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Influence of hinge line gap on aerodynamic forces acting on a harmonically oscillating thin profile in an incompressible flow Part I

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BY APPLYING the method of strongly singular integral equations the solution of the Birnbaum-Possio equations is derived for a system of two profiles (profile with a control surface) lying on one straight line parallel to the direction of flow at infinity. The solutions are then transformed to a form in which the pressure distributions and aerodynamical coefficients may explicitly be expressed in terms of the elementary functions and canonical forms of elliptic integrals. Only some of the integrals (concerning the wake in the gap and behind the profile) require numerical calculations. The influence of the size of the gap on the pressure distributions and aerodynamical coefficients (with various values of the frequency coefficient) is illustrated by graphs.

Posługując się metodami równań całkowych silnie-osobliwych, otrzymano rozwiązanie równania Birnbauma-Possio dla układu dwóch profili (profilu ze sterem) leżących na jednej prostej, równoległej do kierunku przepływu w nieskończoności. Następnie przekształcono je do postaci, w której rozkłady ciśnień i współczynniki aerodynamiczne dają się wyrazić jawnie za pomocą funkcji elementarnych i kanonicznych postaci całek eliptycznych. Tylko nieliczne całki (dotyczące śladu wirowego w szczelinie i za profilem) wymagają obliczeń numerycznych. Wpływ wielkości szczeliny na rozkłady ciśnień i współczynniki aerodynamiczne (dla różnych wartości współczynnika częstości) zilustrowano wykresami.

Послуживаясь методами интегральных сильно сингулярных уравнений, получено решение уравнения Бирнбаума-Поссио для системы двух профилей (профиля с рулем) лежащих на одной прямой, паралельной направлению течения в бесконечности. Затем преобразовано оно к виду, в котором распределения давлений и аэродинамические коэффициенты можно явным образом выразить при помощи элементарных функций и канонических видов эллиптических интегралов. Только немногие интегралы (касающиеся вихревого следа в щели и за профилем) требуют численных расчетов. Влияние величины щели на распределения давлений и аэродинамические коэффициенты (для разных значений коэффициента частоты) иллюстрировано графиками.

1. Introduction

FUNDAMENTAL theoretical results concerning the pressure distribution on a harmonically oscillating profile in an incompressible flow were obtained in the thirties by THEODORSEN, KÜSSNER, SCHWARTZ and others [1, 2]. A comparison of these results with experimental data demonstrates the practical applicability of the model used. The methods of analysis of aerodynamic forces distributions on three-dimensional lifting surfaces developed later are based on similar assumptions and utilize the same scheme of linearization of the boundary conditions at the lifting surfaces and in the wake. The mathematical formulation of the problem is reduced to a singular integral equation relating the prescribed normal velocity distribution to the unknown distribution of pressure differences between both sides of the lifting surface. The singularity of the kernel gives the square root singularity of the functions describing the distribution of pressures on the leading edge and its derivative on the trailing edge. The edges of the control surfaces are the discontinuity lines of the boundary condition (normal velocity distribution). For the function representing the solution of the integral equation, these lines are the logarithmic singularity lines. In the case when the lifting surface contains narrow gaps enabling the flow perpendicular to the surface, considerable irregularities of pressure distribution appear in the neighbourhood of the gaps, thus making the numerical calculations more difficult.

The scheme of a profile with a gap is shown in Fig. 1. The normal velocity distribution $w(x, t)$ is prescribed along both segments modelling the profile with a control surface. If the gap $\delta = \beta - \alpha$ is large enough, the system may be treated as consisting of two profiles;

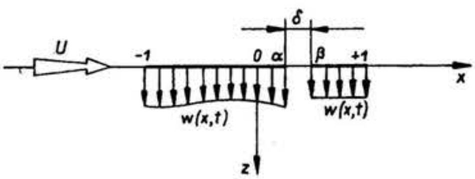

FIG. 1.

on the trailing edges of each of them the Kutta-Joukovski condition is satisfied, while on the leading edges square root singularities appear in the pressure distributions. With decreasing gaps the aerodynamic interaction between the two sections of the profile is increased, what results in considerable changes of the pressure distributions, mainly in the vicinity of points α and β. With $\delta \to 0$ the weak singularity on the control surface (at β) transforms in the limit into a logarithmic singularity when $w(\alpha, t) \neq w(\beta, t)$, or into a regular point when the function $w(x, t)$ is continuous. The changes at the point α are even more considerable since the zero pressure difference for $\delta > 0$ is replaced either by a logarithmic singularity or by a (non-zero) finite value of the distribution function.

Two particular cases of the model shown in Fig. 1 were examined in detail. The first one was concerned with the situation when $\alpha = \beta$ since then the model is reduced to the classical one without a gap. The second case is connected with the investigations of aerodynamic interference of two profiles [3] and concerns large gaps (e.g. when $\delta > 1$). The most difficult case for the analysis yet, at the same time, the most interesting one (from the point of view of determining the aerodynamic forces on a profile with a control surface) corresponds to $0 < \delta \leqslant 1$: very few papers are known to deal with this problem.

In the paper by WHITE and LANDAHL [4] the stationary case of pressure distributions at a gap are investigated by means of fitting the asymptotic solutions. In the monograph by SEDOV [5] closed-form solutions are given for the forces acting on each segment of the profile in a stationary flow and for the apparent mass coefficients (in a flow with constant circulation). The complete, linearized non-stationary model of flow about a system of thin profiles with chords lying on a straight line, the effect of wakes being taken into account, was considered in [6, 7]. Various methods of determining the closed-form solutions for harmonically oscillating profiles were given there, but no effective method of calculation was devised.

A comparison of the measurements of the pressure distributions on vibrating profiles with the results obtained by the classical methods of calculation leads to certain discrepancies which may be supposed to be due to the existence of the gap[8]. It becomes necessary to devise a method of evaluating the aerodynamic forces which would be useful also for arbitrarily small gaps. The only non-stationary solutions available [5] are concerned with a very particular case of motion with a constant circulation.

The present paper retains the full linearization of the model of the phenomenon, but presents the rigorous solution of the problem of the effect of a gap on the pressure distribution acting on the harmonically oscillating profile with the control surface, and on the aerodynamic coefficients of the profile. The problem is solved by the method of singular integral equations (based on [7]), the solutions being then transformed to a form analogous to the well-known solution concerning a profile without gap [1, 2, 9]. Both the pressure distribution over the profile and the aerodynamic coefficients are expressed in terms of elementary functions and canonical forms of elliptic integrals. Only a few integrals depending on the geometry of the system and on the frequency coefficient (and connected with the wakes) require numerical procedures. The method of calculating the aerodynamic coefficients is convenient for immediate applications in the analysis of flutter of the profile.

Linearization of the boundary conditions in the vicinity of the gap represents a simplification which sometimes may not be justified. For instance, in the case of a narrow gap one cannot exclude a considerable influence of the thickness of the profile. This problem has not been studied so far but, on the other hand, it may be conjectured that the simplified linear model (representing a direct generalization of the classical model [1, 2]) may, in spite of that, make it possible to obtain certain technically important information as to the effect of the flow through the gap upon the aerodynamic forces. However, in the case of a thick profile the parameter δ should be interpreted as a certain "effective" gap width, and not as a real geometrical dimension to be measured in the existing structure.

