mathbf{1}$ can be replaced by $p\mathbf{1}$.

Taking into account Eqs. (2.4), (2.10), and (3.1), we arrive at the following stress components:(¹)

(3.6)
$$
\begin{aligned}
T^{13} &= \bar{\alpha}_1 \dot{f}_1 \varphi' + \bar{\alpha}_2 \ddot{f}_1 \varphi',\\
T^{23} &= \bar{\bar{\alpha}}_1 \dot{f}_2 \psi' + \bar{\bar{\alpha}}_2 \ddot{f}_2 \psi',\\
T^{12} &= (\alpha_6 \dot{f}_1 \dot{f}_2 + \alpha_7 \ddot{f}_1 \ddot{f}_2)\varphi'\psi' + (\bar{\alpha}_8 \ddot{f}_1 \dot{f}_2 + \bar{\bar{\alpha}}_8 \dot{f}_1 \ddot{f}_2)\varphi'\psi',\\
T^{11} &= \alpha_0 + \bar{\alpha}_3 \dot{f}_1^2 \varphi'^2 + \bar{\alpha}_4 \ddot{f}_1^2 \varphi'^2 + 2\bar{\alpha}_5 \dot{f}_1 \ddot{f}_1 \varphi'^2 + 2\bar{\alpha}_9 \dot{f}_1^2 \ddot{f}_1^2 \varphi'^4,\\
T^{22} &= \alpha_0 + \bar{\bar{\alpha}}_3 \dot{f}_2^2 \psi'^2 + \bar{\bar{\alpha}}_4 \ddot{f}_2^2 \psi'^2 + 2\bar{\bar{\alpha}}_5 \dot{f}_2 \ddot{f}_2 \psi'^2 + 2\bar{\bar{\alpha}}_9 \dot{f}_2^2 \ddot{f}_2^2 \psi'^4,\\
T^{33} &= \alpha_0 + 2\bar{\alpha}_5 \dot{f}_1 \ddot{f}_1 \varphi'^2 + 2\bar{\bar{\alpha}}_5 \dot{f}_2 \ddot{f}_2 \psi'^2,
\end{aligned}
$$

where α_i are the new functionals of the arguments indicated in Eq. (3.5), and, moreover, we have denoted $\dot{f}_i = \dot{f}_i(t)$, $\ddot{f}_i = \ddot{f}_i(t)$ etc. It is quite clear that all the stress components are independent of X and Y.

In a similar way, an expanded constitutive equation can be written for the motion described by Eq. (2.2). Then, the number of material functionals α_i amounts to twelve. By way of illustration we quote only those stress components which enter into the dynamic equations of motion (cf. Sect. 4). We have, for example,

(3.7)
$$
\begin{aligned}
T^{13} &= \bar{\alpha}_1 \dot{f}_1 \varphi' + \bar{\bar{\alpha}}_1 \dot{f}_2 \psi' + \bar{\alpha}_2 \ddot{f}_1 \varphi' + \bar{\bar{\alpha}}_2 \ddot{f}_2 \psi',\\
T^{23} &= -\bar{\alpha}_1 \dot{f}_1 \psi' + \bar{\bar{\alpha}}_1 \dot{f}_2 \varphi' - \bar{\alpha}_2 \ddot{f}_1 \psi' + \bar{\bar{\alpha}}_2 \ddot{f}_2 \varphi',\\
T^{33} &= \alpha_0 + (\bar{\alpha}_3 \dot{f}_1^2 + \bar{\bar{\alpha}}_2 \dot{f}_2^2)(\varphi'^2 + \psi'^2) + (\bar{\alpha}_4 \ddot{f}_1^2 + \bar{\bar{\alpha}}_4 \ddot{f}_2^2)(\varphi'^2 + \psi'^2)\\
&\quad + 2(\bar{\alpha}_5 \dot{f}_1 \ddot{f}_1 + \bar{\bar{\alpha}}_5 \dot{f}_2 \ddot{f}_2)(\varphi'^2 + \psi'^2) + 2(\bar{\alpha}_9 \dot{f}_1^2 \ddot{f}_1^2 + \bar{\bar{\alpha}}_9 \dot{f}_2^2 \ddot{f}_2^2)(\varphi'^2 + \psi'^2)\\
&\quad + 2(\alpha_{10} \dot{f}_1^2 \dot{f}_2^2 + \alpha_{11} \ddot{f}_1^2 \ddot{f}_2^2)(\varphi'^2 + \psi'^2),
\end{aligned}
$$

where the new functionals α_i are again of the forms (3.5).

