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Modified theory of viscoplasticity
Application to advanced flow and instability phenomena

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THE MAIN objective of the present paper is to formulate the theory of viscoplasticity which can be applicable in a study of the influence of strain rate effects on the instability of plastic flow. We propose to describe the interaction of the intrinsic plastic failure (plastic instability by necking) with the formation and propagation of cracks by assuming new constitutive equations for an elastic-viscoplastic material. The model of the material proposed takes into account imperfections. It is conjectured that a new term responsible for the description of imperfections in the evolution equations proposed can describe effects caused by the interaction of cracks at inclusions with the intrinsic plastic failure. This modified theory satisfies also the requirement that during the deformation process the response of a material thereby modelled becomes elastic-plastic for the assumed static value of the effective strain rate. The material functions and constants are determined on the basis of available experimental data obtained under the condition of dynamic loading for rate sensitive plastic materials. The initial-boundary value problems of straining cylindrical specimens with constant velocities $\dot{U}_1, \dot{U}_2, \ldots, \dot{U}_n$ are formulated. Numerical calculations are performed and discussed. New experimental investigations useful for determining of the imperfection functions are proposed. These imperfection functions play an important role in the description of instability phenomena and, in particular, during the necking process.

Głównym celem pracy jest sformułowanie teorii lepkoplastyczności, która mogłaby być zastosowana do zbadania wpływu efektów prędkości odkształcenia na niestateczność plastycznego płynięcia. Zaproponowano opis współdziałania plastycznego zniszczenia (plastyczna lokalizacja lub niestateczność podczas szyjkowania) z tworzeniem i rozprzestrzenianiem się szczelin, przyjmując nowe równania konstytutywne dla materiału sprężysto-lepkoplastycznego. Proponowany model materiału uwzględnia imperfekcje. Przypuszcza się, że nowy wyraz opisujący imperfekcje w przyjętych równaniach ewolucji może opisać efekty spowodowane oddziaływaniem szczelin na wtrąceniach z plastyczną lokalizacją. Zmodyfikowana teoria spełnia również warunek, że podczas procesu deformacji reakcja modelowanego materiału staje się sprężysto-plastyczna dla przyjętej statycznej wartości efektywnej prędkości odkształcenia. Na podstawie dostępnych rezultatów doświadczalnych otrzymanych w warunkach dynamicznego obciążenia dla materiałów wrażliwych na prędkość odkształcenia określono funkcje i stałe materiałowe. Sformułowano problemy początkowo-brzegowe odkształcania walcowych próbek ze stałymi prędkościami $\dot{U}_1, \dot{U}_2, \ldots, \dot{U}_n$. Otrzymano i przedyskutowano rezultaty numeryczne. Zaproponowano nowe badania experymentalne pożyteczne do określenia funkcji opisujących imperfekcje w materiale. Funkcje te odgrywają podstawową rolę w opisie zjawisk niestateczności, a w szczególności przy opisie procesu szyjkowania.

Главной целью работы является формулировка теории вязкопластичности, которая могла бы быть применена для исследования влияния эффектов скорости деформации на неустойчивость пластического течения. Предложено описание взаимодействия пластического разрушения (пластическая локализация или неустойчивость во время образования горловины) с образованием и распространением щелей, принимая новые определяющие уравнения для упруго-вязкопластического материала. Предложенная модель материала учитывает имперфекции. Предполагается, что новый член, описывающий имперфекции в принятых уравнениях эволюции, может описывать эффекты, вызванные взаимодействием щелей на включениях с пластической локализацией. Модифицированная теория удовлетворяет тоже условию, что во время процесса деформации реакция моделированного материала становится упруго-пластической для принятого статического значения эффективной скорости деформации. На основе доступных экспериментальных результатов, полученных в условиях динамического нагружения для материалов чувствительных на скорость деформации, определены функции и материальные постоянные. Сформулированы начально-граничные задачи деформации цилиндрических образцов с постоянными скоростями $\dot{U}_1, \dot{U}_2, \ldots, \dot{U}_n$. Получены и обсуждены численные результаты. Предложены новые экспериментальные исследования полезные для определения функций, описывающих имперфекции в материале. Эти функции играют основную роль в описании явлений неустойчивости, в частности при описании процесса образования горловины.

1. Introduction

MANY recent theoretical and experimental investigations have been focused on the instability phenomena of plastic flow. The intrinsic plastic failure or the localization of plastic flow may be treated as a prelude to fracture initiation and therefore is a matter of great interest.

It is very well recognized and confirmed experimentally that there are two main modes of localization of plastic deformations, namely necking and the localization in the direction of pure shear(1).

A comprehensive consideration of localization of plastic deformation into a shear band as an instability of plastic flow and a precursor to rupture has been presented by J. R. RICE [30].

A critical review of the theories of strain localization with reference to bifurcation behaviour and the growth of initial imperfections has been given by A. NEEDLEMAN and J. R. RICE [22]. These authors have shown that the oneset of localization does depend critically on the assumed constitutive law. Their analysis has been based on the classical rate independent elastic-plastic equations and departures from this idealized model that include the effect of (i) yield surface vertices, (ii) deviations from plastic "normality" and (iii) the dilatational plastic flow due to the nucleation and growth of voids.

Experimental investigations have shown that the intrinsic plastic failure (necking or instability in the direction of pure shear) is strongly dependent upon the strain rate effects. P. J. WRAY [31] has demonstrated that at high strain rates it is cracks at inclusions that interact with the intrinsic plastic failure (2).

The main objective of the present paper is to study the influence of strain rate effects of the necking phenomenon(3). We propose to describe the interaction of the intrinsic plastic failure (plastic instability by necking) with the formation and propagation of cracks by assuming new constitutive equations for an elastic-viscoplastic material. The model of the material proposed takes into account imperfections. It is conjectured that a new term responsible for the description of imperfections in the evolution equations proposed can describe effects caused by the interaction of cracks at inclusions with the intrinsic plastic failure.

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(1) A second basic mode of localization has been investigated experimentally by A. K. CHAKRABARTI and J. W. SPRETNAK [5]. They have examined two aspects of the phenomenon of plastic instability in the direction of pure shear, namely that the condition od maximum in true flow stress is necessary for the localization of plastic deformation along characteristics, and that fracture is initiated and propagates along characteristics.

(2) For a thorough discussion of the experimental results see Section 2.1.

(3) This problem has been the subject of consideration in the recent papers by O. T. BRUHNS [2] under the condition of rigid-viscoplastic approximation, by J. W. HUTCHINSON and K. W. NEALE [14] and by G. K. GHOSH [10, 11] for the simplified one-dimensional case of loading.

It is a very well recognized fact that the most probable mode of localization to occur is that of necking which is usually connected with the fluctuations in the cross-sectional area. It is supposed that this leads to a maximum load criterion for the instability condition (cf. A. K. CHAKRABARTI and J. W. SPRETNAK [5]).