2. Formulation of the problem

The profile with a control surface is placed in a uniform flow of an incompressible and inviscid fluid; the profile performs harmonic oscillations about the mean position (Fig. 1). The undisturbed flow velocity is U. The flow is described by the equation of continuity, Euler's equation of motion and by the boundary conditions at the profile and in infinity. In the case of a thin profile and small amplitude of vibration, the linearization of the equations and the boundary conditions at the profile are permissible. The influence of the history of motion (being a characteristic feature of non-stationary flows) is manifested here by the existence of the wake convected behind the profile at the velocity U of the unperturbed flow. The x-coordinate in Fig. 1 is normed by assuming the semi-chord b of the profile as a unit of length. According to the assumptions, the boundary conditions at the profile are prescribed on the segments $(-1, \alpha)$ and $(\beta, +1)$ of the x-axis. The segments (α, β) and $(+1, \infty)$ of the x-axis are the lines of discontinuity of velocities which constitute the wake convected at the velocity U. The vertical velocity at the profile is prescribed, and at an arbitrary instant of time t it may be given in the form

$$w(x, t) = w(x)e^{i\nu t},$$

$w(x)$ being the (complex) vibration amplitude. The problem consists in determining the distribution of pressure differences between the upper and lower profile surfaces

$$\Delta p(x, t) = \Delta p(x)e^{i\nu t}.$$

The formulation presented here is based on the same assumptions as the classical formulation for solid profiles [1, 2] and the only difference consists in the fact that the boundary conditions are prescribed not on the entire segment $(-1, +1)$ but along the line L consisting of two segments $(-1, \alpha)$ and $(\beta, +1)$. As a result, the prescribed function $w(x)$ is also here related to the solution $\Delta p(x)$ by means of the Birnbaum - Possio equation [2, 9]

$$w(x) = \omega \int_L K(s)\gamma(\xi)d\xi, \tag{2.1}$$

the (singular) kernel being given by

$$K(s) = \frac{1}{2\pi s} - \frac{ie^{-is}}{2\pi}\left\{Ci(|s|)+i\left[Si(s)+\frac{\pi}{2}\right]\right\}. \tag{2.2}$$

Here $\omega = \nu b/U$ is the frequency coefficient, $s = \omega(x-\xi)$ and $\gamma(x)$ is a function connected with the pressure distribution by means of the formula

$$\Delta p(x, t) = -\varrho U\gamma(x)e^{i\nu t}. \tag{2.3}$$

Here ϱ is the density of the medium. Along the trailing edge of each segment the Kutta-Joukovski condition $\gamma(\alpha) = \gamma(1) = 0$ is fulfilled thus ensuring the uniqueness of solution of the integral equation (2.1).

3. Solution of the Birnbaum-Possio equation

The kernel (2.2) of Eq. (2.1) contains, in addition to the pole in the first term, an additional logarithmic singularity of the cosine integral function $Ci(|s|)$. The typical methods of solving the singular integral equations [10] are based on the formulation and solution of an auxiliary Riemann boundary-value problem of the analytical function constructed according to the singular part of the kernel. In the particular case of Eq. (2.1) in the formulation of the Riemann problem, account must be taken of the multivalued logarithmic term of the kernel.

The variables x and ξ in Eq. (2.1) assume real values. In order to formulate the Riemann boundary-value problem, the right-hand side of Eq. (2.1) is analytically continued onto the entire complex plane $z = x+jy$ (x is the only variable occuring in the physical model). The imaginary unit $j = \sqrt{-1}$ appearing in the analytic continuation is not connected with the imaginary unit $i = \sqrt{-1}$ introduced to the kernel (2.2) in order to express the harmonic time-dependence, and hence $ij \neq -1$. Taking into account the form of the kernel and separating the singular terms, we obtain

$$\Phi(z) = \frac{1}{2\pi j} \int\limits_L \left\{ \frac{1}{\xi - z} + i\omega e^{i\omega(\xi - z)} [\ln(\xi - z) + P(\xi - z)] \right\} \gamma(\xi) d\xi, \tag{3.1}$$

where $P(z)$ is a certain integral function of the variable $z = x+jy$. The first term of $\Phi(z)$ is expressed by a Cauchy-type integral and hence it represents a function analytical in the entire z-plane cut along the line L. The second term of the integrand in Eq. (3.1) possesses branchpoints in $\xi = z$ and at infinity. Cutting the complex plane $z = x+jy$ along the x-axis for $-1 < x < \infty$ and selecting the proper branch of the logarithm, the following limiting values may be obtained for $z \to x \pm j0$:

$$\begin{aligned} &\ln(\xi - z) \to \ln|x - \xi| & & \xi > x, \\ &\ln(\xi - z) \to \ln|x - \xi| \pm \pi j & \text{for} \quad & \xi < x, \end{aligned}$$

this secures the uniqueness of $\Phi(z)$ in the z-plane. Denoting the limiting values of $\Phi(z)$ on the x-axis by $\Phi^+(x) = \Phi(x+j0)$ and $\Phi^-(x) = \Phi(x-j0)$ and using the Plemelj-Sochock, formulae [10] for the Cauchy integral in Eq. (3.1), we arrive, on the basis of Eq. (2.1) to the following Riemann boundary-value problem along the entire x-axis:

$$\begin{aligned} &\Phi^+(x) - \Phi^-(x) = 0 & & -\infty < x < -1, \\ &\Phi^+(x) + \Phi^-(x) = 2jw(x) & & -1 \leqslant x \leqslant \alpha, \\ &\Phi^+(x) - \Phi^-(x) = -i\omega e^{-i\omega x}\Omega_1 & \text{for} \quad & \alpha < x < \beta, \\ &\Phi^+(x) + \Phi^-(x) = 2jw(x) & & \beta \leqslant x \leqslant 1, \\ &\Phi^+(x) - \Phi^-(x) = -i\omega e^{-i\omega x}\Omega_2 & & 1 < x < \infty \end{aligned} \tag{3.2}$$

with the following notations:

$$\Omega_1 = \int\limits_{-1}^{\alpha} e^{i\omega\xi} \gamma(\xi) d\xi \quad \text{and} \quad \Omega_2 = \Omega_1 + \int\limits_{\beta}^{1} e^{i\omega\xi} \gamma(\xi) d\xi .$$

Moreover, the relations following from the form (3.1) of $\Phi(z)$ are satisfied on L,

$$\begin{aligned} &\Phi^+(x) - \Phi^-(x) = \gamma(x) - i\omega e^{-i\omega x} \int\limits_{-1}^{x} e^{i\omega\xi} \gamma(\xi) d\xi & & -1 \leqslant x \leqslant \alpha, \\ & & \text{for} & \\ &\Phi^+(x) - \Phi^-(x) = \gamma(x) - i\omega e^{-i\omega x} \Big[\Omega_1 + \int\limits_{\beta}^{x} e^{i\omega\xi} \gamma(\xi) d\xi \Big] & & \beta \leqslant x \leqslant 1. \end{aligned} \tag{3.3}$$

The constants Ω_1 and Ω_2 are determined from the Kutta–Joukovski conditions $\gamma(\alpha) = \gamma(1) = 0$.

