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# Stability of a wall jet

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EXPERIMENTAL data were obtained for a plane laminar wall jet at very low velocities (Re ≫ 90). The velocity profiles gave good results as compared to the theoretical work of M. B. GLAUERT [2]. A calculation of the stability of a two-dimensional wall jet seemed necessary. There is a previous work by CHUN and SCHWARZ [3] giving very few details on the method they used. We have applied the Galerkin method to this problem. It was our aim to demonstrate that the stability curve of a wall jet is easily obtained; one limit of Glauert's curve is taken as infinite. By decomposing Glauert's curve into two appropriate functions, no difficulty is encountered in joining the two. The perturbation function is chosen in order to give simple integrations. The accuracy was sufficient to determine the instability critical point and we find, for a given profile, a critical Reynolds number slightly higher than the one obtained by Chun and Schwarz (where $Re_{\tilde{\delta}_c} = 57$). An experimental determination of the maximum amplification point ($Re_{\tilde{\delta}} = 300$ for a wave number $\alpha = 1.85$) would be suitable.

Otrzymano dane eksperymentalne dotyczące płaskiego strumienia przyściennego przy bardzo małych prędkościach (Re ≫ 90). Otrzymane profile prędkości zgadzają się dobrze z teoretycznymi wynikami GLAUERTA [2]. Stwierdzono konieczność analizy stateczności dwuwymiarowego strumienia przyściennego. Istnieje wcześniejsza praca CHUNA i SCHWARTZA [3], w której podano bardzo niewiele szczegółów dotyczących zastosowanej przez nich metody. Do zagadnienia zastosowaliśmy metodę Galerkina. Naszym celem było wykazanie, że krzywą stateczności strumienia przyściennego można łatwo otrzymać; jedną z granic krzywej Glauerta przyjęto za nieskończoną. Przy rozkładzie jej na dwie odpowiednie krzywe nie napotyka się trudności z ich połączeniem. Wybrano funkcję perturbacyjną w ten sposób, by otrzymać proste całkowanie. Dokładność była dostateczna do określenia krytycznego punktu niestateczności i, dla danego profilu, krytyczna liczba Reynoldsa okazała się nieco wyższa niż wartość otrzymana przez Chuna i Schwartza (gdzie $Re_{\tilde{\delta}_c} = 57$). Byłoby pożądane określenie doświadczalne punktu maksymalnego wzmocnienia ($Re_{\tilde{\delta}} = 300$ dla liczby falowej $\alpha = 1.85$).

Получены экспериментальные данные, касающиеся плоского пограничного потока при очень малых скоростях (Re ≫ 90). Полученные профили скорости хорошо совпадают с теоретическими результатами Глауэрта [2]. Констатирована необходимость анализа устойчивости двумерного пограничного потока. Существует более ранняя работа Хуна и Шварца [3], в каторой приведено очень немного подробностей, касающихся применяемого ими метода. В проблеме применен метод Галеркина. Нашей целью являлось показание, что кривую устойчивости пограничного потока можно легко получить; одна из границ кривой Глауэрта прицята бесконечной. При разложении ее на две соответствующие кривые не встречается трудностей с их соединением. Пертурбационная функция подобрана таким образом, чтобы получить простое интегрирование. Точность была достаточной для определения критической точки неустойчивости и критическое число Рейнольдса, для данного профиля, оказалось немного больше чем значение полученное Хуном и Шварцом (где $Re_{\tilde{\delta}_c} = 57$). Полезным было бы экспериментальное определение точки максимального усиления ($Re_{\tilde{\delta}} = 300$, для волнового числа $\alpha = 1{,}85$).

## Notations

Letters with the tilde "~" indicate dimensional variables.