Further simplifications in Eqs. (3.6) and (3.7) can be achieved either for $f_1(\tau) = f_2(\tau)$ or for a particular dependence on time.

4. Dynamical equations and governing equations

The dynamical equation of motion can be written in the following form (cf. [1]):

(4.1)
$$\operatorname{div}\mathbf{T} - \operatorname{grad} P = \varrho \ddot{\mathbf{x}},$$

where $\ddot{\mathbf{x}}$ is the acceleration vector, $\varrho(t)$ — variable density of a fluid and P denotes a potential of conservative body forces. Since all the stress components may depend on the variable z only (through the functions φ' and ψ'), it is reasonable to assume that also $P = P(z, t)$.

Substituting from Eqs. (2.17) or (2.18) into Eq. (4.1) and calculating the inertia terms, we obtain

(¹) The functionals $\bar{\alpha}_i$ and $\bar{\bar{\alpha}}_i$ $(i = 0, \ldots, 9)$ in Eqs. (3.6) are not exactly the same as those used in the general equation (3.4). To avoid further multiplication of symbols we have included certain terms even with respect to φ', ψ' into the new functionals.

$$\frac{\partial}{\partial z}[\mathrm{t}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi'] - \varrho \ddot{f}_1 \varphi = 0,$$

(4.2)
$$\frac{\partial}{\partial z}[\mathrm{t}(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\psi'] - \varrho \ddot{f}_2 \psi = 0,$$

$$\frac{\partial}{\partial z}[\mathrm{s}_0(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2) + \mathrm{s}_2(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi'^2 + \mathrm{s}_2(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\psi'^2] - \frac{\partial P}{\partial z} = 0$$

for the motion described by Eq. (2.1), or

$$\frac{\partial}{\partial z}[\mathrm{t}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\varphi' + \mathrm{t}(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\psi'] - \bar{g}\ddot{f}_1 \varphi - \bar{g}\ddot{f}_2 \psi = 0,$$

(4.3)
$$\frac{\partial}{\partial z}[-\mathrm{t}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)\psi' + \mathrm{t}(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)\varphi'] - \varrho \ddot{f}_2 \varphi - \varrho \ddot{f}_1 \psi = 0,$$

$$\frac{\partial}{\partial z}[\mathrm{s}_0(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2) + \mathrm{s}_2(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)(\varphi'^2 + \psi'^2)$$
$$+ \mathrm{s}_2(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2)(\varphi'^2 + \psi'^2)] - \frac{\partial P}{\partial z} = 0$$

for the more general motion (2.2). Here, the symbol $\varkappa^2$ stands for φ'^2, ψ'^2 or $\varphi'^2 + \psi'^2$, respectively.

For compressible simple fluids, the above system of coupled differential equations can be satisfied only in the following cases:

a) there are no body forces ($P \equiv 0$), then φ'^2, ψ'^2 in Eqs. (4.2) or $\varphi'^2 + \psi'^2$ in Eqs. (4.3) must be constant with respect to z;

b) there are body forces ($P \neq 0$) of such a form that the third equations in Eqs. (4.2) or (4.3) are satisfied.

It is worth noting that the case a) is valid for circular shearings discussed elsewhere [8, 9, 10]. For these flows, the solutions of Eqs. (4.3) which are periodic with respect to z and satisfy appropriate initial and boundary conditions correspond to progressive or standing circularly polarized plane shear waves.

For incompressible simple fluids, the above system of differential equations can always be satisfied for arbitrary potentials $P(z, t)$. The first two equations (4.2) or (4.3) with appropriate initial and boundary conditions can, at least in a numerical way, be solved for φ and ψ. The corresponding third equations determine the hydrostatic pressure functions p.