The procedure of solving Eq. (2.1) may then be divided into the following stages:

1) Solution of the boundary-value problem (3.2). The function $\Phi(z)$ is expressed in terms of $w(x)$ and the constants Ω_1, Ω_2.

2) Insertion of $\Phi(z)$ into Eq. (3.3) and solution of the resulting system of the Volterra integral equations.

3) Determination of the constants Ω_1, Ω_2 from the Kutta – Joukovski conditions.

Stage 2) presents no serious difficulties since, with the notation

$$\varphi(x) = \Phi^+(x) - \Phi^-(x) \quad \text{for} \quad x \in L,$$

the simple set of integral equations (3.3) is solved to yield

$$(3.4) \quad \begin{aligned} \gamma(x) &= \varphi(x) + i\omega \int_{-1}^{x} \varphi(\xi)\, d\xi & & -1 < x < \alpha, \\ & & \text{for} & \\ \gamma(x) &= \varphi(x) + i\omega \Big[\Omega_1 e^{-i\omega\beta} + \int_{\beta}^{x} \varphi(\xi)\, d\xi\Big] & & \beta < x < 1. \end{aligned}$$

The solution of the boundary-value problem may be obtained directly by known methods [10] and to this end it is convenient to write it in the form

$$(3.5) \qquad \Phi^+(x) = G(x)\Phi^-(x) + g(x),$$

the coefficient $G(x)$ and the term $g(x)$ being given by

$$(3.6) \quad \begin{aligned} &G(x) = -1 \quad g(x) = 2jw(x) & & & x \in L, \\ &G(x) = +1 \begin{cases} g(x) = 0 \\ g(x) = -i\omega e^{-i\omega x}\Omega_1 \\ g(x) = -i\omega e^{-i\omega x}\Omega_2 \end{cases} & \text{for} & & \begin{aligned} -\infty < x < -1, \\ \alpha < x < \beta, \\ 1 < x < \infty. \end{aligned} \end{aligned}$$

The canonical solution of the homogeneous problem (obtained from Eq. (3.5) by setting $g(x) = 0$) in the class of functions bounded at the points $z = \alpha$ and $z = 1$ is given by the expression

$$(3.7) \qquad X(z) = \frac{(z-1)^{\frac{1}{2}}(z-\alpha)^{\frac{1}{2}}}{(z+1)^{\frac{1}{2}}(z-\beta)^{\frac{1}{2}}} = \sqrt{\frac{R_2(z)}{R_1(z)}}\,.$$

Here

$$R_1(z) = (z+1)(z-\beta) \quad \text{and} \quad R_2(z) = (z-1)(z-\alpha).$$

It is assumed in addition that Eq. (3.7) defines this branch of holomorphic function in the complex plane cut along L which has the following expansion with respect to the decreasing powers of z in the neighbourhood of the point at infinity:

$$X(z) = \sqrt{\frac{R_2(z)}{R_1(z)}} = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots.$$

The limiting values of $X(z)$ on L are purely imaginary, and due to the assumption $X^+(x) = X(x)$, also $X^-(x) = -X(x)$. Using the canonical solution $X(z)$ we immediately arrive at the known form of solution of Eq. (3.5),

$$\Phi(z) = \frac{X(z)}{2\pi j} \int_{-\infty}^{+\infty} \frac{g(\xi)\, d\xi}{X^+(\xi)(\xi - z)} + X(z)C.$$

Here C is an arbitrary constant. From the definition (3.1) it follows that $\Phi(-\infty) = 0$ and hence, if the solution is required to vanish at infinity, we should assume $C = 0$.

The auxiliary function $\varphi(x) = \Phi^+(x) - \Phi^-(x) = 2\Phi^+(x)$ appearing in Eq. (3.4) has now the following form:

$$\varphi(x) = \frac{2}{\pi j}\sqrt{\frac{R_2(x)}{R_1(x)}}\left\{\int_L j\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}\frac{w(\xi)}{\xi-x}d\xi - \frac{i\omega}{2}\left[\Omega_1\int_\alpha^\beta\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}\frac{e^{-i\omega\xi}}{\xi-x}d\xi + \Omega_2\int_1^\infty\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}\frac{e^{-i\omega\xi}}{\xi-x}d\xi\right]\right\}. \tag{3.8}$$

It is evidently real-valued with respect to the imaginary unit j since

$$\frac{1}{j}\sqrt{\frac{R_2(x)}{R_1(x)}} = \sqrt{\frac{1-x}{1+x}\frac{x-\alpha}{x-\beta}} \quad \text{for} \quad x \in L$$

and

$$j\sqrt{\frac{R_1(\xi)}{R_2(\xi)}} = \sqrt{\frac{1+\xi}{1-\xi}\frac{\xi-\beta}{\xi-\alpha}} \quad \text{for} \quad \xi \in L$$

and also

$$\sqrt{\frac{R_1(\xi)}{R_2(\xi)}} = \sqrt{\frac{\xi+1}{\xi-1}\frac{\xi-\beta}{\xi-\alpha}} \quad \text{for} \quad \xi \notin L.$$

The final stage in solving Eq. (2.1) is the determination of the constants Ω_1 and Ω_2. Substituting the values $\gamma(\alpha) = \gamma(1) = 0$ in Eqs. (3.3) and (3.4), we obtain the two necessary equations

$$\int_{-1}^{\alpha}\varphi(x)dx = e^{-i\omega\alpha}\Omega_1 \quad \text{and} \quad \int_\beta^1 \varphi(x)dx = e^{-i\omega}\Omega_2 - e^{-i\omega\beta}\Omega_1. \tag{3.9}$$

The expressions (3.4) and (3.8) together with Eqs. (3.9) completely determine the solution of the Birnbaum – Possio equation (2.1), though their direct application to the determination of $\gamma(x)$ for a prescribed distribution $w(x)$ would be very difficult. The next stage of the procedure should then consist in eliminating the auxiliary function $\varphi(x)$ and the indefinite integrals in Eq. (3.4), and also in further simplification of Eqs. (3.9).

No assumptions concerning the size $\delta = \beta - \alpha$ of the gap were made in solving Eq. (2.1). It is then possible to assume, in particular, that $\delta \to 0$ and to obtain the known solution for the profile without the gap. To this end it is sufficient to observe that on the basis of the first equation of the set (3.9) both expressions (3.4) assume in the limit the same form, and in the formula (3.8) the integral over the gap vanishes. The only necessary constant Ω_2 may be evaluated from the equation obtained as the sum of Eqs. (3.9).

A certain physical interpretation may be ascribed to the individual terms of the solution of the Birnbaum – Possio equation. The function $\varphi(x)$ determines the total vorticity, on the profile with a control surface, while $\gamma(x)$ is termed, according to the definition, the bound vorticity. The relations (3.4) express, due to this interpretation, the law of conservation of circulation.