$\tilde{x}, \tilde{y}$ coordinates measured along and normal to surface,

$\tilde{U}$ horizontal velocity component,
$\tilde{U}_{max}$ maximum velocity in the jet at any section,
$U = \frac{\tilde{U}}{U_{max}}$,
$u, v$ horizontal and vertical components of the disturbance velocity,
$\eta = B\frac{\tilde{y}}{\tilde{x}b}$ transformed coordinate,
$B, b$ constants,
$f'$ transformed velocity component,
$h = \sqrt{f'}$,
$\tilde{\delta}$ spreading of the jet where $\tilde{U} = \frac{\tilde{U}_{max}}{2}$,
$m$ maximum limiting value of $y$ for $U_2 = 0$,
$s$ value of $y$ for $U$ maximum,
$\nu$ kinematic viscosity,
$N$ matrix order,
$D = \frac{\partial}{\partial y}$,
$\alpha$ wave number,
$c$ complex wave velocity.

## 1. Introduction

IN A PREVIOUS work [1] we compared our low velocity experimental results with the theoretical work of M. B. GLAUERT [2]. As a continuation, we calculate here the stability conditions of a plane wall jet.

A previous calculation by CHUN and SCHWARZ [3] offers very few details on the method employed. Galerkin's method used quite successfully for similar problems seemed most appropriate.

This work is theoretical; an experimental verification will follow.

## 2. Previous works

The considered plane wall jet is shown in Fig. 1. The distance $X_0$ is the length necessary to establish the jet on a plane, this distance is equal to 10 $H$ where $H$ is the nozzle height.

M. B. Glauert has studied theoretically the velocity profiles of both plane and radial laminar wall jets. Starting from the Navier-Stokes equations and with a stream function $\psi$, he admits the existence of a self-preserving relation which depends only on the variable

$\eta = B\frac{\tilde{y}}{\tilde{x}^b}$

$$\tilde{U} = A\tilde{x}^a f'(\eta),$$

where $\tilde{U}$ is the horizontal velocity component following the plate axis.

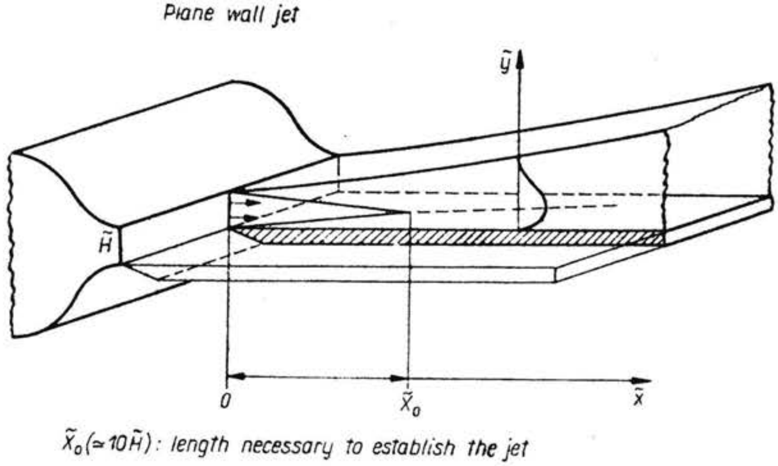


FIG. 1.

With the boundary and initial conditions he obtains the velocity profiles:

$$\tilde{U} = A\tilde{x}^{1/2} f'(\eta)$$

$a$, $b$, $A$, $B$ are constants.

He obtains the following equation:

$$\eta = \mathrm{Log}\frac{\sqrt{h^2+h+1}}{1-h} + \sqrt{3}\,\mathrm{Arctg}\left(\frac{h\sqrt{3}}{2+h}\right),$$

where $h^2 = f$; $hh' = f'/2$

Glauert's curve is shown in Fig. 2.

