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Appropriate solution and its application to problems in fluid dynamics II. Accuracy of approximate solution

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THE NOTION of an appropriate solution and accuracy of approximate solutions are defined in the paper. These notions are applied to the theoretical test for the accuracy of M^2-expension solution of two-dimensional compressible flow past a profile in a uniform stream. It is shown that the basic equation is appropriate under a certain condition, and this property is utilized to check the accuracy of the M^2-expansion solutions for various types of profiles, leading to error estimates in terms of Mach number M and the ratio of specific heats γ.

W pracy zdefiniowano pojęcie rozwiązania właściwego oraz określono dokładność przybliżonego rozwiązania równania. Pojęcia te zastosowano do sprawdzianu teoretycznego dokładności rozwinięcia M^2 przedstawiającego rozwiązanie dwuwymiarowego, ściśliwego opływu profilu w strumieniu jednorodnym. Pokazano, że podstawowe równanie jest właściwe przy spełnieniu pewnego warunku, a własność tę wykorzystać można do sprawdzenia dokładności rozwiązań w postaci rozwinięć M^2 dla różnych typów profili, co prowadzi do oszacowania błędów za pomocą liczby Macha M i ciepła właściwego γ.

В работе определено понятие правильного решения и определена точность приближенного решения уравнения. Эти понятия применены к теоретической проверке точности развложения M^2, представляющего решение двумерного сжимаемого обтекания профиля в однородном потоке. Показано, что основное уравнение является правильным при удовлетворении некоторому условию, а это свойство можно использовать для проверки точности решений в виде разложений M^2 для разных типов профилей, что приводит к оценке ошибок при помощи числа Маха M и удельной теплоемкости γ.

1. Introduction

THE PURPOSE of this paper is to give an application of the concept of "appropriateness" introduced in the previous work [1], to an estimate of accuracy of an approximate solution.

The appropriateness and the accuracy here is defined as follows.

Let X, Y be normed spaces, $D \subset X$, $T: D \to Y$, $y_0 \in Y$ and $\varepsilon, \varepsilon' > 0$ be given, and let $S(y_0, \varepsilon') = \{x \mid \|Tx - y_0\| < \varepsilon', x \in D\}$. The equation $Tx = y_0$ is called $(\varepsilon, \varepsilon')$-appropriate, if $\|x - x'\| < \varepsilon$ for any $x, x' \in S(y_0, \varepsilon')$; and any $x \in S(y_0, \varepsilon')$ of an $(\varepsilon, \varepsilon')$-appropriate $Tx = y_0$ is called its $(\varepsilon, \varepsilon')$-appropriate solution.

The solution set $S(y_0, \varepsilon')$ is introduced to enlarge the conception of solution from x of $Tx = y_0$ to ones of $Tx \approx y_0$ in compliance with the situation that an equation for problems in science and technology is usually an idealized approximation to a certain natural law, which can be written down more precisely by an expression such as $Tx \approx y_0$. The appropriateness is to meet the requirement of a mathematical idealization of the real problem, which can not be reasonable without having the property of "stability" and "uniqueness", representing the feature that a corresponding physical phenomenon in reality should appear determinately.

Consider an approximate solution x_a of an equation $Tx = y_0$ and let

$$S = S(y_0, ||Tx_a - y_0||), \qquad \delta = \sup_{x, x' \in S} ||x - x'||.$$

The accuracy of approximate solution is then defined as $||Tx_a - y_0|| + \delta$ or more concisely by δ itself when the magnitude of $||Tx_a - y_0||$ is the same as or less than that of δ.

Note that δ gives a conventional error bound of the approximate solution if a true solution of $Tx = y_0$ exists. Moreover, it is evident that $\delta \leqslant 2\varepsilon$ for any $(\varepsilon, ||Tx_a - y_0||)$ — appropriate solution x_a of $Tx = y_0$.

As an example, we consider below the accuracy of the classical M^2-expansion solution of a two-dimensional compressible flow past a profile in a uniform stream. For this, the basic case of the flow past a circular cylinder without circulation is considered first in Sect. 2, and it is then extended in Sect. 3 to the cases of more general profiles.

2. M^2-expansion solution of flow past a circular cylinder without circulation

We consider here the basic case of the unit flow past a circular cylinder of unit radius without circulation [2].

