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A nonlocal theory for nematic liquid crystals

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CONTINUING previous works on the subject and considering the conditions to which liquid crystals may be subjected, we present in this work a nonlocal continuum model for nematic liquid crystals. To this end we first list the balance equations which are applicable to nonlocal interactions. These are: conservation of mass, balance of linear and angular momenta, conservation of energy and the entropy inequality. Using these conservation laws and thermodynamical restrictions, a set of nonlinear and linear constitutive relations are derived for nonlocal nematic liquid crystals, and several special cases are discussed. Finally, to illustrate the present derivation the solution of shear flow problem and related discussions are presented.

Przedstawiony w tej pracy nielokalny model kontynualny dla ciekłych kryształów typu nematycznego stanowi kontynuację poprzednich prac na ten temat i uwzględnia warunki jakim mogą być poddane ciekłe kryształy. W tym celu przytacza się równania zachowania stosowalne w przypadku oddziaływań nielokalnych. Są to równania zachowania masy, pędu i momentu pędu i energii oraz nierówność entropii. Za pomocą tych praw zachowania oraz ograniczeń termodynamicznych wyprowadzono układ nieliniowych i liniowych równań stanu dla nielokalnych ciekłych kryształów nematycznych i omówiono szereg przykładów szczególnych. Dla ilustracji przedstawiono rozwiązanie zagadnienia ścinania plastycznego wraz z dyskusją.

Представленная в этой работе нелокальная континуальная модель жидких кристаллов, нематического типа, составляет продолжение предыдущих работ на эту тему и учитывает условия, каким могут быть подвергнуты жидкие кристаллы. С этой целью приводятся уравнения сохранения, применяемые в случае нелокальных взаимодействий. Это уравнения сохранения массы, импульса, момента импульса и энергии, а также неравенство энтропии. С помощью этих законов сохранения и термодинамических ограничений, введена система нелинейных и линейных уравнений состояния для нелокальных жидких нематических кристаллов и обсужден ряд частных случаев. Для иллюстрации представлено решение задачи пластического сдвига совместно с обсуждением.

1. Introduction

BECAUSE of its important technological applications in medicine and electronics, the subject of liquid crystals has been extensively studied by several researchers, both from the viewpoints of the molecular theory and continuum approximations. After the initial approach of OSEEN [1, 2], ANZELIUS [3] and FRANK [4], in a series of papers ERICKSEN [5–7] and LESLIE [8, 9] have proposed a continuum theory of liquid crystals, in which the director theory of Ericksen has been utilized. A number of other approaches to the subject were made by other authors, e.g. DAVISON [10, 11] MARTIN *et al.* [12], HELFRICH [13, 14], AERO and BULYGIN [15], LEE and ERINGEN [16, 17] and ERINGEN and LEE [18]. An assessment of the subject matter was recently made by STEPHEN and STRALEY [19] and ERICKSEN [20] where extensive references to the literature on the subject are to be found. Each theory has its own merits and a limit of applicability and it hardly seems necessary to comment on the nature and the differences of various theories available today.

The above-mentioned continuum theories of liquid crystals are based upon the long wavelength (or low frequency) approximations (cf. FOSTER *et al.* [21]). In other words, these theories, the so-called local theories, do not take into account the long-range intermolecular forces or phenomena associated with high frequency waves. However, due to the feature of places where the liquid crystals are used, in many instances the liquid crystals are subjected to high frequency waves and/or electromagnetic forces which are of long-range in nature. As pointed out by ERINGEN [22] and DEMIRAY [23], in such cases the local theories of materials, in general, cannot be used to describe the electro-mechanical behaviour of the body involved. So that, in such situation the nonlocal theories that take long-range effects into account must be developed. As a matter of fact, nonlocal effects resulting from long-range intermolecular forces and short time effects are briefly touched on in the papers by OSEEN [2] and FOSTER [21].

As a sequence of the above observations, in this paper we introduce a nonlocal director model for nematic non-heat conducting liquid crystals. In doing this, the local balance equations of Ericksen and Leslie are generalized to the contributions of nonlocal residuals. These are: conservation of mass, balance of linear and angular momenta, conservation of energy and entropy inequality. Furthermore, a set of nonlinear and linear constitutive equations satisfying certain symmetry requirements are derived and several special cases are discussed. Finally, to illustrate the theory the problem of shear flow of nematic liquid crystals is studied.

2. Kinematics

We consider the motion of a continuum consisting of rod-like molecules (particles). Referred to the same set of fixed Cartesian axes, the position x_i and the orientation d_i, at time t, of a particle which is initially at X_i with the relative orientation D_i, are given by

$$x_i = x_i(X_k, t), \quad d_i = d_i(X_k, t). \tag{2.1}$$

The components of the macro-velocity v_i and the director velocity w_i of a particle are defined by

$$v_i \equiv \frac{D}{Dt} x_i, \quad w_i \equiv \frac{D}{Dt} d_i, \tag{2.2}$$

where the operator $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k}$ is used to denote the material time derivative.

Adopting the notation used by LESLIE [8] and ERICKSEN [5], we define the following quantities:

$$\begin{gathered} D_{ij} \equiv \frac{1}{2}(v_{i,j}+v_{j,i}), \quad \omega_{ij} \equiv \frac{1}{2}(v_{i,j}-v_{j,i}), \\ N_i \equiv w_i - \omega_{ik} d_k, \qquad N_{ij} \equiv w_{i,j} - \omega_{ik} d_{k,j}. \end{gathered} \tag{2.3}$$

We now consider the motions of the continuum which differ from that given by Eq. (2.1) only by superposed rigid body motions,

$$\begin{aligned} x_i^*(t) &= Q_{ij}(t) x_j(t) - c_i(t), \\ d_i^*(t) &= Q_{ij}(t) d_j(t), \end{aligned} \tag{2.4}$$

where $Q_{ij}(t)$ is a proper orthogonal tensor function of time t. The associated quantities defined in Eq. (2.3) will be denoted by the same symbol with an asterisk above. One can readily show that

$$\begin{aligned} D^*_{ij} &= Q_{ik}Q_{jl}D_{kl}, \quad \omega^*_{ij} = Q_{ik}Q_{jl}\omega_{kl}+\Omega_{ij}, \\ N^*_i &= Q_{il}N_l, \qquad N^*_{ij} = Q_{ik}Q_{jl}N_{kl}, \quad \Omega_{ij} = Q_{im}\frac{DQ_{jm}}{Dt}. \end{aligned} \tag{2.5}$$

Thus it is seen that D_{ij}, N_i and N_{ij} are form invariant under the motions (2.4) of the spatial frame of reference but not ω_{ij}.

3. Conservation laws and entropy inequality

In order to describe the nonlocal dynamical behaviour of liquid crystals, we adopt the conservation laws proposed by ERICKSEN [5] with proper modifications. For completeness of the subject a brief outline of the derivation of the conservation laws is given below.

For a material volume V bounded by a surface A, we assume that

$$\frac{D}{Dt}\int_V \varrho\left(\frac{1}{2}v_iv_i+\frac{1}{2}Jw_iw_i+U\right)dV = \int_V \varrho(r+F_iv_i+G_iw_i)dV + \int_A (t_iv_i+s_iw_i-h)dA, \tag{3.1}$$

where ϱ is the mass density, U integral energy per unit mass, r heat supply per unit mass, t_i surface force per unit area, s_i director surface force per unit area, h the heat flux per unit area and time, F_i body force per unit mass, G_i an intrinsic director body force per unit mass, and J the constant microinertia.