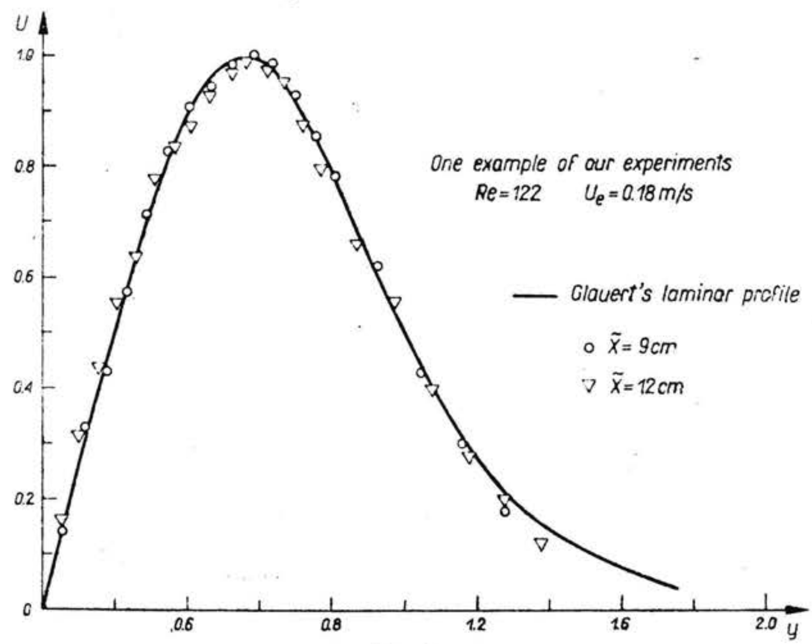


FIG. 2.

## 3. Experimental work

We compared Glauert's theoretical results to the experimental data obtained in a wind tunnel with a continuous but low flow rate plane jet (Fig. 2). The velocity profiles were measured with a hot wire anemometer. We started from a relatively low density (Re = 94) and continued up to Re = 550. Our results agree well with the theoretical results of Glauert, especially for the region near the wall.

## 4. Application of Galerkin's method

### 4.1. Selected laminar profiles

Starting from Glauert's results, we apply Galerkin's method to the Orr-Sommerfeld equation in order to determine the stability criteria. This equation requires $U(y)$ (we suppose an undimensional flow) to be known explicitly; this is not the case for Glauert's equation $\eta = f(h)$. We were thus required to approximate the theoretical curve of the velocity profiles by another simple curve $f'(\eta)$ .We were forced to cut the curve (Fig. 2) into two parts: the first part $U_1$ is homologous to a third degree profile of a laminar boundary layer (from zero to the peak of the curve). For the second part $U_2$, two different profiles are used in order to approach Glauert's curve with a maximum of precision.

We obtain two profiles:

profile 1 with $U_1 = ay+by^3 = 2.708y-2.942y^3$,

(Fig. 3) $U_2 = c+de^{-uy}+fe^{-vy} = 89.79e^{-3.68y} - 100.71e^{-4.048y}$,

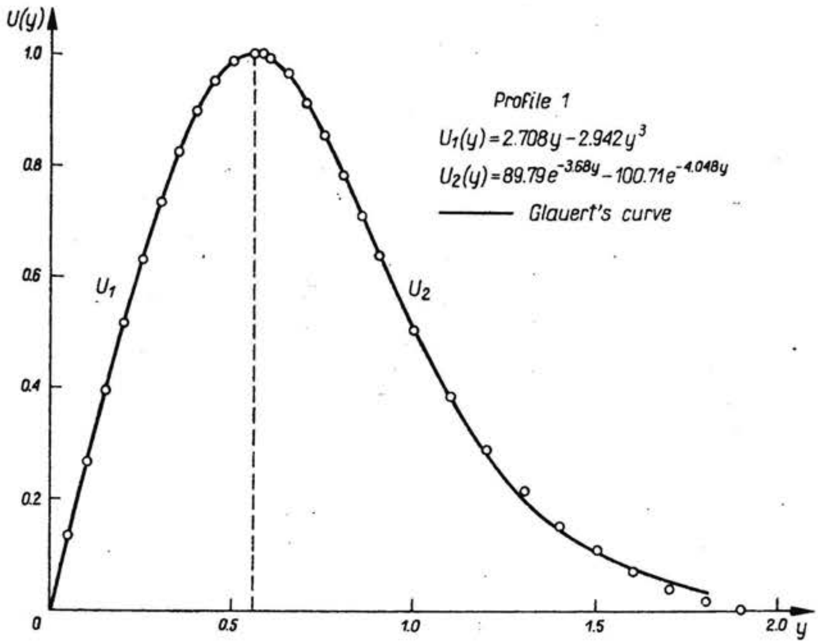


FIG. 3.

