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Influence of hinge line gap on aerodynamic forces acting on a harmonically oscillating thin profile in an incompressible flow Part II

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BY APPLYING the method of strongly singular integral equations, the solution of the Birnbaum-Possio equations is derived for a system of two profiles (profile with a control surface) lying on one straight line parallel to the direction of flow at infinity. The solutions are then transformed to a form in which the pressure distributions and aerodynamical coefficients may explicitly be expressed in terms of the elementary functions and canonical forms of elliptic integrals. Only some of the integrals (concerning the wake in the gap and behind the profile) require numerical calculations. The influence of the size of the gap on the pressure distributions and aerodynamical coefficients (with various values of the frequency coefficient) is illustrated by graphs.

Posługując się metodami równań całkowych silnie-osobliwych, otrzymano rozwiązanie równania Birnbauma-Possio dla układu dwóch profili (profilu ze sterem) leżących na jednej prostej, równoległej do kierunku przepływu w nieskończoności. Następnie przekształcono je do postaci, w której rozkłady ciśnień i współczynniki aerodynamiczne dają się wyrazić jawnie za pomocą funkcji elementarnych i kanonicznych postaci całek eliptycznych. Tylko nieliczne całki (dotyczące śladu wirowego w szczelinie i za profilem) wymagają obliczeń numerycznych. Wpływ wielkości szczeliny na rozkłady ciśnień i współczynniki aerodynamiczne (dla różnych wartości współczynnika częstości) zilustrowano wykresami.

Послуживаясь методами интегральных сильно сингулярных уравнений, получено решение уравнения Бирнбаума-Поссио для системы двух профилей (профиля с рулем) лежащих на одной прямой, параллельной направлению течения в бесконечности. Затем преобразовано оно к виду, в котором распределения давлений и аэродинамические коэффициенты можно явным образом выразить при помощи элементарных функций и канонических видов эллиптических интегралов. Только немногие интегралы (касающиеся вихревого следа в щели и за профилем) требуют численных расчетов. Влияние величины щели на распределения давлений и аэродинамические коэффициенты (для разных значений коэффициента частоты) иллюстрировано графиками.

7. Evaluation of aerodynamic forces on a profile with a control surface

IT HAS ALREADY been mentioned before that if the boundary conditions (that is the functions $w(x)/U$) may be described by polynomials along each of the segments, then there exists a simple method of calculation of the integrals appearing in Eqs. (5.2). In order to comply with the notations generally used [9], the parameters α and β determining the gap will now be replaced with $e = \beta$ and $\delta = \beta - \alpha$. In further considerations the fundamental role will be played by a sequence of constants (elliptic integrals)

$$U_n = \frac{1}{j} \int_C \frac{(x-e)^n}{\sqrt{R(x)}} dx \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{7.1}$$

satisfying the recurrence condition

$$(7.2)\qquad U_{n+3} = -\frac{2n+3}{n+2}\left(e+\frac{\delta}{2}\right)U_{n+2}+\frac{n+1}{n+2}[1-e(e+2\delta)]U_{n+1} + \frac{2n+1}{n+2}\frac{\delta}{2}(1-e^2)U_n,$$

which may be derived by the method described for example, in [14]. The integration contour C in Eq. (7.1) is either the complete line L or the segment $(e, 1)$. In the first case the constants (7.1) are denoted by U_n', and in the second case — by U_n''. The recurrence formulae (7.2) are identical in both cases but differ by the first terms of the sequences. Direct calculation yields

$$(7.3)\qquad \begin{aligned} U_0' &= 0, & U_0'' &= -gK,\\ U_1' &= -\pi, & U_1'' &= (1+e)gK-2g\Pi(\lambda_2^2, k),\\ U_2' &= \pi\left(e+\frac{\delta}{2}\right), & U_2'' &= -eU_1''-2\frac{E}{g}+\delta g\Pi(\lambda_2^2, k). \end{aligned}$$

The sequences of the constants U_n' and U_n'' form the basis for the definition of the additional sequences V_n', V_n'' and W_n', W_n'':

$$(7.4)\qquad \begin{aligned} V_n &= U_{n+1}+(1+e)U_n\\ W_n &= U_{n+1}-(1-e)U_n \end{aligned}\quad \text{for}\quad n = 0, 1, 2, \ldots.$$

The first two integrals appearing in Eq. (5.2) may be (with $w(x)/U = (x-e)^n$) expressed, after lengthy transformations, in terms of U_n and V_n,

$$\sqrt{\frac{R_2(x)}{R_1(x)}}\int\limits_C \sqrt{\frac{R_1(\xi)}{R_2(\xi)}}(\xi-e)^n\frac{d\xi}{\xi-x} = [2U_0\Phi_2(x)-V_0\Phi_1(x)](x-e)^n-\sum_{m=0}^{n-1}V_{n-m}(x-e)^m\Phi_1(x)$$

and

$$\int\limits_C \Lambda^*(v, u)(\xi-e)^n d\xi = \frac{1}{n+1}\Big\{[2U_0\Phi_2(x)-V_0\Phi_1(x)](x-e)^{n+1} -2U_{n+1}\Phi_2(x)-\sum_{m=0}^{n-1}V_{n-m}(x-e)^{m+1}\Phi_1(x)\Big\},$$

where

$$(7.5)\qquad \Phi_1(x) = \frac{1}{j}\sqrt{\frac{R_2(x)}{R_1(x)}} = \sqrt{\frac{1-x}{1+x}\left(1-\frac{\delta}{x-e}\right)}$$

and

$$(7.6)\qquad \Phi_2(x) = \frac{1}{gK}\frac{cnu_1\, dnu_1}{snu_1}\Pi(k^2sn^2u_1, k) = \frac{x-e}{x-1}\frac{1}{gK}\Pi\left(\frac{1+x}{1-x}\frac{1-e}{1+e}, k\right)\Phi_1(x).$$