In what follows, we shall discuss only particular solutions of the system (4.2). For incompressible simple fluids this system, after taking into account Eqs. (3.6), can be expressed in the alternative form:

$$\frac{\partial}{\partial z}[\bar{\alpha}_1 \dot{f}_1 \varphi' + \bar{\alpha}_2 \ddot{f}_1 \varphi'] - \varrho \ddot{f}_1 \varphi = 0,$$

(4.4)
$$\frac{\partial}{\partial z}[\bar{\bar{\alpha}}_1 \dot{f}_2 \psi' + \bar{\alpha}_2 \ddot{f}_2 \psi'] - \varrho \ddot{f}_2 \psi = 0,$$

$$\frac{\partial}{\partial z}[-p + 2\bar{\alpha}_5 \dot{f}_1 \ddot{f}_1 \varphi'^2 + 2\bar{\bar{\alpha}}_5 \dot{f}_2 \ddot{f}_2 \psi'^2] - \frac{\partial P}{\partial z} = 0.$$

It is seen that the terms in square brackets in Eqs. (4.2) and (4.4) determine the shear stresses in the flows considered. The first two equations (4.4) become independent of each other and linear with respect to φ and ψ only in the following cases:

i) the constitutive equations considered are such that the shear response of a fluid is linear; then either $\mathfrak{t}(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2)$ or $\bar{\alpha}_i$ and $\bar{\bar{\alpha}}_i$, $i = 1, 2$, do not depend on φ'^2 and ψ'^2, i.e. on the variable z;

ii) the amounts of shear determined by the functions φ' and ψ' are sufficiently small to disregard higher order terms as compared with those of the first order.

The case i) covers all fluid models with linear shear response, e.g. Newtonian fluids, second order fluids, finite linear viscoelasticity models etc., and may be used without any further restrictions imposed on the amounts of shear.

In some cases it is useful to introduce the following generalized viscosity functions:

$$\eta_1 = \eta_1(t, \varkappa^2) = \frac{T^{13}}{\dot{f}_1 \varphi'} = \frac{1}{\dot{f}_1} \mathfrak{t}\big(\bar{g}_1(s), \bar{g}_2(s); \varkappa^2\big) = \bar{\alpha}_1 + \frac{\ddot{f}_1}{\dot{f}_1} \bar{\alpha}_2,$$

(4.5)

$$\eta_2 = \eta_2(t, \varkappa^2) = \frac{T^{23}}{\dot{f}_2 \psi'} = \frac{1}{\dot{f}_2} \mathfrak{t}\big(\bar{g}_2(s), \bar{g}_1(s); \varkappa^2\big) = \bar{\bar{\alpha}}_1 + \frac{\ddot{f}_2}{\dot{f}_2} \bar{\bar{\alpha}}_2,$$

where the notation from Eqs. (2.17) and (3.6) has been applied.

Under the assumption of linear shear response or for small amounts of shear, the first two equations (4.4) simplify to

(4.6)
$$\eta_1 \dot{f}_1 \varphi'' - \varrho \ddot{f}_1 \varphi = 0,$$
$$\eta_2 \dot{f}_2 \psi'' - \varrho \ddot{f}_2 \psi = 0,$$

where primes and dots denote the derivatives with respect to z and t, respectively.

5. Oscillatory shearings and plane shear waves

Now we shall discuss in greater detail the case in which $f_1(\tau)$ and $f_2(\tau)$ are harmonic functions of time, viz.