4. The set of equations determining the constants Ω_1, Ω_2

In order to determine the constants Ω_1, Ω_2, we may use Eqs. (3.9) or any linear combination of these equations. In particular, by summing up both equations we obtain an equation which is more convenient for calculations,

$$\int_L \varphi(x)dx = e^{-i\omega}\Omega_2-(e^{-i\omega\beta}-e^{-i\omega\alpha})\Omega_1.$$

Taking account of the form (3.8) of $\varphi(x)$ and changing the order of integration, the following equation is obtained

$$\text{(4.1a)}\quad i\omega\left[\int_\alpha^\beta \sqrt{\frac{R_1(\xi)}{R_2(\xi)}}\, e^{-i\omega\xi}d\xi\right]\Omega_1+i\omega\left[\int_1^\infty \left(\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}-1\right)e^{-i\omega\xi}d\xi + \frac{e^{i\omega}}{i\omega}\right]\Omega_2 = 2j\int_L \sqrt{\frac{R_1(\xi)}{R_2(\xi)}}\, w(\xi)d\xi$$

with a structure similar to that corresponding to the profile without the gap [9]. In spite of the fact that we have not succeeded in expressing the integrals occurring in Eq. (4.1a) in terms of known transcendental functions, we state that they represent relatively simple single integrals of elementary functions.

In order to derive the second equation in a similarly simple form like Eq. (4.1a), a change of variables proves to be useful. Since in Eqs. (3.9) elliptic integrals appear, the transformation should be based on elliptic integrals. Let us introduce two parameters

$$\text{(4.2a)}\qquad \lambda_1 = \frac{(\alpha+1)^{\frac{1}{2}}}{(\alpha-1)^{\frac{1}{2}}} = \frac{1}{j}\sqrt{\frac{1+\alpha}{1-\alpha}},\qquad \lambda_2 = \frac{(\beta-1)^{\frac{1}{2}}}{(\beta+1)^{\frac{1}{2}}} = j\sqrt{\frac{1-\beta}{1+\beta}},$$

the moduls k and a complementary modulus k',

$$\text{(4.2b)}\qquad k = \lambda_1\lambda_2 = \sqrt{\frac{1+\alpha}{1-\alpha}\,\frac{1-\beta}{1+\beta}},\qquad k' = \sqrt{1-k^2} = \sqrt{\frac{2(\beta-\alpha)}{(1+\alpha)(1-\beta)}}\,.$$

The new variables $u_1(x)$ and $v_1(\xi)$ are defined by the transformations

$$\text{(4.3a)}\qquad \lambda_1 snu_1 = \frac{(x+1)^{\frac{1}{2}}}{(x-1)^{\frac{1}{2}}},\qquad \lambda_2 snv_1 = \frac{(\xi-1)^{\frac{1}{2}}}{(\xi+1)^{\frac{1}{2}}},$$

where $snu \equiv sn(u,k)$ is the Jacobi's elliptic function [11, 12]. The transformations (4.3) are not unique since the function sn is doubly periodic. To secure the single-valued transformations, the range of variability of u_1 and v_1 is confined to the rectangles

$$0 \leqslant \mathrm{Re}u_1 \leqslant K,\qquad -K' \leqslant \mathrm{Im}u_1 \leqslant K',$$
$$0 \leqslant \mathrm{Re}v_1 \leqslant K,\qquad -K' \leqslant \mathrm{Im}v_1 \leqslant K',$$

where $K = K(k)$ is the complete elliptic integral of the first kind for the modulus k, and $K' = K(k')$. Extending the real variable x to the entire complex plane $z = x+jy$. it is easily verified that the first condition of the set (4.3a) transforms the upper half-plane of the variable z into the upper half of the rectangle mentioned before, Fig 2. Simultaneously, the line $-\infty < x < +\infty$ is mapped onto the periphery of the rectangle $(0, K, K+jK', jK')$, its edges corresponding to the ends of the segments of L. And, similarly, the second condition (4.3a) generalized to the complex plane $\zeta = \xi + j\eta$ transforms the upper half-plane

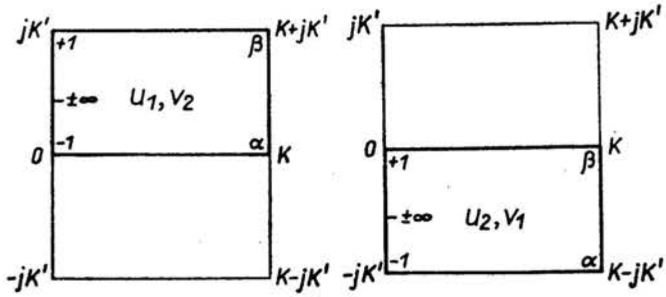

FIG. 2.

of variable ζ into the interior of the rectangle $(0, K, K-jK', -jK')$, and the line $-\infty < \xi < +\infty$ into its periphery. The variable u_1 is real-valued for $x \in [-1, \alpha]$, and the variable v_1 — for $\xi \in [\beta, 1]$.

Equally useful as u_1, v_1 could also be the variables $u_2(x)$ and $v_2(\xi)$ defined by the relations

$$snu_2 = \frac{1}{k\,snu_1}, \quad snv_2 = \frac{1}{k\,snv_1}. \tag{4.3b}$$

Using the correspondence $x \to u_2$ and $\xi \to v_2$ shown in Fig. 2, we obtain

$$u_2 = u_1 - jK', \quad v_2 = v_1 + jK'. \tag{4.3c}$$

The points at infinity of the variables x and ξ correspond, owing to Eqs. (4.3), to the following values of u and v:

$$snu_1 = \frac{1}{\lambda_1} \quad \text{or} \quad snu_2 = \frac{1}{\lambda_2} \quad \text{for} \quad x = \pm\infty$$

and

$$snv_1 = \frac{1}{\lambda_2} \quad \text{or} \quad snv_2 = \frac{1}{\lambda_1} \quad \text{for} \quad \xi = \pm\infty.$$

The variable u_2 is real-valued for $x \in [\beta, 1]$, while v_2 — for $\xi \in [-1, \alpha]$.

Both transformations $x \leftrightarrow u$ and $\xi \leftrightarrow v$ are equivalent and in particular cases we should select the one which leads to simpler considerations or results. The range of variables u and v not only secures the uniqueness of the transformations (4.3) but also makes it possible to write the expression occurring in Eq. (3.8) in one form, valid for the entire range of variables x and ξ

$$\sqrt{\frac{R_2(x)}{R_1(x)}} = j\sqrt{\frac{1-\lambda_2^2}{1-\lambda_1^2}}\,\frac{cnu_1}{snu_1\,dnu_1} = j\sqrt{\frac{1-\lambda_2^2}{1-\lambda_1^2}}\,\frac{snu_2\,dnu_2}{cnu_2}, \tag{4.4}$$

$$\sqrt{\frac{R_1(\xi)}{R_2(\xi)}} = \frac{1}{j}\sqrt{\frac{1-\lambda_1^2}{1-\lambda_2^2}}\,\frac{cnv_1}{snv_1\,dnv_1} = \frac{1}{j}\sqrt{\frac{1-\lambda_1^2}{1-\lambda_2^2}}\,\frac{snv_2\,dnv_2}{cnv_2}.$$

Here *sn*, *cn* and *dn* are the Jacobian elliptic functions for the modulus $k = \lambda_1\lambda_2$.