A comprehensive numerical analysis of necking in circular cylindrical bars of an elastic-plastic material has been presented by A. NEEDLEMAN [21]. The latter formulated two boundary-value problems for a circular cylindrical bar in uniaxial tension(4). In the first of them the ends of the specimen are assumed to be cemented to rigid grips, while in the second they remain shear free. In both cases the bar is strained parallel to its main axis with a constant velocity $\dot{U}$.

This velocity $\dot{U}$ in the numerical computations is assumed to change from $0.04 \cdot 0.0072 L_0$ cm/s to $0.9 \cdot 0.0072\ L_0$ cm/s. So, the mean strain rate varies from $2.88 \cdot 10^{-4} s^{-1}$ to $6.4 \cdot 10^{-3} s^{-1}$ and is only of a statical nature.

Basing on the analysis of the results obtained by A. NEEDLEMAN [21], we observed that the investigation of strain rate effects on the necking phenomenon could be performed by a solution of a sequence of similar boundary-value problems (to that posed by Needleman) under the assumption that the material of a bar has elastic-viscoplastic properties with imperfections included and the specimens are strained with the constant velocities $\dot{U}_1, \dot{U}_2, \dot{U}_3 \ldots, \dot{U}_n$, respectively.

The spectrum of velocities $\dot{U}_1, \dot{U}_2, \dot{U}_3, \ldots, \dot{U}_n$ has to be assumed such that it causes changes of the mean strain rate in a sufficiently large range.

Particular attention is also devoted to the modification of the constitutive equations for an elastic-viscoplastic material to satisfy the requirement that during the deformation process in which the effective strain rate is equal to the static value (e.g., the second invariant of the strain rate tensor $I_2 = I_2^s$) the response of a material becomes elastic-plastic.

For such modified constitutive equations we can obtain the NEEDLEMAN results as a limit case of our processes under the assumption that the velocity $\dot{U}_1 = \dot{U}_{static} = \dot{U}$ (assumed by NEEDLEMAN).

In Sect 3.2 it is shown how to determine the material functions and the material constants basing on available experimental data for rate sensitive plastic materials. However, this procedure does not shed light on the determination of the functions responsible for the description of imperfections. Hence the additional purpose of this paper is to show what kind of experimental investigations are needed to obtain data which can provide a reasonable basis for determing of the new functions.

2. Analysis of instability and fracture modes

2.1. Discussion of experimental investigations

Let us start from the analysis of the failure of metals deformed under a unidirectional tensile mode of loading. It can be expected that in the temperature-strain rate spectrum

(4) A similar problem for a rectangular bar has been recently investigated by M. A. BURKE and W. D. NIX [3].

different failure modes may occur. This conjecture is justified by the fact that different mechanisms of plastic flow should lead to different fracture modes of a material.

A comprehensive discussion of the failure modes which can occur in a uniaxial tensile test of a polyphase, polycrystalline material at various strain rates at elevated temperature has been presented by P. J. WRAY [31]. The particular results in the temperature range (0.56–0.70) ϑ_m (where ϑ_m is the solidus temperature of 1370°C) and in the strain rate range $2.08\times10^{-5}-1.66\times10^{-1}$ s^{-1} for Type 316 stainless steel as a representative material have been given. In the strain rate-temperature plane four different modes of failure have

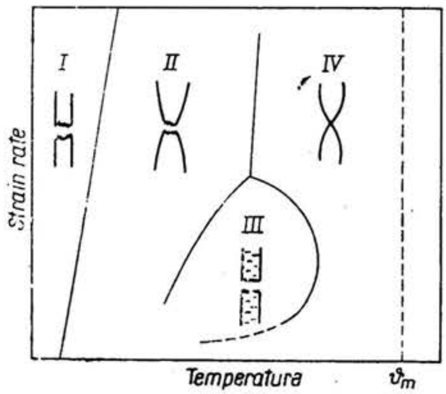

FIG. 1.

been observed, Fig. 1, Mode I being that of brittle fracture, Mode II ductile fracture, Mode III creep rupture and Mode IV the intrinsic plastic failure.

The general conclusions of the investigations performed by P. J. WRAY [31] are as follows. The failure of a polycrystalline material at elevated temperature may be treated as the interaction of the intrinsic plastic failure with the formation of cracks at second phase particles and grain interfaces. The intrinsic plastic failure is strongly dependent upon the strain rate in the temperature range from 0.56 ϑ_m to 0.70 ϑ_m (650 to 870°C). At low strain rates cracks at grain interfaces lead to fracture well before intrinsic failure can commence, and hence there is no interaction of intrinsic failure with the internal cracks. At intermediate strain rates cracks form at grain interfaces within the neck, and thereby interrupt the intrinsic plastic failure process, in some cases resulting in a shear fracture mode. At high strain rates it is cracks at inclusions that interact with the intrinsic plastic failure. For low temperature and very large strain rates brittle fracture mode is observed.

T. B. COX and J. R. LOW [6] have investigated the mechanisms of plastic fracture in high purity and commercial 18 Ni 200 grade maraging steels. They have found that there are three generally recognized stages of plastic fracture, namely void initiation, void growth and void coalescence. The most important microstructural features governing the plastic fracture of these alloys are the void nucleating, second phase particles. The sizes of non-metallic inclusions are an important aspect of the fracture resistance of these alloys since the investigation has demonstrated that void nucleation occurs more readily at the larger inclusions and that void growth proceeds more rapidly from the larger inclusions.

W. PAVINICH and R. RAJ [23] have performed a systematic experimental study of fracture in materials which contain hard second-phase particles. Creep fracture experiments have been carried out on copper-silica alloys in the strain rate range of 10^{-4} to 10^{-7} s^{-1} and in the temperature range of 400 to 800°C. Three modes of fracture have been observed, namely transgranular necking fracture at high strain rates, fracture by the propagation of intergranular cracks initiated at the surface at intermediate strain rates and intergranular fracture by grain boundary cavitation throughout the entire specimen cross-section at low strain rates. The transition between the fracture modes was shown to shift systematically with temperature, strain rate and the microstructure.

2.2. Theoretical propositions

A discussion of experimental investigations has shown that the fracture problem even for the one-dimensional case becomes very complex. However, we are convinced that the description of viscoplastic materials within the internal state variable structure provides the framework for interpreting and handling the fracture phenomena of metals. Particularly, the introduction of a set of the internal state variables $\{\mathbf{E}_p(t), \varkappa(t), \xi(t)\}$ which are interpreted as the inelastic strain tensor, the work hardening parameter and the concentration of defects, respectively, can be a very convenient tool for this study (cf. P. PERZYNA [24, 27]).

Considering the main features of fracture phenomena observed experimentally, we propose to interpret the scalar state variable $\xi(t)$ as the measure of concentration of defects, inclusions and imperfections in the material.