(5.1)
$$f_1(\tau) = \exp i\omega_1 \tau, \quad f_2(\tau) = \exp i\omega_2 \tau,$$

where ω_1 and ω_2 are real positive angular frequencies. Since, according to Eq. (2.9),

(5.2)
$$\bar{g}_i(s) = e^{i\omega_i t}(e^{i\omega_i s} - 1), \quad g_i(s) = e^{i\omega_i s} - 1, \quad i = 1, 2,$$

the generalized viscosities defined by Eqs. (4.5) depend only on ω_1 and ω_2 if the assumption of linear shear response holds. Instead of Eqs. (4.6) we obtain

(5.3)
$$i\omega_1 \eta_1^*(\omega_1, \omega_2)\varphi'' + \varrho\omega_1^2 \varphi = 0,$$
$$i\omega_2 \eta_2^*(\omega_2, \omega_1)\psi'' + \varrho\omega_2^2 \psi = 0,$$

where η_1^* and η_2^* can be considered as the mechanical impedances or the generalized dynamic viscosities for two oscillations superposed in two mutually perpendicular planes. On using representations of the Rivlin-Ericksen type, it results from Eqs. (4.5) that

(5.4)
$$\eta_1^*(\omega_1, \omega_2) = \bar{\alpha}_1^* + i\omega_1 \bar{\alpha}_2^*,$$
$$\eta_2^*(\omega_2, \omega_1) = \bar{\bar{\alpha}}_1^* + i\omega_2 \bar{\bar{\alpha}}_2^*,$$

where α^* are also dynamic functions of two angular frequencies ω_1 and ω_2.

A general solution of the system (5.3) can be written in the form

$$\varphi(z) \quad \text{or} \quad \psi(z) = A_i \exp(\beta_i + i\gamma_i)z + B_i \exp(-\beta_i - i\gamma_i)z, \quad i = 1 \text{ or } 2, \tag{5.5}$$

where A_i and B_i are integration constants, and

$$(\beta_i + i\gamma_i)^2 = \frac{i\varrho\omega_i}{\eta_i^*} = \frac{-\varrho\omega_i\eta_i'' + i\varrho\omega_i\eta_i'}{\eta_i'^2 + \eta_i''^2}, \tag{5.6}$$

where the real (primed) and imaginary (doubly primed) parts of functions η_i^*, $i = 1, 2$, are defined by

$$\begin{aligned} \eta_1^* &= \eta_1' - i\eta_1'', \quad \eta_1' = \bar{\alpha}_1' - \omega_1\bar{\alpha}_2'', \quad \eta_1'' = -\bar{\alpha}_1'' - \omega_1\bar{\alpha}_2', \\ \eta_*^2 &= \eta_2' - i\eta_2'', \quad \eta_2' = \bar{\bar{\alpha}}_1' - \omega_2\bar{\bar{\alpha}}_2'', \quad \eta_2'' = -\bar{\bar{\alpha}}_1'' - \omega_2\bar{\bar{\alpha}}_2'. \end{aligned} \tag{5.7}$$

Periodic solutions of the form (5.5) describe plane sinusoidal waves, standing or propagating along the z-axis with the phase velocities $c_i = \omega_i/\gamma_i$. The parameters β_i depending on angular frequencies ω_i characterize growth or decay of the wave amplitude and may be called the coefficients of attenuation or damping. The parameters γ_i determine a dispersion of waves and may be called the phase shifts or the wave numbers.

Equation (5.6) leads to the following useful relations (cf. [12, 1]):

$$\beta_i^2 = \frac{\varrho\omega_i}{2\eta_i'}\left[\frac{1}{\sqrt{1+\xi_i^2}} - \frac{\xi_i}{1+\xi_i^2}\right] = \frac{\varrho\omega_i\xi_i}{2\eta_i''}\left[\frac{1}{\sqrt{1+\xi_i^2}} - \frac{\xi_i}{1+\xi_i^2}\right], \tag{5.8}$$

$$\gamma_i^2 = \frac{\varrho\omega_i}{2\eta_i'}\left[\frac{1}{\sqrt{1+\xi_i^2}} + \frac{\xi_i}{1+\xi_i^2}\right] = \frac{\varrho\omega_i\xi_i}{2\eta_i''}\left[\frac{1}{\sqrt{1+\xi_i^2}} + \frac{\xi_i}{1+\xi_i^2}\right], \tag{5.9}$$

where

$$\xi_i = \frac{\eta_i''}{\eta_i'} = \frac{1}{\operatorname{tg}\delta_i}, \quad i = 1, 2. \tag{5.10}$$

The second forms of Eqs. (5.8) and (5.9) are valid only for $\xi_i \neq 0$. By an analogy to the theory of linear viscoelasticity δ_i can be considered as the generalized loss angles (cf. [13]).