Equations (4.1) yield in the present case the set of equations

$$i\omega I_4(\omega)\Omega_1 + i\omega J_4(\omega)\Omega_2 = A_n,$$

(7.5)

$$\left[\frac{1+e}{2}\frac{E}{K}I_1(\omega) - I_2(\omega) - i\omega I_4(\omega)\right]\Omega_1 + \left[\frac{1+e}{2}\frac{E}{K}J_1(\omega) - \frac{\delta}{2}J_3(\omega)\right]\Omega_2 = B_n,$$

where

$$A_n = 2j\int_C \sqrt{\frac{R_1(\xi)}{R_2(\xi)}}(\xi-e)^n d\xi = -2V_{n+1},$$

and

$$B_n = 2\int_C \left[\sqrt{\frac{1-\lambda_1^2}{1-\lambda_2^2}}\frac{cnv_1\, dnv_1}{snv_1}\frac{1}{K}\Pi(k^2 sn^2 v_1, k) - j\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}\right](\xi-e)^n d\xi$$

$$= \frac{1+e}{2}\delta g^2\left(W_{n+1} + \frac{2n+1}{n+1}\frac{1+e}{2}W_n\right) + (1+e)\frac{U_{n+1}}{n+1}\frac{E}{K}.$$

Once the coefficients A_n, B_n are found, Eqs. (7.7) may be used to determine Ω_1, Ω_2 and the pressure distribution

$$\Delta p(x) = \frac{\varrho U^2}{2}f_n(x), \tag{7.8}$$

where

$$f_n(x) = \frac{4}{\pi}\left(1+\frac{i\omega}{n+1}\right)[2U_0\Phi_2(x) - V_0\Phi_1(x)](x-e)^n \tag{7.9}$$

$$-\frac{4}{\pi}\left\{V_n - \frac{i\omega}{2}[I_2(\omega)\Omega_1 + J_2(\omega)\Omega_2]\right.$$

$$\left.+\sum_{m=1}^{n-1}\left(V_{n-m} + \frac{i\omega}{n+1}V_{n-m+1}\right)(x-e)^m + \frac{i\omega}{n+1}(x-e)^n V_1\right\}\Phi_1(x)$$

$$+\frac{4}{\pi}\left[\frac{2}{n+1}U_{n+1} + I_1(\omega)\Omega_1 + J_1(\omega)\Omega_2\right]\Phi_2(x).$$

If the boundary condition $w(x)/U = (x-e)^n$ is prescribed along the entire line L, then the sequence of constants U_n' $(n = 0, 1, \ldots, n+1)$ should be substituted into the above expression (to be denoted later by $f_n'(x)$), while if it applies only to the second segment (at the first segment $w(x) = 0$), then the corresponding function $f_n''(x)$ will be obtained from the sequence $U_n''(n = 0, 1, \ldots, n+1)$.

Determination of the generalized forces on the profile is simple since the necessary integrals are easily expressed in terms of the elements of the sequence U_m and W_m, and namely

(7.10)

$$\int_C (x-e)^m\Phi_1(x)dx = W_{m+1} + \delta W_m,$$

$$\int_C (x-e)^m\Phi_2(x)dx = \frac{1+e}{2}\left(W_{m+1} + \frac{2m+1}{m+1}\frac{\delta}{2}W_m\right) + \frac{U_{m+1}}{m+1}\frac{1}{g^2}\frac{E}{K}.$$

The method of notation of the results used here enables us to pass directly to the limit $\delta \to 0$. The recurrence formula (7.2) reduces after substitution $\delta = 0$ to a formula relating three consecutive terms of the sequence. The initial elements are (with $\delta = 0$)

$$\begin{aligned} U_0' &= 0, & U_0'' &= \infty, \\ U_1' &= -\pi, & U_1'' &= -\arccos e, \\ U_2' &= e\pi, & U_2'' &= e\arccos e - \sqrt{1-e^2}. \end{aligned}$$

It is seen that only the constant U_0'' is unbounded for $\delta \to 0$; however, it occurs only in a linear combination with V_0'' and

$$\lim_{\delta\to 0}[2U_0''\Phi_2(x) - V_0''\Phi_1(x)] = \frac{1}{2}\ln\left|\frac{1-xe+\sqrt{1-x^2}\sqrt{1-e^2}}{1-xe-\sqrt{1-x^2}\sqrt{1-e^2}}\right| + \sqrt{\frac{1-x}{1+x}}\arccos e.$$