This parameter is of a similar nature to that introduced in the phenomenological theory of creep rupture by L. M. KACHANOV [15] and Y. N. RABOTNOV [28] (cf. also J. B. MARTIN and F. A. LECKIE [20], D. R. HAYHURST and F. A. LECKIE [12, 18] and D. R. HAYHURST, F. A. LECKIE and C. J. MORRISON [13]) and which was interpreted as a measure of the cracking or damage in the material during the process of creep phenomenon.

Unidirectional tensile mode of loading experimental investigation of metals suggest that the elongation at fracture decreases with a decrease of temperature, and it decreases with an increase of the strain rate. It is also observed that the magnitude of the elongation at fracture does depend on the concentration of defects, inclusions and imperfections, namely it decreases with an increase of the concentration of defects.

This confirms our indication that by proper interpretation of the scalar parameter $\xi(t)$ and proper assumption of its evolution equation we can supply important information about fracture phenomena of the material considered.

It is noteworthy that the description proposed will have practical value provided we have, for the material under consideration, experimental data sufficient for determining all the material functions and material constants.

3. Model equations for an elastic-viscoplastic material

3.1. Constitutive equations with internal state variables

In what follows we shall consider only pure mechanical processes. Let us assume that the intrinsic state σ of a particle X consists of its local configuration $\mathbf{E}(t)$ and its method of preparation $\boldsymbol{\omega}(t)$, i.e.

$$\sigma = \big(\mathbf{E}(t), \boldsymbol{\omega}(t)\big), \tag{3.1}$$

where $\mathbf{E}(t)$ denotes the strain tensor and $\boldsymbol{\omega}(t)$ is the internal state vector. It is postulated that the internal state vector $\boldsymbol{\omega}(t)$ can be assumed in the form

$$\boldsymbol{\omega}(t) = \big(\mathbf{E}_p(t), \varkappa(t), \xi(t)\big), \tag{3.2}$$

where $\mathbf{E}_p(t)$ denotes the inelastic strain tensor, $\varkappa(t)$ is an isotropic work-hardening parameter and $\xi(t)$ is interpreted as a scalar measure of the concentration of defects, inclusions and imperfections.

It will be shown that the parameter $\xi(t)$ and its evolution constitute the crucial point in the description of instability phenomena in elasto-viscoplastic processes (cf. also P. PERZYNA [27]).

The constitutive equation for the Piola-Kirchhoff stress tensor $\mathbf{T}(t)$ is assumed in the form as follows:

$$\mathbf{T}(t) = \hat{\mathbf{T}}(\sigma). \tag{3.3}$$

The tensorial material function $\hat{\mathbf{T}}$ is assumed to be differentiable with respect to all components of the intrinsic state σ.

The evolution equation for the internal state vector $\boldsymbol{\omega}(t)$ is postulated in the form

$$\dot{\boldsymbol{\omega}}(t) = \hat{\boldsymbol{\Omega}}\big(\mathbf{E}(t), \dot{\mathbf{E}}(t), \boldsymbol{\omega}(t), \boldsymbol{\zeta}(t)\big), \qquad t \in [0, d_{\mathrm{P}}] \tag{3.4}$$

with the initial value as follows

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 = (\mathbf{E}_p^0, \varkappa^0, \xi^0), \tag{3.5}$$

where d_{P} denotes the duration of the process considered.

The vectorial function $\hat{\boldsymbol{\Omega}}$ depends on the strain tensor $\mathbf{E}(t)$, the strain rate tensor $\dot{\mathbf{E}}(t)$, the internal state vector $\boldsymbol{\omega}(t)$ and the control vector $\boldsymbol{\zeta}(t)$. It is assumed

$$\boldsymbol{\zeta} = \langle \varphi, \psi \rangle \quad \text{and} \quad \boldsymbol{\zeta} \in U, \tag{3.6}$$

where

$$U = \Big\{\boldsymbol{\zeta}\colon \max|\varphi| < M, \max|\psi| < M, \lim_{\mathrm{I}_2 \to \mathrm{I}_2^s} \varphi = 0, \lim_{\mathrm{I}_2 \to \mathrm{I}_2^s} \psi = 0, \tag{3.7}$$
$$\varphi(0) = 0, \quad \psi(0) = 0, \quad \varphi(\cdot) = 0 \quad \text{and} \quad \psi(\cdot) = 0 \quad \text{for} \quad \mathrm{I}_2 < \mathrm{I}_2^s\Big\}$$

and we introduced the denotations as follows([5])

$$\mathrm{I}_2 = (\mathrm{II}_{\dot{\mathbf{E}}})^{1/2}, \quad \mathrm{I}_2^s = (\mathrm{II}_{\dot{\mathbf{E}}_s})^{1/2}. \tag{3.8}$$

([5]) The second invariant I_2^s is called the static value of strain rate measure. We define I_2^s as such value for which, in a test under combined stress conditions with $\mathrm{I}_2 \leqslant \mathrm{I}_2^s$, there is no rate sensitivity effect observed. In other words, rate sensitivity effects are not experimentally detected.

The reason why we introduce the control vector $\boldsymbol{\zeta}$ in the evolution equation (3.4) is twofold. Firstly, it helps to describe the properties of a material in a range of strain rates near the static value (say $I_2 = I_2^s$). Secondly, it will play a very impottant role in determining of the material functions and material constants according to available experimental results for strain rate-sensitive plastic materials.

Physical analysis based on the dynamics of dislocations suggested that two mechanisms, namely the thermally activated process and the phonon damping effects, are most important for explaining the strain rate sensitivity of metals (cf. P. PERZYNA [26]).

Assuming these two physical mechanisms as fundamental, we can postulate a particular form of the evolution equation (3.4) as follows:

$$\begin{aligned} \dot{\mathbf{E}}_p(t) &= \frac{\gamma_0}{\varphi(\cdot)}\left\langle \Phi\left(\frac{f(\cdot)}{\varkappa(t)}-1\right)\right\rangle \partial_{\mathbf{T}(t)} f(\cdot), \\ \dot{\varkappa}(t) &= \mathrm{tr}[\hat{\mathbf{K}}(\sigma)\dot{\mathbf{E}}_p(t)], \qquad (3.9) \\ \dot{\xi}(t) &= \mathrm{tr}[\hat{\Xi}_1(\sigma)\dot{\mathbf{E}}_p(t)]+\psi(\cdot)\hat{\Xi}_2(\sigma), \end{aligned}$$

where

$$f(\cdot) = f(\mathbf{T}(t), \mathbf{E}_p(t), \xi(t)) \tag{3.10}$$

denotes the quasi-static yield function (loading function), and the symbol $\langle[\,]\rangle$ is understood according to the definition

$$\langle[\ ]\rangle = \begin{cases} 0 & \text{if} \quad f(\cdot) \leqslant \varkappa(t), \\ [\ ] & \text{if} \quad f(\cdot) > \varkappa(t). \end{cases} \tag{3.11}$$

3.2. Determination of the material functions

To determine the material functions and constants involved in the theory proposed we shall use available experimental data obtained in dynamical tests.