It would be quite interesting to discuss the behaviour of phase shifts and coefficients of damping in the full range of angular frequencies, i.e. from zero to infinity. Such an analysis for one-dimensional sinusoidal waves in a second order fluid has been presented by TRUESDELL [12, 1]. Since in our case an explicit dependence of ξ_i on ω_i is not known, we shall rather analyse β_i and γ_i as functions of the parameters ξ_i. We shall omit, moreover, the indices $i = 1, 2$ and overbars denoting each of the component motions. Although the latter simplifications are formally equivalent to the assumption that $\omega_1 = \omega_2 = \omega$ or to the cases in which either ω_1 or ω_2 is constant, some information can be obtained for motions with two distinct frequencies, since the material characteristics with index 1 (or single overbar) are such functions of ω_1 as those with index 2 (or double overbar) of ω_2.

To begin with, we note that for numerous viscoelastic fluids like dilute and uncross-linked polymer solutions, light oils with small amounts of polymeric additives etc., it is reasonable to assume that

$$\lim_{\omega \to 0} \xi(\omega) = 0, \quad \lim_{\omega \to \infty} \xi(\omega) = \infty. \tag{5.11}$$

The above assumptions mean that for very low frequencies or very long times the fluid considered behaves like a purely viscous fluid while for very high frequencies or very short times its behaviour is almost purely elastic. In other words, for $\omega \to 0$ the tangent of loss angle tends to infinity, while for $\omega \to \infty$ it is negligibly small (cf. [13]).

Thus the limits in Eqs. (5.11) and the formulae (5.8) and (5.9) imply also that

$$\lim_{\omega\to 0}\beta^2 = 0, \quad \lim_{\omega\to\infty}\beta^2 = 0 \quad \text{or finite}, \tag{5.12}$$

$$\lim_{\omega\to 0}\gamma^2 = 0, \quad \lim_{\omega\to\infty}\gamma^2 = \lim_{\omega\to\infty}\frac{\varrho\omega}{\eta''(\omega)} = \lim_{\omega\to\infty}\frac{\varrho\omega^2}{G'(\omega)}, \tag{5.13}$$

where $G' = \eta''/\omega$ can be considered as the real part of the generalized complex modulus. If the last limit is finite, the wave length is constant for ultrasonic frequencies; if γ^2 tends to infinity the wave length tends to zero for sufficiently high frequencies. It turns out, moreover, that in any case γ^2 increases monotonically while β^2 may reach a maximum value for $\xi = 1/\sqrt{3}$, i.e. for $\operatorname{tg}\delta = \sqrt{3}$. It is also seen from Eq. (5.8) that

$$\beta^2_{\max} = \frac{\varrho\omega_{cr}}{8\eta''(\omega_{cr})} = \frac{\varrho\omega^2_{cr}}{8G'(\omega_{cr})}, \tag{5.14}$$

where the critical value of angular frequency results from the equation

$$\eta'(\omega_{cr}) = \sqrt{3}\,\eta''(\omega_{cr}) = \sqrt{3}\,\frac{G'(\omega_{cr})}{\omega_{cr}}. \tag{5.15}$$

In agreement with Truesdell's remarks [12] any critical value of angular frequency determines a certain characteristic time of a fluid: $\theta = 2\pi/\omega_{cr}$. On the other hand, the existence of a critical frequency characterizing the strongest damping effects is usually connected with a passage from purely liquid state to highly elastic state (cf. [14]).

It is worth-while to remind that for Newtonian fluids

$$\beta^2 = \gamma^2 = \frac{\varrho\omega}{2\eta_0}, \tag{5.16}$$

where the Newtonian viscosity can be identified with $\eta'(0) = \text{const}$. In this case, β^2 as well as γ^2 tend to infinity with the increasing frequency ω.