The second equation determining the constants Ω_1, Ω_2 is conveniently assumed as a linear combination of the expressions (3.9),

$$\int_{-1}^{\alpha}\varphi(x)dx - \frac{2g}{\pi}\Pi(\lambda_1^2, k)\int_L \varphi(x)dx$$

$$= e^{-i\omega\alpha}\Omega_1 + \frac{2g}{\pi}\Pi(\lambda_1^2, k)\,(e^{-i\omega\beta} - e^{-i\omega\alpha})\Omega_1 - \frac{2g}{\pi}\Pi(\lambda_1^2, k)e^{-i\omega}\Omega_2$$

with the notation

$$g = \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)} = \frac{2}{\sqrt{(1-\alpha)(1+\beta)}},$$

and with

$$\Pi(\lambda_1^2, k) = \int_0^K \frac{du}{1-\lambda_1^2 sn^2 u}$$

being the complete elliptic integral of the third kind. On using the relations

$$2g\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}\,\frac{\xi-\alpha}{\xi+1} = -2j\,\frac{cnv_1\,dnv_1}{snv_1}$$

and

$$\frac{1}{j}\int_{-1}^{\alpha}\sqrt{\frac{R_2(x)}{R_1(x)}}\,\frac{dx}{\xi-x} = 2g\left[\Pi(\lambda_1^2, k) - \frac{\xi-\alpha}{\xi+1}\Pi(k^2 sn^2 v_1, k)\right],$$

we obtain, after rearrangements, the equation

$$\text{(4.5)}\quad i\omega\Omega_1\int_{\alpha}^{\beta}\left[j\,\frac{cnv_1\,dnv_1}{snv_1}\Pi(k^2 sn^2 v_1, k) + g\Pi(\lambda_1^2, k)\right]e^{-i\omega\xi}d\xi$$

$$+\Omega_1\left[\frac{\pi}{2}e^{-i\omega\alpha} + g\Pi(\lambda_1^2, k)\,(e^{-i\omega\beta} - e^{-i\omega\alpha})\right]$$

$$+i\omega\Omega_2\int_1^{\infty}\left[j\,\frac{cnv_1\,dnv_1}{snv_1}\Pi(k^2 sn^2 v_1, k) + g\Pi(\lambda_1^2, k)\right]e^{-i\omega\xi}d\xi$$

$$-\Omega_2 g\Pi(\lambda_1^2, k)e^{-i\omega} = -2\int_L \frac{cnv_1\,dnv_1}{snv_1}\Pi(k^2 sn^2 v_1, k)\,w(\xi)d\xi.$$

In order to simplify the further transformations, the Jacobi's Z-function is introduced [11, 12]. If $\xi \notin (-1, \alpha)$, then

$$\frac{cnv_1\,dnv_1}{snv_1}\Pi(k^2 sn^2 v_1, k) = K\frac{cnv_1\,dnv_1}{snv_1} + KZ(v_1) = KZ(v_2) + j\frac{\pi}{2}.$$

Both notations are equivalent. If $\xi \in(-1, \alpha)$, the integral $\Pi(k^2 sn^2 v_1, k)$ is singular and, considering its principal value, the constant term $j\pi/2$ should be subtracted from these expressions. On the basis of the definition of the Zeta function we have

$$\frac{dZ(v)}{dv} = dn^2 v - \frac{E}{K},$$

here $E = E(k)$ is the complete elliptic integral of the second kind. Taking into account the limiting values of the Z-function

$$Z(0) = Z(K) = 0, \quad Z(K+jK') = -j\frac{\pi}{2K},$$

Eq. (4.5) may be integrated by parts. The integral taken over the gap yields

$$i\omega \int\limits_{\alpha}^{\beta} \left[j\frac{cnv_1\, dnv_1}{snv_1} \Pi(k^2 sn^2 v_1, k) + g\Pi(\lambda_1^2, k)\right] e^{-i\omega\xi} d\xi$$

$$= i\omega \int\limits_{\alpha}^{\beta} \left[jKZ(v_2) - \frac{\pi}{2} + g\Pi(\lambda_1^2, k)\right] d\left(\frac{e^{-i\omega\xi}}{-i\omega}\right)$$

$$= -\frac{\pi}{2} e^{-i\omega\alpha} - g\Pi(\lambda_1^2, k)\,(e^{-i\omega\beta} - e^{-i\omega\alpha}) + jK \int\limits_{v_2(\alpha)}^{v_2(\beta)} \left(dn^2 v_2 - \frac{E}{K}\right) e^{-i\omega\xi} dv_2.$$

Similar transformations of the second integral in Eq. (4.5) are slightly more complicated due to its infinite upper limit. The identity

$$\frac{cnv_1\, dnv_1}{snv_1} = \frac{cnv_1}{snv_1\, dnv_1} - k^2 \frac{snv_1\, cnv_1}{dnv_1}$$

enables the first part of the integrand to be transformed to a form more suitable for calculations. Taking into account the limits

$$\lim_{\xi\to\infty} \frac{cnv_1\, dnv_1}{snv_1} = -\frac{g}{j}, \quad \lim_{\xi\to\infty} \Pi(k^2 sn^2 v_1, k) = \Pi(\lambda_1^2, k)$$

and

$$\lim_{\xi\to\infty} jKZ(v_1) = g[\Pi(\lambda_1^2, k) - K], \quad \lim_{\xi\to\infty} k^2 \frac{snv_1\, cnv_1}{dnv_1} = j\lambda_1^2 \sqrt{\frac{1-\lambda_2^2}{1-\lambda_1^2}}$$

we obtain

$$i\omega \int\limits_{1}^{\infty} \left[j\frac{cnv_1\, dnv_1}{snv_1} \Pi(k^2 sn^2 v_1, k) + g\Pi(\lambda_1^2, k)\right] e^{-i\omega\xi} d\xi$$

$$= i\omega \int\limits_{1}^{\infty} \left[jK\frac{cnv_1\, dnv_1}{snv_1} + jKZ(v_1) + g\Pi(\lambda_1^2, k)\right] e^{-i\omega\xi} d\xi$$

$$= i\omega K \sqrt{\frac{1-\lambda_2^2}{1-\lambda_1^2}} \int\limits_{1}^{\infty} \left(-j\sqrt{\frac{1-\lambda_1^2}{1-\lambda_2^2}} \frac{cnv_1}{snv_1\, dnv_1} - 1\right) e^{-i\omega\xi} d\xi - K\sqrt{\frac{1-\lambda_2^2}{1-\lambda_1^2}}\, e^{-i\omega}$$

$$+g\Pi(\lambda_1^2, k)e^{-i\omega}+jK\int\limits_{v_1(1)}^{v_1(\infty)}\left[\left(1-\frac{E}{K}\right)-k^2\frac{cn^2v_1}{dn^2v_1}\right]e^{-i\omega\xi}dv_1.$$