It will be shown that the determination of the material function Φ, the control function φ and the material constants can be based on both the combined and one-dimensional loading tests.

Let us develop first the procedure for the dynamical test under combined loading.

U. S. LINDHOLM [19] performed experimental tests for a number of metals under combined stress conditions by using a new pneumatic machine. Of particular interest for us are the results for mild steel plotted the square root of the second invariant of the deviation of the stress tensor as a function of the square root of the second invariant of the strain rate tensor.

Similar results have been obtained by M. R. D. RANDALL and J. D. CAMPBELL [29] for a low-carbon steel. The authors used a new hydraulically-operated medium strain rate testing machine, capable of loading tubular specimens in simultaneous tension and shear.

From these works for particular metals we have the experimental curves as follows

$$(\mathrm{II}_S)^{1/2} = \mathscr{G}(\mathrm{I}_2)\big|_{(\mathrm{II}_E)^{1/2}=\mathrm{const}}, \tag{3.12}$$

where

$$\mathbf{S} = \mathbf{T} - \left(\frac{1}{3}\,\mathrm{T}_{ii}\right)\mathbf{I} \tag{3.13}$$

is the deviation of the stress tensor **T**.

To obtain reasonable results which can be useful for practical applications, we have to introduce some simplifications to the evolution equations postulated.

We assume

$$f(\cdot) = (\mathrm{II_S})^{1/2} \tag{3.14}$$

and

$$\varphi(\cdot) = \varphi\left(\frac{\mathrm{I}_2(t)}{\mathrm{I}_2^s} - 1\right), \quad \psi(\cdot) = \psi\left(\frac{\mathrm{I}_2(t)}{\mathrm{I}_2^s} - 1\right). \tag{3.15}$$

The first assumption (3.14) is suggested by experimental results. The second (3.15) is justified by the initial-boundary-value problem we intend to formulate. This problem will have a quasi-static nature, so we are interested to have an efficient description in a wide range of small strain rates, near the static value, i.e. $\mathrm{I}_2(t) = \mathrm{I}_2^s$.

As an optimum criterion for the best fitting of experimental data by theoretical results we introduce the functional

$$\mathscr{S}\big(\boldsymbol{\omega}(t), \varphi(\cdot)\big) = \max_{\substack{t\in[o,d_p] \\ \mathrm{I}_2(t)\in(\mathrm{I}_2^s, \mathrm{I}_2^{\max}]}} \left\| J_2(t) - \mathscr{G}(\mathrm{I}_2)\big|_{(\mathrm{II_{\dot{E}}})^{1/2}=\mathrm{const}} \right\|, \tag{3.16}$$

where

$$J_2(t) = \varkappa(t)\left\{1 + \Phi^{-1}\left[\frac{\left(\mathrm{tr}\,(\dot{\mathbf{E}}_p(t))^2\right)^{1/2}}{\gamma_0}\,\varphi\left(\frac{\mathrm{I}_2(t)}{\mathrm{I}_2^s} - 1\right)\right]\right\} \tag{3.17}$$

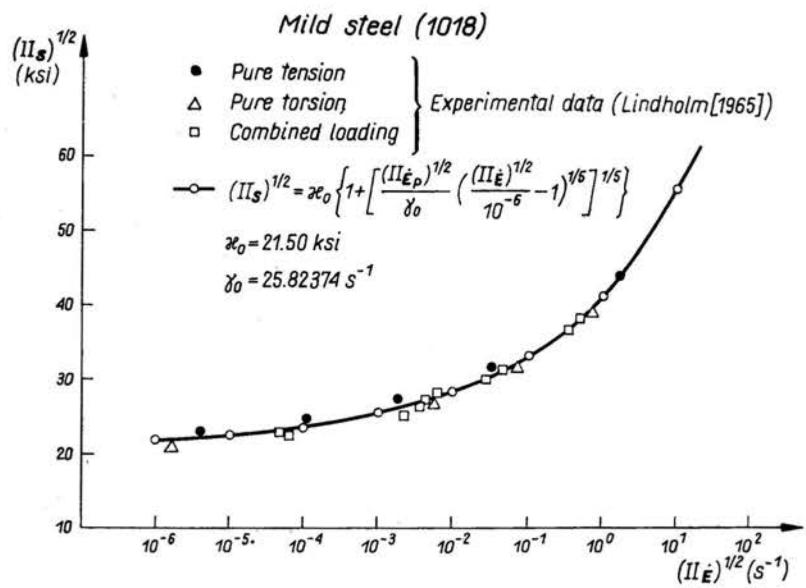

FIG. 2.

and

$$J_2(t) = (\mathrm{II_S})^{1/2}, \quad \varkappa(t) = \varkappa^0 = \text{const}. \tag{3.18}$$

To determine the material function Φ, the control function φ and the constant γ_0, we shall minimize the functional (3.16).

The results obtained for mild steel and low-carbon steel are plotted in Figs. 2 and 3, respectively.

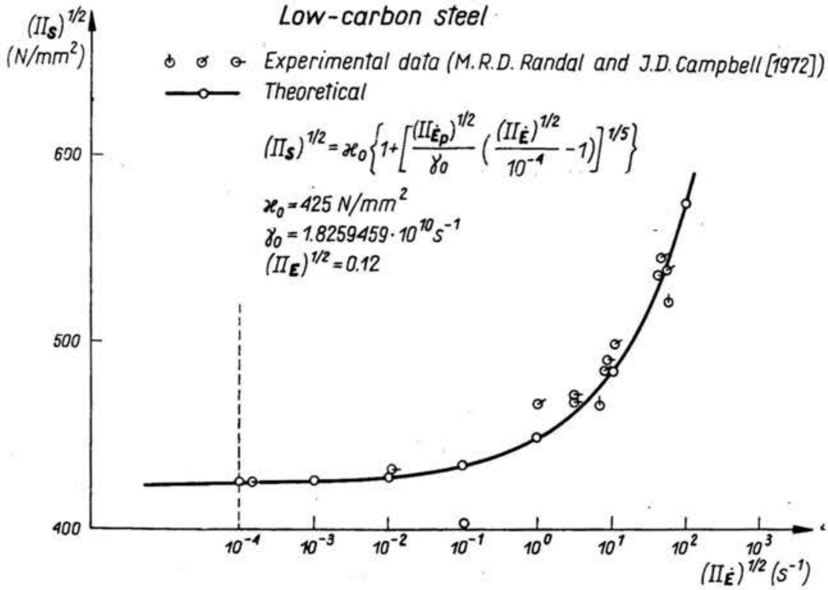

FIG. 3.

A similar procedure can be developed basing on the results obtained in dynamical tests performed under one-dimensional loading.

