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The overshoot in entry flow(*)

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ENTRY flow is re-examined analytically to determine whether the axial-velocity overshoots found numerically are spurious or not. For large Reynolds number and uniform entry an overshoot is found inside each boundary layer, while for irrotational entry it lies at the edge. A model equation is used in the latter case to demonstrate the existence of overshoots for all Reynolds numbers.

Przeanalizowano ponownie problem przepływu wlotowego w celu ustalenia, czy ustalone numerycznie przeskoki nie mają charakteru przypadkowego. Przy dużych liczbach Reynoldsa i wlocie jednorodnym stwierdzono przeskoki w obrębie wszystkich warstw przyściennych, podczas gdy przy wlocie bezwirowym przeskok występuje przy krawędzi. W tym ostatnim przypadku zastosowano równanie modelowe dla wykazania istnienia przeskoków dla dowolnych liczb Reynoldsa.

Повторно проанализирона проблема впускного течения с целью установления не имеют ли численно найденные перескоки случайного характера? При больших числах Рейнольдса и однородном впуске перескоки обнаружены в области всех пограничных слоев, в то время, когда при безвихревом впуске перескок выступает при грани. В этом последнем случае применено модоельное уравнение для указания существования перескоков для произвольных чисел Рейнольдса.

1. Introduction

THE DEVELOPMENT of the velocity profile in a semi-infinite channel is re-examined for both uniform entry and irrotational entry. Motivation comes from the numerical computations of BRANDT and GILLIS [2] who found (for uniform entry) that overshoots develop and move toward the centerline; there they merge into a single maximum characteristic of the ultimate Poiseuille profile. It was thought that such a phenomenon would contradict the boundary-layer theory if the velocity maxima were trapped in the boundary layers as the Reynolds number R became arbitrarily large. Accordingly, attempts were made to show that the overshoots were real (i.e. not a consequence of numerical error) and to determine (numerically) whether they lay inside or outside the boundary layers (cf. ABARBANEL, BENNETT, BRANDT and GILLIS [1]).

Such overshoots are in fact an inevitable consequence of continuity for any R if the influence of the two plates that form the channel spreads gradually into the center: the velocity defect near a wall must be compensated by an adjacent overshoot. Then the boundary-layer theory, even for a single plate, should demonstrate the phenomenon and we show that it does. In either case the overshoot is buried in the second-order theory, lying inside the boundary layer for uniform entry and at the edge for irrotational entry.

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To be sure that the influence of the channel walls could not spread instantaneously to the center, Abarbanel *et al.* (loc. cit.) obtained an exact solution for uniform entry in the Stokes limit $R \to 0$, corresponding for finite R to flow near the wall in the immediate neighborhood of the entrance. They found the overshoot, but would not have done so had they considered irrotational entry. Extension to finite R, say through the Oseen approximation, is clearly desirable but proves to be very complicated; so we have considered the model equations

$$\mathbf{v} = -\nabla p + R^{-1}\nabla^2\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0 \tag{1.1}$$

instead. The solution for irrotational entry can be written in closed form (by means of a Fourier transform), and the presence of an overshoot at each station is easily demonstrated for every value of R.

The phenomenon does not appear for entry into a cascade of channels since the walls are able to influence the incident flow before it enters a channel.

2. Second-order boundary-layer analysis

Wilson [5] has treated the boundary layers of entry flow in a channel so that we could quote his results in the limit of a channel of infinite breadth. However, since they contain various minor errors and the limit is not taken, we derived the following formulas directly. Li [3] has given the details.

i) Uniform entry. The boundary conditions on the streamfunction are

$$\psi_x(x,0) = \psi_y(x,0) = 0, \quad \psi_y(0,y) = 1, \quad \psi_x(0,y) = 0. \tag{2.1}$$