A diagram illustrating the above discussed variability of β^2 and γ^2 is shown schematically in Fig. 1. The abcissae denote the values of ξ; the corresponding values of ω are indicated only for orientation.

It may happen, however, that for other viscoelastic fluids like more condensed and cross-linked polymer solutions, polymeric gels etc., the function $\eta''(\omega)$ does not tend to zero for very small angular frequencies but approaches some finite value $\eta''(0)$. Then, according to Eqs. (5.7) and (5.10), we have

$$\lim_{\omega\to 0}\xi(\omega) = \lim_{\omega\to 0} - \frac{\alpha_1''(\omega)+\omega\alpha_2'(\omega)}{\alpha_1'(\omega)-\omega\alpha_2'(\omega)} = \lim_{\omega\to 0} - \frac{\alpha_1''(\omega)}{\alpha_1'(\omega)}. \tag{5.17}$$

The above result means that for $\omega = 0$, ξ is equal to some finite, rather small quantity. It is also seen from Eqs. (5.8) and (5.9) that for $\omega = 0$, both β^2 and γ^2 vanish, while for $\xi = 0$ we have

$$\beta_0^2 = \gamma_0^2 = \frac{\varrho\omega_0}{2\big(\alpha_1'(\omega_0)-\omega_0\alpha_2''(\omega_0)\big)} = \frac{-\varrho\alpha_1''(\omega_0)}{2\big(\alpha_1'(\omega_0)\alpha_2'(\omega_0)+\alpha_1''(\omega_0)\alpha_2''(\omega_0)\big)}, \tag{5.18}$$

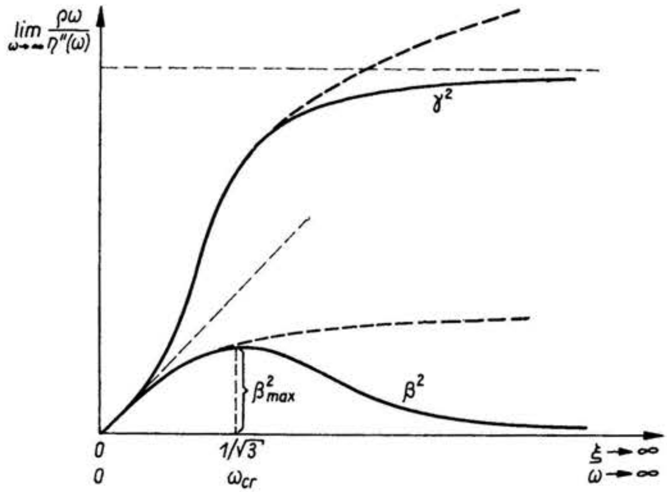

FIG. 1.

where ω_0 denotes a real positive root of the equation(²):

$$\alpha_1''(\omega_0)+\omega_0\alpha_2'(\omega_0) = 0. \tag{5.19}$$

Since, moreover, β^2 and γ^2 are such functions of ξ that

$$\beta^2(-\xi) = \gamma^2(\xi) \tag{5.20}$$

and vice versa, and γ^2 may have only a maximum for $\xi = -1/\sqrt{3}$, it is expected that the limit value of ξ determined by Eq. (5.17) is greater than $-1/\sqrt{3}$. This fact implies

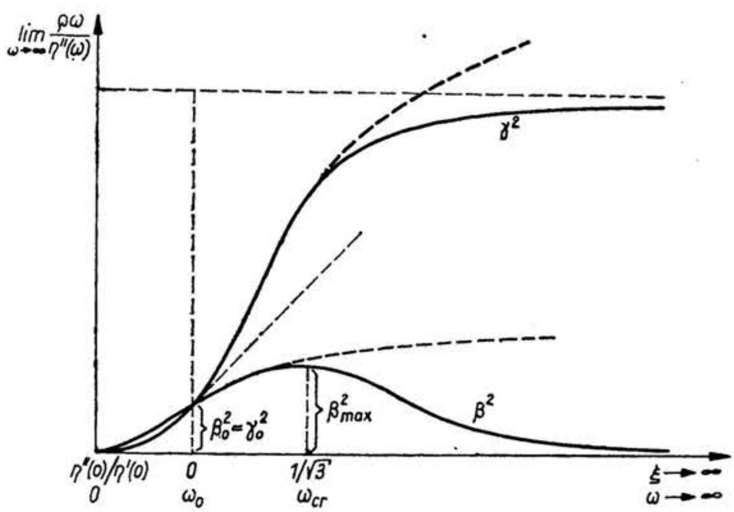

FIG. 2.