For this case we have as an optimum criterion the following functional:

$$\mathscr{T}(\boldsymbol{\omega}(t), \varphi(\cdot)) = \max_{\substack{t\in[o,d_p] \\ \dot{\mathrm{E}}\in(\dot{\mathrm{E}}_s,\, \dot{\mathrm{E}}_{max}]}} \|\mathrm{T}(t) - \hat{\mathscr{G}}(\dot{\mathrm{E}}(t))|_{\mathrm{E}=\text{const}}\|, \tag{3.19}$$

where

$$\mathrm{T}(t) = \varkappa_0^*\left\{1+\Phi^{-1}\left[\frac{\dot{\mathrm{E}}_p}{\gamma_0^*}\varphi\left(\frac{\dot{\mathrm{E}}}{\dot{\mathrm{E}}_s}-1\right)\right]\right\} \tag{3.20}$$

and

$$\mathrm{T}(t) = \hat{\mathscr{G}}(\dot{\mathrm{E}}(t))|_{\mathrm{E}=\text{const}} \tag{3.21}$$

represents the experimental curve obtained for $\mathrm{E} = \text{const}$.

We shall use experimental data obtained by different authors for various metals.

J. D. C. CAMPBELL and W. G. FERGUSON [4] performed experiments in which the shear flow stress of mild steel was measured at temperatures from 195 to 713 K and strain

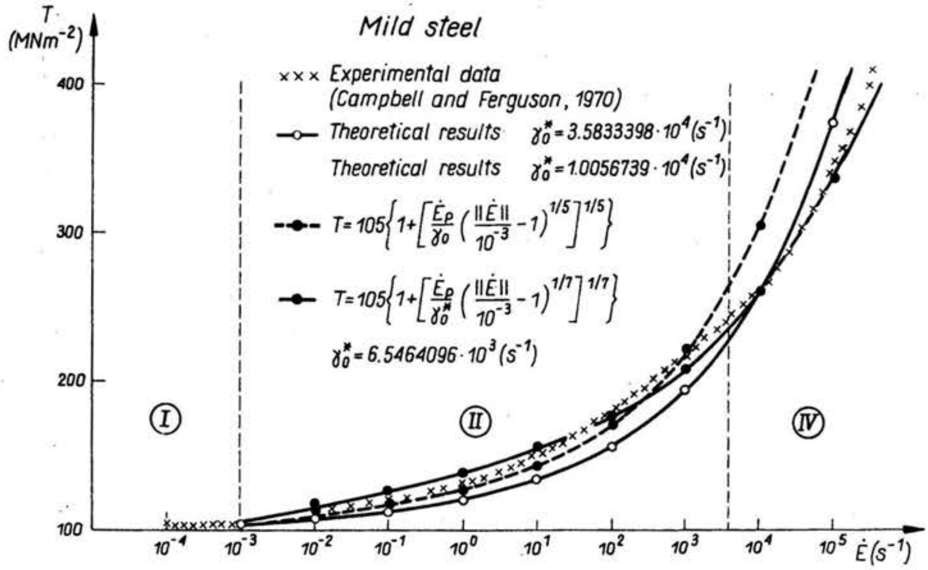

FIG. 4.

rates from 10^{-3} to 4×10^4 s^{-1}. In Fig. 4 the theoretical assumptions are compared with experimental results for room temperature (293 K).

M. F. KANNINEN, A. K. MUKHERJEE, A. R. ROSENFIELD and G. T. HAHN [16] presented experimental results obtained by different authors for plain carbon steel in the range of strain rates from 10^{-4} to 10^4 s^{-1}. These data are compared with theoretical results in Fig. 5.

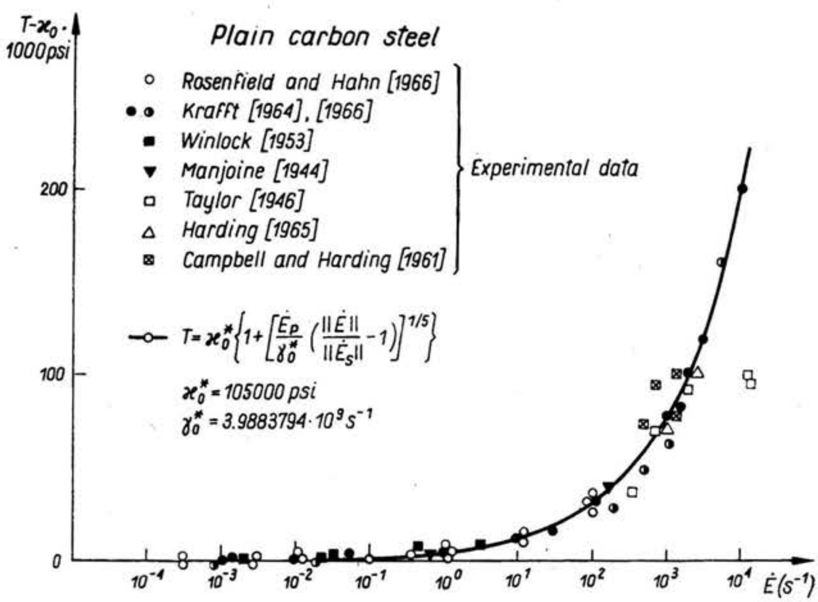

FIG. 5.

C. H. KARNES and E. A. RIPPERGER [17] performed tests in which specimens of high-purity polycrystalline aluminium were subjected to strain rates from 10^{-4} to 4×10^3 s^{-1} in order to determine the nature of the strain rate sensitivity. The specimens used in this test were cold worked to varying degrees from an annealed condition to an engineering strain of 50 per cent. A comparison of experimental data for the annealed specimen with theoretical results is shown in Fig. 6.

A similar comparison of theoretical results with copper data obtained by A. R. DOWLING, J. HARDING and J. D. CAMPBELL [7] by using a punch shear test is presented in Fig. 7 and with data for Cu crystal tested in compression by J. W. EDINGTON [8] in Fig. 8.

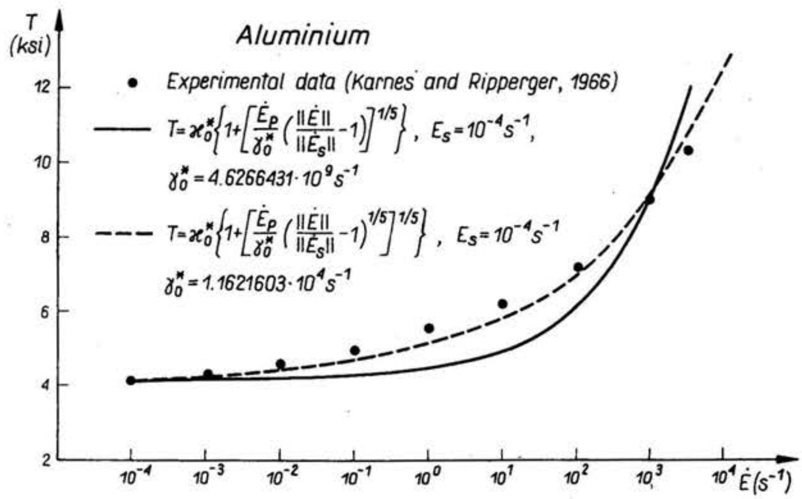

FIG. 6.