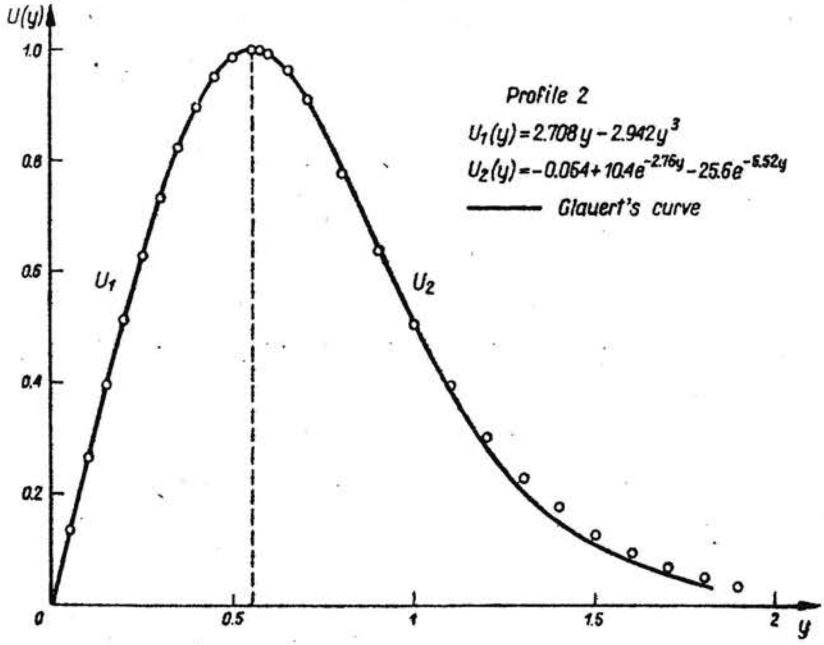


FIG. 4.

profile 2 with $U_1 = 2.708y-2.942y^3,$

(Fig. 4) $U_2 = -0.054+10.4e^{-2.76y}-25.6e^{-5.52y}.$

This last profile agrees best with our experimental results.

We chose two profiles to study the influence of the upper boundary of the jet.

### 4.2. On the Sommerfeld equation

For a plane wall jet, from the Navier-Stokes equations and the continuity equation using the hypothesis $U = U(y)$, we obtain

$$\left(\frac{\partial}{\partial t}+\frac{\partial U}{\partial x}\right)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial_2 v}{\partial y^2}\right)-\frac{\partial v}{\partial x}\frac{\partial^2 U}{\partial y^2} = \frac{1}{\mathrm{Re}_{\bar{g}}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2(v),$$

$v$ is the vertical component of the perturbation velocity. The stream function is given by Tollmien Schlichting,

$$\Psi(x, y) = \varphi(y)e^{i\alpha(x-ct)},$$

where

$$u = \frac{\partial\Psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial\Psi}{\partial x},$$

$\alpha$ is the wave number, $\varphi(y)$ is the perturbation function, $c$ is the complex wave velocity, $c$ is decomposed into $c_r+ic_i$, $c_r$ is the wave velocity, $\alpha c_i$ is the attenuation or amplification of the wave with time.

The Orr-Sommerfeld equation is

$$(U-c)\,(D^2-\alpha^2)\big(\varphi(y)\big)-D^2U\cdot\varphi(y)=\frac{1}{i\alpha\mathrm{Re}_{\mathfrak{z}}}\,(D^2-\alpha^2)\big(\varphi(y)\big).$$

The boundary conditions are

$$U=0,\quad u=v=0,\quad \varphi(0)=0,\quad \varphi'(0)=0\quad \text{at}\quad y=0,$$
$$U=0,\quad u=v=0,\quad \varphi(\infty)=0,\quad \varphi'(\infty)=0\quad \text{at}\quad y\to\infty.$$