We start with the equation of a two-dimensional compressible flow [3],

$$\frac{\partial F}{\partial \bar{z}} = \frac{1-\varrho}{2} \frac{\partial}{\partial \bar{z}} (F + \bar{F}), \tag{2.1}$$

where $z = x + iy$, F is the complex velocity potential and ϱ is the dimensionless density based on the density at infinity. For a polytropic gas of adiabatic index γ (> 1),

$$\varrho = \left[1 - \frac{\gamma - 1}{2} M^2(|q|^2 - 1)\right]^{\frac{1}{\gamma - 1}}, \tag{2.2}$$

where M is the Mach number of the uniform flow at infinity, and q is the complex velocity defined here as

$$q = \frac{\partial}{\partial \bar{z}} (F + \bar{F}). \tag{2.3}$$

The boundary conditions are $\mathrm{Im}(F) = \text{const}$ ($= 0$, say) on the surface of the body, and

$$\partial F / \partial z \to 1 \quad \text{as} \quad z \to \infty.$$

In addition to the above, we postulate that the flow velocity q belongs to a certain class of functions, which bears the characteristics of M^2-expansion solutions of the present case. Introduce for this a normed space $X = \{f\}$, where $f(r, \theta)$ is a function given by

$$f(r, \theta) = \sum_{m=-\infty}^{\infty} f_m(r) e^{im\theta}$$

with complex valued, continuous $f_m(r)$, and

$$||f|| = \sum_{m=-\infty}^{\infty} \sup_{r\geqslant 1} |f_m(r)| < \infty.$$

Now let q_0 be the complex velocity of the incompressible flow of the present case or $q_0 = 1-\bar{z}^{-2}$, and let

$$q = q_0+Q, \quad Q \in D, \quad D = \{Q|rQ \in X, \ \mathrm{Re}(Qe^{-i\theta})_{r=1} = 0\}. \tag{2.5}$$

Observe that q in Eq. (2.5) satisfies the boundary condition in Eq. (2.4), and the class of q's includes any M^2-expansion solution to the flow velocity past a circular cylinder without circulation [4].

It can be shown then that

$$(1-\varrho)q \in D, \text{ and } Q = N\left\{\frac{1}{2}(1-\varrho)q\right\}, \tag{2.6}$$

where $N: D \to D$ is a bounded linear operator defined by $Q = Nh$ through $Q = q-q_0$, $q = \dfrac{\partial}{\partial \bar{z}}(F+\bar{F}), \dfrac{\partial F}{\partial \bar{z}} = h \in D$.

Define $T: D \to D$ by

$$TQ = Q-N\left\{\frac{1}{2}(1-\varrho)q\right\} \tag{2.7}$$

and rewrite Eq. (2.6) as $TQ = 0$.

The equation $TQ = 0$ is proved to be $(2\varepsilon'/G_A(A_0, M^2, \gamma), \varepsilon')$-appropriate, where $G_A(A, M^2, \gamma)$ is the derivative of a given function $G(A, M^2, \gamma)$ with $A = 2+||rQ||$ and A_0 is the smallest root of $G(A, M^2, \gamma) = 2+\varepsilon'$ for a given $M^2, \gamma, \varepsilon'$. Thus the accuracy δ of an approximate solution $\bar{Q}$ of $TQ = 0$ is given by

$$\delta \leqslant 2||rT\bar{Q}||/G_A(A_0, M^2, \gamma). \tag{2.8}$$

Now the M^2-expansion solution $q^{(n)}$ of order n can be written as

$$q^{(n)} = q_0+Q^{(n)}, \quad Q^{(n)} = \sum_{i=0}^{n} M^{2i}R^{(i)}, \quad n = 0, 1, \ldots,$$

where $R^{(0)} = 0$ and $R^{(i)}$ for $i \geqslant 1$ are determined successively from $R^{(0)}$ using Eq. (2.6). The accuracy δ of the n-th order approximation $Q^{(n)}$ is given by Eq. (2.8) as

$$\delta \leqslant \delta^{(n)} \equiv 2||rTQ^{(n)}||/G_A((A_0, M^2, \gamma), \tag{2.9}$$

which can be expressed, in principle, in terms of γ, M and $A^{(n)}$ $(= 2+||rQ^{(n)}||)$ as

$$\delta^{(n)} \leqslant \tilde{\delta}^{(n)}(A^{(n)}, M^2, \gamma). \tag{2.10}$$