Now consider a motion of the type (2.4) in which

$$c_i(t) = a_it, \quad Q_{ij} = \delta_{ij},$$

where a_i is an arbitrary constant vector. Assuming that $\varrho, U, F_i, G_i, t_i, s_i, r$ and h remain form invariant under such a motion, Eq. (3.1) becomes

$$\frac{D}{Dt}\int_V \varrho\left[\frac{1}{2}(v_i+a_i)(v_i+a_i)+\frac{1}{2}Jw_iw_i+U\right]dV = \int_V \varrho[r+F_i(v_i+a_i)+G_iw_i]dV+\int_A [t_i(v_i+a_i)+s_iw_i-h]dA. \tag{3.2}$$

Hence, from Eqs. (3.1) and (3.2) it follows that

$$\frac{D}{Dt}\int_V \varrho\left(v_ia_i+\frac{1}{2}a_ia_i\right)dV = \int_V \varrho F_ia_idV+\int_A t_ia_idA \tag{3.3}$$

and, since a_i is arbitrary,

$$\frac{D}{Dt}\int_V \varrho dV = 0, \quad \frac{D}{Dt}\int_V \varrho v_idV = \int_V \varrho F_idV+\int_A t_idA. \tag{3.4}$$

Assuming that without loss in generality the mass is locally conserved, from Eqs. (3.4), with the usual smoothness assumptions

$$\frac{\partial \varrho}{\partial t}+(\varrho v_i)_{,i}=0, \quad \varrho\frac{Dv_i}{Dt}=\varrho F_i+\sigma_{ji,j}-\hat{f}_i, \tag{3.5}$$

$$t_i=\sigma_{ji}n_j, \quad \int\limits_V \hat{f}_i dV=0, \tag{3.6}$$

where σ_{ij} are the components of the stress tensor, n_i is the unit outward normal to the surface and $\hat{f}_i$ is the nonlocal body force (or the rate of nonlocal linear momentum) per unit volume. From these equations one sees that the classical linear momentum equation for local bodies is generalized to include a term $(\hat{f}_i)$ corresponding to nonlocal interactions.

Introducing Eqs. (3.5) and (3.6) into the master equation (3.1), one obtains

$$\int\limits_V \left(J\varrho w_i\frac{Dw_i}{Dt}+\varrho\frac{DU}{Dt}\right)dV=\int\limits_V(\varrho r+G_iw_i+\hat{f}_iv_i+\sigma_{ij}D_{ij}+\sigma_{ji}\omega_{ij})dV+\int\limits_A(s_iw_i-h)\,dA. \tag{3.7}$$

Adopting the arguments introduced by LESLIE [8], we may assume that the following local equations are valid:

$$\varrho_J\frac{Dw_i}{Dt}=\pi_{ji,j}+\varrho G_i-g_i, \tag{3.8}$$

where g_i is the total intrinsic director body force (both local and nonlocal) and π_{ij} is the director stress tensor defined by

$$s_i=\pi_{ji}n_j. \tag{3.9}$$

Substituting Eqs. (3.8) and (3.9) into Eq. (3.7), the energy equation becomes

$$\int\limits_V\varrho\frac{DU}{Dt}dV=\int\limits_V(\varrho r+\hat{f}_iv_i+g_iw_i+\sigma_{ij}D_{ij}+\sigma_{ji}\omega_{ij}+\pi_{ji}w_{i,j})dV-\int\limits_A h\,dA. \tag{3.10}$$

Consider a second motion of the type (2.4) in which

$$c_i(t)=0, \quad Q_{ij}(t)=\delta_{ij}, \quad \frac{dQ_{ij}}{dt}=a_{ij}=-a_{ji},$$

where a_{ij} is a constant and skew-symmetric second-order tensor. Assuming that ϱ, U, $\hat{f}_i$, g_i, σ_{ij}, π_{ij}, r and h are form invariant under such a motion, Eq. (3.10) takes the following form:

$$\int\limits_V\varrho\frac{DU}{Dt}dV=\int\limits_V(\varrho r+\sigma_{ij}D_{ij}+\pi_{ji}N_{ij}+\hat{f}_iv_i+g_iN_i)dV-\int\limits_A h\,dA+\int\limits_V\omega_{ij}(\sigma_{ji}+g_id_j+\pi_{ki}d_{j,k})dV+a_{ij}\int\limits_V(\sigma_{ji}+g_id_j+\pi_{ki}d_{j,k}+\hat{f}_ix_j)dV. \tag{3.11}$$

Equation (3.11) must be valid for all arbitrary variations of a_{ij}. Thus we must have

$$\sigma_{ji}+d_j g_i+d_{i,k}\pi_{ki}+x_j\hat{f}_i = \hat{R}_{ji} \tag{3.12}$$

such that

$$\int_V (\hat{R}_{ij}-\hat{R}_{ji})dV = 0, \tag{3.13}$$

where $\hat{R}_{ij}$ is the rate of the nonlocal generalized momentum residual tensor. Introduction of Eq. (3.12) into Eq. (3.11) yields

$$\int_V \varrho\frac{DU}{Dt}dV = \int_V (\varrho r+\sigma_{ij}D_{ij}+\pi_{ji}N_{ij}+\hat{f}_i v_i+g_i N_i+\hat{P}_{ij}\omega_{ji}-q_{i,i})dV, \tag{3.14}$$

where the heat flux vector q_i and the tensor $\hat{P}_{ij}$ are defined by

$$h = q_i n_i, \quad \hat{p}_{ij} = \hat{R}_{ij}-x_i\hat{f}_j. \tag{3.15}$$

Hence, from Eq. (3.14) the localized form of the energy equation follows:

$$\varrho\frac{DU}{Dt} = \varrho r+\sigma_{ji}D_{ij}+\pi_{ji}N_{ij}+\hat{f}_i v_i+g_i N_i+\hat{P}_{ji}\omega_{ij}-q_{i,i}+\hat{e}. \tag{3.16}$$

Here $\hat{e}$ is the nonlocal energy residual and subject to

$$\int_V \hat{e}dV = 0. \tag{3.17}$$

In conjunction with the above conservation laws we consider an entropy inequality of the form

$$\frac{D}{Dt}\int_V \varrho S dV-\int_V \frac{\varrho r}{T}dV+\int_A \left(\frac{q_i}{T}\right)n_i dA \geqslant 0. \tag{3.18}$$

Here S is the entropy per unit mass, T is the absolute temperature. Then the localized form of the entropy inequality reads

$$\varrho\frac{DS}{Dt}-\frac{\varrho r}{T}+\frac{q_{i,i}}{T}-\frac{q_i T_{,i}}{T^2}+\hat{n} \geqslant 0, \tag{3.19}$$

where $\hat{n}$ is the rate of the nonlocal entropy residual and subject to

$$\int_V \hat{n}dV = 0. \tag{3.20}$$

Introducing Helmholtz free energy F

$$F \equiv U-TS \tag{3.21}$$

with the aid of Eq. (3.16), the inequality becomes

$$\sigma_{ji}D_{ij}+\pi_{ji}N_{ij}+\hat{f}_i v_i+g_i N_i+\hat{P}_{ji}\omega_{ij}-\frac{q_i T_{,i}}{T}-\varrho\left(\frac{DF}{Dt}+S\frac{DT}{Dt}\right)+(T\hat{n}+\hat{e}) \geqslant 0. \tag{3.22}$$