(²) For such a root to exist it is necessary that $\alpha_2'(\omega_0) < 0$. This condition is similar to that for second order fluids for which always $\alpha_2 = \text{const} < 0$ (cf. e.g. [1]).

a monotonic increase of β^2 and γ^2 in the interval from 0 (for $\omega = 0$) to $\beta_0^2 = \gamma_0^2$ (for $\omega = \omega_0$) and, moreover, $\beta^2 > \gamma^2$. At the point $\xi = 0$ (or $\omega = \omega_0$) where a breakage of internal structure takes place, the fluid considered behaves as purely viscous.

A diagram illustrating the above discussed situation is shown in Fig. 2. It does not seem that other possibilities, different from those presented in Figs. 1 and 2, may occur at all for the case considered. For two shear waves with distinct angular frequencies ω_1 and ω_2 or for one-dimensional longitudinal waves, however, the whole picture may look in a quite different way. For longitudinal waves, for instance, two or more maxima on $\beta^2(\xi)$ and $\gamma^2(\xi)$ curves are possible (cf. [15]).

When the waves considered are caused by some sinusoidal disturbances applied in any of xy-planes, a maximum value of the amount of shear $\varkappa^2 = \varphi'^2+\psi'^2$ essentially depends on ω_1 and ω_2. For the sake of simplicity, we assume that angular frequencies of two component motions are equal, i.e. $\omega_1 = \omega_2 = \omega$. Bearing in mind that, in general, φ' and ψ' are complex functions, we obtain from Eqs. (5.5)

$$\varkappa^2_{\max} = |\varphi'^2+\psi'^2| = C^2(\beta^2+\gamma^2), \tag{5.21}$$

where $C^2 = (A_1-B_1)^2+(A_2-B_2)^2$ is composed of the integration constants involved in Eq. (5.5). Substituting for β^2 and γ^2 from Eqs. (5.8) and (5.9), we finally arrive at

$$\varkappa^2_{\max} = \frac{1}{2} C^2 \frac{\varrho\omega}{\sqrt{\eta'^2+\eta''^2}} = \frac{1}{2} C^2 \frac{\varrho\omega}{\sqrt{(\alpha_1'-\omega\alpha_2'')^2+(\alpha_1''+\omega\alpha_2')^2}}. \tag{5.22}$$

Taking into account the variability of the parameter ξ defined by Eq. (5.10), it is seen that

$$\lim_{\omega\to 0} \varkappa^2_{\max} = \lim_{\omega\to 0} \frac{1}{2} C^2 \frac{\varrho\omega}{\eta'(\omega)} = 0, \tag{5.23}$$

$$\lim_{\omega\to\infty} \varkappa^2_{\max} = \lim_{\omega\to\infty} \frac{1}{2} C^2 \frac{\varrho\omega}{\eta''(\omega)} = -\frac{1}{2} C^2 \frac{\varrho}{\alpha_2'(\omega)}, \tag{5.24}$$

where the last limit may be finite.

The corresponding Newtonian maximum amount of shear is

$$\varkappa^2_{\max N} = \frac{1}{2} C^2 \frac{\varrho\omega}{\eta_0}, \tag{5.25}$$

where $\eta_0 = \eta'(0) = \text{const.}$

The result (5.23) proves that for very low angular frequencies the maximum amount of shear tends to zero as for Newtonian fluids. On the contrary, the result (5.24) shows that for very high or ultrasonic frequencies the maximum amount of shear in vicoelastic fluids may tend to some finite value, while in Newtonian fluids it increases proportionally to the square root of the angular frequency. Thereby, we can conclude that viscoelastic fluids at high frequencies may be sheared less than Newtonian fluids of similar viscosities subjected to the same initial disturbances. On the other hand, such disturbances in visco-elastic fluids can propagate at longer distances as compared with those in Newtonian fluids, since for very high frequencies the damping effects are weaker (cf. [12]).