The boundary-layer expansion is

$$\psi = R^{-1/2}(2x)^{1/2} f_1(\eta) + R^{-3/4} x^{1/4} f_2(\eta) \ldots, \tag{2.2}$$

where

$$R = U/\nu, \quad \eta = R^{1/2} y/(2x)^{1/2}. \tag{2.3}$$

Here f_1 is the Blasius function and f_2 satisfies

$$f_2''' + f_1 f_2'' + \frac{1}{2} f_1' f_2' + \frac{1}{2} f_1'' f_2 = 0, \tag{2.4}$$

$$f_2(0) = f_2'(0) = 0, \quad f_2(\eta) \sim 2^{5/4}\beta\sqrt{\eta} \quad \text{as} \quad \eta \to \infty. \tag{2.5}$$

Here $\beta = \lim_{\eta\to\infty} (\eta - f_1) = 1.21678\ldots$. For the inviscid flow outside the boundary layer the corresponding expansion is

$$\psi = y + R^{-1/2}\psi_2 + \ldots, \tag{2.6}$$

where

$$\psi_2 = \sqrt{2\beta}\left\{(2y)^{1/2} - [(x^2+y^2)^{1/2}+y]^{1/2}\right\}. \tag{2.7}$$

We are concerned with the axial velocity ψ_y, which the expansions (2.2) and (2.6) give correct to $O(R^{-1/4})$ and $O(R^{-1/2})$, respectively. As noted by Wilson [5], there exists an infinite number of terms between the first-order boundary-layer solution and

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the usual correction of relative order $O(R^{-1/2})$. To obtain ψ_y in the boundary layer correct to $O(R^{-1/2})$ would therefore require an infinite number of terms, without which a composite expansion correct to $O(R^{-1/2})$ cannot be formed. Fortunately, the overshoot is exhibited by f_2 so that the boundary-layer expansion to the order shown in the expansion (2.2) suffices. For large η it gives

$$\psi_y \sim 1+(2Rx)^{-1/4}\beta\eta^{-1/2}-\exp, \tag{2.8}$$

where exp is an exponentially small term contributed by f_1, and the second term on the right comes from f_2. Clearly, ψ_y is greater than 1 for sufficiently large η, so that the axial velocity must have an overshoot as $R \to \infty$. Figure 1 demonstrates the overshoot.

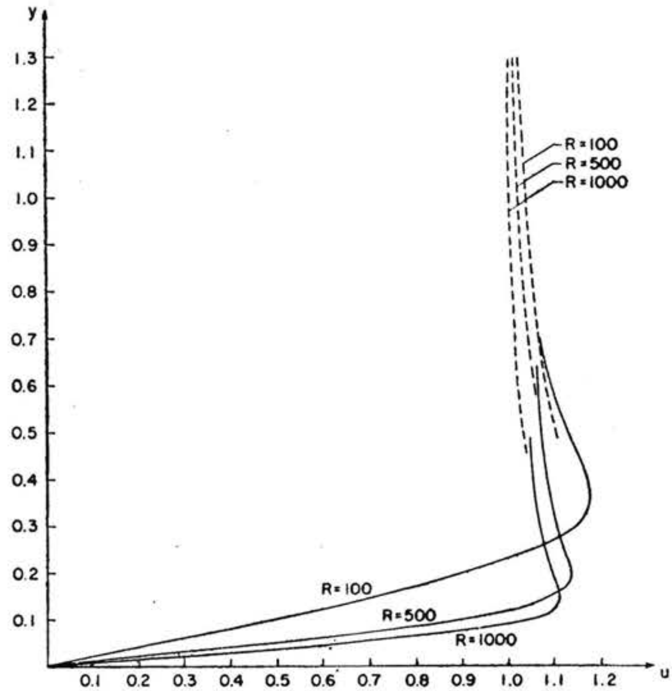

FIG. 1. Graphs of the inner ——— and outer — — — expansions of the axial velocity u at $x = 1$ for uniform entry.