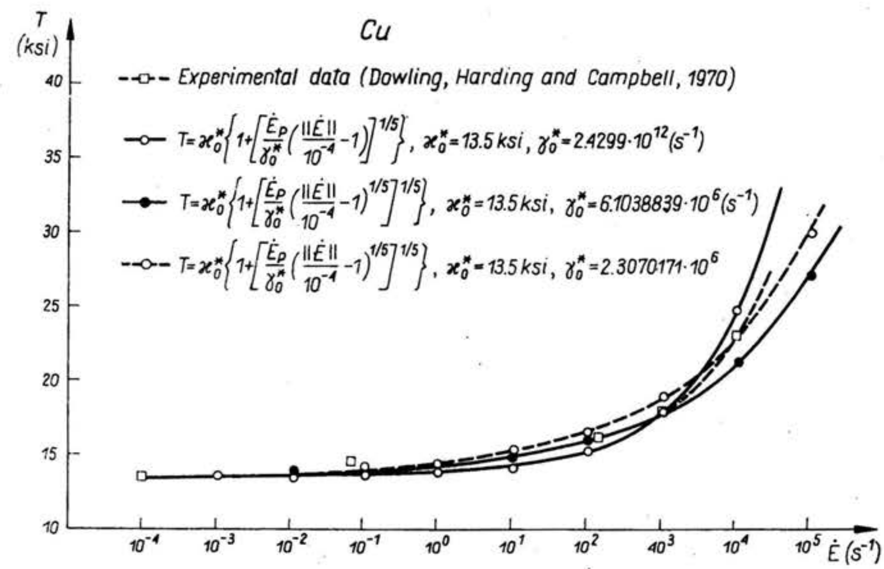

FIG. 7.

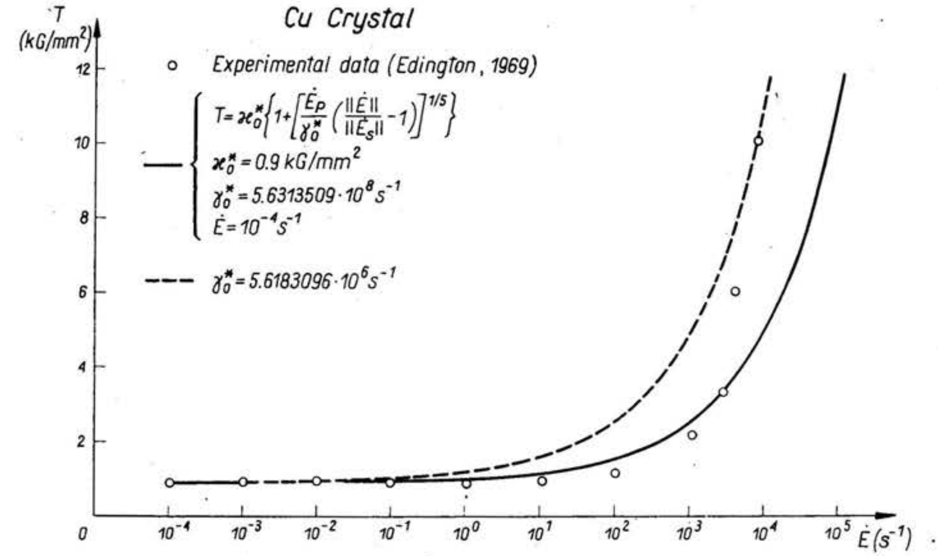

FIG. 8.

On analyzing the constitutive equation (3.3) and the evolution equations (3.9), we can easily observe that the material functions $\hat{\mathbf{T}}$, $\hat{\mathbf{K}}$, $\hat{\Xi}_1$, $\hat{\Xi}_2$ and the control function ψ are still undetermined.

3.3. Elastic-plastic response

It is noteworthy that for(6)

$$\|\dot{\mathbf{E}}(t)-\dot{\mathbf{E}}_s\| = 0 \Rightarrow \mathbf{I}_2(t) = \mathbf{I}_2^s \tag{3.22}$$

the evolution equations (3.9) lead to the following results:

$$\begin{aligned} \dot{\mathbf{E}}_p(t) &= \Lambda \partial_{\mathbf{T}(t)} f(\cdot), \quad f(\cdot) = \varkappa(t), \quad \mathrm{tr}(\partial_{\mathbf{T}} f \dot{\mathbf{T}}) > 0, \\ \dot{\varkappa}(t) &= \mathrm{tr}[\hat{\mathbf{K}}(\sigma)\dot{\mathbf{E}}_p(t)], \\ \dot{\xi}(t) &= \mathrm{tr}[\hat{\Xi}_1(\sigma)\dot{\mathbf{E}}_p(t)], \end{aligned} \tag{3.23}$$

where

$$\Lambda = \{\mathrm{tr}[(\hat{\mathbf{K}} - \partial_{\mathbf{E}_p} f - \partial_{\xi} f \hat{\Xi}_1)\,\partial_{\mathbf{T}} f]\}^{-1} \mathrm{tr}(\partial_{\mathbf{T}} f \dot{\mathbf{T}}). \tag{3.24}$$

The constitutive equation (3.3) together with the evolution equations (3.23) describe a work-hardening elastic-plastic material.

Since the tensorial material functions $\hat{\mathbf{T}}$, $\hat{\mathbf{K}}$ and $\hat{\Xi}_1$ are the same for the elastic-visco-plastic response of a material as well as for the elastic-plastic range, we can assume these functions based on the results for the theory of plasticity. Hence, the main question to be asked is the determination of the scalar imperfection function $\hat{\Xi}_2$ and the scalar control function ψ.

(6) The norm $\|\cdot\|$ is understood as the natural norm in the space of strain rate tensors $\dot{\mathbf{E}}$.

3.4. Rate type constitutive equations

For the formulation of the initial-boundary-value problem it will be useful to have the rate type material structure. It is easy to prove (cf. P. PERZYNA [25]) that the constitutive equation (3.3) and the evolution equations (3.9) lead to the following rate type evolution equation:

$$\dot{\mathbf{T}}(t) = \hat{\boldsymbol{\beta}}_0 + \hat{\boldsymbol{\beta}}_1[\dot{\mathbf{E}}(t)], \tag{3.25}$$

where

$$\begin{aligned}\hat{\boldsymbol{\beta}}_0 = \frac{\gamma_0}{\varphi\left(\frac{\mathrm{I}_2}{\mathrm{I}_2^s}-1\right)}\left\langle\Phi\left(\frac{f(\cdot)}{\varkappa(t)}-1\right)\right\rangle\{\partial_{\mathbf{E}_p}\hat{\mathbf{T}}[\partial_{\mathbf{T}}f]+\partial_\varkappa\hat{\mathbf{T}}\,\mathrm{tr}[\hat{\mathbf{K}}\partial_{\mathbf{T}}f]\\ +\partial_\xi\hat{\mathbf{T}}\,\mathrm{tr}[\hat{\boldsymbol{\Xi}}_1\partial_{\mathbf{T}}f]+\psi\left(\frac{\mathrm{I}_2}{\mathrm{I}_2^s}-1\right)\partial_\xi\hat{\mathbf{T}}\hat{\boldsymbol{\Xi}}_2,\end{aligned} \tag{3.26}$$

$$\hat{\boldsymbol{\beta}}_1 = \partial_{\mathbf{E}}\hat{\mathbf{T}},$$

valid in a range of strain rates $\mathrm{I}_2 > \mathrm{I}_2^s$.