#### 4.3. The Galerkin method

The function $\varphi(y)$ is approximated by a series $\sum_{k=1}^{n} a_k\varphi_k$, where $\varphi_k$ are given functions of $y$. From Sommerfeld's equation, we substitute

$$\mathscr{F}=(D^2-\alpha^2)^2-i\alpha\mathrm{Re}_{\mathfrak{z}}\,[(U-c)\,(D^2-\alpha^2)-D^2U].$$

It is not possible to assure that $\mathscr{F}=0$ by replacing $\varphi$ by its approximation. However, we can resolve the system

$$\int_0^\infty \varphi_l\Big(\sum_{k=1}^{n} a_k\varphi_k\Big)\,dy=0.$$

We obtain $n$ equations with $n$ unknowns $a_1, a_2 \dots a_n$. $\mathscr{F}$ is a linear operator.

We can write

$$\sum_{k=1}^{n} a_k\int_0^\infty \varphi_l\mathscr{F}(\varphi_k)dy=0,$$

$$\mathscr{F}=\mathscr{G}+c\,\mathscr{H},$$

with

$$\mathscr{G}=(D^2-\alpha^2)^2-i\alpha\mathrm{Re}_{\mathfrak{z}}[U(D^2-\alpha^2)-D^2U],$$

and

$$\mathscr{H}=i\alpha\mathrm{Re}_{\mathfrak{z}}(D^2-\alpha^2).$$

The system becomes

$$\sum_{k=1}^{n} a_k\Big[\int_0^\infty \varphi_l\mathscr{G}(\varphi_k)dy+c\int_0^\infty \varphi_l\mathscr{H}(\varphi_k)\Big]\,dy=0.$$

We named $[A_{kl}]$ and $[B_{kl}]$ matrices such that

$$[A_{kl}]=\int_0^\infty \varphi_l\mathscr{G}(\varphi_k)dy,$$

$$[B_{kl}]=-\int_0^\infty \varphi_l\mathscr{H}(\varphi_k)dy.$$

The system has the following simplified form:

$$\sum_{k=1}^{n} a_k(A_{kl}-cB_{kl})=0.$$

This system has a non-zero solution only if

$$\det([A_{kl}]-c[B_{kl}]) = 0 \quad \text{if} \quad [B_{kl}] \neq 0$$

we can write

$$\det([A_{kl}]\cdot[B_{kl}]^{-1}-c(I)) = 0.$$

This equation determines the eigenvalues of the matrix product $[A_{kl}]\,[B_{kl}]^{-1}$. At least, $n$ complex values satisfying this condition exist. If $c_i$ is negative, the flow is stable; if one of these values is positive, the flow is unstable. If one of these values is zero while all the others are negative, the flow has a neutral stability.

Galerkin's method is quite accurate, convergence depends on the choice of the perturbation functions $\varphi_k$.

The term $A_{kl}$ is split into 5 parts:

$$A_{kl} = I^1_{kl}+i\alpha \mathrm{Re}_{\bar{\delta}}[-I^2_{kl}-I^3_{kl}+I^4_{kl}+I^5_{kl}]$$

with

$$I^1_{kl} = \int\limits_0^\infty \varphi_l(D^2-\alpha^2)^2(\varphi_k)dy = \int\limits_0^\infty \varphi_l(D^4\varphi_k-2\alpha^2D^2\varphi_k+\alpha^4\varphi_k)dy,$$

$$I^2_{kl} = \int\limits_0^s \varphi_l U_1(y)(D^2\varphi_k-\alpha^2\varphi)dy,$$

$$I^3_{kl} = \int\limits_s^\infty \varphi_l U_2(y)\,(D^2\varphi_k-\alpha^2\varphi_k)dy,$$

$$I^4_{kl} = \int\limits_s^\infty \varphi_l D^2\big(U_1(y)\big)\varphi_k dy,$$

$$I^5_{kl} = \int\limits_s^\infty \varphi_l D^2\big(U_2(y)\big)\varphi_k dy.$$

The term $B_{kl}$ is

$$B_{kl} = i\alpha \mathrm{Re}_{\bar{\delta}} I^6_{kl}$$

with

$$I^6_{kl} = \int\limits_0^\infty \varphi_l(D^2\varphi_k-\alpha^2\varphi_k)dy,$$

$s$ = value of $y$ for the peak value of $U$.