i) Irrotational entry. All the boundary conditions (2.1) apply except the last which is repalced by

$$\psi_{xx}(0, y) = 0. \tag{2.9}$$

The solution for ψ_2 is now

$$\psi_2 = -\sqrt{2\beta}[(x^2+y^2)^{1/2}-y]^{1/2} \tag{2.10}$$

and the boundary-layer expansion is

$$\psi = R^{-1/2}(2x)^{1/2}f_1(\eta)+R^{-1}f_2(\eta)+ \ldots, \tag{2.11}$$

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where f_2 satisfies

(2.12) $$f_2''' + f_1 f_2'' + f_1' f_2' = \beta,$$

(2.13) $$f_2(0) = f_2'(0) = 0, \quad f_2'(\eta) \sim \beta \quad \text{as} \quad \eta \to \infty.$$

For large η we now find

(2.14) $$\psi_y \sim 1 + (2Rx)^{-1/2}\beta - \exp,$$

which has a maximum value, $1+(2Rx)^{-1/2}\beta$, at the edge of the boundary layer. To see that the axial velocity does indeed have its overshoot there, we need only note that

(2.15) $$\psi_{2y} = \beta\left[\frac{(x^2+y^2)^{1/2}-y}{2R(x^2+y^2)}\right]^{1/2}$$

has its maximum, $\beta(2Rx)^{-1/2}$, at $y=0$.

From these results we can form the composite expansion of ψ_y correct to $O(R^{-1/2})$:

(2.16) $$(\psi_y)_c = f_1'(\eta) + (2Rx)^{-1/2} f_2'(\eta) + \beta\left[\sqrt{\frac{(x^2+y^2)^{1/2}-y}{2R(x^2+y^2)}} - \frac{1}{\sqrt{2Rx}}\right].$$

Various profiles are shown in Fig. 2.

Investigation of the leading-edge region proceeds via a coordinate perturbation (Van Dyke, [4]). For uniform entry, the governing Stokes problem is solved by Abarbanel *et*

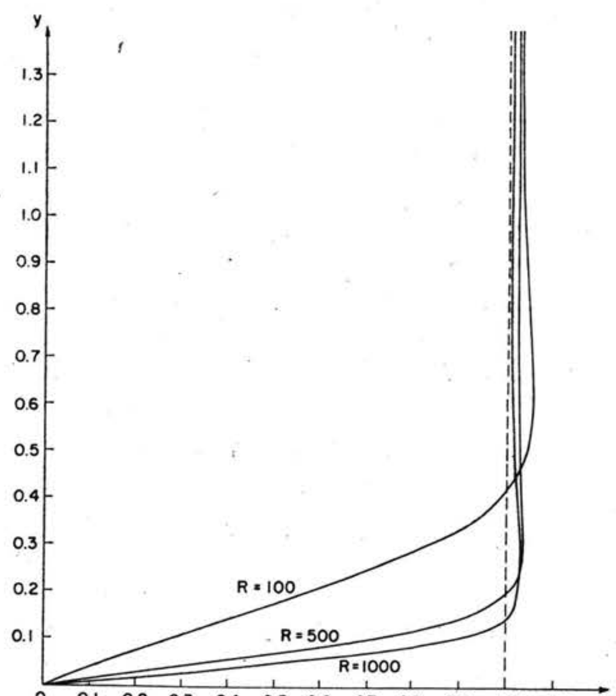

Fig. 2. Graphs of the composite expansion of the axial velocity u at $x = 1$ for irrotational entry.

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al. [1], who find a maximum at an $O(R^{-1})$ distance from the wall. The corresponding problem for the irrotational entry is

(2.17) $$\nabla^4\psi = 0 \quad \text{for} \quad x, y > 0$$

with ψ subject to the boundary conditions of (ii) and x, y are measured the $O(R^{-1})$ scale. The solution to Eq. (2.17) is still of the form used by ABARBANEL *et al.*, namely

(2.18) $$\psi(x, y) = xF(\eta) \quad \text{with} \quad \eta = y/x,$$

but now

(2.19) $$F(\eta) = (2/\pi)\eta\arctan\eta.$$

This gives the axial velocity

(2.20) $$\psi_y = F'(\eta) = \frac{2}{\pi}\left(\arctan\eta + \frac{\eta}{1+\eta^2}\right),$$

which increases monotonically from 0 to 1 as y increases from 0 to ∞ keeping x fixed. We conclude that the overshoot does not occur in the $O(R^{-1})$ leading-edge region.