6. Speed of wave propagation and ultrasonic velocity

Acceleration and sound waves in viscoelastic fluids can also be considered as the propagating surfaces of second order discontinuities as shown by COLEMAN, GURTIN, HERRERA and TRUESDELL [16]. For the case of one-dimensional transverse waves, the speed of propagation (the intrinsic velocity of propagation) is determined by the formula

$$U_{\perp}^2 = \frac{G_t}{\varrho}, \tag{6.1}$$

where ϱ is the density of a fluid, and the instantaneous tangent modulus G_t is defined as follows:

$$G_t = \frac{\partial}{\partial F(t)} \mathop{\mathsf{E}}_{s=0}^{\infty} \left(F_r^t(s);\ F(t)\right), \tag{6.2}$$

where $F(t) = F^t(0)$ is the present value of the deformation gradient, and $F_r^t(s)$ denotes the past history, i.e. the restriction of $F^t(s)$ to the open interval $(0, \infty)$.

Assuming for simplicity that $f_1(\tau) = f_2(\tau) = f(\tau)$, we see that the shear stress components in Eqs. (2.17) and (3.6) imply that

$$G_t = \frac{\partial}{\partial \varphi' f}\left[\mathfrak{t}\left(\bar{g}(s); \varkappa^2\right)\varphi'\right] = \frac{1}{f}\mathfrak{t}\left(\bar{g}(s); \varkappa^2\right) = \frac{\dot{f}}{f}\left(\alpha_1 + \frac{\ddot{f}}{\dot{f}}\alpha_2\right) = \frac{\dot{f}}{f}\eta(t; \varkappa^2) \tag{6.3}$$

and an analogous expression with ψ'. Here, $\varphi' f$ denotes the shear gradient and η — the generalized viscosity function defined by Eqs. (4.5).

For an acceleration wave which since time $t = 0$ has been propagating into a region having been at rest in a fixed reference configuration, the instantaneous response is determined by the initial value of the stress relaxation function $G(t)$ (cf. [16]). Therefore, on taking $G(0) = G_t|_{t=0}$, we have from Eq. (6.3)

$$G(0) = \lim_{\omega\to\infty}|i\omega\eta^*(\omega)| = \lim_{\omega\to\infty}\omega\sqrt{\eta'^2+\eta''^2}, \tag{6.4}$$

where Eqs. (5.7) have been used. Substituting the above result into Eq. (6.1), we finally arrive at

$$U_{\perp}^2 = \frac{1}{\varrho}\lim_{\omega\to\infty}\omega\sqrt{\eta'^2+\eta''^2} = \frac{1}{\varrho}\lim_{\omega\to\infty}\omega\eta''(\omega) = \frac{G'(\infty)}{\varrho}. \tag{6.5}$$

If the last limit exists, the speed of propagation is finite, otherwise it tends to infinity as for Newtonian fluids.

On the other hand, it has been proved in [16] that for sinusoidal progressive waves $U_{\perp} = c_{\infty}$, where c_{∞}:

$$c_{\infty}^2 = \lim_{\omega\to\infty} c^2(\omega) = \lim_{\omega\to\infty}\frac{\omega^2}{\gamma^2(\omega)} \tag{6.6}$$

denotes the ultrasonic velocity. Taking into account Eq. (5.9) for $\omega \to \infty$, we obtain exactly the result expressed by Eq. (6.5).

It is also worth-while to observe that for fluids with linear shear responses, the instantaneous second order modulus defined by the second derivative of the functional $\mathsf{E}(F_r^t(s);\ F(t))$ with respect to $F(t)$ is always equal to zero (cf. [16]). This means that the amplitudes of shear waves cannot grow unlimitedly in a finite time.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

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