This inequality must be valid for all *independent processes.*

4. Constitutive equations

For nonlocal nematic liquid crystals adopting the principle of equipresence [24] we assume that the constitutive dependent variables

$$F,\ S,\ q_i,\ \sigma_{ij},\ \pi_{ij},\ g_i,\ \hat{f}_i,\ \hat{P}_{ij}\hat{e},\ \hat{n}, \tag{4.1}$$

at any particle x at time t are all single-valued functions of(1)

$$\varrho,\ T,\ d_i,\ d_{i,j},\ w_i,\ v_i,\ v_{i,j},\ T_{,i},\ x_i \tag{4.2}$$

and single-valued functionals in

$$\varrho',\ T',\ d'_i,\ d'_{i,j},\ w'_i,\ v'_i,\ v'_{i,j},\ T'_{,i},\ x'_i. \tag{4.3}$$

Equivalently, from Eqs. (2.3) and (2.4) they are functions, or functionals, in the following quantities:

$$\begin{gathered}\varrho,\ T,\ d_i,\ d_{i,j},\ N_i,\ D_{ij},\ T_{,i},\\ \varrho',\ T',\ d'_i,\ d'_{i,j},\ N'_i,\ D'_{ij},\ T'_{,i},\ \Omega'_{ij},\ \omega'_i,\ r'_i,\end{gathered} \tag{4.4}$$

where Ω'_{ij}, ω'_i and r'_i are defined by

$$\Omega'_{ij} \equiv \omega'_{ij}-\omega_{ij},\quad \omega'_i \equiv v'_i-v_i+(x'_r-x_r)\omega_{ri},\quad r'_i = x'_i-x_i. \tag{4.5}$$

It can easily be shown that the above quantities are objective under orthogonal transformation of the spatial frame of reference. Thus the general functional form of a dependent variable reads

$$F = F(\varrho, T, d_i, d_{i,j}, N_i, D_{ij}, T_{,i}; \varrho', T', d'_i, d'_{i,j}, N'_i, D'_{ij}, T'_{,i}, \Omega'_{ij}, \omega'_i, r'_i). \tag{4.6}$$

We are here primarily concerned with liquid crystals of the nematic type. Hence, following FRANK [4] we assume that the constitutive functionals are invariant under reflections through planes containing the director. This excludes the liquid crystals of the cholesteric type. However, the implication of this assumption is that our constitutive functionals become locally isotropic rather than hemitropic, i.e. Eq. (4.5) holds for both proper and improper orthogonal tensors. Also, we assume that d_i and $-d_i$ are physically indistinguishable and thus

$$\begin{gathered}F\to -F,\quad S\to -S,\quad q_i\to -q_i,\quad \sigma_{ij}\to -\sigma_{ij},\quad \pi_{ij}\to -\pi_{ij},\\ g_i\to -g_i,\quad \hat{f}_i\to -\hat{f}_i,\quad \hat{P}_{ij}\to -\hat{P}_{ij},\quad \hat{e}\to -\hat{e},\quad \hat{n}\to -\hat{n},\end{gathered} \tag{4.7}$$

if

$$d_i\to -d_i,\quad d_{ij}\to -d_{i,j},\quad N_i\to -N_{ij},\quad N_{i,j}\to -N_{ij}. \tag{4.8}$$

These constitutive equations are further restricted by the principle of entropy inequality. If one takes the material derivative of Eq. (4.6) and substitute it into Eq. (3.22), the first implication of the second law of thermodynamics is(2)

$$F = F(\varrho, T, d_i, d_{i,j}; \varrho', T', d'_i, d'_{i,j}, r'_i). \tag{4.9}$$

(1) In general, one should also include $w_{i,j}$ as an independent constitutive variable. Following Leslie, we have not included this term which characterizes the directorial viscosity.

(2) In its complete generality the function F depends on D_{ij}, D'_{ij}, N_{ij}, N'_{ij}, N_i, N'_i, Ω'_{ij} and ω'_i in a special way, e.g.

$$\frac{\partial F}{\partial A_{kl}}+\int_A \varrho'\left(\frac{\delta F}{\delta A'_{kl}}\right)^{*} dV' = 0,$$

Noting that F is a functional in the primed quantities, the material time derivative of Eq. (4.9) may be given by (cf. Demiray [23])

$$\begin{aligned}(4.10)\quad \dot{F} = & -\varrho\frac{\partial F}{\partial \varrho}v_{k,k}+\frac{\partial F}{\partial T}\frac{DT}{Dt}+\frac{\partial F}{\partial d_i}(N_i+\omega_{ik}d_k)+\frac{\partial F}{\partial d_{i,j}}(N_{ij}-d_{i,k}D_{kj}\\ & +\omega_{ik}d_{k,j}+\omega_{jk}d_{i,k})+\int\limits_{V}\Big[-\varrho'\frac{\delta F}{\delta\varrho}v'_{k,k}+\frac{\delta F}{\delta T'}\frac{DT'}{Dt}+\frac{\delta F}{\delta d'_i}(N'_i+\omega'_{ik}d'_k)\\ & +\frac{\delta F}{\delta d'_{i,j}}(N'_{ij}-d'_{i,k}D'_{kj}+\omega'_{ik}d'_{k,j}+\omega'_{jk}d'_{i,k})+\frac{\delta F}{\delta r'_i}(v'_i-v_i)\Big]dV',\end{aligned}$$

where we have used the following relations:

$$(4.11)\quad \begin{aligned}&\frac{Dd_i}{Dt}=N_i+\omega_{ik}d_k,\qquad \frac{D}{Dt}(d_{i,j})=N_{ij}+\omega_{ik}d_{k,j}+\omega_{jk}d_{i,k}-d_{i,k}D_{kj},\\ &\frac{D}{Dt}=-\varrho v_{k,k}.\end{aligned}$$

Here the symbol $\delta F/\delta(\ )'$ is used to denote the Fréchet gradient of F with respect to that of associated quantity.

Introducing the following abbreviated notations

$$(4.12)\quad \begin{aligned}&\pi\equiv\varrho^2\frac{\partial F}{\partial\varrho},\qquad \eta\equiv\frac{\partial F}{\partial T},\qquad \sigma_i\equiv\varrho\frac{\partial F}{\partial d_i},\qquad K_{ij}=\varrho\frac{\partial F}{\partial d_{j,i}},\\ &\pi'\equiv\varrho'\varrho\frac{\delta F}{\delta\varrho'},\qquad \eta'\equiv\frac{\delta F}{\delta T'},\qquad \sigma'_i\equiv\varrho\frac{\delta F}{\delta d'_i},\qquad K'_{ij}\equiv\varrho\frac{\delta F}{\delta d'_{j,i}},\\ &\tau'_i\equiv\varrho\frac{\delta F}{\delta r'_i}\end{aligned}$$

into Eq. (4.10), we have

$$\begin{aligned}(4.13)\quad \dot{F} = & -\frac{\pi}{\varrho}v_{k,k}+\eta\frac{DT}{Dt}+\frac{1}{\varrho}\sigma_i(N_i+\omega_{ik}d_k)+\frac{1}{\varrho}K_{ji}(N_{ij}-d_{i,k}D_{kj}\\ & +\omega_{ik}d_{k,j}+\omega_{jk}d_{i,k})+\frac{1}{\varrho}\int\limits_{V}\Big[-\frac{\pi'}{\varrho}v'_{k,k}+\eta'+\sigma'_i(N'_i+\omega'_{ik}d'_k)\\ & +K'_{ji}(N'_{ij}-d'_{i,k}D'_{kj}+\omega'_{ik}d'_{k,j}+\omega'_{jk}d'_{i,k})+\tau'_i(v'_i-v_i)\Big]dV'.\end{aligned}$$