Thus, for uniform entry the overshoot lies in the leading-edge region and the subsequent boundary layer; for irrotational entry it lies outside both (though at the edge of the boundary layer).

3. A model equation for all Reynolds numbers

BRANDT and GILLIS [2] integrated the complete steady-state Navier-Stokes equations for the inlet region of the channel numerically and found overshoots in the velocity profile for all values of the Reynolds number. We have noted the inevitability of such a phenomenon when it is assumed that wall effects cannot spread instantaneously. Since analytical confirmation of this assumption cannot be obtained from the Navier-Stokes equations, we turned next to their Oseen approximation. The analysis became very complicated, if not intractable, so that model equations (1.1), retaining the essential features, were substituted. These equations are Ossen's equations with v replacing $\partial v/\partial x$. For a two-dimensional flow, the introduction of the streamfunction ψ reduces them to

(3.1) $$(\nabla^2 - k^2)\nabla^2\psi = 0 \quad \text{with} \quad k = R^{1/2}.$$

To this we append the boundary conditions for the irrotational entry to which we shall confine ourselves.

By means of a Fourier-sine transform the solution may be written explicitly as

(3.2) $$\psi = y + \frac{2}{\pi}\int_0^\infty \frac{e^{-y\sqrt{\xi^2+k^2}} - e^{-y\xi}}{\sqrt{\xi^2+k^2}} \frac{\sin\xi x}{\xi} d\xi,$$

so that

(3.3) $$\psi_y = 1 + \frac{2}{\pi}\int_0^\infty \frac{\xi e^{-\xi y} - \sqrt{\xi^2+k^2}e^{-y\sqrt{\xi^2+k^2}}}{\sqrt{\xi^2+k^2}-\xi} \frac{\sin\xi x}{\xi} d\xi.$$

From this result it may be shown that, for all values of k (i.e. R),

(3.4) $$\psi_{yy}(x, y) \sim \frac{-2x}{\pi}\int_0^\infty \frac{\xi^2 e^{-\xi y}}{\sqrt{\xi^2+k^2}-\xi} d\xi < 0 \quad \text{as} \quad y \to \infty,$$

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so that ψ_y is decreasing for sufficiently large y. It follows that the axial velocity must have an overshoot, though we were unable to show that it is unique. The details of the above analysis will not be given here, but appear in LI [3]. We conclude that the wall effect is unable to spread instantaneously for any value of R.

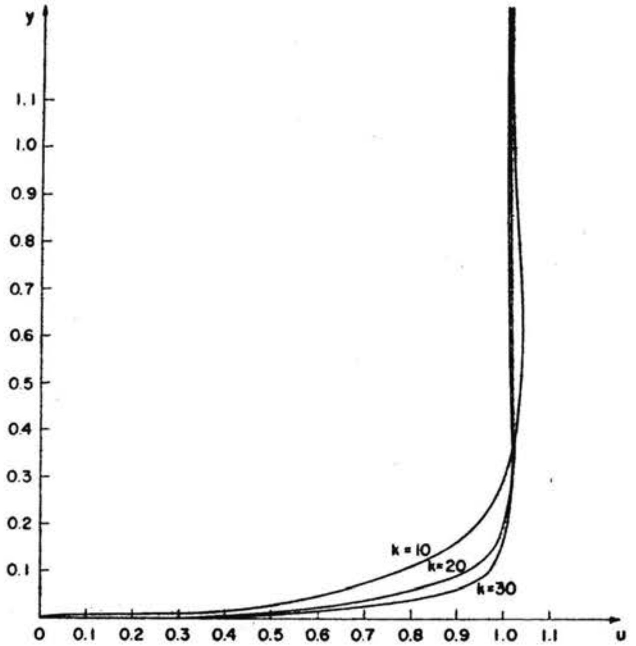

FIG. 3. Axial velocity u at $x = 1$ for the model problem.

Figure 3 shows velocity profiles for various values of k, for each of which there appears to be a unique maximum.

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