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The unified theory of variational principles in nonlinear elasticity

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THE PRESENT paper deals with the most important variational principles in nonlinear elasticity — the classical potential energy theorem, the complementary energy theorem and two other complementary theorems much discussed in the literature recently (Levinson's principle and Fraeijs de Veubeke's principle). It is shown that all these principles and their generalized versions can be derived from the virtual work principle in a unifying manner. The various principles joined then in a common frame constitute an organic complex. The flow diagram will convey the reader a picture of the interconnexion between the various principles.

W pracy rozważa się najważniejsze zasady wariancyjne nieliniowej teorii sprężystości — klasyczne twierdzenie o energii potencjalnej, twierdzenie o energii uzupełniającej i dwa inne twierdzenia uzupełniające, szeroko dyskutowane we współczesnej literaturze: twierdzenie Levinsona i Fraeijsa de Veubeke. Wykazuje się, że wszystkie wymienione zasady i ich uogólnione wersje wyprowadzić można w sposób jednolity z zasady prac wirtualnych. Zasady te, po ujęciu we wspólne ramy, stanowią organiczną całość. Zamieszczony w pracy schemat blokowy pozwoli czytelnikowi w sposób jasny uzmysłowić sobie wzajemne zależności między poszczególnymi zasadami.

В работе рассматриваются самые важные вариационные принципы нелинейной теории упругости — классическая теорема о потенциальной энергии, теорема о дополнительной энергии и две другие дополнительные теоремы широко обсуждаемые в современной литературе: теоремы Левинсона и Фрейза де Веубека. Показывается, что все перечисленные принципы и их обобщенные варианты можно вывести единым образом из принципа виртуальных работ. Эти принципы, после помещения их в общие рамки, составляют единую целость. Помещенная в работе структурная схема позволит читателю осмыслить взаимные зависимости между отдельными принципами.

1. Introduction

THE PROBLEM of whether there exists in the nonlinear elasticity theory a complementary energy theorem involving only the stress variable as in the linearized theory, has been widely discussed in recent years. The complementary principles hitherto proposed can be divided into three groups. The representative of the first group is Reissner's principle based on the Kirchhoff stress tensor [1–5]. It has been regarded, however, as a theorem not in its truly complementary form because it also involves the displacement vector as an independent variable besides the Kirchhoff tensor. The next group is Levinson's principle [6–15] which is based only on the Piola stress tensor $\boldsymbol{\tau}$ and once passed for the truly complementary principle. Unfortunately, this principle is not always valid because the Piola stress-deformation gradient relation can be inverted provided, even in the simplest case of isotropy, $\boldsymbol{\tau}^* \cdot \boldsymbol{\tau}$(1) have at every point of the body distinct eigenvalues (however, it is

(1) "*" denotes the conjugation of a tensor.

difficult in advance to know whether this requirement is fulfilled). OGDEN [14] has given a criterion to choose the suitable branch, nevertheless, the multi-valued inversion involved (for isotropy at least four distinct branches) would raise difficulties in application. Fraeijs de Veubeke's principle based on the polar decomposition is always valid [10–15]. But here, besides the Piola stress tensor, the rotation tensor is also involved as an independent argument. Again, it does not come up to the expectation for a truly complementary form. It seems that the appropriate formulation of the complementary theorem requires a further study.

The aim of this paper is to show that all these variational principles can be derived from the virtual work principle in a unifying manner, the interconnection of the particular principles being given.

2. Mathematical formalism

In this paper the two-point tensor field method will be used [16, 17]. Let the reference configuration $\mathscr{R}$ and the actual configuration r of the body $\mathscr{B}$ be referred to two independent curvilinear coordinate systems $\{X^A\}$ and $\{x^i\}$, respectively. Their corresponding basis vectors and matric tensors are

$$\begin{array}{llll} \mathbf{G}_A, & \mathbf{G}^A, & G_{AB}, & G^{AB}, \\ \mathbf{g}_i, & \mathbf{g}^i, & g_{ij}, & g^{ij}. \end{array}$$

The deformation of the body $\mathscr{B}$ from $\mathscr{R}$ to r can be presented in the form

$$\mathbf{x} = \mathbf{x}(\mathbf{X}) \quad \text{or} \quad x^i = x^i(X^A), \tag{2.1}$$

i.e. the generic particle $\mathbf{X}$ of the body assumes the position $\mathbf{x}$ in the configuration r.

All vectors with $\mathbf{X}$ as the application point form a Euclidean space of 3 dimensions, denoted also by $\mathscr{R}$, causing no confusion. Any element of space $\mathscr{R}$ can be expressed in a linear combination of $\mathbf{G}_A$ or $\mathbf{G}^A$, for example, $\mathbf{V} = V^A\mathbf{G}_A = V_A\mathbf{G}^A$. We have, for instance, $\mathbf{v} = v^i\mathbf{g}_i = v_i\mathbf{g}^i$ similarly for space r. Consider now four tensor product spaces: $\mathscr{R}\otimes\mathscr{R}$, $\mathscr{R}\otimes r$, $r\otimes\mathscr{R}$, $r\otimes r$, the elements of which can be presented in a linear combination of $\mathbf{G}_A\otimes\mathbf{G}_B$, $\mathbf{G}_A\otimes\mathbf{g}_i$, $\mathbf{g}_i\otimes\mathbf{G}_A$, $\mathbf{g}_i\otimes\mathbf{g}_j$ (the covariant basis vectors can be replaced by the contravariant one. To avoid having to write numerous tensor product symbols, we adopt the convenient Gibbs dyadic notation), for example:

$$\begin{array}{ll} \mathbf{R} = R^{AB}\mathbf{G}_A\mathbf{G}_B, & \mathbf{S} = S^{Ai}\mathbf{G}_A\mathbf{g}_i, \\ \mathbf{T} = T^{iA}\mathbf{g}_i\mathbf{G}_A, & \mathbf{U} = U^{ij}\mathbf{g}_i\mathbf{g}_j. \end{array}$$

By introducing the dot product operation the elements of the tensor product space (of 9 dimensions) become the linear transformations (or the tensors) from 3-dimensional space into 3-dimensional space, for example:

$$\begin{aligned} \mathbf{W} &= \mathbf{R}\cdot\mathbf{V} = (R^{AB}\mathbf{G}_A\mathbf{G}_B)\cdot(V_D\mathbf{G}^D) = R^{AB}V_B\mathbf{G}_A = W^A\mathbf{G}_A, \\ \mathbf{w} &= \mathbf{T}\cdot\mathbf{V} = (T^{iA}\mathbf{g}_i\mathbf{G}_A)\cdot(V_B\mathbf{G}^B) = T^{iA}V_A\mathbf{g}_i = w^i\mathbf{g}_i. \end{aligned}$$

The elements from $\mathscr{R}\otimes r$ or $r\otimes\mathscr{R}$ are two point tensors. The order of dyads is essential.

"*" denotes the conjugation: $\mathbf{R}^* = R^{AB}\mathbf{G}_B\mathbf{G}_A$, $\mathbf{T}^* = T^{iA}\mathbf{G}_A\mathbf{g}_i$. The unit elements (or identity tensors) of $\mathcal{R}\otimes\mathcal{R}$ and $r\otimes r$ are

$$\overset{\ll}{\mathbf{I}} = G_{AB}\mathbf{G}^A\mathbf{G}^B = \delta^A_B\mathbf{G}_A\mathbf{G}^B = \ldots,$$
$$\overset{\gg}{\mathbf{I}} = g_{ij}\mathbf{g}^i\mathbf{g}^j = \delta^i_j\mathbf{g}_i\mathbf{g}^j = \ldots,$$

whereas the unit elements of $\mathcal{R}\otimes r$ and $r\otimes\mathcal{R}$

$$\overset{<>}{\mathbf{I}} = g_A^{\cdot i}\mathbf{G}^A\mathbf{g}_i = g_{Ai}\mathbf{G}^A\mathbf{g}^i = \ldots,$$
$$\overset{><}{\mathbf{I}} = g_i^{\cdot A}\mathbf{g}^i\mathbf{G}_A = g^{iA}\mathbf{g}_i\,\mathbf{G}_A = \ldots$$

are the so-called shifters. With the aid of shifters we can transform a two-point tensor into a usual one, and vice versa. Replacing the tensor product between the dyadic basis vectors by a dot product, we obtain the trace of the tensor

$$\operatorname{tr}\mathbf{R} \overset{\text{df}}{=} R^{AB}\mathbf{G}_A\cdot\mathbf{G}_B = R^A_{\cdot A},$$
$$\operatorname{tr}\mathbf{S} \overset{\text{df}}{=} S^A_{\cdot i}\mathbf{G}_A\cdot\mathbf{g}^i = g^i_A S^A_{\cdot i}.$$

The trace of a tensor is a scalar and the double dot product of two tensors also

$$\mathbf{R}:\mathbf{S} \overset{\text{df}}{=} \operatorname{tr}(\mathbf{R}^*\cdot\mathbf{S}).$$

The absolute differentials of the tensor fields $\mathbf{R}$ and $\mathbf{U}$ are

$$d\mathbf{R} = (\mathbf{R}\nabla)\cdot d\mathbf{X},$$
$$d\mathbf{U} = (\mathbf{U}\mathring{\nabla})\cdot d\mathbf{x},$$

where

$$\mathbf{R}\nabla = R^{BC}_{\ \ ;A}\mathbf{G}_B\mathbf{G}_C\mathbf{G}^A,$$
$$\mathbf{U}\mathring{\nabla} = U^{jk}_{\ \ ;i}\mathbf{g}_j\mathbf{g}_k\mathbf{g}^i$$

are the absolute derivatives. $(\ )_{;A}$ and $(\ )_{;i}$ denote the covariant differentiation in $\mathcal{R}$ and in r, respectively. In the case of double tensor fields, the absolute derivatives must be understood as total absolute derivatives.

$$\mathbf{R}\cdot\nabla = R^{BC}_{\ \ ;A}\mathbf{G}_B\mathbf{G}_C\cdot\mathbf{G}^A = R^{BA}_{\ \ ;A}\mathbf{G}_B,$$
$$\mathbf{R}\times\nabla = R^{BC}_{\ \ ;A}\mathbf{G}_B\mathbf{G}_C\times\mathbf{G}^A$$

are the divergence and curl of $\mathbf{R}$, respectively. The absolute derivatives of the unit tensor and shifter are zero. In the further discussion we assume that all quantities have been reduced in advance with the help of the shifter to one-point tensors and $\mathbf{I}$ will denote $\overset{\ll}{\mathbf{I}}$.