The requirement that F is form invariant under rigid body rotation implies that the tensor

$$(4.14)\quad T_{ij}\equiv\sigma_i d_j+K_{ik}d_{k,j}+K_{ki}d_{j,k}+\int\limits_{V}\tau'_{ij}dV',$$

where

$$(4.15)\quad \tau'_{ij}\equiv\sigma'_i d'_j+K'_{ik}d'_{k,j}+K'_{ki}d_{j,k}+\tau'_i(x'_j-x_j)$$

where A_{kl} (or A'_{kl}) represents one of the above variables on which F depends. But it is rather difficult to find the solution for F that satisfies the foregoing functional equation. One of the solutions of this equation is that F is independent of these variables, such as Eq. (4.9).

should be symmetric in its indices (i, j). Substituting Eq. (4.13) into Eq. (3.22), then integrating the result over the volume of the body and using the condition (3.17), one obtains

$$(4.16)\quad \int_V \Big\{\Big[\sigma_{ij}+\Big(\pi+\int_V \pi^* dV'\Big)\delta_{ji}+\Big(K_{jk}+\int_V K^*_{jk}dV'\Big)d_{k,i}\Big]D_{ij}$$
$$+\Big(\pi_{ji}-K_{ji}-\int_V K^*_{ji}dV'\Big)+\Big(g_i-\sigma_i-\int_V \sigma^*_i dV'\Big)N_i+\Big[\hat{f}_i-\int_V (\tau^*_i-\tau_i)dV'\Big]v_i$$
$$+\Big[\hat{P}_{ji}-d_j\Big(\sigma_i+\int_V \sigma^*_i dV'\Big)-d_{j,k}\Big(K_{ki}+\int_V K^*_{ki}dV'\Big)-\Big(K_{ik}+\int_V K^*_{ik}dV'\Big)d_{k,j}\Big]\omega_{ij}$$
$$-\Big(\varrho S+\varrho\eta+\int_V \eta^* dV'\Big)\frac{DT}{Dt}+\Big(T\hat{n}-\frac{q_i T_{,i}}{T}\Big)\Big\}dV \geqslant 0,$$

where the starred quantities ( )* indicate the conjugate functions obtained from the original functions by interchanging the primed and unprimed variables, i.e.

$$(4.17)\qquad [G(\mathbf{X}', \mathbf{X})]^* \equiv G(\mathbf{X}, \mathbf{X}').$$

This inequality must be valid for all arbitrary variations of independent constitutive variables. For the sake of simplicity, in the subsequent analysis we write

$$(4.18)\qquad \begin{aligned} &\sigma_{ij} = \sigma^e_{ij}+\sigma^D_{ij}, \quad \pi_{ij} = \pi^e_{ij}+\pi^D_{ij}, \quad \hat{P}_{ij} = \hat{P}^e_{ij}+p^D_{ij},\\ &\hat{f}_i = \hat{f}^e_i+\hat{f}^D_i, \quad g_i = g^e_i+g^D_i, \end{aligned}$$

where

$$(4.19)\qquad \sigma^e_{ij} \equiv -\Big(\pi+\int_V \pi^* dV'\Big)\delta_{ij}-\pi^e_{ik}d_{k,j},$$

$$(4.20)\qquad \pi^e_{ij} \equiv K_{ij}+\int_V K_{ij}dV',$$

$$(4.21)\qquad \hat{P}^e_{ij} \equiv d_i g^e_j+d_{i,k}\pi^e_{kj}+\pi^e_{jk}d_{k,i},$$

$$(4.22)\qquad \hat{f}^e_i \equiv \int_V (\tau^*_i-\tau'_i)dV',$$

$$(4.23)\qquad g^e_i \equiv \sigma_i+\int_V \sigma^*_i dV'$$

are respectively the elastic (or reversible) part of the stress tensor, the director stress tensor, the rate of nonlocal generalized momentum residual, nonlocal body force residual, and the intrinsic director body force. The attached letter (D) is used to indicate the irreversible parts of the corresponding quantities.

Substituting Eqs. (4.19)–(4.23) into Eq. (4.16) and noting that the resulting inequality is linear in $\frac{DT}{Dt}$ and N_{ij}, we obtain

$$(4.25)\qquad S = -\eta-\frac{1}{\varrho}\int_V \eta^* dV', \quad \pi^D_{ij} \equiv 0.$$

The remaining parts of the inequality (4.16) take the following form:

$$(4.25)' \qquad \int\limits_V \left(\sigma^D_{ji}D_{ij}+\hat{f}^D_i v_i+g^D_i N_i+\hat{P}^D_{ji}\omega_{ij}+T\hat{n}-\frac{q_i T_{,i}}{T}\right)dV \geqslant 0.$$

If the heat conduction is neglected, i.e. $T = T(t)$ for all $X' \in B$, from Eq. (4.25) one has

$$(4.26) \qquad q_i \equiv 0,$$

$$(4.27) \qquad \int\limits_V (\sigma^D_{ji}D_{ij}+\hat{f}^D_i v_i+g^D_i N_i+\hat{P}^D_{ji}\omega_{ij})\,dV \geqslant 0.$$

The irreversible parts are subjected to the following restrictions:

$$(4.28) \qquad \sigma^D_{ij} \equiv \hat{f}^D_i \equiv g^D_i \equiv \hat{P}^D_{ij} \equiv 0, \quad \text{when} \quad D_{ij} \equiv N_{ij} \equiv N_i \equiv v_i \equiv \omega_{ij} \equiv 0.$$

Furthermore, as can be seen from Eq. (4.25), when $D_{ij} \equiv N_{ij} \equiv N_i \equiv 0$, i.e. the body undergoes rigid body motions, we must have

$$(4.29) \qquad \int\limits_V \hat{f}^D_i dV = 0, \quad \int\limits_V (\hat{R}^D_{ij}-\hat{R}^D_{ij})\,dV = 0.$$

Since $\hat{R}^e_{ij}$ and $\hat{f}^e_i$ are independent of D_{ij}, N_{ij} and N_i, from Eq. (3.6), (3.13) and (4.29) it follows that

$$(4.30) \qquad \int\limits_V \hat{f}^e_i dV = 0, \quad \int\limits_V (\hat{R}^e_{ij}-R^e_{ji})\,dV = 0,$$

where

$$(4.31) \qquad \hat{R}^e_{ij} = x_i \int\limits_V (\tau^*_j-\tau'_j)\,dV'+d_i g^e_j+d_{i,k}\pi^e_{kj}+\pi^e_{jk}d_{k,i}.$$

If Eqs. (4.10) and (4.18)–(4.26) are substituted into Eq. (3.22), this yields

$$(4.32) \qquad \sigma^D_{ji}D_{ij}+\hat{f}^D_i v_i+g^D_i N_i+\hat{P}^D_{ji}\omega_{ij}\hat{\varepsilon}+E \geqslant 0,$$

where

$$(4.33) \qquad \hat{\varepsilon} \equiv T\hat{n}+\hat{e}, \qquad \int\limits_V \hat{\varepsilon}\,dV = 0;$$

$$(4.34) \qquad E \equiv \int\limits_V \Big[(-\pi^* v_{k,k}+\pi' v'_{k,k})+(-K^*_{jk}d_{k,i}D_{ij}+K'_{jk}d'_{ki}D'_{ij})+(K^*_{ji}N_{ij}-K'_{ji}N'_{ij})$$
$$+(\tau^*_i v_i-\tau'_i v'_i)+(\sigma^*_i N_i-\sigma'_i N'_i)+(\sigma^*_i d_j\omega_{ij}-\sigma^*_i d'_j\omega'_{ij})+(K^*_{ki}d_{j,k}\omega_{ij}$$
$$-K'_{ki}d'_{j,k}\omega'_{ij})+(K^*_{ik}d_{kj}\omega_{ij}-K'_{ik}d'_{k,j}\omega'_{ij})+\left(\eta^*\frac{DT}{Dt}-\eta'\frac{DT'}{Dt}\right)\Big]dV'.$$