In addition,

$$\lim_{\delta\to 0}\Phi_1(x) = \sqrt{\frac{1-x}{1+x}} \quad \text{and} \quad \lim_{\delta\to 0}\Phi_2(x) = \frac{1+e}{2}\sqrt{\frac{1-x}{1+x}}.$$

Thus Eqs. (7.7) and (7.9) may be directly applied to arbitrarily small gaps δ. If the boundary condition $w(x)/U = P_n(x)$ is prescribed in the form of an arbitrary polynomial of order n, it may be written as $P_n(x) = \sum_{m=0}^{n} a_m(x-e)^m$ and the pressure distribution is the linear combination

$$\Delta p(x) = \frac{\varrho U^2}{2}\sum_{m=0}^{n} a_m f_m(x).$$

8. Degrees of freedom and generalized forces

If the profile and the control surface form rigid segments, the displacement of the system may be described by means of four generalized coordinates shown in Fig. 3. The coordinates $h(t)$ and $z(t)$ correspond to translations of the profile and the control surface, respectively while $\alpha(t)$ is the rotation of that system about the axis placed at the point with the coordinate a, and $\beta(t)$ denotes the rotation of the control surface about its nose.

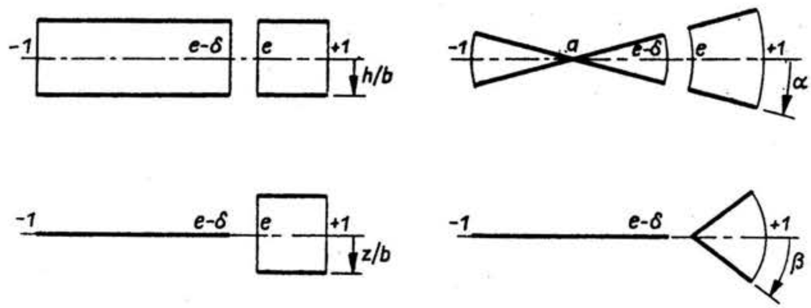

FIG. 3.

For an arbitrary motion of a profile with a control surface, the definition of generalized coordinates leads to the following formulation of the boundary condition:

$$\frac{w(x)}{U} = i\omega\frac{h}{b} + [1+i\omega(x-a)]\alpha + \begin{cases} 0 & \text{for} \quad -1 < x < e-\delta, \\ i\omega\dfrac{z}{b} + [1+i\omega(x-e)]\beta & \phantom{\text{for}} \quad e < x < +1. \end{cases}$$

Here h, α, z, β are the (complex) amplitudes of the generalized coordinates.

The elementary pressure distributions corresponding to unit generalized coordinates are the following:

$$\Delta p_h(x) = \frac{\varrho U^2}{2} i\omega f_0'(x),$$

$$\Delta p_\alpha(x) = \frac{\varrho U^2}{2}\{[1+i\omega(e-a)]f_0'(x)+i\omega f_1'(x)\},$$

$$\Delta p_z(x) = \frac{\varrho U^2}{2} i\omega f_0''(x),$$

$$\Delta p_\beta(x) = \frac{\varrho U^2}{2}[f_0''(x)+i\omega f_1'')x)],$$

the functions f_0', f_0'', f_1', f_1'' being found from Eq. (7.9). The pressure distribution in an arbitrary motion of the profile is obtained by superposing the elementary distributions

$$\Delta p(x) = \Delta p_h(x)\frac{h}{b} + \Delta p_\alpha(x)\alpha + \Delta p_z(x)\frac{z}{b} + \Delta p_\beta(x)\beta.$$

Each generalized coordinate corresponds to a suitable generalized force, and namely, in the case of the coordinates h and z these are the resultant forces L and P acting on the profile with a control surface and the control itself, respectively, while the generalized forces M and T correspond to the coordinates and represent the respective couples. Hence we may write

$$L = b\int_L \Delta p(x)dx = \frac{\varrho U^2}{2} b\left(L_h\frac{h}{b} + L_\alpha\alpha + L_z\frac{z}{b} + L_\beta\beta\right),$$

$$M = b^2\int_L (x-a)\Delta p(x)dx = \frac{\varrho U^2}{2} b^2\left(M_h\frac{h}{b} + M_\alpha\alpha + M_z\frac{z}{b} + M_\beta\beta\right),$$

$$P = b\int_e^1 \Delta p(x)dx = \frac{\varrho U^2}{2} b\left(P_h\frac{h}{b} + P_\alpha\alpha + P_z\frac{z}{b} + P_\beta\beta\right),$$

$$T = b^2\int_e^1 (x-e)\Delta p(x)dx = \frac{\varrho U^2}{2} b^2\left(T_h\frac{h}{b} + T_\alpha\alpha + T_z\frac{z}{b} + T_\beta\beta\right).$$

The sixteen aerodynamical coefficients $L_h, L_\alpha, \ldots, T$ (being functions of the frequency coefficient ω and of the geometry of the system) are evaluated, by means of Eqs. (7.10), in closed forms. The known solutions for a stationary flow and the attached mass coefficients [5] constitute particular cases of the general solution given here, and they may be obtained by disregarding all but the first (or second) terms in the expression (5.2) and by suitable simplification of the expression (7.9).