3. Strain and stress

Denote the displacement vector of the particle $\mathbf{X}$ in the deformation (2.1) by

$$\mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}. \tag{3.1}$$

The deformation gradient

$$\mathbf{F} = \mathbf{x}\nabla = \mathbf{I} + \mathbf{u}\nabla \tag{3.2}$$

has a unique polar decomposition

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U}, \tag{3.3}$$

in which $\mathbf{U}$ is the (right) stretch tensor and $\mathbf{R}$ is the rotation tensor (proper orthogonal $\mathbf{R}^* \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{R}^* = \mathbf{I}$). $\mathbf{U}$ is positive-definite, for which the following relation is valid:

$$\mathbf{U}^2 = \mathbf{F}^* \cdot \mathbf{F} \overset{\mathrm{df}}{=} \mathbf{C}. \tag{3.4}$$

$\mathbf{C}$ is the so-called right Cauchy-Green tensor. Further, the Almansi strain tensor will also be used:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I}) = \frac{1}{2}[\mathbf{u}\nabla + \nabla\mathbf{u} + (\nabla\mathbf{u}) \cdot (\mathbf{u}\nabla)], \tag{3.5}$$

where

$$\nabla\mathbf{u} = (\mathbf{u}\nabla)^*.$$

The Cauchy stress tensor $\mathbf{t}$ has a direct physical interpretation in describing the stress state. When it acts on the unit vector $\mathbf{n}$ in the configuration r, it gives the contact force of the unit area element with the normal vector $\mathbf{n}$:

$$\mathbf{t} \cdot \mathbf{n} = \mathbf{t}_n. \tag{3.6}$$

$\mathbf{t}$ satisfies the Cauchy (force) equilibrium equation (body forces being absent in order to save the space):

$$\mathbf{t} \cdot \overset{\circ}{\nabla} = 0 \tag{3.7}$$

and the moment equilibrium condition

$$\mathbf{t} = \mathbf{t}^*. \tag{3.8}$$

Equation (3.8) means that $\mathbf{t}$ is symmetric. For the purpose of the Lagrangian formulation, the Piola stress $\boldsymbol{\tau}$ and the Kirchhoff stress tensor $\mathbf{T}$ have been introduced

$$\boldsymbol{\tau} \cdot \mathbf{N} = \mathbf{F} \cdot (\mathbf{T} \cdot \mathbf{N}) = \sigma_n \mathbf{t}_n = \mathbf{T}_N, \tag{3.9}$$

in which $\mathbf{N}$ is the unit normal vector in the configuration $\mathscr{R}$ and σ_n is the area ratio of the surface element after and before deformation. These stress tensors are interrelated with each other by

$$\boldsymbol{\tau} = J\mathbf{t} \cdot \overset{-1}{\mathbf{F}^*} = \mathbf{F} \cdot \mathbf{T}, \tag{3.10}$$

$$\mathbf{T} = \overset{-1}{\mathbf{F}} \cdot \boldsymbol{\tau} = J\overset{-1}{\mathbf{F}} \cdot \mathbf{t} \cdot \overset{-1}{\mathbf{F}^*}, \tag{3.11}$$

$$\mathbf{t} = j\mathbf{F} \cdot \mathbf{T} \cdot \mathbf{F}^* = j\boldsymbol{\tau} \cdot \mathbf{F}^*, \tag{3.12}$$

where the superposed (-1) denotes inverse, j is the volume ratio of a volume element before and after deformation and $J = 1/j$. The Piola tensor $\boldsymbol{\tau}$ satisfies the Boussinesq (force) equilibrium equation

$$\boldsymbol{\tau} \cdot \nabla = 0 \tag{3.13}$$

and the Kirchhoff tensor $\mathbf{T}$ satisfies the Kirchhoff (force) equilibrium equation:

$$(\mathbf{F} \cdot \mathbf{T}) \cdot \nabla = 0. \tag{3.14}$$

It is evident that a symmetric tensor $\mathbf{T}$ satisfies identically the moment equilibrium condition, whereas this condition for $\boldsymbol{\tau}$ is

(3.15) $$\boldsymbol{\tau}\cdot\mathbf{F}^* = \mathbf{F}\cdot\boldsymbol{\tau}^*.$$

The symmetry of $\mathbf{t}$ and $\mathbf{T}$ is always assumed in the subsequent considerations. Moreover, from the viewpoint of work the Jaumann tensor $\mathbf{S}$ has been introduced, which plays an important role in the formulation of complementary principles:

(3.16) $$\mathbf{S} = \frac{1}{2}(\mathbf{T}\cdot\mathbf{U}+\mathbf{U}\cdot\mathbf{T}).$$

Substitution of

$$\mathbf{T} = \overset{-1}{\mathbf{F}}\cdot\boldsymbol{\tau} = \overset{-1}{\mathbf{U}}\cdot\mathbf{R}^*\cdot\boldsymbol{\tau} = \boldsymbol{\tau}^*\cdot\mathbf{R}\cdot\overset{-1}{\mathbf{U}}$$

into Eq. (3.16) leads to another form of the Jaumann tensor

(3.17) $$\mathbf{S} = \frac{1}{2}(\boldsymbol{\tau}^*\cdot\mathbf{R}+\mathbf{R}^*\cdot\boldsymbol{\tau}).$$

4. Conjugate variables and Legendre transformation

The mathematical statement of elasticity is the one to one stress-strain correspondence. The hyperelastic material possesses a stored-energy function Σ (per unit volume in the reference configuration), which may be regarded as a function either of any strain measure, for example $\mathbf{E}$, $\mathbf{U}$ etc. or of the deformation gradient $\mathbf{F}$. But Σ depends on $\mathbf{F}$ only through a strain measure. For the sake of brevity, the symbol δ will be frequently replaced by a superposed dot. In any virtual displacement $\dot{\mathbf{u}} \equiv \delta\mathbf{u}$ the virtual work absorbed by the body, i.e. the increment of the stored-energy function, is given by the formula

(4.1) $$\dot{\Sigma} = J\mathbf{t}:(\dot{\mathbf{u}}\mathring{\nabla}).$$

The consecutive substitution of Eq. $(3.12)_{2,1}$ into Eq. (4.1) leads to

(4.2) $$\dot{\Sigma} = (\boldsymbol{\tau}\cdot\mathbf{F}^*):(\dot{\mathbf{u}}\mathring{\nabla}) = \boldsymbol{\tau}:[(\dot{\mathbf{u}}\mathring{\nabla})\cdot\mathbf{F}] = \boldsymbol{\tau}:[(\dot{\mathbf{u}}\mathring{\nabla})\cdot(\mathbf{x}\nabla)]$$
$$= \boldsymbol{\tau}:(\dot{\mathbf{u}}\nabla) = \boldsymbol{\tau}:[(\mathbf{x}+\dot{\mathbf{u}})\nabla-\mathbf{x}\nabla] = \boldsymbol{\tau}:\dot{\mathbf{F}},$$

and

(4.3) $$\dot{\Sigma} = (\mathbf{F}\cdot\mathbf{T}):\dot{\mathbf{F}} = \mathbf{T}:(\mathbf{F}^*\cdot\dot{\mathbf{F}}) = \mathbf{T}:\frac{1}{2}\,\delta(\mathbf{F}^*\cdot\mathbf{F}) = \mathbf{T}:\dot{\mathbf{E}}.$$

In extending the notion of generalized coordinates and generalized forces from analytic mechanics to mechanics of deformable bodies, any strain measure may be regarded as a generalized coordinate and the generalized stress obtained from the expression for $\dot{\Sigma}$, [18, 19]. Taking the stretch tensor $\mathbf{U}$ and substituting Eqs. (3.5) and (3.4) into Eq. (4.3), we obtain

(4.4) $$\dot{\Sigma} = \mathbf{T}:\frac{1}{2}(\mathbf{U}\cdot\dot{\mathbf{U}}+\dot{\mathbf{U}}\cdot\mathbf{U}) = \frac{1}{2}(\mathbf{T}\cdot\mathbf{U}+\mathbf{U}\cdot\mathbf{T}):\dot{\mathbf{U}} = \mathbf{S}:\dot{\mathbf{U}}.$$

$\mathbf{S}$ being just the Jaumann stress tensor as remarked above.