It is seen from Eq. (4.34) that E has the property of nonlocal residuals of a scalar nature, viz. the integral of E over the volume of the body vanishes. The requirement that E should be form invariant under rigid body motions implies that the tensor

$$(4.35) \qquad \Sigma_{ij} \equiv \int\limits_V (\tau^*_{ij}-\tau'_{ij})\,dV'$$

must be symmetric with respect to its indices (i,j). At this point it might be pertinent to introduce

$$\hat{\varepsilon} = \hat{\varepsilon}^D - E, \tag{4.36}$$

where $\hat{\varepsilon}^D$ satisfies the condition (3.17), and will be termed as the *irreversible* part of the nonlocal energy residual. Substituting Eq. (4.36) into Eq. (4.32), there follows the entropy inequality governing the irreversible part of the constitutive relations

$$\sigma^D_{ji} D_{ji} + \hat{f}^D_i v_i + g^D_i N_i + \hat{P}^D_{ji}\omega_{ij} + \hat{\varepsilon}^D \geqslant 0. \tag{4.37}$$

Furthermore, assuming that $\hat{\varepsilon}^D$ is objective and requiring that the entropy inequality is form invariant under all arbitrary motions of the spatial frame of reference, we obtain

$$\hat{f}^D_i \equiv 0, \quad \hat{P}^D_{ij} \equiv 0. \tag{4.38}$$

Hence Eq. (4.37) becomes

$$\sigma^D_{ji} D_{ij} + g^D_i N_i + \hat{\varepsilon}^D \geqslant 0. \tag{4.39}$$

Since we have assumed that the temperature is space-uniform, without loss in generality, we may set $\hat{\varepsilon}^D \equiv 0$. Then the simplified form of entropy inequality takes the following form:

$$\sigma^D_{ji} D_{ij} + g^D_i N_i \geqslant 0. \tag{4.40}$$

From this inequality we may conclude that

$$\sigma^D_{ij} \equiv g^D_i \equiv 0, \quad \text{when} \quad D_{ij} \equiv N_i \equiv 0. \tag{4.41}$$

These restrictions will be used as we formulate the linear constitutive equations.

5. Incompressible liquid with director of constant magnitude

In the liquid crystal theory it is generally assumed that the fluid is incompressible and the director rotates about its center of gravity without changing its magnitude. When the director is constrained to be of fixed length, it is convenient to absorb its magnitude into other fluid properties and to consider the vector d_i as a unit vector. In this particular case the following relations are valid:

$$\begin{aligned} d_i d_i &= 1, \quad d_i d_{i,j} = 0, \\ d_i N_i &= 0, \quad d_i N_{ij} + N_i d_{i,j} = 0, \end{aligned} \tag{5.1}$$

$$v_{i,i} = 0 \quad \text{(incompressibility)}. \tag{5.2}$$

Due to the relations $(5.1)_1$ and (5.2), the free energy F can be replaced by

$$F \to F + \frac{p}{\varrho_0^2} + (\varrho - \varrho_0) + \frac{\gamma}{\varrho}(d_i d_i - 1) + \frac{\beta_j}{\varrho_0} d_i d_{i,j}, \tag{5.3}$$

where ϱ_0 is the constant fluid mass density, p is the hydrostatic pressure, and γ and β_i are the arbitrary scalar and vector. These quantities are to be determined from the field equations and the boundary conditions.

If Eq. (5.3) is substituted into Eq. (4.6) and noting that F is independent of ϱ and the magnitude of d_i, after some manipulations we obtain

$$\pi^e_{ij} = \beta_i d_i + \hat{\pi}^e_{ij}, \quad \hat{\pi}^e_{ij} \equiv K_{ij} + \int_V K^*_{ij} dV, \tag{5.4}$$

$$\sigma^e_{ij} = -p\delta_{ij} - \hat{\pi}^e_{ik} d_{k,j}, \tag{5.5}$$

$$g^e_i = \gamma d_i + \beta_j d_{i,j} + \hat{g}^e_i, \quad \hat{g}^i_e \equiv \sigma_i + \int_V \sigma^*_i dV', \tag{5.6}$$

$$\hat{f}^e_i = \int_V (\tau^*_i - \tau'_i) dV', \tag{5.7}$$

$$\hat{P}^e_{ij} = d_i \hat{g}^e_j + d_{i,k} \hat{\pi}^e_{kj} + \hat{\pi}^e_{jk} d_{k,i}. \tag{5.8}$$

Here p, γ and β_i are, in general, functions of $\mathbf{x}$ but functionals in $\mathbf{x}'$. However, to simplify the problem, in the rest of the study, we will assume that these quantities are only functions of $\mathbf{x}$ and t.

Thus far we have kept the generality of the problem and assumed that F is an arbitrary but continuous functional in d'_i, $d'_{i,j}$ and r'_i (assuming that $\varrho = T =$ constant). For the purpose of simplifying the problem further we assume that F is given by

$$\varrho_0 F = \int_V \psi(d_i, d_{i,j}; d'_i, d'_{i,j}, r'_i) dV', \tag{5.9}$$

where ψ is a continuous function of the primed quantities as well as unprimed ones. Introducing Eq. (5.9) into Eqs. (5.4)–(5.8) we obtain

$$\hat{\pi}^e_{ij} = \int_V \frac{\partial \Sigma}{\partial d_{j,i}} dV', \quad \pi^e_{ij} = \hat{\pi}^e_{ij} + \beta_i d_j, \tag{5.10}$$

$$\sigma^e_{ij} = -p\delta_{ij} - \hat{\pi}^e_{ik} d_{k,j}, \tag{5.11}$$

$$g^e_i = \gamma d_i + \beta_j d_{i,j} + \hat{g}^e_i, \quad \hat{g}^e_i \equiv \int_V \frac{\partial \Sigma}{\partial d_i} dV', \tag{5.12}$$

$$\hat{f}^i_e = \int_V \frac{\partial \Sigma}{\partial r'_i} dV', \tag{5.13}$$

$$\hat{P}^e_{ij} = d_i \hat{g}^e_j d_{i,k} + \hat{\pi}^e_{kj} + \hat{\pi}^e_{jk} d_{k,i}, \tag{5.14}$$

where

$$\Sigma \equiv \psi + \psi^*. \tag{5.15}$$

Here we note that, because of the particular form we have selected for the free energy functional, the Fréchet gradients of F are simply replaced by the ordinary partial derivatives of ψ.