Returning back to the integration variable ξ, multiplying Eq. (4.5) by a constant factor $-\frac{1}{K}\sqrt{\frac{1-\lambda_1^2}{1-\lambda_2^2}}$ and rearranging the terms, we arrive at the following form of the second equation determining the constants Ω_1, Ω_2:

$$\text{(4.1b)}\qquad \left\{\frac{1+\beta}{2}\frac{E}{K}\int\limits_{\alpha}^{\beta}\frac{e^{-i\omega\xi}}{\sqrt{R(\xi)}}d\xi-\int\limits_{\alpha}^{\beta}\frac{\xi-\beta}{\xi-1}\frac{e^{-i\omega\xi}}{\sqrt{R(\xi)}}d\xi\right\}\Omega_1$$

$$+\left\{\frac{1+\beta}{2}\frac{E}{K}\int\limits_{1}^{\infty}\frac{e^{-i\omega\xi}}{\sqrt{R(\xi)}}d\xi-\frac{\beta-\alpha}{2}\int\limits_{1}^{\infty}\frac{\xi+1}{\xi-\alpha}\frac{e^{-i\omega\xi}}{\sqrt{R(\xi)}}d\xi\right.$$

$$\left.+i\omega\left[\int\limits_{1}^{\infty}\left(\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}-1\right)e^{-i\omega\xi}d\xi+\frac{e^{-i\omega}}{i\omega}\right]\right\}\Omega_2$$

$$=\frac{2}{K}\sqrt{\frac{1-\lambda_1^2}{1-\lambda_2^2}}\int\limits_{L}\frac{cnv_1\,dnv_1}{snv_1}\Pi(k^2sn^2v_1, k)\,w(\xi)d\xi.$$

Here $\sqrt{R(\xi)} = \sqrt{R(\xi+j0)}$ is the limiting value of the function

$$\sqrt{R(z)} = (z+1)^{\frac{1}{2}}(z-1)^{\frac{1}{2}}(z-\alpha)^{\frac{1}{2}}(z-\beta)^{\frac{1}{2}}.$$

Equations (4.1a) and (4.1b) tend at $\delta\to 0$ to the same form since

$$\lim_{k\to 1}\frac{1}{K}\sqrt{\frac{1-\lambda_1^2}{1-\lambda_2^2}}\,\frac{cnv_1\,dnv_1}{snv_1}\Pi(k^2sn^2v_1, k) = \lim_{\alpha\to\beta}j\sqrt{\frac{R_1(\xi)}{R_2(\xi)}} = \sqrt{\frac{1+\xi}{1-\xi}}.$$

If the function $w(\xi)$ is represented by polynomials on all segments of L, the right-hand side integrals in Eqs. (4.1) may be evaluated analytically and written in terms of transcendental functions (canonical forms of elliptic integrals).

5. Pressure distribution on the first segment (profile)

The expression (3.4) determining the pressure distribution on the profile with a control surface may be transformed following the procedure used in the case of a profile without the gap, [9]. In order to simplify the integral occuring in the first expression of (4.3), it is convenient to consider the following linear combination of integrals:

$$\text{(5.1)}\qquad \int\limits_{-1}^{x}\varphi(x)dx-\frac{u_1}{K}\int\limits_{-1}^{\alpha}\varphi(x)dx-\frac{2g}{\pi}\left[\Pi(u_1, \lambda_1^2, k)-\frac{u_1}{K}\Pi(\lambda_1^2, k)\right]\int\limits_{L}\varphi(x)dx$$

$$=\int\limits_{-1}^{x}\varphi(x)dx-\frac{u_1}{K}e^{-i\omega\alpha}\Omega_1-\frac{2g}{\pi}\left[\Pi(u_1, \lambda_1^2, k)-\frac{u_1}{K}\Pi(\lambda_1^2, k)\right][e^{-i\omega}\Omega_2$$

(5.1) [cont.]
$$-(e^{-i\omega\beta}-e^{-i\omega\alpha})\Omega_1] = \frac{2}{\pi}\int_L \Lambda(u_1, v_1)w(\xi)d\xi$$
$$+\frac{i\omega}{\pi}\Omega_1\int_\alpha^\beta \left\{j\Lambda(u_1, v_1)-2g\left[\Pi(u_1, \lambda_1^2, k)-\frac{u_1}{K}\Pi(\lambda_1^2, k)\right]\right\}e^{-i\omega\xi}d\xi$$
$$+\frac{i\omega}{\pi}\Omega_2\int_1^\infty \left\{j\Lambda(u_1, v_1)-2g\left[\Pi(u_1, \lambda_1^2, k)-\frac{u_1}{K}\Pi(\lambda_1^2, k)\right]\right\}e^{-i\omega\xi}d\xi.$$

Here
$$\Lambda(u_1, v_1) = -2\frac{cnu_1 dnu_1}{snu_1}\Pi(u_1, k^2sn^2v_1, k)-\frac{u_1}{K}\Pi(k^2sn^2v_1, k).$$

The function $\Pi(u, \lambda^2, k)$ denotes the elliptic integral of the third kind defined by the formula
$$\Pi(u, \lambda^2, k) = \int_0^u \frac{dv}{1-\lambda^2 sn^2 v}.$$

The function $\Lambda(u, v)$ plays in the case considered the role of $\Lambda_1(x, \xi)$ and $\Lambda_2(x, \xi)$ in the classical case of a profile without the gap [9]. The fundamental properties of $\Lambda(u, v)$ were derived and collected in the Appendix (published with Part II of the present paper). It can be used to perform the necessary integration by parts:
$$\frac{i\omega}{\pi}\int_\alpha^\beta \left\{j\Lambda(u_1, v_1)-2g\left[\Pi(u_1, \lambda_1^2, k)-\frac{u_1}{K}\Pi(\lambda_1^2, k)\right]\right\}d\left(\frac{e^{-i\omega\xi}}{-i\omega}\right)$$
$$= -\frac{u_1}{K}e^{-i\omega\alpha}+\frac{2g}{\pi}\left[\Pi(u_1, \lambda_1^2, k)-\frac{u_1}{K}\Pi(\lambda_1^2, k)\right](e^{-i\omega\beta}-e^{-i\omega\alpha})$$
$$+\frac{1}{\pi}j\int_{v_1(\alpha)}^{v_1(\beta)} e^{-i\omega\xi}\frac{\partial}{\partial v_1}\Lambda(u_1, v_1)dv_1$$

and
$$\frac{i\omega}{\pi}\int_1^\infty \left\{j\Lambda(u_1, v_1)-2g\left[\Pi(u_1, \lambda_1^2, k)-\frac{u_1}{K}\Pi(\lambda_1^2, k)\right]\right\}d\left(\frac{e^{-i\omega\xi}}{-i\omega}\right)$$
$$= -\frac{2g}{\pi}\left[\Pi(u_1, \lambda_1^2, k)-\frac{u_1}{K}\Pi(\lambda_1^2, k)\right]e^{-i\omega}+\frac{1}{\pi}j\int_{v_1(1)}^{v_1(\infty)} e^{-i\omega\xi}\frac{\partial}{\partial v_1}\Lambda(u_1, v_1)dv_1.$$