In the given four expressions for virtual work, only $\dot{\mathbf{E}}$ and $\dot{\mathbf{U}}$ are the increments of strain measures. Hence, also only the associated **T** and **S** are consistent with the requirement of the above-mentioned notion of generalized stress. In regarding **E** and **U** as the argument of Σ, from Eqs. (4.3) and (4.4) we have

$$\mathbf{T}(\mathbf{E}) = \frac{d\Sigma}{d\mathbf{E}} \tag{4.5}$$

and

$$\mathbf{S}(\mathbf{U}) = \frac{d\Sigma}{d\mathbf{U}}, \tag{4.6}$$

respectively. In view of one to one stress-strain correspondence, **T(E)** and **S(U)** are invertible in the global sense. The pairing of variables like those are called conjugate stress-strain variables [19]. For every pair of conjugate variables, through Legendre transformation, a corresponding complementary energy density may be defined:

$$\Sigma^c(\mathbf{T}) = \mathbf{T}:\mathbf{E}(\mathbf{T}) - \Sigma[\mathbf{E}(\mathbf{T})], \tag{4.7}$$

$$\tilde{\Sigma}^c(\mathbf{S}) = \mathbf{S}:\mathbf{U}(\mathbf{S}) - \Sigma[\mathbf{U}(\mathbf{S})]. \tag{4.8}$$

We also have

$$\mathbf{E}(\mathbf{T}) = \frac{d\Sigma^c}{d\mathbf{T}}, \tag{4.9}$$

$$\mathbf{U}(\mathbf{S}) = \frac{d\tilde{\Sigma}^c}{d\mathbf{S}}. \tag{4.10}$$

The identity

$$\mathbf{S}:\mathbf{U} = (\mathbf{T}\cdot\mathbf{U}):\mathbf{U} = \mathbf{T}:\mathbf{U}^2 = 2\mathbf{T}:\mathbf{E} + \operatorname{tr}\mathbf{T}$$

yields

$$\tilde{\Sigma}^c = \Sigma^c + \mathbf{T}:\mathbf{E} + \operatorname{tr}\mathbf{T}. \tag{4.11}$$

For commodity Σ^c and $\tilde{\Sigma}^c$ will conventionally be referred to as the first and second complementary energy, respectively. It will be seen that the transformation (4.7) leads to Reissner's principle.

Regarding Σ as a composed function of **F**, Eq. (4.2) yields

$$\boldsymbol{\tau}(\mathbf{F}) = \frac{d\Sigma}{d\mathbf{F}}. \tag{4.12}$$

The deformation gradient **F** involves not only strain, but also rotation. Hence the Piola tensor $\boldsymbol{\tau}$ associated with **F** in Eq. (4.2) does not meet wholely the notion requirement of a generalized stress. But due to the form (4.2), to a certain extent, $\boldsymbol{\tau}$ and **F** may also be called conjugate variables [19]. The rotational part of $\dot{\mathbf{F}}$ does not contribute to $\dot{\Sigma}$. This may be seen from the following expression:

$$\begin{aligned} \boldsymbol{\tau}:\dot{\mathbf{F}} &= \boldsymbol{\tau}:(\mathbf{R}\cdot\dot{\mathbf{U}} + \dot{\mathbf{R}}\cdot\mathbf{U}) = (\boldsymbol{\tau}^*\cdot\mathbf{R}):\dot{\mathbf{U}} + \boldsymbol{\tau}:(\dot{\mathbf{R}}\cdot\mathbf{R}^*\cdot\mathbf{F}) \\ &= \frac{1}{2}(\boldsymbol{\tau}^*\cdot\mathbf{R} + \mathbf{R}^*\cdot\boldsymbol{\tau}):\dot{\mathbf{U}} + (\boldsymbol{\tau}\cdot\mathbf{F}^*):(\dot{\mathbf{R}}\cdot\mathbf{R}^*) = \mathbf{S}:\dot{\mathbf{U}} + J\mathbf{t}:(\dot{\mathbf{R}}\cdot\mathbf{R}^*). \end{aligned} \tag{4.13}$$

The first term corresponds to the increment of pure strain while the last to the rotation. As a result of the antisymmetry of $\dot{\mathbf{R}} \cdot \mathbf{R}^*$ and the symmetry of $\mathbf{t}$, the last term equals zero. In comparing with Eq. (4.4) this conclusion is quite obvious. $\mathbf{F}$ determines uniquely the strain and then the stress state, but a stress state may correspond to the same strain state under various rotations. The multivalued inverse of $\boldsymbol{\tau}(\mathbf{F})$ is entirely apprehensible. When $\boldsymbol{\tau}(\mathbf{F})$ is invertible, in view of

$$\mathbf{S}:\mathbf{U} = (\boldsymbol{\tau}^* \cdot \mathbf{R}):\mathbf{U} = \boldsymbol{\tau}:(\mathbf{R} \cdot \mathbf{U}) = \boldsymbol{\tau}:\mathbf{F}, \tag{4.14}$$

the transformation (4.8) can be rewritten as

$$\tilde{\Sigma}^c(\boldsymbol{\tau}) = \boldsymbol{\tau}:\mathbf{F}(\boldsymbol{\tau}) - \Sigma[\mathbf{F}(\boldsymbol{\tau})] \tag{4.15}$$

and then we have

$$\mathbf{F}(\boldsymbol{\tau}) = \frac{d\Sigma^c}{d\boldsymbol{\tau}}. \tag{4.16}$$

It should be remarked that $\tilde{\Sigma}^c(\boldsymbol{\tau})$ can be a composed function of $\boldsymbol{\tau}$ only through some stress measure independent of rotation, such as $\boldsymbol{\tau}^* \cdot \boldsymbol{\tau}$. The transformation (4.15) leads to Levinson's principle.

In case of the lack of invertibility of $\boldsymbol{\tau}(\mathbf{F})$, in conformity with Eq. (3.17) and the invertibility of $\mathbf{S}(\mathbf{U})$, the right-hand side of Eq. (4.8) depends on $\boldsymbol{\tau}$ and $\mathbf{R}$, regarded simultaneously as independent variables. The second complementary energy may then be written as

$$\tilde{\Sigma}^c[\mathbf{S}(\boldsymbol{\tau}, \mathbf{R})] = \boldsymbol{\tau}:[\mathbf{R} \cdot \mathbf{U}(\mathbf{S})] - \Sigma[\mathbf{U}(\mathbf{S})], \tag{4.17}$$

which will lead to Fraeijs de Veubeke's principle.

5. Principle of virtual work

Let the body $\mathscr{B}$ be in equilibrium and occupy in the configuration $\mathscr{R}$ a domain V with a regular boundary A. Upon a portion A_u of A the surface displacements are prescribed: $\mathbf{u}|_{A_u} = \mathring{\mathbf{u}}$ while upon the remainder A_t the surface tractions per unit area in the configuration $\mathscr{R}$ are assigned: $\mathbf{T_N}|_{A_t} = \mathring{\mathbf{T}}_\mathbf{N}$, where $\mathbf{N}$ is the exterior unit normal vector to the boundary in $\mathscr{R}$. We shall confine ourselves to the dead loading.

Let the solution to the elasticity problem be called the real displacement field $\mathbf{u}$ and the real Piola stress field $\boldsymbol{\tau}$. The nomenclature "real" is used in order to be distinguished from the following notions:

1) Kinematically admissible displacement fields $\overset{*}{\mathbf{u}}$: which are those that are sufficiently smooth and satisfy the geometric boundary conditions

$$\overset{*}{\mathbf{u}}|_{A_u} = \mathring{\mathbf{u}}. \tag{5.1}$$

2) Statically admissible stress fields $\underset{*}{\boldsymbol{\tau}}$: which are those that satisfy the force equilibrium equation

$$\underset{*}{\boldsymbol{\tau}} \cdot \boldsymbol{\nabla} = 0 \tag{5.2}$$

and the prescribed traction boundary conditions

(5.3) $$\underset{*}{\boldsymbol{\tau}}\cdot \mathbf{N}|_{A_t} = \mathring{\mathbf{T}}_{\mathbf{N}}.$$

For any independent admissible displacement field $\overset{*}{\mathbf{u}}$ and admissible stress field $\underset{*}{\boldsymbol{\tau}}$, the following integral relation is valid

(5.4) $$\int_{A_u} \mathring{\mathbf{u}}\cdot \underset{*}{\boldsymbol{\tau}}\cdot \mathbf{N}dA + \int_{A_t} \overset{*}{\mathbf{u}}\cdot \mathring{\mathbf{T}}_{\mathbf{N}}dA = \oint_{A} \overset{*}{\mathbf{u}}\cdot \underset{*}{\boldsymbol{\tau}}\cdot \mathbf{N}dA = \int_{V} (\overset{*}{\mathbf{u}}\cdot \underset{*}{\boldsymbol{\tau}})\cdot \nabla dV = \int_{V} \underset{*}{\boldsymbol{\tau}}:(\overset{*}{\mathbf{u}}\nabla)\, dV,$$

in which Eqs. (5.1)–(5.3) have been taken into account. The relation (5.4) has been supplied first by Vorobyev, and then repeated by Novozhilov in his treatise [20].

Consecutive substitution of $\mathbf{u}+\dot{\mathbf{u}}$ and $\mathbf{u}$ for $\overset{*}{\mathbf{u}}$ in Eq. (5.4) and subtraction of the resulting equations from each other lead, for the real stress, to the virtual displacement principle

(5.5) $$\int_{A_t} \dot{\mathbf{u}}\cdot \mathring{\mathbf{T}}_{\mathbf{N}}dA = \int_{V} \boldsymbol{\tau}:(\dot{\mathbf{u}}\nabla)dV.$$

In a similar manner, from Eq. (5.4) we can obtain the virtual stress principle

(5.6) $$\int_{A_u} \mathring{\mathbf{u}}\cdot \dot{\boldsymbol{\tau}}\cdot \mathbf{N}dA = \int_{V} \dot{\boldsymbol{\tau}}:(\mathbf{u}\nabla)dV.$$

The virtual displacement principle is the starting point for derivation of the total potential energy theorem while the various complementary energy theorems can be derived from the virtual stress principle. Both principles originating from Eq. (5.4) may be conventionally called the principles of virtual work.