6. Quasi-linear theory of nonlocal nematic liquid crystals

In this section we will develop a set of constitutive equations that is linear in $d_{i,j}$, D'_{ij}, Ω'_{ij} and N'_i but an arbitrary function of d_i, d'_i and r'_i. To this end we assume that the function Σ may be expressed as

$$\Sigma = \Sigma_0(d_i, d'_i, s') + C'_{ijkl} d_{i,j} d'_{k,l}, \tag{6.1}$$

where

$$s' = |\mathbf{r}'|^2, \quad C'_{ijkl} = C'_{ijkl}(d_m, d'_n, s'). \tag{6.2}$$

In general, the functions Σ_0 and C'_{ijkl} depend upon r'_i rather than on s', but for isotropic materials such an assumption is permissible on the grounds that the orientational dependence of liquid crystals is characterized by the vector d_i (or d'_i), not by r'_i. Furthermore, as stated by OSEEN [2] and FRANK [4], only the splay, bending and twist (in general the gradient of d_i) of far-distant thread-like particles, but not the relative rotation of them, affect the behaviour of a particle considered. This statement is tantamount to the assumption that Σ_0 and C'_{ijkl} are of the form

$$\Sigma_0 \Sigma_0(d_i, s'), \quad C'_{ijkl} = C'_{ijkl}(d'_i, s'). \tag{6.3}$$

Introducing Eq. (6.1) into Eqs. (5.10)–(5.14), the constitutive equations read

$$\hat{\pi}^e_{ij} = \int_V C'_{jikl} d'_{k,l} dV', \tag{6.4}$$

$$\sigma^e_{ij} = -p\delta_{ij} - \hat{\pi}^e_{ik} d_{k,j}, \tag{6.5}$$

$$\hat{g}^e_i = \int_V \frac{\partial \Sigma_0}{\partial d_i} dV', \tag{6.6}$$

$$\hat{f}^e_i = -2 \int_V \left[\frac{\partial \Sigma}{\partial s'} + \frac{\partial C'_{mnkl}}{\partial s'} d_{m,n} d'_{k,l} \right] r'_i dV', \tag{6.7}$$

$$\hat{P}^e_{ij} = d_i \hat{g}^e_j + d_{i,k} + \hat{\pi}^e_{kj} + \hat{\pi}^e_{jk} d_{k,i}. \tag{6.8}$$

From Eqs. (4.40) and (4.41) the reversible part of the constitutive equation reads

$$\sigma^D_{ij} = \int_V (\Sigma'_{ijkl} D'_{kl} + F'_{ijk} N'_k) dV', \quad \Sigma'_{ijkl} = \Sigma'_{ijlk}, \tag{6.9}$$

$$g^D_i = \int_V (\alpha'^0_{ij} N'_j + \alpha'^1_{ikl} D'_{kl}) dV', \qquad \alpha'^1_{ilk} = \alpha'^1_{ilk}, \tag{6.10}$$

where Σ'_{ijkl}, F'_{ijk}, α'^0_{ij} and α'^1_{ikl} are functions of d'_i, and s'.

The constitutive equations are further restricted by the axiom of objectivity. This axiom states that the constitutive functionals of nematic liquid crystals are form invariant under a full group of orthogonal transformation of the spatial frame of references, e.g.

$$\sigma^*_{ij} = Q_{im} Q_{jm} \sigma_{mn}, \tag{6.11}$$

where

$$Q_{im}Q_{jm} = Q_{mi}Q_{mj} = \delta_{ij}, \quad \det \mathbf{Q} = \mp 1.$$

Similar forms are valid for other dependent variables provided that their vectorial and scalar character are taken into account.

Applying this principle to Eq. (6.1)–(6.10) and noting that $d_i d_i = 1$, the following appropriate relations are obtained:

(6.12) $\Sigma_0 = \Sigma_0(s'),$

(6.13) $C'_{ijkl} \equiv \alpha'_0 \delta_{ij}\delta_{kl} + \alpha'_1 \delta_{ik}\delta_{jl} + \alpha'_2 \delta_{il}\delta_{jk} + \alpha'_3 \delta_{ik} d'_j d'_l + \alpha'_4 \delta_{jk} d'_i d'_l + \alpha'_5 \delta_{kl} d_i d_j,$

(6.14) $$\Sigma'_{ijkl} \equiv \frac{1}{2}\beta'_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{1}{2}\beta'_2(\delta_{ik} d'_j d'_l + \delta_{il} d'_j d'_k) + \frac{1}{2}\beta'_3(\delta_{jk} d'_i d'_l + \delta_{jl} d'_i d'_k) + \beta'_4 d'_i d'_j d'_k d'_l,$$

(6.15) $F'_{ilk} = f'_1 \delta_{ik} d'_j + f'_2 \delta_{jk} d'_i,$

(6.16) $\alpha'_{ijk} = \frac{1}{2}\gamma'_0(\delta_{ij} d'_k + \delta_{ik} d'_j),$

(6.17) $\alpha'^0_{ij} = c'_0 \delta_{ij},$

where $\alpha'_0, \alpha'_1, \ldots, c'_0$ and c'_1 are all functions of $s' = |\mathbf{r}'|^2$. Here one must note that F when $d_i \to D_i$ (where D_i is the initial value of d_i), $d_{i,j} \to 0$, $D_{ij} \to 0$, $N_{ij} \to 0$, and $N_i \to 0$, the constitutive variables $\pi_{ij}, \sigma_{ij}, g_i, \hat{P}_{ij}$ and $\hat{f}_i$ must vanish. From Eqs. (6.4)–(6.8) this argument implies that

(6.19) $$\int_V \frac{\partial \Sigma_0}{\partial s'} r'_i dV' = 0.$$

This is possible if and only if

(6.19)′ $$\Sigma_0(s') = \Sigma_0 = \text{constant}.$$

Thus, substituting Eqs. (6.12)–(6.17) into Eqs. (6.1)–(6.10), the linearized form of the constitutive relations follows:

(6.20) $$\Sigma = \Sigma_0 + \alpha'_0 d_{r,r} d'_{m,m} + \alpha'_1 d_{k,l} d'_{k,l} + \alpha'_2 d_{l,k} d'_{k,l} + \alpha'_3 d_{k,l} d'_{k,} d'_j d'_l + \alpha'_4 d_{l,k} d'_{k,l} d'_i d'_l + \alpha'_5 d_{l,j} d'_i d'_j d'_{m,m},$$

(6.21) $$\hat{\pi}^e_{ij} = \int_V (\alpha'_0 d'_{m,m}\delta_{ij} + \alpha'_1 d'_{j,l} + \alpha'_2 d'_{i,j} + \alpha'_3 d'_{i,l} d'_j d'_l + \alpha'_4 d'_{j,l} d'_i d'_l + \alpha'_5 d'_{m,m} d'_i d'_j) dV',$$

(6.22) $\sigma^e_{ij} = -p\delta_{ij},$

(6.23) $\hat{g}^e_i = 0, \quad \hat{g}_i = 0, \quad \hat{P}^e_{ij} = 0,$

(6.24) $$\sigma^D_{ij} = \int_V (\beta'_1 D'_{ij} + \beta'_2 D'_{il} d'_i d'_j + \beta'_3 d'_i D'_{jl} d'_l + \beta'_4 d'_i d'_j D'_{kl} d'_k d'_l + f_1 N'_i d'_j + f'_2 d'_i N'_j) dV',$$

(6.25) $$g^D_i = \int_V (c'_0 N'_i + \gamma'_0 D'_{ik} d'_k) dV'.$$

It might easily be seen that the condition $(4.30)_2$ is spontaneously satisfied by Eq. $(6.23)_3$. Moreover, Eqs. (6.24) and (6.25) are further restricted by the principle of objectivity and the condition (3.13). However, because of the simplifying assumptions we have made at the outset of linear constitutive relations, the condition (3.13) is not exactly satisfied. In order to fulfill this requirement approximately, we must have

$$c'_0 f'_2 - f'_1, \quad \gamma'_0 \equiv \beta'_3 - \beta'_2. \tag{6.26}$$

The constitutive equations (6.24) and (6.25) are further restricted by the principle of entropy inequality. Substituting these relations into Eq. (4.40), we obtain

$$\int_V (\beta'_1 D'_{ij} D_{ij} + \beta'_2 D'_{ji} d'_l d'_i D_{ij} + \beta'_3 d'_j D'_{il} d'_l D_{ij} + \beta'_4 d'_i d'_j D_{ij} D'_{kl} d'_k d'_l + f'_1 N'_j d'_i D_{ij} + f'_2 d'_j N'_i D'_{ij} + C'_0 N'_i N_i + \gamma'_0 D_{ik} d'_k N'_i) dV' \geqslant 0. \tag{6.27}$$

This inequality must be valid *for all arbitrary variations* of D'_{ij}, D_{ij}, N_i and N'_i.