Actual computation to determine $\tilde{\delta}^{(n)}$ is roughly equivalent to the derivation of the next approximation $Q^{(n+1)}$ from $Q^{(n)}$. The magnitudes of $\tilde{\delta}^{(0)}$ and $\tilde{\delta}^{(1)}$ computed for two cases $\gamma = 2$ and $\gamma = 1.4$ are shown graphically in Fig. 1, where their relative values $\tilde{\delta}^{(0)}/||q_0||$ and $\tilde{\delta}^{(1)}/||q_0||$ expressed as percentages are plotted against the Mach number M.

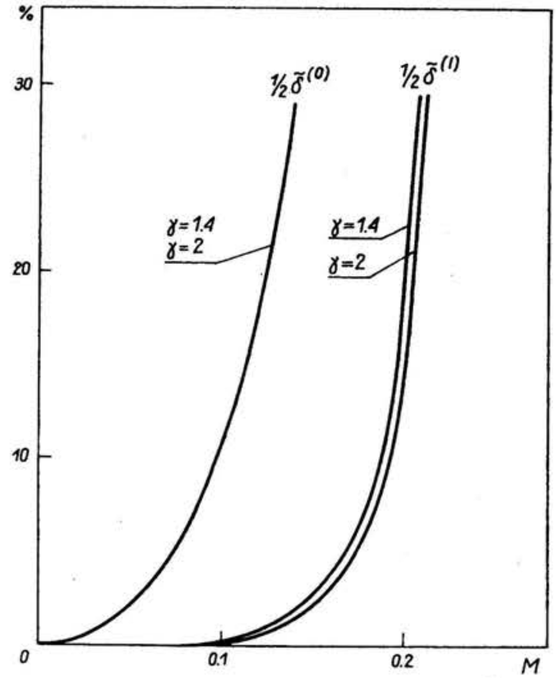

FIG. 1. Relative accuracies of the zero-th and the first order M^2-expansion solutions.

3. M^2-expansion solution of a flow past an arbitrary cylinder

The above procedure for the determination of the accuracy of the M^2-expansion solution to a circular cylinder shown in the previous section is extended here to incorporate a cylinder of a more general profile P and a circulation around it.

Let us assume that the region outside the profile P in the z-plane is conformally mapped onto the region outside a unit circle with the center at the origin in the Z-plane by the transformation function $z = P(Z)$:

$$z = b_{-1}Z+b_0+b_1Z^{-1}+\ldots, \qquad b_{-1} = ae^{i\delta}. \tag{3.1}$$

Then the complex velocity q_0 of the incompressible flow past a profile P is given generally as

$$q_0 = \overline{\frac{dF_0}{dZ}} = \overline{\frac{dF_0}{dz}\Big/\frac{dz}{dZ}} \tag{3.2}$$

with

$$F_0 = a\{e^{i(\delta-\alpha)}Z+e^{-i(\delta-\alpha)}Z^{-1}\}+i\varkappa\log Z,$$

where α is the angle with the x-axis of the direction of the undisturbed flow at infinity, $2\pi\varkappa$ is the amount of circulation around the profile.

Introduce here the same space X and D as above in Eq. (2.5) in the Z-plane and assume for the complex velocity q that $q = q_0+Q$, with

(3.3) $$q_0, |Z|Q, \quad dz/dZ \in X.$$

It should be noted that the assumption in Eq. (3.3) excludes some singular profiles such as a flat plate.

With these assumptions, the equation $TQ = 0$ in Eq. (2.7) can be extended to $T'Q = 0$ with

$$T'Q = Q - \overline{\left(\frac{dz}{dZ}\right)}^{-1} N\left\{\frac{1}{2}(1-\varrho)q\,\overline{\frac{dz}{dZ}}\right\},$$

which can be shown to have a similar property as to the appropriateness to that of Eq. (2.7). This property is utilized for the accuracy of the flow past various profiles with circulation.

Actual computations are done for the zero-th approximate solutions to cases including a circular cylinder with circulation, an elliptic cylinder and Joukowski aerofoil. The results show the general tendency that the larger a profile deforms from the circle, and the larger the amount of circulation is, the less the accuracy becomes.

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