9. Examples of numerical calculations

To illustrate the effect of a gap on the pressure distributions and the magnitudes of aerodynamical coefficients, the results of calculations concerning a profile with a 40% control surface ($e = 0.2$) are shown in Figs. 4-9 for various values of the frequency coefficients ω and various gaps δ. The graphs illustrate the distributions of the pressure coefficient jump on the profile, that is of the parameter

$$c_p = \frac{\Delta p}{\frac{\varrho U^2}{2}}$$

for various modes of motion (degrees of freedom). In the upper parts of the graphs the

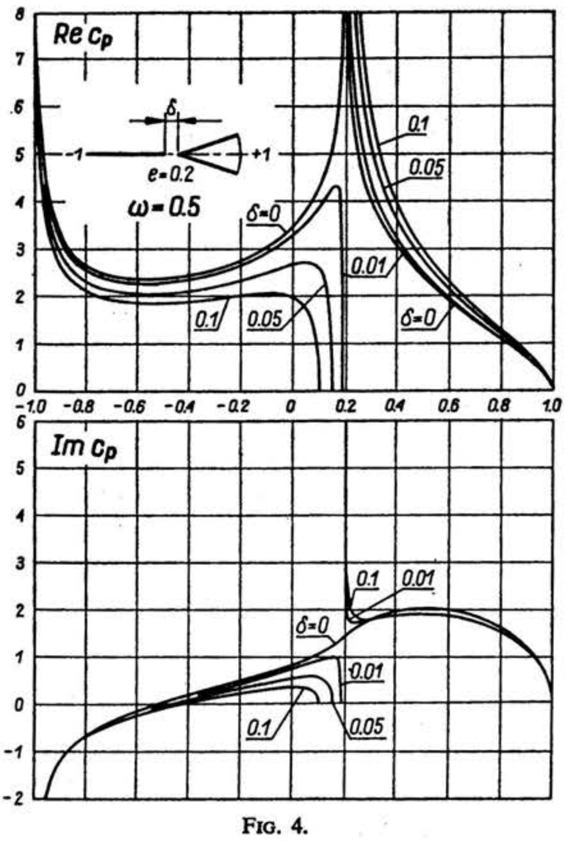

FIG. 4.

7 Arch. Mech. Stos. 4/80

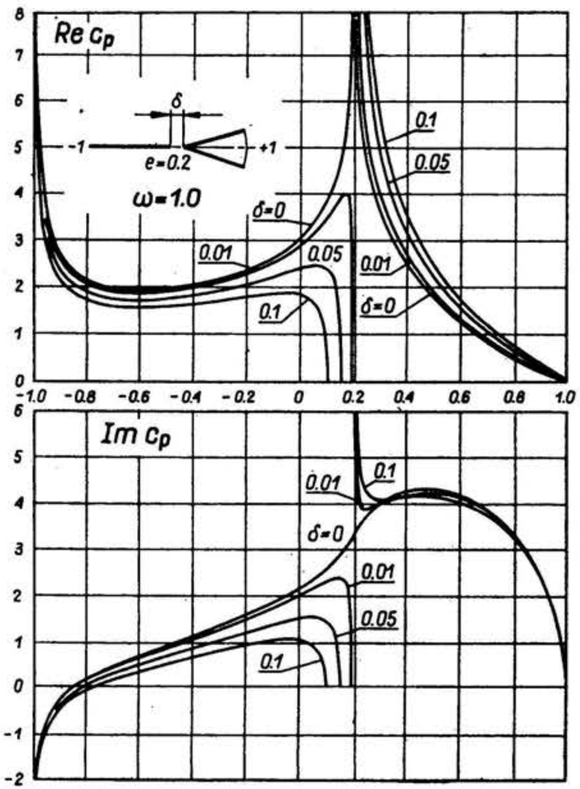

FIG. 5.

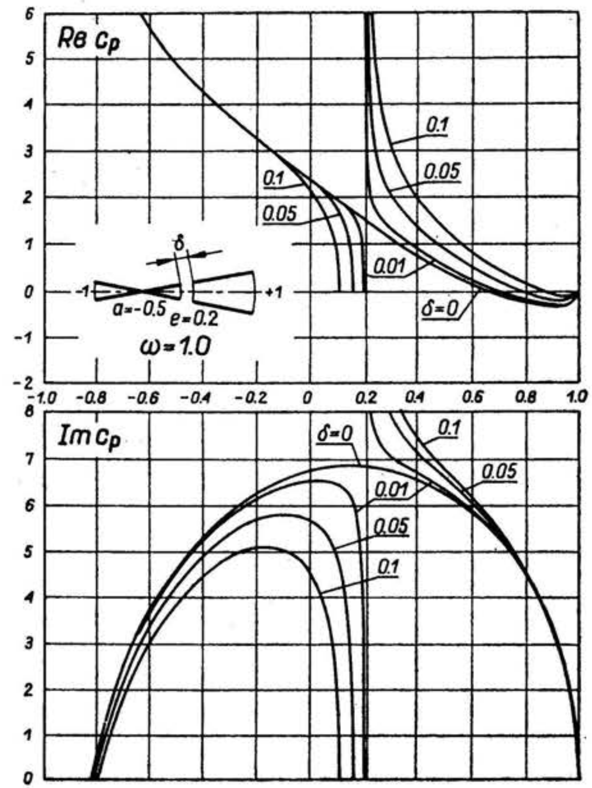

FIG. 6.