6. Classical variational principles

Inserting Eq. (4.2) into the virtual displacement principle (5.5), we obtain

(6.1) $$\int_{V} \dot{\Sigma}dV - \int_{A_t} \dot{\mathbf{u}}\cdot \mathring{\mathbf{T}}_{\mathbf{N}}dA = 0$$

and, consequently,

(6.2) $$\delta\Big(\int_{V} \Sigma dV - \int_{A_t} \mathbf{u}\cdot \mathring{\mathbf{T}}_{\mathbf{N}}dA\Big) = 0,$$

since $\mathring{\mathbf{T}}_{\mathbf{N}}$ is a dead loading. The functional

(6.3) $$\Pi(\mathbf{u}) = \int_{V} \Sigma(\mathbf{F})dV - \int_{A_t} \mathbf{u}\cdot \mathring{\mathbf{T}}_{\mathbf{N}}dA$$

defined for the class of admissible displacements is called the total potential energy. The formula (6.2) states that the real displacement makes the total potential energy stationary. Conversely, it will be shown that an admissible stress field $\boldsymbol{\tau}$ which satisfies the moment equilibrium condition corresponds, through the constitutive relation (4.12), to the admissible displacement field $\mathbf{u}$ which makes Π stationary. To this end, using

$$\dot{\Sigma} = \frac{d\Sigma}{d\mathbf{F}} : \dot{\mathbf{F}} = \boldsymbol{\tau} : (\dot{\mathbf{u}}\nabla) = (\dot{\mathbf{u}} \cdot \boldsymbol{\tau}) \cdot \nabla - \dot{\mathbf{u}} \cdot (\boldsymbol{\tau} \cdot \nabla) \tag{6.4}$$

and $\dot{\mathbf{u}}|_{A_u} = 0$, we compute the variation of $\Pi(\mathbf{u})$:

$$\dot{\Pi} = \int_V [(\dot{\mathbf{u}} \cdot \boldsymbol{\tau}) \cdot \nabla - \dot{\mathbf{u}} \cdot (\boldsymbol{\tau} \cdot \nabla)] dV - \int_{A_t} \dot{\mathbf{u}} \cdot \mathring{\mathbf{T}}_{\mathbf{N}} dA \tag{6.5}$$

$$= \int_{A_t} \dot{\mathbf{u}} \cdot (\boldsymbol{\tau} \cdot \mathbf{N} - \mathring{\mathbf{T}}_{\mathbf{N}}) dA - \int_V \dot{\mathbf{u}} \cdot (\boldsymbol{\tau} \cdot \nabla) dV.$$

From $\dot{\Pi} = 0$ and the arbitrariness of $\dot{\mathbf{u}}$ in V and on A_t it follows that

$$\boldsymbol{\tau} \cdot \nabla = 0 \quad \text{in} \quad V, \tag{6.6}$$

$$\boldsymbol{\tau} \cdot \mathbf{N} = \mathring{\mathbf{T}}_{\mathbf{N}} \quad \text{on} \quad A_t. \tag{6.7}$$

Supposing $\Sigma(\mathbf{F})$ is a composed function of $\mathbf{F}$ through $\mathbf{C} = \mathbf{F}^* \cdot \mathbf{F}$, we have

$$\boldsymbol{\tau} = \frac{d\Sigma}{d\mathbf{F}} = 2\mathbf{F} \cdot \frac{d\Sigma}{d\mathbf{C}} \quad \text{and} \quad \boldsymbol{\tau}^* = 2\frac{d\Sigma}{d\mathbf{C}} \cdot \mathbf{F}^*, \tag{6.8}$$

the symmetry of $d\Sigma/d\mathbf{C}$ having been used. From Eq. (6.8) it follows that

$$\boldsymbol{\tau} \cdot \mathbf{F}^* = \mathbf{F} \cdot \boldsymbol{\tau}^* \quad \text{in} \quad V. \tag{6.9}$$

This stress field $\boldsymbol{\tau}$ is real because it not only corresponds to an admissible displacement field $\mathbf{u}$, but also satisfies the conditions (6.6), (6.7) and (6.9). The variational principle associated with the functional (6.3) is called the stationary principle of total potential energy.

Finally, for the real displacement and real stress field, with the aid of the divergence theorem, Eq. (6.3) can be rewritten as

$$\Pi = \int_V \Sigma dV - \oint_A \mathbf{u} \cdot \boldsymbol{\tau} \cdot \mathbf{N} dA + \int_{A_u} \mathring{\mathbf{u}} \cdot \boldsymbol{\tau} \cdot \mathbf{N} dA = \int_V [\Sigma - \boldsymbol{\tau} : (\mathbf{u}\nabla)] dA + \int_{A_u} \mathring{\mathbf{u}} \cdot \boldsymbol{\tau} \cdot \mathbf{N} dA.$$

By taking the following expression

$$\boldsymbol{\tau} : (\mathbf{u}\nabla) = (\mathbf{F} \cdot \mathbf{T}) : (\mathbf{u}\nabla) = (\mathbf{u}\nabla) : \mathbf{T} + [(\nabla\mathbf{u}) \cdot (\mathbf{u}\nabla)] : \mathbf{T}$$

$$= \frac{1}{2}[\nabla\mathbf{u} + \nabla\mathbf{u} + (\nabla\mathbf{u}) \cdot (\mathbf{u}\nabla)] : \mathbf{T} + \frac{1}{2}[(\nabla\mathbf{u}) \cdot (\mathbf{u}\nabla)] : \mathbf{T} = \mathbf{E} : \mathbf{T} + \frac{1}{2}[(\nabla\mathbf{u}) \cdot (\mathbf{u}\nabla)] : \mathbf{T}$$

into account, the total potential energy in the real state can be expressed in terms of real displacement and real stress:

$$\Pi'(\mathbf{u}, \mathbf{T}) = \int_V \left\{ \Sigma(\mathbf{E}) - \mathbf{T} : \mathbf{E} - \frac{1}{2}[(\nabla\mathbf{u}) \cdot (\mathbf{u}\nabla)] : \mathbf{T} \right\} dV + \int_{A_u} \mathring{\mathbf{u}} \cdot \mathbf{F} \cdot \mathbf{T} \cdot \mathbf{N} dA. \tag{6.10}$$

The meaning of Π' will be explained later.

We proceed now to derive the stationary principle of complementary energy, i.e. Reisner's principle, from the virtual stress principle (5.6). Using the first complementary energy defined in Eq. (4.7) and taking account of Eqs. (4.9) and (3.5), we have

$$(6.11)\qquad \dot{\boldsymbol{\tau}}:(\mathbf{u}\nabla) = \delta(\mathbf{F}\cdot\mathbf{T}):(\mathbf{u}\nabla) = (\dot{\mathbf{F}}\cdot\mathbf{T}):(\mathbf{u}\nabla)+(\mathbf{F}\cdot\dot{\mathbf{T}}):(\mathbf{u}\nabla)$$
$$= [(\mathbf{u}\nabla)\cdot\dot{\mathbf{F}}]:\mathbf{T}+[(\nabla\mathbf{u})\cdot\mathbf{F}]:\dot{\mathbf{T}} = [(\nabla\mathbf{u})\cdot(\dot{\mathbf{u}}\nabla)]:\mathbf{T}+[\nabla\mathbf{u}+(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\dot{\mathbf{T}}$$
$$= \frac{1}{2}\,\delta[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T}+\frac{1}{2}\,[\mathbf{u}\nabla+\nabla\mathbf{u}+2(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\dot{\mathbf{T}}$$
$$= \mathbf{E}:\dot{\mathbf{T}}+\frac{1}{2}\,\delta\{[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T}\} = \delta\left\{\Sigma^c+\frac{1}{2}\,[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T}\right\}.$$

Substituting Eq. (6.11) into Eq. (5.6), we obtain

$$(6.12)\qquad \delta\left\{\int_V\left[\Sigma^c+\frac{1}{2}((\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)):\mathbf{T}\right]dV-\int_{A_u}\mathbf{u}\cdot\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}dA\right\} = 0.$$

Here the displacement $\mathbf{u}$ and stress $\mathbf{T}$ occur simultaneously. The functional

$$(6.13)\qquad \Pi^c(\mathbf{T},\mathbf{u}) = \int_V\left\{\Sigma^c(\mathbf{T})+\frac{1}{2}\,[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T}\right\}dV-\int_{A_u}\mathring{\mathbf{u}}\cdot\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}dA$$

defined for admissible displacement fields([2]) and admissible stress fields([3]) is called the total complementary energy. The formula (6.12) states that the real displacement and real stress make the total complementary energy stationary. The converse is also true: the admissible stress $\mathbf{T}$ and admissible displacement $\mathbf{u}$ which make $\Pi^c(\mathbf{T},\mathbf{u})$ stationary correspond to each other and $\mathbf{u}$ satisfies the boundary condition (5.1). Thus they constitute the solution to the problem. In order to prove the converse, it is necessary to release the variable $\mathbf{T}$ from constraints (the force equilibrium equation in V). Incorporating also the surface traction condition into the functional, $\mathbf{T}$ becomes completely free of any restrictions. The introduction of the Lagrangian multipliers $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ (vector fields defined in V and on A_t, respectively) furnishes a functional without subsidiary conditions:

$$(6.14)\qquad \overset{*}{\Pi}{}^c = \int_V\left\{\Sigma^c(\mathbf{T})+\frac{1}{2}\,[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T}+\boldsymbol{\eta}\cdot[(\mathbf{F}\cdot\mathbf{T})\cdot\nabla]\right\}dV$$
$$-\int_{A_u}\mathring{\mathbf{u}}\cdot\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}dA+\int_{A_t}\boldsymbol{\xi}\cdot(\mathring{\mathbf{T}}_{\mathbf{N}}-\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N})dA,$$

([2]) The presence of the prescribed boundary value $\mathring{\mathbf{u}}$ in the surface integral on A_u makes it possible to omit the condition (5.1) required for the admissible displacements, because this condition will appear as the natural boundary condition of the variational problem.