## 5. Results

Calculations are performed on a IBM 360/168 computer using double precision. The object consists in observing the variation of the imaginary parts of the eigenvalues of the product $[A_{kl}]$. $[B_{kl}]^{-1}$ as a function of the Reynolds number $\mathrm{Re}_{\bar{\delta}}$ evolution for a given $\alpha$. This method permits us to detect rapidly the negative to positive transition of an imaginary part, and to locate the curve $\alpha$, $\mathrm{Re}_{\tilde{\delta}_*}$, corresponding to $c_i = 0$, in other words, the stability curve.

a) We first choose the function $\varphi_k(y) = y^2 e^{-ky}$. It satisfies the boundary condition and, when associated with the functions of the two curve parts similar to the one of Glauert, permits easy integration.

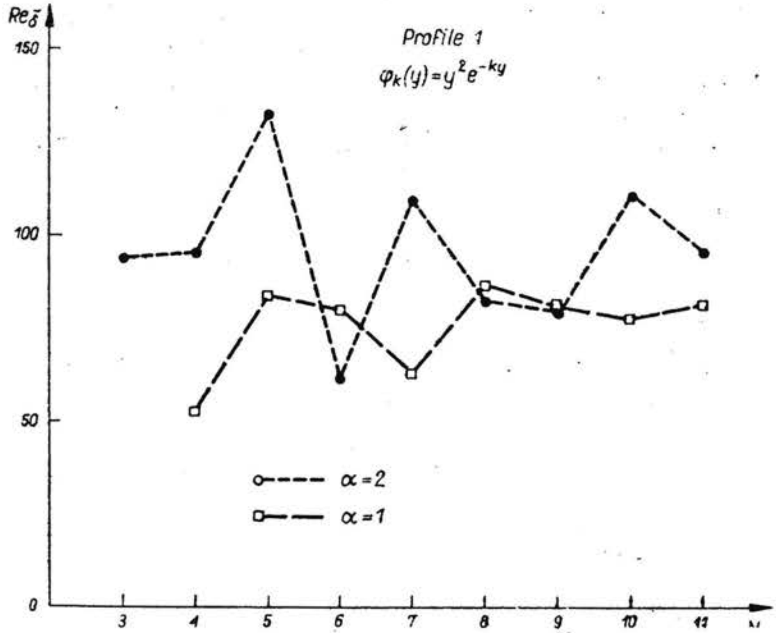


FIG. 5.

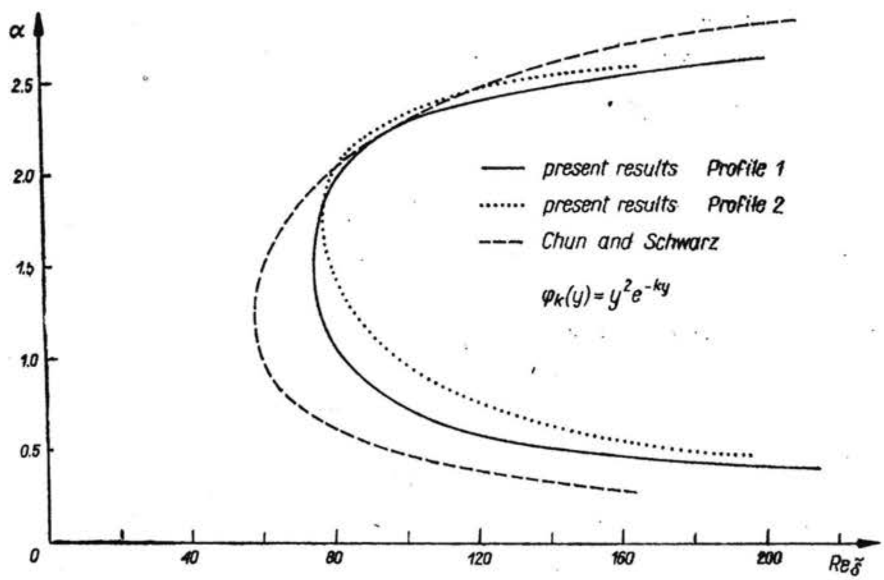


FIG. 6.