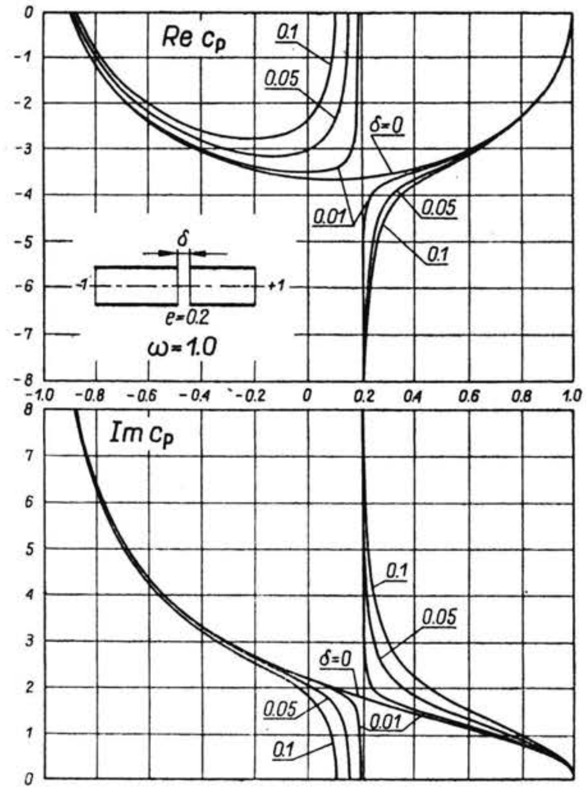
Re cp
Im cp
δ=0
0.1
0.05
0.01
δ
e=0.2
-1
+1
ω=1.0
0
-1
-2
-3
-4
-5
-6
-7
-8
8
7
6
5
4
3
2
1
-1.0
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1.0

FIG. 7.

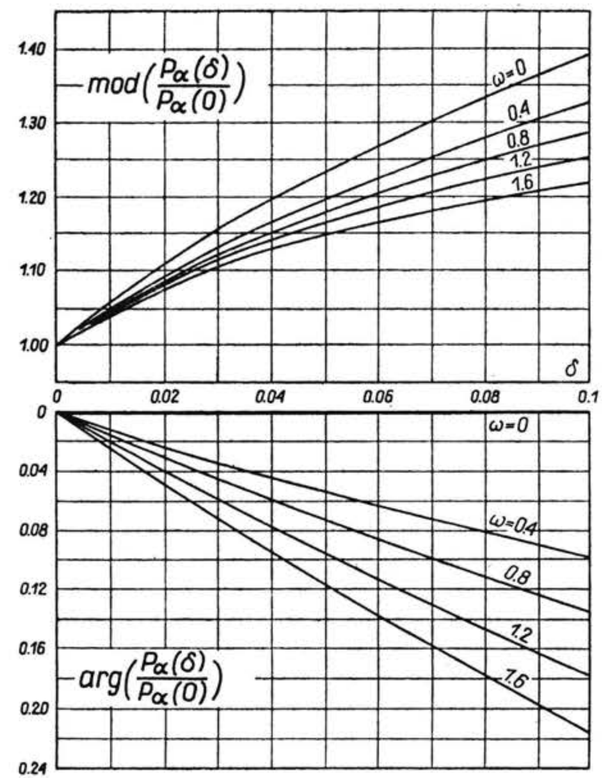
mod(Pα(δ)/Pα(0))
arg(Pα(δ)/Pα(0))
ω=0
0.4
0.8
1.2
1.6
ω=0.4
7.6
δ
1.40
1.30
1.20
1.10
1.00
0
0.02
0.04
0.06
0.08
0.1
0.04
0.08
0.12
0.16
0.20
0.24

FIG. 8.

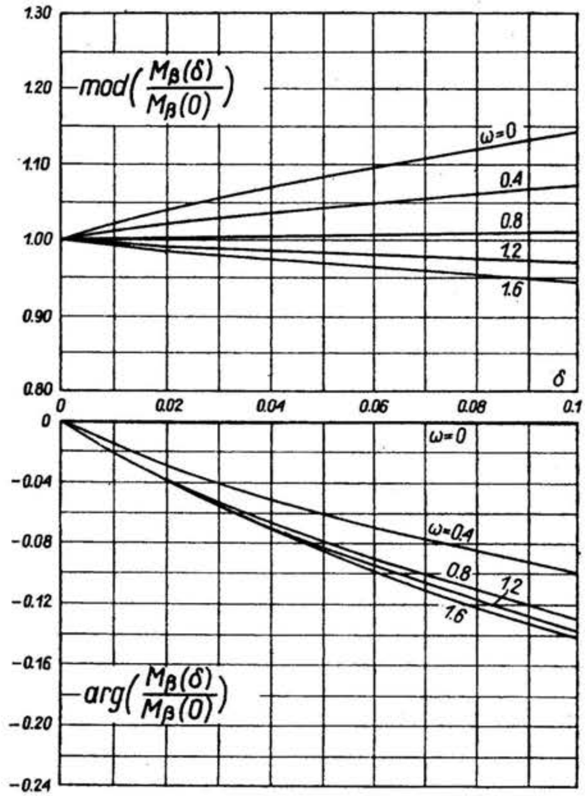

FIG. 9.