([3]) The definition of admissible stresses $\underset{*}{\mathbf{T}}$ undergoes also some modification: $\underset{*}{\mathbf{T}}$ are those stresses which are symmetric and satisfy the equilibrium equation

$$(\overset{*}{\mathbf{F}}\cdot\mathbf{T})\cdot\nabla = 0$$

and the boundary condition

$$\overset{*}{\mathbf{F}}\cdot\underset{*}{\mathbf{T}}\cdot\mathbf{N}|_{A_t} = \mathring{\mathbf{T}}_{\mathbf{N}}.$$

It must be remarked that $\underset{*}{\mathbf{T}}$ is always considered simultaneously with some $\overset{*}{\mathbf{u}}$.

in which $\mathbf{T}$, $\mathbf{u}$, $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are independent free variables. Now temporarily keeping $\mathbf{T}$, $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ unchanged, we calculate the variation of $\overset{*}{\Pi}{}^c$:

$$(6.15)\quad \delta\overset{*}{\Pi}{}^c = \int_V \{[(\nabla\mathbf{u})\cdot(\dot{\mathbf{u}}\nabla)]:\mathbf{T}+\boldsymbol{\eta}\cdot[(\dot{\mathbf{F}}\cdot\mathbf{T})\cdot\nabla]\}dV$$

$$-\int_{A_u}\mathring{\mathbf{u}}\cdot\dot{\mathbf{F}}\cdot\mathbf{T}\cdot\mathbf{N}dA-\int_{A_t}\boldsymbol{\xi}\cdot\dot{\mathbf{F}}\cdot\mathbf{T}\cdot\mathbf{N}dA$$

$$=\int_V [(\mathbf{u}-\boldsymbol{\eta})\nabla]:\dot{\boldsymbol{\tau}}dV+\int_{A_u}(\boldsymbol{\eta}-\mathring{\mathbf{u}})\cdot\dot{\boldsymbol{\tau}}\cdot\mathbf{N}dA+\int_{A_t}(\boldsymbol{\eta}-\boldsymbol{\xi})\cdot\dot{\boldsymbol{\tau}}\cdot\mathbf{N}dA,$$

where $\dot{\boldsymbol{\tau}} = \dot{\mathbf{F}}\cdot\mathbf{T}$ is the increment of the Piola tensor field $\boldsymbol{\tau}$ caused by $\dot{\mathbf{u}}$. $\boldsymbol{\tau}$ may be regarded as entirely arbitrary. From $\delta\overset{*}{\Pi}{}^c = 0$ follows the interpretation of the Lagrangian multipliers $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$:

$$(6.16)\qquad (\mathbf{u}-\boldsymbol{\eta})\nabla = 0 \quad \text{in} \quad V,$$

$$(6.17)\qquad \boldsymbol{\eta} = \mathring{\mathbf{u}} \quad \text{on} \quad A_u,$$

$$(6.18)\qquad \boldsymbol{\xi} = \boldsymbol{\eta} \quad \text{on} \quad A_t.$$

According to Eq. (6.16), the vector field $\boldsymbol{\eta}$ can differ from the displacement field at the most by a constant field. By virtue of Eq. (6.17) and continuity it follows that $\boldsymbol{\eta} = \mathbf{u}$ in V. Furthermore, Eq. (6.18) implies $\boldsymbol{\xi} = \mathbf{u}$ on A_t. Elimination of $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ from Eq. (6.14) yields

$$(6.19)\quad \overset{*}{\Pi}{}^c(\mathbf{T},\mathbf{u}) = \int_V \left\{\Sigma^c(\mathbf{T})+\frac{1}{2}[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T}+\mathbf{u}\cdot[(\mathbf{F}\cdot\mathbf{T})\cdot\nabla]\right\}dV$$

$$-\int_{A_u}\mathring{\mathbf{u}}\cdot\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}dA-\int_{A_t}\mathbf{u}\cdot(\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}-\mathring{\mathbf{T}}_{\mathbf{N}})dA.$$

This is a functional with two free variables $\mathbf{T}$ and $\mathbf{u}$. Now with the aid of $\overset{*}{\Pi}{}^c(\mathbf{T},\mathbf{u})$, we proceed to prove the converse statement. Rewriting $\overset{*}{\Pi}{}^c(\mathbf{T},\mathbf{u})$ as

$$(6.20)\quad \overset{*}{\Pi}{}^c(\mathbf{T},\mathbf{u}) = \int_V \left\{\Sigma^c(\mathbf{T})+\frac{1}{2}[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T}-(\mathbf{u}\nabla):(\mathbf{F}\cdot\mathbf{T})\right\}dV$$

$$+\int_{A_u}(\mathbf{u}-\mathring{\mathbf{u}})\cdot\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}dA+\int_{A_t}\mathbf{u}\cdot\mathring{\mathbf{T}}_{\mathbf{N}}dA$$

and calculating its variation

$$(6.21)\quad \delta\overset{*}{\Pi}{}^c = \int_V \left\{\frac{d\Sigma^c}{d\mathbf{T}}:\dot{\mathbf{T}}+\frac{1}{2}[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\dot{\mathbf{T}}+[(\nabla\mathbf{u})\cdot(\dot{\mathbf{u}}\nabla)]:\mathbf{T}\right.$$

$$\left.-(\dot{\mathbf{u}}\nabla):(\mathbf{F}\cdot\mathbf{T})-(\mathbf{u}\nabla):(\dot{\mathbf{F}}\cdot\mathbf{T})-(\mathbf{u}\nabla):(\dot{\mathbf{F}}\cdot\mathbf{T})\right\}dV$$

$$+\int_{A_u}\dot{\mathbf{u}}\cdot\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}dA+\int_{A_u}(\mathbf{u}-\mathring{\mathbf{u}})\cdot\delta(\mathbf{F}\cdot\mathbf{T})\cdot\mathbf{N}dA+\int_{A_t}\dot{\mathbf{u}}\cdot\mathring{\mathbf{T}}_{\mathbf{N}}dA$$

$$= \int_V \left\{ \left[\frac{d\Sigma^c}{d\mathbf{T}} - \frac{1}{2} (\mathbf{u}\nabla + \nabla\mathbf{u} + (\nabla\mathbf{u}) \cdot (\mathbf{u}\nabla)) \right] : \dot{\mathbf{T}} + [(\mathbf{F}\cdot\mathbf{T})\cdot\nabla]\cdot\dot{\mathbf{u}} \right\} dV$$

$$+ \int_{A_u} (\mathbf{u} - \mathring{\mathbf{u}}) \cdot \delta(\mathbf{F}\cdot\mathbf{T}) \cdot \mathbf{N} dA - \int_{A_t} \dot{\mathbf{u}} \cdot (\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N} - \mathring{\mathbf{T}}_{\mathbf{N}}) dA,$$

we finally arrive at

(6.22) $$(\mathbf{F}\cdot\mathbf{T})\cdot\nabla = 0 \quad \text{in} \quad V,$$

(6.23) $$\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N} = \mathring{\mathbf{T}}_{\mathbf{N}} \quad \text{on} \quad A_t,$$

(6.24) $$\mathbf{u} = \mathring{\mathbf{u}} \quad \text{on} \quad A_u,$$

(6.25) $$\frac{d\Sigma^c}{d\mathbf{T}} = \frac{1}{2} [\mathbf{u}\nabla + \nabla\mathbf{u} + (\nabla\mathbf{u}) \cdot (\mathbf{u}\nabla)] \quad \text{in} \quad V.$$

Equation (6.25) confirms that $\mathbf{T}$ and $\mathbf{u}$ which make $\overset{*}{\Pi}{}^c(\mathbf{T}, \mathbf{u})$ (and then $\Pi^c(\mathbf{T}, \mathbf{u})$) stationary correspond to each other through the constitutive relation (4.9). Thus $\mathbf{T}$ and $\mathbf{u}$ solve the problem.

Making use of Eq. (4.7) and comparing Eqs. (6.10) and (6.13), we find $\Pi^c = -\Pi'$, i.e. for the real displacement and real stress there exists a complementary relation between the total potential energy and total complementary energy:

(6.26) $$\Pi + \Pi^c \overset{r}{=} 0.$$

Here r denotes the equality which holds only by the real state. It should be observed that $\Pi(\mathbf{u})$ and $\Pi^c(\mathbf{T}, \mathbf{u})$ are two distinct functionals and Eq. (6.26) states merely that they have an equal value at the stationary point.

7. Generalized variational principles

Both classical principles are variational problems with subsidiary conditions and would be inconvenient in application. Consequently, the idea of generalized variation has been developed. The Lagrangian multipliers incorporate directly the constraints into the original functional to furnish an associated functional without subsidiary conditions. In fact, we have already done something of this kind in proving Reissner's principle.