On substituting these results in Eq. (5.1) and rearranging the terms, we obtain the simple relation
$$\int_{-1}^x \varphi(x)dx = \frac{2}{\pi}\int_L \Lambda(u_1, v_1)w(\xi)d\xi$$

$$+\frac{1}{\pi}j\left[\Omega_1\int\limits_{v_1(\alpha)}^{v_1(\beta)} e^{-i\omega\xi}\frac{\partial}{\partial v_1}\Lambda(u_1, v_1)dv_1+\Omega_2\int\limits_{v_1(1)}^{v_1(\infty)} e^{-i\omega\xi}\frac{\partial}{\partial v_1}\Lambda(u_1, v_1)dv_1\right].$$

The second and third integrals in Eq. (3.8) may be transformed by means of the formula

$$\sqrt{\frac{R_2(x)}{R_1(x)}}\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}\frac{d\xi}{\xi-x}=2\frac{cnu_1dnu_1}{snu_1}\frac{dv_1}{1-k^2sn^2u_1sn^2v_1}+\sqrt{\frac{R_2(x)}{R_1(x)}}\frac{\xi+1}{\sqrt{R(\xi)}}d\xi$$

to yield the required form of the expression determining the distribution of pressure

$$(5.2)\quad \gamma(x)=\frac{2}{\pi}\sqrt{\frac{R_2(x)}{R_1(x)}}\int\limits_L\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}\frac{w(\xi)}{\xi-x}d+i\omega\frac{2}{\pi}\int\limits_L\Lambda^*(v, u)w(\xi)d\xi$$

$$-\frac{i\omega}{\pi}\Omega_1\left[\frac{1}{j}\sqrt{\frac{R_2(x)}{R_1(x)}}\int\limits_\alpha^\beta\frac{\xi+1}{\sqrt{R(\xi)}}e^{-i\omega\xi}d\xi-\frac{2}{g}\frac{cnu_1dnu_1}{snu_1}\frac{1}{K}\Pi(k^2sn^2u_1, k)\right.$$

$$\left.\times\int\limits_\alpha^\beta\frac{e^{-i\omega\xi}}{\sqrt{R(\xi)}}d\xi\right]-\frac{i\omega}{\pi}\Omega_2\left[\frac{1}{j}\sqrt{\frac{R_2(x)}{R_1(x)}}\int\limits_1^\infty\frac{\xi+1}{\sqrt{R(\xi)}}e^{-i\omega\xi}d\xi\right.$$

$$\left.-\frac{2}{g}\frac{cnu_1dnu_1}{snu_1}\frac{1}{K}\Pi(k^2sn^2u_1, k)\int\limits_1^\infty\frac{e^{-i\omega\xi}}{\sqrt{R(\xi)}}d\xi\right].$$

In this formula the definition of the function $\Lambda^*(v, u)$ given in the Appendix has been utilized. The first integral in Eq. (5.2) concerns the stationary solution, the second one is due to the apparent mass. The term proportional to Ω_1 expresses the effect of the wake in the gap between the profile and the control surface. The last term is due to the action of the wake behind the trailing edge of the control surface.

For $\alpha\to\beta$ the solution (5.2) is transformed to the classical solution for the profile without the gap. It is proved in the Appendix that with $k = 1$ (i.e for $\alpha = \beta$)

$$\Lambda(u, v)=\Lambda_1(x, \xi)=\frac{1}{2}\ln\left|\frac{1-x\xi+\sqrt{1-x^2}\sqrt{1-\xi^2}}{1-x\xi-\sqrt{1-x^2}\sqrt{1-\xi^2}}\right|,$$

and, simultaneously,

$$\sqrt{\frac{R_2(x)}{R_1(x)}}\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}=\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+\xi}{1-\xi}}.$$

The third term in Eq. (5.2) vanishes for $\alpha\to\beta$, while the fourth one tends to the limit

$$-\frac{i\omega}{\pi}\Omega_2\sqrt{\frac{1-x}{1+x}}\int\limits_1^\infty\frac{e^{-i\omega\xi}}{\sqrt{\xi^2-1}}d\xi=-\frac{\omega}{2}\Omega_2\sqrt{\frac{1-x}{1+x}}H_0^{(2)}(\omega).$$

It is then found that for $\alpha = \beta$

$$\gamma(x)=\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\int\limits_{-1}^{+1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{w(\xi)}{\xi-x}d\xi+i\omega\frac{2}{\pi}\int\limits_{-1}^{+1}\Lambda_1(x, \xi)w(\xi)d\xi$$

$$-\frac{\omega}{2}\Omega_2\sqrt{\frac{1-x}{1+x}}H_0^{(2)}(\omega).$$

From each equation of the set (4.1) it follows that

$$2\int\limits_{-1}^{+1}\sqrt{\frac{1+\xi}{1-\xi}}\,w(\xi)d\xi = i\omega\Omega_2\left[\int\limits_{1}^{\infty}\left(\sqrt{\frac{\xi+1}{\xi-1}}-1\right)e^{-i\omega\xi}d\xi+\frac{e^{-i\omega}}{i\omega}\right]$$

$$= -i\omega\frac{\pi}{2}\,\Omega_2[H_1^{(2)}(\omega)+iH_0^{(2)}(\omega)].$$

After eliminating the constant Ω_2, the known solution is finally obtained:

$$(5.3)\qquad \gamma(x) = \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\int\limits_{-1}^{+1}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{w(\xi)}{\xi-x}\,d\xi+i\omega\frac{2}{\pi}\int\limits_{-1}^{+1}\Lambda_1(x,\xi)w(\xi)d\xi$$

$$-\frac{2}{\pi}[1-C(\omega)]\sqrt{\frac{1-x}{1+x}}\int\limits_{-1}^{+1}\sqrt{\frac{1+\xi}{1-\xi}}\,w(\xi)d\xi,$$

where

$$C(\omega) = \frac{H_1^{(2)}(\omega)}{H_1^{(2)}(\omega)+iH_0^{(2)}(\omega)}$$

is the Theodorsen function containing all the necessary information on the history of the motion.