distributions of the real part ($\text{Re}\, c_p$) are given, that is of the component with phase complying with the variation of the corresponding generalized coordinate. Lower parts of the graphs refer to the imaginary part distributions ($\text{Im}\, c_p$) of the pressure coefficient jump, i.e. to the component with the shift $\pi/2$. Figures 4 and 5 are concerned with the same degree of freedom (displacement of the control surface) but with different frequency coefficients ($\omega = 0.5$ or $= 1.0$). The real part distribution c_p depends in this case only slightly on the changes of the frequency coefficient and has a form similar to that corresponding to the stationary case [4]. The imaginary part ($\text{Im}\, c_p$) which in the stationary case equals zero, increases with the increasing frequency coefficient, the effect of the gap being manifested by a distribution discontinuity. Figures 6 and 7 illustrate the effect of the gap in the case when the profile moves with an undisplaced control surface. If $\delta = 0$, all the distributions (beyond the leading edge) are continuous but (even for an arbitrarily small gap) a singularity in the c_p-distribution appears at the nose of the control surface, and the pressure difference (Kutta-Joukovski condition) vanishes at the boundary of the gap.

Consequently, the changes in the pressure difference at the profile affects the values of the aerodynamical coefficients which characterize the corresponding generalized forces. In Figs. 8 and 9 we can see examples of the influence of the gap on the values of certain aerodynamical coefficients. Separately shown are: (upper parts of the graphs) variation of the modulus of the ratio of the given coefficient to its value at $\delta = 0$, and (lower parts) variation of the argument denoting the phase shift produced by the gap.

Appendix. Properties of the function $\Lambda(u, v)$

The function $\Lambda(u, v)$ is defined by the formula

$$\Lambda(u,v) = -2\frac{cnv\,dnv}{snv}\left[\Pi(u, k^2sn^2v, k) - \frac{u}{K}\Pi(k^2sn^2v, k)\right]. \tag{A.1}$$

Simple transformations make it possible to write it in the form

$$\Lambda(u,v) = -2\int_0^u \frac{2k^2snv\,cnv\,dnv\,sn^2\mu}{1-k^2sn^2v\,sn^2\mu}\,d\mu + 2\frac{u}{K}\int_0^K \frac{k^2snv\,cnv\,dnv\,sn^2\mu}{1-k^2sn^2v\,sn^2\mu}\,d\mu. \tag{A.2}$$

Using known (cf. [12]) relations, each of the integrals in Eq. (A.2) may be expressed in terms of the Theta and Zeta Jacobi functions. $Z(v) = \dfrac{\theta'(v)}{\theta(v)}$

$$\int_0^u \frac{k^2snv\,cnv\,dnv\,sn^2\mu}{1-k^2sn^2v\,sn^2\mu}\,d\mu = \frac{1}{2}\ln\frac{\theta(u-v)}{\theta(u+v)} + uZ(v).$$

If $u = K$, then

$$\int_0^K \frac{k^2snv\,cnv\,dnv\,sn^2\mu}{1-k^2sn^2v\,sn^2\mu}\,d\mu = \frac{1}{2}\ln\frac{\theta(K-v)}{\theta(K+v)} + KZ(v) = KZ(v).$$

Substituting these relations into Eq. (A.2), we obtain a particularly simple expression

$$\Lambda(u,v) = \ln\frac{\theta(u+v)}{\theta(u-v)}, \tag{A.3a}$$

which holds true for arbitrary u and v with the only important reservation that both integrals in Eq. (A.2) should be non-singular. Theta functions being even, it follows that $\Lambda(u, v) = \Lambda(v, u)$. Equation (A.3a) may also be used to obtain the limiting values of $\Lambda(u, v)$ at the points v corresponding to the ends of the line L since

$$\Lambda(u,0) = \Lambda(u,K) = 0,$$

$$\begin{aligned} &\Lambda(u,jK') = j\pi\left(1-\frac{u}{K}\right), \qquad \Lambda(u,-jK') = -j\pi\left(1-\frac{u}{K}\right), \\ &\Lambda(u,K+jK') = -j\pi\frac{u}{K}, \qquad \Lambda(u,K-jK') = j\pi\frac{u}{K}. \end{aligned} \tag{A.4a}$$

If at least one of the integration contours in Eq. (A.2) contains such a point μ_0 that $1-k^2sn^2v\,sn^2\mu_0 = 0$, then the corresponding integral (or both) should be interpreted in the sense of the Cauchy principal value. From the condition $sn^2\mu_0 = 1/k^2sn^2v$ it follows that $\mu_0 = v+jK'$. Taking now into account the range of variability of the arguments of the function $\Lambda(u, v)$ shown in Fig. 2, we can find out that this result is true only if one of the arguments of $\Lambda(u, v)$ is real and the other one belongs to one of the segments

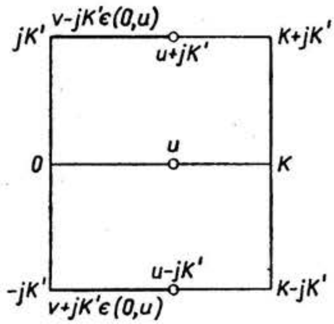

FIG. 10.