Let us now derive the generalized functional from the total potential energy $\Pi(\mathbf{u})$, Eq. (6.3). $\mathbf{u}$ is the independent variable in $\Pi(\mathbf{u})$, but the stored-energy function Σ involved depends on $\mathbf{u}$ only through some strain measure, say $\mathbf{E}$ in Eq. (3.5). Incorporating Eq. (3.5) and the displacement boundary condition (5.1) into Π, the Lagrangian multipliers $\boldsymbol{\sigma}$ (the tensor field defined in V) and $\boldsymbol{\xi}$ (the vector field defined on A_u) furnish

(7.1) $$\overset{*}{\Pi} = \int_V \Sigma(\mathbf{E}) dV + \int_V \left\{ \frac{1}{2} [\mathbf{u}\nabla + \nabla\mathbf{u} + (\nabla\mathbf{u}) \cdot (\mathbf{u}\nabla)] - \mathbf{E} \right\} : \boldsymbol{\sigma} dV$$

$$- \int_{A_t} \mathbf{u} \cdot \mathring{\mathbf{T}}_{\mathbf{N}} dA + \int_{A_u} (\mathbf{u} - \mathbf{u}) \cdot \boldsymbol{\xi} dA,$$

in which $\mathbf{E}$, $\mathbf{u}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\xi}$ are independent variables. Temporarily keeping $\boldsymbol{\sigma}$ and $\boldsymbol{\xi}$ unchanged and using Eq. (4.3), we calculate the variation of $\overset{*}{\Pi}$:

$$(7.2)\qquad \delta\overset{*}{\Pi} = \int_V \{(\mathbf{T}-\boldsymbol{\sigma}):\dot{\mathbf{E}}+(\dot{\mathbf{u}}\nabla):\boldsymbol{\sigma}+(\dot{\mathbf{u}}\nabla):[(\mathbf{u}\nabla)\cdot\boldsymbol{\sigma}]\}dV - \int_{A_t} \dot{\mathbf{u}}\cdot\mathring{\mathbf{T}}_{\mathbf{N}}dA - \int_{A_u}\dot{\mathbf{u}}\cdot\boldsymbol{\xi}dA.$$

Using

$$\int_V \{(\dot{\mathbf{u}}\nabla):\boldsymbol{\sigma}+(\dot{\mathbf{u}}\nabla):[(\mathbf{u}\nabla)\cdot\boldsymbol{\sigma}]\}dV = \int_V (\dot{\mathbf{u}}\nabla):(\mathbf{F}\cdot\boldsymbol{\sigma})dV$$

$$= \oint_A \dot{\mathbf{u}}\cdot\mathbf{F}\cdot\boldsymbol{\sigma}\cdot\mathbf{N}dA - \int_V \dot{\mathbf{u}}\cdot[(\mathbf{F}\cdot\boldsymbol{\sigma})\cdot\nabla]dV$$

as a result of the application of the divergence theorem, Eq. (7.2) becomes

$$(7.3)\qquad \delta\overset{*}{\Pi} = \int_V \{(\mathbf{T}-\boldsymbol{\sigma}):\dot{\mathbf{E}}-[(\mathbf{F}\cdot\boldsymbol{\sigma})\cdot\nabla]\cdot\dot{\mathbf{u}}\}dV + \int_{A_t}(\mathbf{F}\cdot\boldsymbol{\sigma}\cdot\mathbf{N}-\mathring{\mathbf{T}}_{\mathbf{N}})\cdot\dot{\mathbf{u}}dA$$
$$+\int_{A_u}(\mathbf{F}\cdot\boldsymbol{\sigma}\cdot\mathbf{N}-\boldsymbol{\xi})\cdot\dot{\mathbf{u}}dA.$$

From the first term in the volume integral and the last surface integral, the interpretation of Lagrangian multipliers follows:

$$(7.4)\qquad \boldsymbol{\sigma} = \mathbf{T} \quad \text{in} \quad V,$$

$$(7.5)\qquad \boldsymbol{\xi} = \mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N} \quad \text{on} \quad A_u.$$

As a result, we arrive at the following expression:

$$(7.6)\qquad \overset{*}{\Pi}(\mathbf{u},\mathbf{T}) = \int_V \left\{\Sigma(\mathbf{E})-\left[\mathbf{E}-\frac{1}{2}(\mathbf{u}\nabla+\nabla\mathbf{u}+(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla))\right]:\mathbf{T}\right\}dV$$
$$-\int_{A_t}\mathbf{u}\cdot\mathring{\mathbf{T}}_{\mathbf{N}}dA - \int_{A_u}(\mathbf{u}-\mathring{\mathbf{u}})\cdot\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}dA.$$

Here $\mathbf{E}$ is not an independent variable because it depends on $\mathbf{T}$ through the constitutive relation (4.5). The functional $\overset{*}{\Pi}(\mathbf{u},\mathbf{T})$ with the two free variables $\mathbf{u}$ and $\mathbf{T}$ may be called the generalized total potential energy. Computing the variation of $\overset{*}{\Pi}(\mathbf{u},\mathbf{T})$

$$(7.7)\qquad \delta\overset{*}{\Pi} = \int_V \left\{\left[\frac{1}{2}(\mathbf{u}\nabla+\nabla\mathbf{u}+(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla))-\mathbf{E}\right]:\dot{\mathbf{T}}-[(\mathbf{F}\cdot\mathbf{T})\cdot\nabla]\cdot\dot{\mathbf{u}}\right\}dV$$
$$+\int_{A_t}(\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}-\mathring{\mathbf{T}}_{\mathbf{N}})\cdot\dot{\mathbf{u}}dA + \int_{A_u}(\mathring{\mathbf{u}}-\mathbf{u})\cdot\delta(\mathbf{F}\cdot\mathbf{T})\cdot\mathbf{N}dA,$$

we can assert that $\mathbf{u}$, $\mathbf{T}$ and $\mathbf{E}$ $\left(\mathbf{E}\text{ depends on }\mathbf{T}\text{ through }\mathbf{T}=\dfrac{d\Sigma}{d\mathbf{E}}\right)$, which make $\overset{*}{\Pi}(\mathbf{u},\mathbf{T})$ stationary, satisfy

$$\mathbf{u} = \mathring{\mathbf{u}} \quad \text{on} \quad A_u, \tag{7.8}$$

$$\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N} = \mathring{\mathbf{T}}_{\mathbf{N}} \quad \text{on} \quad A_t, \tag{7.9}$$

$$(\mathbf{F}\cdot\mathbf{T})\cdot\nabla = 0 \quad \text{in} \quad V, \tag{7.10}$$

$$\mathbf{E} = \frac{1}{2}[\mathbf{u}\nabla+\nabla\mathbf{u}+(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)] \quad \text{in} \quad V. \tag{7.11}$$

Equation (7.11) confirms that **T** through the constitutive relation (4.5) corresponds to **u** while Eqs. (7.8)–(7.10) assert that the corresponding pair **u** and **T** are simultaneously the admissible displacement and stress and then provide a solution to our problem. The variational problem associated with the free functional $\overset{*}{\Pi}(\mathbf{u}, \mathbf{T})$ may be called the stationary principle of generalized potential energy.

Making use of

$$\frac{1}{2}[\mathbf{u}\nabla+\nabla\mathbf{u}+(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T} = (\mathbf{u}\nabla):(\mathbf{F}\cdot\mathbf{T})-\frac{1}{2}[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T} \tag{7.12}$$

$$= (\mathbf{u}\cdot\mathbf{F}\cdot\mathbf{T})\cdot\nabla-\mathbf{u}\cdot[(\mathbf{F}\cdot\mathbf{T})\cdot\nabla]-\frac{1}{2}[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T},$$

the divergence theorem and Eq. (4.7) and changing the sign, the generalized total potential energy reduces to an equivalent form:

$$\overset{*}{\Pi}{}^c(\mathbf{u}, \mathbf{T}) = \int_V \left\{\Sigma^c(\mathbf{T})+\frac{1}{2}[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T}+\mathbf{u}\cdot[(\mathbf{F}\cdot\mathbf{T})\cdot\nabla]\right\}dV \tag{7.13}$$

$$-\int_{A_t}\mathbf{u}\cdot(\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}-\mathring{\mathbf{T}}_{\mathbf{N}})dA-\int_{A_u}\mathring{\mathbf{u}}\cdot\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{N}dA.$$

This is just the free functional (6.19), which appeared in proving Reissner's principle. If Eq. (6.19) were called analogically the generalized total complementary energy, then we would have a stationary principle of generalized complementary energy. Between $\overset{*}{\Pi}$ and $\overset{*}{\Pi}{}^c$ there also exists a complementary relation:

$$\overset{*}{\Pi}+\overset{*}{\Pi}{}^c = 0. \tag{7.14}$$

Equation (7.12) (and then Eq. (7.14)) holds for arbitrary **u** and **T**. Therefore, the generalized potential energy theorem and the generalized complementary energy theorem in essence are stationary principles of the same functional (with a different sign) and may be generally referred to as the generalized variational principle.

8. Levinson's principle

Assume now that $\boldsymbol{\tau}(\mathrm{F})$ is invertible. Using the second complementary energy defined in Eq. (4.15), we have

$$\dot{\overset{*}{\Sigma}} = \mathbf{F}:\dot{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}}:(\mathbf{u}\nabla)+\mathrm{tr}\,\dot{\boldsymbol{\tau}}. \tag{8.1}$$

Substitution of Eq. (8.1) into the virtual stress principle (5.6) yields

$$\delta\Big\{\int_V [\tilde{\Sigma}^c(\boldsymbol{\tau})-\mathrm{tr}\,\boldsymbol{\tau}]dV-\int_{A_u}\mathring{\mathbf{u}}\cdot\boldsymbol{\tau}\cdot\mathbf{N}dA\Big\}=0. \tag{8.2}$$

For the functional

$$\Pi_1^c(\boldsymbol{\tau})=\int_V[\tilde{\Sigma}^c(\boldsymbol{\tau})-\mathrm{tr}\,\boldsymbol{\tau}]dV-\int_{A_u}\mathring{\mathbf{u}}\cdot\boldsymbol{\tau}\cdot\mathbf{N}dA \tag{8.3}$$

defined for the class of admissible stresses, Eq. (8.2) states that the real stress field $\boldsymbol{\tau}$ makes the functional stationary. In the following the converse will be proved: the admissible stress field $\boldsymbol{\tau}$ which makes Π_1^c stationary satisfies the moment equilibrium condition and has an associated displacement field satisfying the geometric boundary condition. Supposing $\tilde{\Sigma}^c(\boldsymbol{\tau})$ is a composed function of $\boldsymbol{\tau}$ through $\mathbf{K}\equiv\boldsymbol{\tau}^*\cdot\boldsymbol{\tau}$, we have

$$\mathbf{F}=\frac{d\tilde{\Sigma}^c}{d\boldsymbol{\tau}}=2\boldsymbol{\tau}\cdot\frac{d\tilde{\Sigma}^c}{d\mathbf{K}}. \tag{8.4}$$