In the case of a finite gap ($\alpha \neq \beta$), the integrals in Eqs. (4.1) and (5.2) depending on the geometry of the system and on the frequency coefficient ω must be evaluated numerically since they could not be expressed in terms of known functions. These are the following four integrals taken over the gap:

$$(5.4)\qquad \begin{aligned}
I_1(\omega) &= \int\limits_{\alpha}^{\beta}\frac{e^{-i\omega\xi}}{\sqrt{R(\xi)}}\,d\xi = -\int\limits_{\alpha}^{\beta}\frac{e^{-i\omega\xi}d\xi}{\sqrt{(1-\xi^2)(\xi-\alpha)(\beta-\xi)}},\\
I_2(\omega) &= \int\limits_{\alpha}^{\beta}\frac{\xi+1}{\sqrt{R(\xi)}}\,e^{-i\omega\xi}d\xi = -\int\limits_{\alpha}^{\beta}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{e^{-i\omega\xi}d\xi}{\sqrt{(\xi-\alpha)(\beta-\xi)}},\\
I_3(\omega) &= \int\limits_{\alpha}^{\beta}\frac{\xi-\beta}{\xi-1}\,\frac{e^{-i\omega\xi}}{\sqrt{R(\xi)}}\,d\xi = -\int\limits_{\alpha}^{\beta}\sqrt{\frac{\beta-\xi}{\xi-\alpha}}\,\frac{e^{-i\omega\xi}d\xi}{(1-\xi)\sqrt{1-\xi^2}},\\
I_4(\omega) &= \int\limits_{\alpha}^{\beta}\sqrt{\frac{R_1(\xi)}{R_2(\xi)}}\,e^{-i\omega\xi}d\xi = \int\limits_{\alpha}^{\beta}\sqrt{\frac{(1+\xi)(\beta-\xi)}{(1-\xi)(\xi-\alpha)}}\,e^{-i\omega\xi}d\xi
\end{aligned}$$

and the four integrals taken over the (infinite) wake behind the trailing edge of the control surface

$$(5.5)\qquad J_1(\omega) = \int\limits_{1}^{\infty}\frac{e^{-i\omega\xi}}{\sqrt{R(\xi)}}\,d\xi = \int\limits_{1}^{\infty}\frac{e^{-i\omega\xi}d\xi}{\sqrt{(\xi^2-1)(\xi-\alpha)(\xi-\beta)}},$$

(5.5) [cont.]
$$J_2(\omega) = \int\limits_1^\infty \frac{\xi+1}{\sqrt{R(\xi)}} e^{-i\omega\xi} d\xi = J_1(\omega) + \int\limits_1^\infty \frac{\xi e^{-i\omega\xi} d\xi}{\sqrt{(\xi^2-1)(\xi-\alpha)(\xi-\beta)}},$$

$$J_3(\omega) = \int\limits_1^\infty \frac{\xi+1}{\xi-\alpha} \frac{e^{-i\omega\xi}}{\sqrt{R(\xi)}} d\xi = \int\limits_1^\infty \sqrt{\frac{\xi+1}{\xi-1}} \frac{e^{-i\omega\xi} d\xi}{(\xi-\alpha)\sqrt{(\xi-\alpha)(\xi-\beta)}},$$

$$J_4(\omega) = \int\limits_1^\infty \left(\sqrt{\frac{R_1(\xi)}{R_2(\xi)}} - 1\right) e^{-i\omega\xi} d\xi + \frac{e^{-i\omega}}{i\omega}$$

$$= \int\limits_1^\infty \left(\sqrt{\frac{(\xi+1)(\xi-\beta)}{(\xi-1)(\xi-\alpha)}} - 1\right) e^{-i\omega\xi} d\xi + \frac{e^{-i\omega}}{i\omega}.$$

The integrals (5.4) may easily be evaluated numerically by means of the Gauss – Jacobi quadratures. The method based on the interpolation formula by Everett (described in [13]) may be applied to the integrals (3.5). The square root singularity of the integrands for $\xi \to 1$ may be removed by using suitable linear combinations of integrals containing the Hankel functions.

In the case of the function $w(\xi)$ represented by polynomials on each of the segments of the line L, all the remaining integrals occuring in Eq. (5.2) are expressible in terms of elementary functions and some other transcendental functions which are easily evaluated (canonical forms of elliptic integrals).

Transformation of Eq. (5.2) to a form analogous to Eq. (5.3) by eliminating the constants Ω_1, Ω_2 according to Eq. (4.1) is not purposeful since in the case of a profile with a gap there exist no functions depending exclusively on the frequency coefficient ω and revealing similar properties as the Theodorsen function $C(\omega)$.

6. Pressure distribution on the second segment (control surface)

Transformation of the second expression of Eq. (3.4) is based on the following combination of integrals:

$$\int\limits_x^1 \varphi(x)dx - \frac{u_2}{K}\int\limits_\beta^1 \varphi(x)dx - \frac{2g}{\pi}\left[\Pi(u_2, \lambda_2^2, k) - \frac{u_2}{K}\Pi(\lambda_2^2, k)\right]\int\limits_L \varphi(x)dx$$

$$= \int\limits_x^1 \varphi(x)dx - \frac{u_2}{K}(e^{-i\omega}\Omega_2 - e^{-i\omega\beta}\Omega_1) - \frac{2g}{\pi}\left[\Pi(u_2, \lambda_2^2, k) - \frac{u_2}{K}\Pi(\lambda_2^2, k)\right]$$

$$\times [e^{-i\omega}\Omega_2 - (e^{-i\omega\beta} - e^{-i\omega\alpha})\Omega_1] = \frac{2}{\pi}\int\limits_L \Lambda(u_2, v_2) w(\xi) d\xi$$

$$+ \frac{i\omega}{\pi}\Omega_1 \int\limits_\alpha^\beta \left\{ j\Lambda(u_2, v_2) - 2g\left[\Pi(u_2, \lambda_2^2, k) - \frac{u_2}{K}\Pi(\lambda_2^2, k)\right]\right\} e^{-i\omega\xi} d\xi$$

$$+\frac{i\omega}{\pi}\Omega_2\int\limits_1^\infty\left\{j\Lambda(u_2,v_2)-2g\left[\Pi(u_2,\lambda_2^2,k)-\frac{u_2}{K}\Pi(\lambda_2^2,k)\right]\right\}e^{-i\omega\xi}d\xi.$$

Performing the integration by parts and rearranging the terms, we obtain

$$\Omega_1 e^{-i\omega\beta}+\int\limits_\beta^x\varphi(x)dx=\Omega_2 e^{-i\omega}-\int\limits_x^1\varphi(x)dx=\frac{2}{\pi}\int\limits_L\Lambda^*(v,u)w(\xi)d\xi$$

$$+\frac{1}{\pi}j\left[\Omega_1\int\limits_{v_2(\alpha)}^{v_2(\beta)}e^{-i\omega\xi}\frac{\partial}{\partial v_2}\Lambda(u_2,v_2)dv_2+\Omega_2\int\limits_{v_2(1)}^{v_2(\infty)}e^{-i\omega\xi}\frac{\partial}{\partial v_2}\Lambda(u_2,v_2)dv_2\right].$$

Since $dv_2 = dv_1$ and, as shown in the Appendix, $\frac{\partial}{\partial v_2}\Lambda(u_2,v_2)=\frac{\partial}{\partial v_1}\Lambda(u_1,v_1)$, the introduction of new variables u_1, v_1 and comparison with the analogous relation for the first segment makes is possible to verify that the right-hand sides of the two relations are identical. Consequently, the forms of expressions defining $\gamma(x)$ must also be identical what proves that Eq. (5.2) determines the distributions of pressures on both segments, that is for arbitrary values of $x \in L$.

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