$(jK', K+jK')$ or $(-jK', K-jK')$. From the correspondence of the variables $x \to u$, $\xi \to v$, it follows that the variables x and ξ belong then to the same segment of the line L; in such a case the relations (A.3a) and (A.4a) cease to hold true.

In order to investigate the case of singular integrals, let us assume the variable u to be real-valued ($\mathrm{Im}\, u = 0$) and $0 < u < K$ (Fig. 10). Due to the definition of the principal value of improper integrals, we obtain

$$\int_0^u \frac{k^2 snv cnv dnv sn^2\mu}{1-k^2sn^2v sn^2\mu}\, d\mu = \frac{1}{2}\ln\frac{\theta(u-v)}{\theta(u+v)} + uZ(v) + \begin{cases} \mp j\dfrac{\pi}{2} & \text{when } v\pm jK' \in (0,u), \\ 0 & \phantom{\text{when }} v\pm jK' \notin (0,u). \end{cases}$$

The sign of the additional term results from the condition of single-valuedness of the function $\ln[\theta(u-v)/\theta(u+v)]$ within the rectangle with corners at jK', $K+jK'$, $K-jK'$, $-jK'$. Similarly,

$$\int_0^K \frac{k^2 snv cnv dnv sn^2\mu}{1-k^2sn^2v sn^2\mu}\, d\mu = KZ(v) + \begin{cases} \mp j\dfrac{\pi}{2} & \text{when } v\pm jK' \in (0,K), \\ 0 & \phantom{\text{when }} v\pm jK' \notin (0,K). \end{cases}$$

These expressions enable us to generalize the relation (A.3a) to the case of singular integrals (under the assumption that $\mathrm{Im}\, u = 0$)

$$\text{(A.3b)} \qquad \Lambda(u, v) = \ln\frac{\theta(u+v)}{\theta(u-v)} + \begin{cases} \pm j\pi\left(1-\dfrac{u}{K}\right) & v\pm jK' \in (0,u), \\ \pm j\pi\dfrac{u}{K} & \text{when} \quad v\pm jK' \in (u,K), \\ 0 & \text{in remaining cases,} \end{cases}$$

and to determine the limiting values

(A.4b)
$$\lim_{v\to jK'} \Lambda(u,v) = \lim_{v\to K+jK'} \Lambda(u,v) = 0 \quad \text{when} \quad v \in (jK', K+jK'),$$
$$\lim_{v\to -jK'} \Lambda(u,v) = \lim_{v\to K-jK'} \Lambda(u,v) = 0 \quad \text{when} \quad v \in (-jK', K-jK').$$

The expressions (A.4) should be interpreted in such a way that if $x \in L$, then the left- and right-hand limits for ξ approaching the ends of this segment of L to which x belongs differ from each other, the „interior" limits being given by Eq. (A.4b), and the „exterior" ones by Eq. (A.4a).

For the variables $u_2 = u_1 - jK'$ and $v_2 = v_1 + jK'$, the following relation holds true:

$$\ln \frac{\theta(u_1+v_1)}{\theta(u_1-v_1)} = \ln \frac{\theta(u_2+v_2)}{\theta(u_2-v_2+2jK')} = \ln\left[-e^{j\frac{\pi}{K}(u_1-v_2+jK')} \frac{\theta(u_2+v_2)}{\theta(u_2-v_2)}\right]$$
$$= \ln \frac{\theta(u_2+v_2)}{\theta(u_2-v_2)} + \pi j\left(\frac{u_1-v_2}{K} \pm 1\right).$$

The sign of the last term should be opposite to the sign of $\mathrm{Re}(u-v)$. Using this relation and the formula (A.3b) we directly obtain

(A.5)
$$\Lambda(u_1, v_1) = \begin{cases} \Lambda(v_2, u_2) \\ \Lambda(v_1, u_1) \end{cases} \text{when} \quad \begin{matrix} \mathrm{Im}(u_1-v_1) = K', \\ \mathrm{Im}(u_1-v_1) \neq K', \end{matrix}$$
$$\Lambda(u_2, v_2) = \begin{cases} \Lambda(v_1, u_1) \\ \Lambda(v_2, u_2) \end{cases} \text{when} \quad \begin{matrix} \mathrm{Im}(v_2-u_2) = K', \\ \mathrm{Im}(v_2-u_2) \neq K'. \end{matrix}$$

From the definition (A.1) of $\Lambda(u, v)$ it follows that

(A.6)
$$\frac{\partial}{\partial u} \Lambda(u,v) = -2\frac{cnv\,dnv}{snv}\left[\frac{1}{1-k^2sn^2u\,sn^2v} - \frac{1}{K}\Pi(k^2sn^2v, k)\right].$$

This expression may also be used in calculating the derivative $\partial\Lambda(u,v)/\partial v$ under simultaneous application of the formula (A.5). In particular, if $x \in L$, then

$$\Pi(k^2sn^2u_1, k) + \Pi(k^2sn^2u_2, k) = K$$

and so

(A.7)
$$\frac{\partial}{\partial v_1}\Lambda(v_1, u_1) = -2\frac{cnu_1\,dnu_1}{snu_1}\left[\frac{1}{1-k^2sn^2u_1\,sn^2v_1} - \frac{1}{K}\Pi(k^2sn^2u_1, k)\right]$$
$$= \frac{\partial}{\partial v_2}\Lambda(v_2, u_2) = -2\frac{cnu_2\,dnu_2}{snu_2}\left[\frac{1}{1-k^2sn^2u_2\,sn^2v_2} - \frac{1}{K}\Pi(k^2sn^2u_2, k^2\right],$$

independently of the assignation of ξ to any of the segments.