Before we proposed further it might be useful to study a special case, namely, the local theory of nematic liquid crystals.

Local theory of nematic liquid crystals

To obtain the local constitutive relations of nematic liquid crystals it will be sufficient to express the functions characterizing the material properties as

$$a'(s') \equiv a\delta(x'-x), \tag{6.28}$$

where a is a constant characterizing the material properties and $\delta(\mathbf{x}'-\mathbf{x})$ is a Dirac delta function.

Introducing Eq. (6.28) into Eqs. (6.20)–(6.25) the following linearized local constitutive equations are obtained:

$$F = \alpha_0 d_{r,r} d_{m,m} + \alpha_1 d_{k,l} d_{k,l} + \alpha_2 d_{k,} d_{l,k} + \alpha_3 d_{k,j} d_{k,l} d_l d_j, \tag{6.29}$$

$$\hat{\pi}^e_{ij} = \alpha_0 d_{m,m} \delta_{ij} + \alpha_1 d_{j,i} + \alpha_2 d_{i,j} + \alpha_3 d_{i,l} d_l d_j, \tag{6.30}$$

$$\sigma^e_{ij} = -p\delta_{ij}, \tag{6.31}$$

$$\sigma^D_{ij} = \beta_1 D_{ij} + \beta_2 D_{il} d_l d_j + \beta_3 d_l D_{jl} d_i + \beta_4 d_i d_j D_{kl} d_k d_l + f_1 N_i d_j + f_2 d_i N_j, \tag{6.32}$$

$$g^D_i = c_0 N_i + \gamma_0 D_{ik} d_k. \tag{6.33}$$

These equations are exactly the same as those given by LESLIE [5]. This consistency provides us with a check-up about the completeness of the theory.

Field equations

The field equations governing the nonlocal mechanical behaviour of nematic liquid crystals may be obtained by substituting Eqs. (6.21)–(6.25) into the balance laws $(3.5)_2$ and (3.8). If this is done, the following integro-differential equations are obtained:

$$-p_{,i} + \int_V [\beta'_1 D'_{ji,j} + \beta'_2 (D'_{jl} d' d'_i)_{,j} + \beta'_3 (d'_j D'_{il} d'_l)_{,j} + \beta'_4 (d'_j d'_i D'_{kl} d'_k d'_l)_{,j} + f_1 (N'_j d'_i)_{,j} + f'_2 (d'_j N'_i)_{,j}] dV' - \int_A t'_{ji} n'_j dS + \varrho \left(f_i - \frac{Dv_i}{Dt} \right) = 0, \tag{6.34}$$

$$(6.35)\quad (\beta_{j,j}-\gamma)d_i+\int_V [\alpha_0' d'_{m,mi}+\alpha_1' d'_{i,jj}+\alpha_2' d'_{j,ji}+\alpha_3'(d'_{j,l}d'_l d'_i)_{,j}+\alpha_4'(d'_{i,l}d'_l d'_j)_{,j}$$
$$+\alpha_5'(d'_{m,m}d'_j d'_i)_{,j}-c_0' N_i'-\gamma_0 D'_{ik}d'_k]dV'-\int_A P'_{ji}n'_k dS+\varrho\left(G_i-j\frac{Dw_i}{Dt}\right)=0,$$

where t'_{ij} and P'_{ij} are defined by

$$(6.36)\quad t'_{ij}=\beta_1' D'_{ij}+\beta_2' D'_{il}d'_l d'_j+\beta_3' d'_i D'_{jl}d'_l+\beta_4' d'_i d'_j D'_{kl}d'_k d'_l+f_1 N'_i d'_j+f_2' d'_i N'_j,$$

$$(6.37)\quad P'_{ij}=\alpha_0 d'_{m,m}\delta_{ij}+\alpha_1' d'_{j,i}+\alpha_2' d'_{i,j}+\alpha_3' d'_{i,l}d'_j d'_l+\alpha_4' d'_{j,l}d'_l d'_i+\alpha_5' d'_{m,m}d'_i d'_j.$$

Here, in obtaining Eqs. (6.34) and (6.35) we have used the following relation:

$$\frac{\partial a'_m}{\partial x_j}=-\frac{\partial a'_m}{\partial x'_j},\qquad a'_m\equiv(\alpha'_i,\beta'_j\,...)$$

and the generalized Green–Gauss theorem to convert some volume integrals into surface integrals. It is interesting to note that the field equations (6.34) and (6.35) contain some surface integral term which represents the effect of surface tension and has no counterpart in the local theory of nematic liquid crystals. These field equations and the constitutive relations provide us with sufficient relations to describe the nonlocal mechanical behaviour of liquid crystals completely. In the next section we will study the solution of the shear flow problem in nematic liquid crystals.

We finally remark that the influence functions α'_i, β'_i and γ'_i are restricted by the axiom of attenuating neighbourhood (cf. ERINGEN [24]). This axiom is a strong continuity requirement arising from the fact that the nonlocal effects diminish rapidly with the distance, e.g. the intermolecular forces are known to die out with distance rapidly. This can be achieved by taking

$$(6.38)\qquad \lim_{\sqrt{s'}\to 0} s^{1/2+\varepsilon}[\alpha'_i(s'),\beta'_j(s'),\gamma_k(s')]=0,\qquad \varepsilon>0.$$

In practice we may select a functional form such as

$$(6.39)\qquad (\alpha'_i,\beta'_i,\gamma'_i)\equiv(\alpha^0_i,\beta^0_i,\gamma^0_i)\frac{1}{\varkappa}\exp(-s'/\varkappa).$$

Here we note that the parameter $\varkappa$ represents the range of nonlocality. It can easily be shown that as $\varkappa\to 0$ the following relation is valid:

$$(6.40)\qquad \lim_{\varkappa\to 0}\left\{\frac{1}{\varkappa}\exp(-s'/\varkappa)\right\}\to\delta(s').$$

Using this property of influence functions in the limit of $\varkappa\to 0$, one can easily obtain the classical local theory of liquid crystals.