From the symmetry of $\dfrac{d\tilde{\Sigma}^c}{d\mathbf{K}}$, the moment equilibrium condition immediately follows:

$$\boldsymbol{\tau}\cdot\mathbf{F}^*=\mathbf{F}\cdot\boldsymbol{\tau}^* \quad \text{in} \quad V. \tag{8.5}$$

In order to be able to calculate the displacement from

$$\mathbf{u}(\mathbf{X})=\int_{\mathbf{X}_0}^{\mathbf{X}}(\mathbf{F}-\mathbf{I})\cdot d\mathbf{X}+\mathring{\mathbf{u}}(\mathbf{X}_0)), \tag{8.6}$$

the quantity $\mathbf{F}$ obtained from Eq. (8.4) must satisfy the integrability condition:

$$\mathbf{F}\times\nabla=0 \quad \text{in} \quad V. \tag{8.7}$$

Instead of Lagrangian multipliers for the purpose of release from constraints, this time we introduce the stress tensor function $\boldsymbol{\Phi}$

$$\boldsymbol{\tau}=\boldsymbol{\Phi}\times\nabla, \tag{8.8}$$

which satisfies identically the force equilibrium equation (3.13). Thus the variation of Π_1^c is

$$\dot{\Pi}_1^c=\int_V(\mathbf{F}-\mathbf{I}):(\dot{\boldsymbol{\Phi}}\times\nabla)dV-\int_{A_u}\mathring{\mathbf{u}}\cdot\dot{\boldsymbol{\tau}}\cdot\mathbf{N}dA. \tag{8.9}$$

By making use of the identity

$$(\mathbf{F}-\mathbf{I}):(\dot{\boldsymbol{\Phi}}\times\nabla)=(\mathbf{F}\times\nabla):\dot{\boldsymbol{\Phi}}+\mathrm{tr}\{[(\mathbf{F}^*-\mathbf{I})\cdot\dot{\boldsymbol{\Phi}}]\times\nabla\} \tag{8.10}$$

and the divergence theorem

$$\int_V\mathrm{tr}[(\mathbf{F}^*\cdot\dot{\boldsymbol{\Phi}})\times\nabla]dV=\oint_A\mathbf{F}:(\dot{\boldsymbol{\Phi}}\times\mathbf{N})dA, \tag{8.11}$$

Eq. (8.9) reduces to

$$\dot{\Pi}_1^c=\int_V(\mathbf{F}\times\nabla):\dot{\boldsymbol{\Phi}}dV+\oint_A(\mathbf{F}-\mathbf{I}):(\dot{\boldsymbol{\Phi}}\times\mathbf{N})dA-\int_{A_u}\mathring{\mathbf{u}}\cdot\dot{\boldsymbol{\tau}}\cdot\mathbf{N}dA. \tag{8.12}$$

From the volume integral the integrability condition (8.7) follows. Thus a vector field $\mathbf{u}(\mathbf{X})$ is determined from Eq. (8.6). This vector field becomes the displacement field provided it satisfies the geometric boundary condition. To this end, substituting the vector field $\mathbf{u}(\mathbf{X})$ into the first surface integral of Eq. (8.12), the integrand function assumes the form

$$(8.13)\qquad (\mathbf{F}-\mathbf{I}):(\dot{\mathbf{\Phi}}\times\mathbf{N}) = (\mathbf{u}\nabla):(\dot{\mathbf{\Phi}}\times\mathbf{N}) = \mathbf{u}\cdot(\dot{\mathbf{\Phi}}\times\nabla)\cdot\mathbf{N}-[(\mathbf{u}\cdot\dot{\mathbf{\Phi}})\times\nabla]\cdot\mathbf{N}.$$

On the basis of the Stokes theorem which states that the flux of the curl of any vector field through a closed surface equals zero, the two surface integrals reduce to

$$(8.14)\qquad \oint_A \mathbf{u}\cdot(\dot{\mathbf{\Phi}}\times\nabla)\cdot\mathbf{N}dA-\int_{A_u}\mathring{\mathbf{u}}\cdot\dot{\boldsymbol{\tau}}\cdot\mathbf{N}dA = \int_{A_u}(\mathbf{u}-\mathring{\mathbf{u}})\cdot\dot{\boldsymbol{\tau}}\cdot\mathbf{N}dA$$

and, consequently,

$$(8.15)\qquad \mathbf{u}|_{A_u} = \mathring{\mathbf{u}},$$

where $\dot{\boldsymbol{\tau}}|_{A_t} = 0$ has been used. Thus the proof of Levinson's principle is completed.

With the help of Lagrangian multipliers, the functional without subsidiary conditions may be similarly arrived at:

$$(8.16)\qquad \overset{*}{\Pi}{}^c_1(\boldsymbol{\tau},\mathbf{u}) = \int_V [\tilde{\Sigma}^c(\boldsymbol{\tau})-\operatorname{tr}\boldsymbol{\tau}+\mathbf{u}\cdot(\boldsymbol{\tau}\cdot\nabla)]dV - \int_{A_u}\mathring{\mathbf{u}}\cdot\boldsymbol{\tau}\cdot\mathbf{N}dA-\int_{A_t}\mathbf{u}\cdot(\boldsymbol{\tau}\cdot\mathbf{N}-\mathring{\mathbf{T}}_{\mathbf{N}})dA,$$

the variation of which is

$$(8.17)\qquad \delta\overset{*}{\Pi}{}^c_1(\boldsymbol{\tau},\mathbf{u}) = \int_V [(\mathbf{F}-\mathbf{I}-\mathbf{u}\nabla):\dot{\boldsymbol{\tau}}+(\boldsymbol{\tau}\cdot\nabla)\cdot\dot{\mathbf{u}}]dV + \int_{A_u}(\mathbf{u}-\mathring{\mathbf{u}})\cdot\dot{\boldsymbol{\tau}}\cdot\mathbf{N}dA-\int_{A_t}\dot{\mathbf{u}}\cdot(\boldsymbol{\tau}\cdot\mathbf{N}-\mathring{\mathbf{T}}_{\mathbf{N}})dA.$$

From $\delta\overset{*}{\Pi}{}^c_1 = 0$, follow

$$(8.18)\qquad \begin{aligned} \boldsymbol{\tau}\cdot\nabla &= 0 && \text{in} \quad V,\\ \mathbf{u}\nabla &= \mathbf{F}-\mathbf{I} && \text{in} \quad V,\\ \boldsymbol{\tau}\cdot\mathbf{N} &= \mathring{\mathbf{T}}_{\mathbf{N}} && \text{on} \quad A_t,\\ \mathbf{u} &= \mathring{\mathbf{u}} && \text{on} \quad A_u. \end{aligned}$$

Equations (8.18), (8.5) and (8.7) assert that $\boldsymbol{\tau}$ and $\mathbf{u}$ are real stress and displacement fields. The variational principle associated with the functional (8.16) may be referred to as the generalized Levinson's principle.

For the real state, the following identity holds:

$$(8.19)\qquad \tilde{\Sigma}^c-\operatorname{tr}\boldsymbol{\tau} = \tilde{\Sigma}^c+\frac{1}{2}[(\nabla\mathbf{u})\cdot(\mathbf{u}\nabla)]:\mathbf{T}$$

and, consequently, the complementary relations:

$$(8.20)\qquad \Pi+\Pi^c_1 \doteq 0, \quad \overset{*}{\Pi}+\overset{*}{\Pi}{}^c_1 \doteq 0.$$

9. Fraeijs de Veubeke's principle

In case of the lack of invertibility of $\boldsymbol{\tau}(\mathbf{F})$, Levinson's principle is invalid. By making use of the second complementaty energy defined by Eq. (4.17)

$$\tilde{\Sigma}^c[\mathbf{S}(\boldsymbol{\tau}, \mathbf{R})] = \boldsymbol{\tau}:[\mathbf{R}\cdot\mathbf{U}(\mathbf{S})] - \Sigma[\mathbf{U}(\mathbf{S})], \tag{9.1}$$

the virtual stress principle leads to Fraeijs de Veubeke's principle based on the polar decomposition. For this case Eq. (8.1) remains valid. Substituting it into the virtual stress principle (5.6), we have

$$\delta\left\{\int_V [\tilde{\Sigma}^c(\mathbf{S}) - \operatorname{tr}\boldsymbol{\tau}]dV - \int_{A_u} \mathring{\mathbf{u}}\cdot\boldsymbol{\tau}\cdot\mathbf{N}dA\right\} = 0. \tag{9.2}$$