For the elastic-plastic response, i.e. in a range of strain rates $\mathrm{I}_2 \leqslant \mathrm{I}_2^s$, the rate type evolution equation has the form

$$\dot{\mathbf{T}}(t) = \hat{\boldsymbol{\beta}}_2[\dot{\mathbf{E}}(t)], \tag{3.27}$$

where the tensorial function $\hat{\boldsymbol{\beta}}_2$ is determined as follows:

$$\hat{\boldsymbol{\beta}}_2 = \hat{\boldsymbol{\beta}}_1 = \partial_{\mathbf{E}}\hat{\mathbf{T}} \quad \text{for} \quad \mathrm{tr}(\partial_{\mathbf{T}}f\dot{\mathbf{T}}] \leqslant 0, \tag{3.28}$$

and for $\mathrm{tr}(\partial_{\mathbf{T}}f\dot{\mathbf{T}}) > 0$ the function $\hat{\boldsymbol{\beta}}_2$ can be obtained from the identity

$$\dot{\mathbf{T}}(t) = \partial_{\mathbf{E}}\hat{\mathbf{T}}[\dot{\mathbf{E}}]+\partial_{\mathbf{E}_p}\hat{\mathbf{T}}\Lambda\partial_{\mathbf{T}}f+\partial_\varkappa\hat{\mathbf{T}}\,\mathrm{tr}[\hat{\mathbf{K}}\partial_{\mathbf{T}}f]\Lambda+\partial_\xi\hat{\mathbf{T}}\,\mathrm{tr}[\hat{\boldsymbol{\Xi}}_1\partial_{\mathbf{T}}f]\Lambda, \tag{3.29}$$

where Λ is given by Eq. (3.24).

To keep our consideration as simple as possible we can assume $\hat{\boldsymbol{\Xi}}_1 \equiv 0$.

In the case when the imperfection function $\hat{\boldsymbol{\Xi}}_1$ does not vanish, we could base our study on recent results for an elastic-plastic material(7).

4. Strain rate effects on the necking phenomenon

4.1. Formulation of the initial-boundary value problem(8)

Let us study the tensile deformation of a circular cylindrical bar of initial length $2L_0$ and initial radius R_0. We assume the cylindrical coordinates r, θ, z. We treat the problem

(7) Conditions for a localization bifurcation with an initial, finite imperfection for an elastic-plastic material by developing the localization theory given by J. RICE [30] have been studied by H. YAMAMOTO [32]. Cf. also the literature on the subject cited there.

as axisymmetric and additionally assume that the deformations are symmetric about the mid plane $z = 0$.

To formulate the basic equations and the initial-boundary-value conditions, we introduce the stress tensor

$$T_*^{ik} = \left(\frac{G}{g}\right)^{1/2} T_c^{ik}, \tag{4.1}$$

where T_c^{ik} are the contravariant components of the Cauchy stress tensor, G is the determinant of the metric tensor G_{ij} in the deformed body, and g is the determinant of the metric tensor g_{ij} in the undeformed body.

Let us define the rate of traction

$$\dot{t}_*^i = (\dot{T}_*^{ik} + \dot{T}_*^{kj} u^i_{,j} + T_*^{kj} \dot{u}^i_{,j}) n_k, \tag{4.2}$$

where $\mathbf{u}$ is the displacement vector and $\mathbf{n}$ the unit normal vector.

The incremental equilibrium equations with body forces neglected have the form

$$(\dot{T}_*^{ik} + \dot{T}_*^{ij} u^k_{,j} + T_*^{ij} \dot{u}^k_{,j})_{,i} = 0. \tag{4.3}$$

The Lagrangian strain rate $\dot{E}^*_{ij}$ is given by

$$\dot{E}^*_{ij} = \frac{1}{2} [\dot{u}_{i,j} + \dot{u}_{j,i} + g_{kl}(\dot{u}^k_{,i} u^l_{,j} + \dot{u}^k_{,j} u^l_{,i})]. \tag{4.4}$$

The constitutive equations of the rate type can be written in the form as follows

$$\begin{aligned} \dot{T}_*^{ij} &= \hat{\hat{\beta}}_0^{ij} + \hat{\hat{\beta}}_{1kl}^{ij} \dot{E}_*^{kl}, \qquad I_2 > I_2^s, \\ T_*^{ij} &= \hat{\hat{\beta}}_{2kl}^{ij} E_*^{kl}, \qquad I_2 \leqslant I_2^s, \end{aligned} \tag{4.5}$$

where the functions $\hat{\hat{\boldsymbol{\beta}}}_0$, $\hat{\hat{\boldsymbol{\beta}}}_1$ and $\hat{\hat{\boldsymbol{\beta}}}_2$ are determined by $\hat{\boldsymbol{\beta}}_0$, $\hat{\boldsymbol{\beta}}_1$ and $\hat{\boldsymbol{\beta}}_2$ given by Eqs. (3.26) and (3.29) after transformation to the new coordinate system and using $\mathbf{T}_*$ and $\dot{\mathbf{E}}^*$ as tensor measures of stress and strain rate.

The boundary conditions are as follows: the assumed symmetry about $z = 0$ requires that at $z = 0$ we have

$$\begin{aligned} &\dot{t}_*^\alpha(r, 0, t) = 0 \qquad (\alpha = 1, 2), \\ &\dot{u}_3(r, 0, t) = 0; \end{aligned} \tag{4.6}$$

at $z = L_0$ for the bar with shear-free ends

$$\begin{aligned} &\dot{t}_*^\alpha(r, L_0, t) = 0, \qquad (\alpha = 1, 2), \\ &\dot{u}_3(r, L_0, t) = \dot{U}_1, \dot{U}_2, \dot{U}_3, \ldots, \dot{U}_n; \end{aligned} \tag{4.7}$$

at $z = L_0$ for the bar with the ends cemented to rigid grips

$$\begin{aligned} &\dot{u}_1(r, L_0, t) = 0, \\ &\dot{u}_3(r, L_0, t) = \dot{U}_1, \dot{U}_2, \dot{U}_3, \ldots, \dot{U}_n; \end{aligned} \tag{4.8}$$

([8]) In the consideration of the boundary-value problem we follow here the exposition presented by A. NEEDLEMAN [21].

for $t \in [0, d_P]$, where d_P denotes presupposed duration of the process considered. The lateral surfaces of the bar are required to remain stress free in both cases.