To verify our program, we tested the stability of a plane front by feeding appropriate parameters. This profile is always stable ($a = 0, b = 0, c = 1, d = 0, f = 0, s = 0$). The imaginary parts of the eigenvalues of $[A_{kl}]$. $[B_{kl}^{-1}]$ are always negative and the real parts are equal to 1. This check was done for $\alpha = 1$, $\mathrm{Re}_{\delta_1} = 10$ and 10 000 for square matrices up to the 9th order; errors appear for higher order approximations. A convergence test on the stability for profile one Fig. 5 was then made; this figure shows the Reynolds number as a function of the matrix order for a given $\alpha$. For $\alpha = 1$, the curve seems to converge correctly up to order 10; for $\alpha = 2$, convergence is not good for $n = 10$. We show the stability curves of the two velocity profiles chosen up to order 9 as well as the curve of Chun and Schwarz (Fig. 6). The critical Reynolds number for the first profile is 73.

b) We tried to find an orthogonal function for $\varphi_k(y)$ to obtain better results. The Laguerre polynomials do not lead towards well-conditioned matrices and thus we opt for a function with orthogonal derivatives,

$$\varphi_k(y) = \sin^2 \frac{k\pi y}{m},$$

$m$ corresponds to the maximum value of $y$ for which $U(y)$ is zero (Fig. 2). So, instead of integrating from $s$ to $\infty$, we integrate from $s$ to $m$. This function, though it does not rigorously satisfy the boundary conditions, offers several advantages and thus calculations are simplified.

For profile 1 (Fig. 3), we consider by extrapolating Glauert's curve that $U(y) = 0$ for $m$ between 1.9 and 2.1. Thus the influence of the three limits $m = 1.9$; $m = 2$ and $m = 2.1$ is investigated.

For profile 2 (Fig. 4) the limit is $m = 1.9$ since $U(y) = 0$ for this value of $y$.

For $m = 2.1$ as a maximum value of $y$, calculation of the percentage of $U(y)$ for $y = 2.1$ in connection with the maximum value of $U(y)$ for $y = 0.554$ (peak) is 1.2%, that is satisfactory.

The test on the stability of a plane front is excellent; for a matrix of order 20 we obtain a real part equal to 1. The convergence test for the profile 1 gives excellent results. In

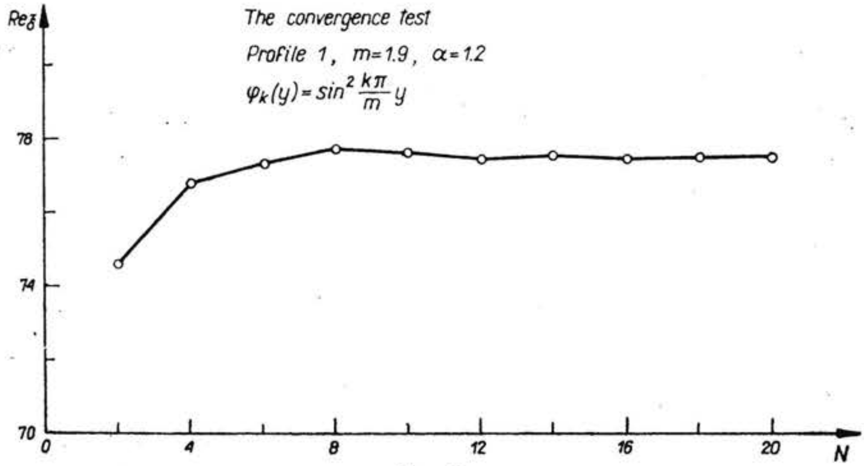


FIG. 7.