7. Shear flow

The purpose of this section is to investigate the shear flow of an incompressible liquid crystal of the nematic type. Assuming that the body forces vanish, the velocity and director fields may be expressed as

$$(7.1)\qquad \begin{aligned} &v_x=u(y), && v_y=v_z=0,\\ &d_x=\cos\theta(y), && d_y=\sin\theta(y), \quad d_z=0.\end{aligned}$$

The statement of the balance equations (6.21) to (6.25) in the Cartesian coordinate system reads

$$\frac{\partial \sigma_{yx}}{\partial y} - \frac{\partial p}{\partial x} = 0, \quad \frac{\partial}{\partial y}(\sigma_{yy}) = 0, \quad \frac{\partial p}{\partial z} = 0, \tag{7.2}$$

$$\frac{\partial}{\partial y}\pi_{yx} + g_x = 0, \quad \frac{\partial}{\partial y}\pi_{yy} + g_y = 0. \tag{7.3}$$

From Eqs. (7.2) it follows that

$$\sigma_{yx} = ay + c, \quad p = p_0 + ax + \bar{\sigma}_{yy}, \tag{7.4}$$

where a, c and p_0 are arbitrary constants and $\bar{\sigma}_{yy}$ is the pressureless part of the corresponding stress components which can be derived from constitutive relations. Here we note that the constant a stands for the magnitude of the pressure gradient along the x axis. Assuming that, for the time being, the pressure gradient is zero, that is $a \equiv 0$, Eqs. (7.4) become

$$\sigma_{yx} = c, \quad p = p_0 + \bar{\sigma}_{yy}. \tag{7.5}$$

From Eqs. (6.24) and $(7.5)_1$, we obtain

$$\int_0^h K(y'-y; y') \frac{du'}{dy'} dy' = c, \tag{7.6}$$

where $\bar{\beta}'_1, \bar{\beta}'_2, \ldots, \bar{f}'_1, \bar{f}'_2$ and $K(y'-y, y)$ are defined by

$$\begin{aligned} &(\bar{\beta}'_i, \bar{f}'_i)(y'-y) \equiv \int_{-\infty}^{\infty} (\beta'_i, f'_j) dx' dy', \\ &K(y'-y; y') \equiv \frac{1}{2} [\bar{\beta}'_1 + (\bar{\beta}'_2 + \bar{f}'_1)\cos^2\theta'(y') \\ &\qquad + (\bar{\beta}'_3 + \bar{f}'_2)\sin^2\theta'(y')] + \bar{\beta}'_4 \sin^2\theta'(y')\cos^2\theta(y') \end{aligned} \tag{7.7}$$

and h represents the depth of the upper surface of the liquid body.

After some elimination Eqs. (7.3) reduce to

$$\int_0^h \left\{ H(\theta'; \theta) \frac{d^2\theta'}{dy'^2} + \frac{\partial H(\theta'; \theta)}{\partial \theta'} \left(\frac{d\theta'}{dy'}\right)^2 - \frac{1}{2}\frac{du'}{dy'} [c'_0 \cos(\theta' - \theta) + \gamma'_0 \cos(\theta' + \theta)] \right\} dy' = 0, \tag{7.8}$$

where $H(\theta', \theta)$ is defined by

$$\begin{aligned} H'(\theta'; \theta) \equiv {} & (\bar{\alpha}'_0 + \bar{\alpha}'_2)\cos\theta' \cos\theta + \bar{\alpha}'_1 \cos(\theta' - \theta) \\ & + (\bar{\alpha}'_3 + \bar{\alpha}'_5)\cos\theta' \sin\theta' \sin(\theta' - \theta) + \bar{\alpha}'_4 \sin^2\theta' \cos(\theta' - \theta). \end{aligned} \tag{7.9}$$

Here $\bar{\alpha}'_1, \bar{\alpha}'_2, \ldots, \bar{\alpha}'_5$ are defined in a similar fashion to quantities in Eq. $(7.6)_1$.

Equations (7.6) and (7.8) give the complete set of integro-differential equations for determining the functions θ and u, provided that a set of properly posed boundary conditions is given.

Flow near a boundary

We look for a solution of the set (7.6) satisfying the boundary conditions

$$
\begin{aligned}
&u(0) = 0, \quad \theta(0) = \theta_1, \\
&\theta(y) \to \theta_0, \quad y \to \infty (h \to \infty),
\end{aligned}
\tag{7.10}
$$

where θ_0 and θ_1 are given constants having values between zero and 2π.

Selecting h to be infinite and integrating Eqs. (7.6) and (7.8) and noting the relation (6.39), we obtain

$$
H(\theta_1, \theta)\varkappa_1 + \int_0^\infty [c_0' \cos(\theta' - \theta) + \gamma_0' \cos(\theta' + \theta)] \frac{du'}{dy'} dy' = 0,
\tag{7.11}
$$

$$
\int_0^\infty G(y' - y; y') u(y') dy' = -c,
\tag{7.12}
$$

where

$$
\begin{aligned}
&G(y' - y; y') = \frac{\partial K(y' - y; y')}{\partial y'}, \\
&\varkappa_1 = 2 \left. \frac{d\theta}{dy} \right|_{y=0}
\end{aligned}
\tag{7.13}
$$

Here $\varkappa_1$ is another constant of integration and should be determined from the boundary conditions (7.10).

Equations (7.11) and (7.12) give two integro-differential equations for the determination of θ' and u'. As is seen, these equations are linear in u' but highly nonlinear in θ' and they cannot be solved by analytical means. A numerical technique must be made use of.

Flow between parallel plates

We now look for a solution of Eqs. (7.6) and (7.8) satisfying the boundary conditions

$$
u(-h) = 0, \quad u(h) = V, \quad \theta(-h) = \theta(h) = 0,
\tag{7.14}
$$

where h and V are constants. This boundary condition corresponds to a flow between two parallel plates at a constant distant $2h$ apart, one of which is at rest and the other moving with uniform velocity V. The orientation at the walls has been set equal to zero, so that the directors are parallel to the plates at the boundaries. As indicated by LESLIE [5], this is a possible boundary condition for the flow of liquid crystals of the nematic type. In view of the condition $(7.14)_3$, it is reasonable to assume that

$$
\theta(-y) = \theta(y), \quad \left. \frac{d\theta}{dy} \right|_{y=0} = 0.
\tag{7.15}
$$

In this case from Eqs. (7.2) the governing integro-differential equations read

$$(7.16)\qquad \int_{-h}^{h} K(y'-y; y')\frac{du'}{dy'}dy' = c,$$

$$(7.17)\qquad \int_{-h}^{h}\left\{H(\theta';\theta)\frac{d^2\theta'}{dy'^2}+\frac{\partial H(\theta';\theta)}{\partial\theta'}\left(\frac{d\theta'}{dy'}\right)^2-\frac{1}{2}\frac{du'}{dy'}[c'_0\cos(\theta'-\theta)+\gamma'_0\cos(\theta'+\theta)]\right\}dy' = 0.$$

Integration of these equations gives

$$(7.18)\qquad \int_{-h}^{h} G(y'-y; y')u'(y')dy' = K(h-y; h)V-c,$$

$$(7.19)\qquad \int_{-h}^{h}[c'_0\cos(\theta'-\theta)+\gamma'_0\cos(\theta'+\theta)]\frac{du'}{dy'}dy' = H_1(\theta)\varkappa_1,$$

where

$$(7.20)\qquad H_1(\theta) = (\overline{\alpha}'_0+\overline{\alpha}'_1+\overline{\alpha}'_2)|_{-h}^{h}\cos\theta$$

and $K_1 = \left.\dfrac{dy}{d\theta}\right|_{y=h}$ which must be determined from the conditions (7.15). These two relations again give two nonhomogeneous integro-differential equations for u' and θ'. It is extremely difficult to solve these equations by analytical means. A numerical technique must rather be used. To make the paper short we will not discuss the solution technique here. A separate paper is planned for the method of solving these integro-differential equations.

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