For the functional

$$\Pi_2^c(\boldsymbol{\tau}, \mathbf{R}) = \int_V \{\tilde{\Sigma}^c[\mathbf{S}(\boldsymbol{\tau}, \mathbf{R})] - \operatorname{tr}\boldsymbol{\tau}\}dV - \int_{A_u} \mathring{\mathbf{u}}\cdot\boldsymbol{\tau}\cdot\mathbf{N}dA \tag{9.3}$$

with the Piola stress tensor $\boldsymbol{\tau}$ and the rotation tensor $\mathbf{R}$ as an independent variables, Eq. (9.2) states that the real stress field $\boldsymbol{\tau}$ and the real rotation field $\mathbf{R}$ make Π_2^c stationary. The converse is true: the admissible stress field $\boldsymbol{\tau}$ and the rotation field $\mathbf{R}$ which make Π_2^c stationary satisfy the moment equilibrium condition and possess a unique associated displacement field $\mathbf{u}$ consistent with geometric boundary conditions. Thus these $\boldsymbol{\tau}$ and $\mathbf{R}$ are the solution to the problem. It should be noted that the moment equilibrium condition is now not guaranteed by the structure of the function $\tilde{\Sigma}^c$ as in Levinson's principle. The associated displacement means the displacement calculated in the following way: starting from $\boldsymbol{\tau}$ and $\mathbf{R}$, by virtue of Eq. (3.17) and the invertibility of $\mathbf{S}(\mathbf{U})$ compute $\mathbf{R}\cdot\mathbf{U}$, and then on the basis of Eqs. (3.2) and (3.3) perform the integration

$$\mathbf{u}(\mathbf{X}) = \int_{\mathbf{X}_0}^{\mathbf{X}} (\mathbf{R}\cdot\mathbf{U} - \mathbf{I})\cdot d\mathbf{X} + \mathring{\mathbf{u}}(\mathbf{X}_0). \tag{9.4}$$

The integrability condition is

$$(\mathbf{R}\cdot\mathbf{U})\times\nabla = 0 \quad \text{in} \quad V. \tag{9.5}$$

Making use of Eqs. (4.10) and (3.17), bearing the symmetry of the stretch tensor $\mathbf{U}$ in mind and introducing the stress tensor function $\boldsymbol{\Phi}$ as in Eq. (8.8), the integrand function in the volume integral of the variation of Π_2^c assumes the form

$$\begin{aligned} \frac{d\tilde{\Sigma}^c}{d\mathbf{S}}:\dot{\mathbf{S}} - \operatorname{tr}\dot{\boldsymbol{\tau}} &= \mathbf{U}:(\dot{\boldsymbol{\tau}}^*\cdot\mathbf{R} + \boldsymbol{\tau}^*\cdot\dot{\mathbf{R}}) - \operatorname{tr}\dot{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}}:(\mathbf{R}\cdot\mathbf{U} - \mathbf{I}) + \boldsymbol{\tau}:(\dot{\mathbf{R}}\cdot\mathbf{U}) \\ &= (\mathbf{R}\cdot\mathbf{U} - \mathbf{I}):(\dot{\boldsymbol{\Phi}}\times\nabla) + \frac{1}{2}(\boldsymbol{\tau}\cdot\mathbf{U}\cdot\mathbf{R}^* - \mathbf{R}\cdot\mathbf{U}\cdot\boldsymbol{\tau}^*):(\dot{\mathbf{R}}\cdot\mathbf{R}^*). \end{aligned} \tag{9.6}$$

Using the identity (8.10) and the divergence theorem (8.11), we obtain the variation of Π_2^c:

$$(9.7)\quad \dot{\Pi}_2^c = \int\limits_V \left\{[\mathbf{R}\cdot\mathbf{U})\times\nabla]:\dot{\mathbf{\Phi}} + \frac{1}{2}(\boldsymbol{\tau}\cdot\mathbf{U}\cdot\mathbf{R}^* - \mathbf{R}\cdot\mathbf{U}\cdot\boldsymbol{\tau}^*):(\dot{\mathbf{R}}\cdot\mathbf{R}^*)\right\}dV$$

$$+ \oint\limits_A (\mathbf{R}\cdot\mathbf{U}-\mathbf{I}):(\dot{\mathbf{\Phi}}\times\mathbf{N})dA - \int\limits_{A_u} \mathring{\mathbf{u}}\cdot\dot{\boldsymbol{\tau}}\cdot\mathbf{N}dA.$$

From the volume integral we arrive at the integrability condition (9.5) and the moment equilibrium condition:

$$(9.8)\qquad \boldsymbol{\tau}\cdot\mathbf{U}\cdot\mathbf{R}^* = \mathbf{R}\cdot\mathbf{U}\cdot\boldsymbol{\tau}^* \quad \text{in} \quad V.$$

If we replace **F** by $\mathbf{R}\cdot\mathbf{U}$, the further procedure to obtain the displacement field $\mathbf{u}(\mathbf{X})$ satisfying the geometric boundary condition is precisely the same as for the case of Levinson's principle. Thus Fraeijs de Veubeke's principle is entirely proved. Similarly, the corresponding generalized functional is

$$(9.9)\quad \overset{*}{\Pi}{}_2^c(\boldsymbol{\tau},\mathbf{R},\mathbf{u}) = \int\limits_V \{\tilde{\Sigma}^c[\mathbf{S}(\boldsymbol{\tau},\mathbf{R})] - \mathrm{tr}\,\boldsymbol{\tau} + \mathbf{u}\cdot(\boldsymbol{\tau}\cdot\nabla)\}dV$$

$$- \int\limits_{A_u} \mathring{\mathbf{u}}\cdot\boldsymbol{\tau}\cdot\mathbf{N}dA - \int\limits_{A_t} \mathbf{u}\cdot(\boldsymbol{\tau}\cdot\mathbf{N} - \mathring{\mathbf{T}}_{\mathbf{N}})dA$$

and, consequetnly, its variation

$$(9.10)\quad \delta\overset{*}{\Pi}{}_2^c = \int\limits_V [(\mathbf{R}\cdot\mathbf{U}-\mathbf{I}-\mathbf{u}\nabla):\dot{\boldsymbol{\tau}} + \frac{1}{2}(\boldsymbol{\tau}\cdot\mathbf{U}\cdot\mathbf{R}^* - \mathbf{R}\cdot\mathbf{U}\cdot\boldsymbol{\tau}^*):(\dot{\mathbf{R}}\cdot\mathbf{R}^*) + (\boldsymbol{\tau}\cdot\nabla)\cdot\dot{\mathbf{u}}]dV$$

$$+ \int\limits_{A_u} (\mathbf{u}-\mathring{\mathbf{u}})\cdot\dot{\boldsymbol{\tau}}\cdot\mathbf{N}dA - \int\limits_{A_t} \dot{\mathbf{u}}\cdot(\boldsymbol{\tau}\cdot\mathbf{N} - \mathring{\mathbf{T}}_{\mathbf{N}})dA.$$

From $\delta\overset{*}{\Pi}{}_2^c = 0$, it follows that

$$(9.11)\quad \begin{aligned} \boldsymbol{\tau}\cdot\nabla &= 0 && \text{in} \quad V,\\ \boldsymbol{\tau}\cdot\mathbf{U}\cdot\mathbf{R}^* &= \mathbf{R}\cdot\mathbf{U}\cdot\boldsymbol{\tau}^* && \text{in} \quad V,\\ \mathbf{u}\nabla &= \mathbf{R}\cdot\mathbf{U}-\mathbf{I} && \text{in} \quad V,\\ \boldsymbol{\tau}\cdot\mathbf{N} &= \mathring{\mathbf{T}}_{\mathbf{N}} && \text{on} \quad A_t,\\ \mathbf{u} &= \mathring{\mathbf{u}} && \text{on} \quad A_u. \end{aligned}$$

In addition to Eq. (9.11) the constitutive relation

$$(9.12)\qquad \mathbf{U} = \mathbf{U}(\mathbf{S}) = \mathbf{U}\left[\frac{1}{2}(\boldsymbol{\tau}^*\cdot\mathbf{R} + \mathbf{R}^*\cdot\boldsymbol{\tau})\right]$$

and the orthogonality of the rotation tensor

$$(9.13)\qquad \mathbf{R}\cdot\mathbf{R}^* = \mathbf{I}.$$

constitute the complete set of relations for our problem. The variational problem associated with the functional $\overset{*}{\Pi}{}_2^c$ may be referred to as the generalized Fraeijs de Veubeke's principle.

Similarly, the following complementary relations can be obtained:

$$(9.14)\qquad \Pi + \Pi_2^c \overset{r}{=} 0, \quad \overset{*}{\Pi} + \overset{*}{\Pi}{}_2^c \overset{r}{=} 0.$$

10. Flow diagramm

The following diagramm (Fig. 1) gives a clear picture illustrating the interconnexion of the diverse variational principles derived from the virtual work principle in unifying manner.

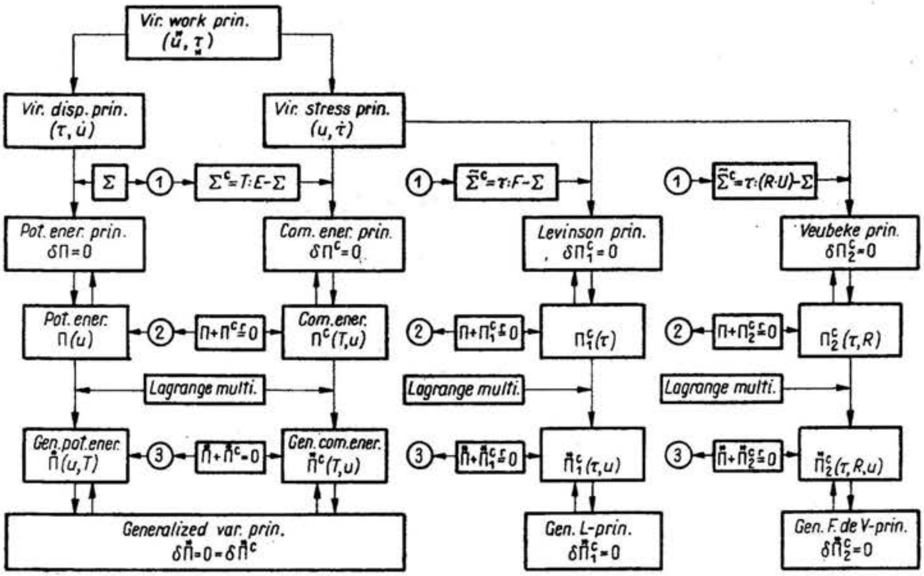

FIG. 1.

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