The initial condition are assumed as follows: all components of the stress tensor $\mathbf{T}_*$ and all components of the displacement vector $\mathbf{u}$ for $t = 0$ vanish.

4.2. Discussion and conclusions

Let the pair $(\mathbf{T}_*, \mathbf{u})$ be a solution of the initial-boundary-value problem formulated.

Let us presuppose that we can perform an experimental investigation for a similar boundary-value problem. In this test specimens of cylindrical bars (a material of which is mild steel) are strained with the velocities $\dot{U}_1, \dot{U}_2, \dot{U}_3, \ldots, \dot{U}_n$, respectively. We measure the load as the function of strain for every test[9]. The results obtained can be plotted as it is shown in Fig. 9 (they are consistent with preliminary, very rough numerical cal-

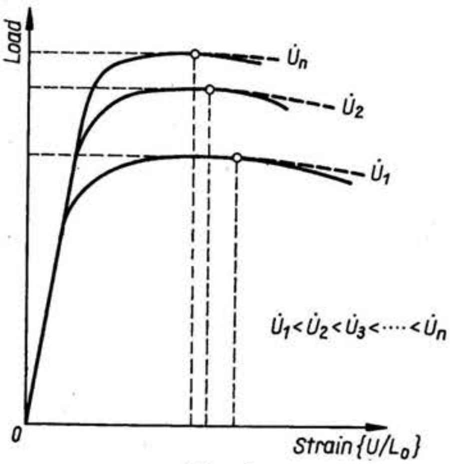

FIG. 9.

culations of the initial-boundary-value problem formulated). If we can control the instability points on load–strain curves for the constant velocities $\dot{U}_i$ $(i = 1, 2, \ldots, n)$, then we can obtain two curves, namely the load at the instability point as the function of the strain rate and the strain at the instability point as the function of the strain rate. These results are plotted in Fig. 10.

Having the results discussed, we can introduce the functional

$$\mathscr{F}(\mathbf{T}_*, \mathbf{u}, \zeta) = \max_{\substack{t \in [0, d_P] \\ I_2 \in (I_2^s, I_2^{\max}]}} \{ \| \mathscr{G}_{1_{\text{theoret}}}(I_2) - \mathscr{G}_{1_{\exp}}(I_2) \|, \| \mathscr{G}_{2_{\text{theoret}}}(I_2) - \mathscr{G}_{2_{\exp}}(I_2) \| \}, \tag{4.9}$$

[9] We can recall a very similar experimental investigation of plastic instability in the direction of pure shear for two types of sheet specimens performed by A. K. CHAKRABARTI and J. W. SPRETNAK [5]. They have found that a maximum in true flow stress is consistent at the onset of instabilities. They have also shown that fracture is propagated consistently along the instability band-matrix interface. It has been observed that variations in specimen geometry produce significant changes in stress state, directions of characteristics and nature of instability phenomena.

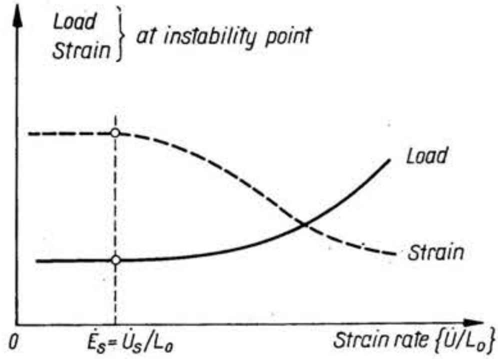

FIG. 10.

where

$$\text{(4.10)} \quad \begin{aligned} &\frac{\text{Load}}{\text{(at the instability point)}} = \mathscr{G}_{1_{\text{theoret}}}(\mathrm{I}_2), \\ &\frac{\text{Strain}}{\text{(at the instability point)}} = \mathscr{G}_{2_{\text{theoret}}}(\mathrm{I}_2), \end{aligned}$$

represent the theoretical curves obtained by means of the numerical method.

The functional (4.9) is understood as an optimal control criterion of the process of straining of cylindrical bar specimens.

The main conception is to determine such an imperfection scalar function $\hat{\Xi}_2$ and the control scalar function ψ that the solution $(\mathbf{T}_*, \mathbf{u})$ minimizes the functional (4.9).

The advantage of the procedure proposed is the unified formulation of the problem for the elastic-viscoplastic range as well as for the elastic-plastic response of a material. This permits to use a similar numerical method in both regions of the solution.

Of course there are some shortcomings of the procedure proposed. Since the problem has been formulated as quasi-statical, we have to restrict our considerations to such values of the velocities $\dot{\mathrm{U}}_i$ $(i = 1, 2, \ldots, n)$ to be sure that our quasi-static approximation is valid.

As the solution $(\mathbf{T}_*, \mathbf{u})$ can be obtained only by means of the numerical method (e.g. similar to that proposed by A. NEEDLEMAN [21]) then the determination of the functions $\hat{\Xi}_2$ and ψ will depend on the accuracy of the method applied and on the ability of the computer.

The main disadvantage is the lack of the experimental data presupposed in this procedure. We hope, however, that kind of experimental investigations is not difficult to perform nowadays.

5. Final comments

The question arises whether the evolution equation (3.9) assumed for the internal state variable $\xi(t)$ is adequate to describe instability effects observed experimentally. This is the main question we would like to answer.

The evolution equation (3.9) has no direct physical foundation. It represents convenient phenomenological assumptions.

Physical foundations connected with the intergranular fracture mode could provide a ground for the justification of the evolution equation assumed.

Recent physical investigations on the growth of an array of grain boundary voids during a deformation process (e.g. creep process) (cf. W. BEERE and M. V. SPEIGHT [1] and G. H. EDWARD and M. F. ASHBY [9]) suggested a coupled diffusion and power-law creep mechanism to explain complex fracture phenomena in metals.

G. H. EDWARD and M. F. ASHBY [9] have developed a coupled mechanism in which each void grows by diffusion, but the void plus its diffusion field is contained within a cage of power-law creeping material. They have proved that the coupled model predicts times and a strain-to-fracture consistent with experimental observation. They have pointed out that there is also the possibility that interface kinetics, not diffusion, control the rates of transport around and out of the growing void.

Basing on this physical suggestion, we have to take into consideration this complicated nature of interface kinetics.

Therefore, we can expect than the evolution equation for the scalar internal state variable $\xi(t)$ interpreted as a measure of concentration of imperfections (e.g. grain boundary voids, the area fraction of holes) will take the form of the transport differential equation or, in a simplified form, the diffusion differential equation.

Of course, the coefficients in this partial differential evolution equation can be determined basing on the physical model assumed or on statistical considerations.

Coupling effects between different cooperative phenomena described by the internal state variables and its evolution equations may model a complicated fracture process in a dissipative material.

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