Fig. 7, $Re_{\tilde{\delta}}$ is plotted versus the matrix order for $m = 1.9$ and $\alpha = 1.2$. We can see the great advantage of the new choice for $\varphi_k(y)$. The curves of neutral stability ($c_i = 0$) give a critical Reynolds number of $Re_{\tilde{\delta}_c} = 77$ for $\alpha = 1.3$ for the profile 1; for the profile 2 $Re_{\tilde{\delta}_c} = 73$ for $\alpha = 1.32$ for $n = 17$. These two profiles represent Glauert's profile well and give results close to each other (Fig. 8).

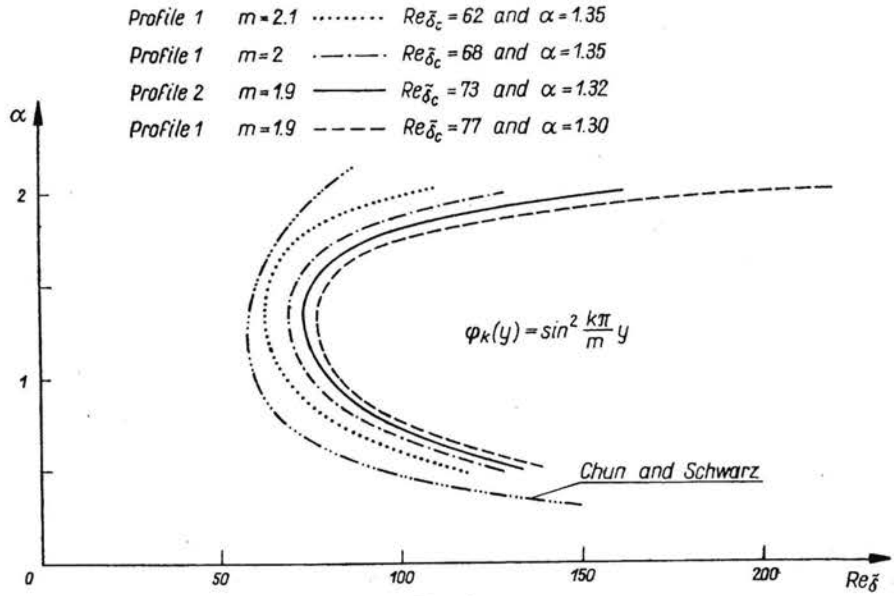


FIG. 8.

The difference between our results and the previous ones (Chun and Schwarz) and (Yutaka Tsuji) can be explained as follows: Chung and Schwarz give $(Re_{\tilde{\delta}})_c = 57$ for $\alpha = 1.18$. They use a different mathematical method and the velocity profile chosen for $U$ is unknown.

Yutaka TSUJI and Yoshinobu MORIKAWA [4] obtain some differences because the chosen profile for $U_{(y)}$ is probably experimental and deviates appreciably from Glauert's curve (second part of the curve).

We still have to chose between profiles one and two. A comparison with our previous experimental results and with thoose of BAJURA [5] leads us to choose profile 2. This profile gives critical values of $\alpha = 1.32$ for $Re_{\tilde{\delta}_c} = 73$.

The function $\varphi_k = \sin\dfrac{2k\pi y}{m}$ is kept.

## 6. Conclusion

We set out to prove that it is possible without much difficulty to determine the stability curve of a plane wall jet by Galerkin's method. The decomposition of Glauert's curve into 2 parts and the matching at the top was no problem mathematically. After inital calculation with a perturbation function satisfying perfectly the boundary conditions but with mediocre results concerning the convergence, a second function whose derivatives are orthogonal was used most successfully.

We can see that the influence of $m$ is very important. We may conclude that it is necessary to limit the effect of mathematic perturbation $\varphi_k(y) = \sin^2 \frac{k\pi y}{m}$ at the main part of the jet.

The precision of our results is sufficient for the critical instability values and we opt for profile 2 because of experimental considerations: $Re_{\tilde{\delta}_c} = 73$ for $\alpha_c = 1.32$ and $\varphi_k = \sin^2 \frac{k\pi y}{m}$.

An experimental study which is in progress will follow this theoretical study.

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