With the notation

(A.8a)
$$\int_L \Lambda^*(v,u)w(\xi)d\xi \equiv \int_{-1}^{\alpha} \Lambda(v_2, u_2)w(\xi)d\xi + \int_{\beta}^{1} \Lambda(v_1, u_1)w(\xi)d\xi,$$

true for any $x \in L$, Eq. (A.5) may be used to obtain

(A.8b)
$$\int_L \Lambda^*(v,u)w(\xi)d\xi = \begin{cases} \int_L \Lambda(u_1, v_1)w(\xi)d\xi & x \in [-1, \alpha], \\ & \text{for} \\ \int_L \Lambda(u_2, v_2)w(\xi)d\xi & x \in [\beta, 1]. \end{cases}$$

The function $\Lambda(u, v)$ is defined by Eq. (A.1) for arbitrary values of $0 \leqslant k \leqslant 1$. Of particular importance is the limiting case for $k \to 1$ corresponding to a profile without the gap ($\delta = 0$). On the basis of the known relations [11] concerning the limiting values of elliptic integrals, it may be written as

$$\lim_{k\to 1} \Pi(u, k^2 sn^2 v, k) = \lim_{k\to 1}\left[\frac{1}{cn^2 v}\left(u - \frac{1}{2} snu \ln\frac{1+snusnv}{1-snusnv}\right)\right].$$

If the integral $\Pi(u, k^2 sn^2 v, k)$ is singular, its principal value should be evaluated by considering the absolute value of the expression under the logarithm sign. In the particular case of $u \to K$,

$$\lim_{k\to 1}\left[\frac{1}{K}\Pi(k^2 sn^2 v, k)\right] = \lim_{k\to 1}\frac{1}{cn^2 v}.$$

Taking into account the additional property of $dnv \to cnv$, we derive, by means of Eq. (A.1), a relation suitable for calculating the limits:

$$\lim_{k\to 1}\Lambda(u, v) = \lim_{k\to 1}\ln\frac{1+snusnv}{1-snusnv}. \tag{A.9}$$

In the case of the singular function $\Pi(u, k^2 sn^2 v, k)$, the absolute value of the expression under the logarithm sign should be taken in the formula (A.9). From the definition (A.3) it follows that

$$snu_1 snv_1 = \frac{1}{K}\frac{(x+1)^{\frac{1}{2}}(\xi-1)^{\frac{1}{2}}}{(x-1)^{\frac{1}{2}}(\xi+1)^{\frac{1}{2}}}$$

and

$$snu_2 snv_2 = \frac{1}{K}\frac{(x-1)^{\frac{1}{2}}(\xi+1)^{\frac{1}{2}}}{(x+1)^{\frac{1}{2}}(\xi-1)^{\frac{1}{2}}}.$$

with $k = 1$ and $x, \xi \in L$ we then obtain

$$snu_1 snv_1 = \sqrt{\frac{1+x}{1-x}}\sqrt{\frac{1-\xi}{1+\xi}} \quad \text{and} \quad snu_2 snv_2 = \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+\xi}{1-\xi}}$$

and so

$$\Lambda(u_1, v_1) = \Lambda(u_2, v_2) = \frac{1}{2}\ln\left|\frac{1-x\xi+\sqrt{1-x^2}\sqrt{1-\xi^2}}{1-x\xi-\sqrt{1-x^2}\sqrt{1-\xi^2}}\right| = \Lambda_1(x, \xi).$$

With $k = 1$ and $x \in L$, $\xi \notin L$ we obtain, however,

$$snu_1 snv_1 = \frac{1}{j}\sqrt{\frac{1+x}{1-x}}\sqrt{\frac{\xi-1}{\xi+1}} \quad \text{and} \quad snu_2 snv_2 = j\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\xi+1}{\xi-1}},$$

where

$$\Lambda(u_1, v_1) = \ln\frac{1-j\sqrt{\frac{1+x}{1-x}}\sqrt{\frac{\xi-1}{\xi+1}}}{1+j\sqrt{\frac{1+x}{1-x}}\sqrt{\frac{\xi-1}{\xi+1}}} = j\left(2\operatorname{arctg}\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\xi+1}{\xi-1}} - \pi\right) = j\Lambda_2(x, \xi)$$

and

$$\Lambda(u_2, v_2) = j[\Lambda_2(x, \xi) + \pi].$$

The functions $\Lambda_1(x, \xi)$ and $\Lambda_2(x, \xi)$ appear in the known method of solution of the Birnbaum equation [9]. From the derived properties it follows that $\Lambda(u, v)$ combines the roles of both functions, even in the more general case of a profile with a gap.

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