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BRIEF NOTES

The stress-singularity situation for the high-frequency Reissner-Sagoci problem in a certain inhomogeneous medium

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AT LOW frequency and also in the static case of the well-known Reissner-Sagoci problem, the surface stress just above the disc and close to the disc edge from within has the usual square-root singularity. At high frequency some earlier researchers found that this stress varies linearly from the centre of the disc and exhibits no spatial singularity at all at the disc edge, quite contrary to the results for the previous cases. Thus there is a discrepancy since the solution to the governing field equations must be a continuous function of the frequency factor on the whole half-line. This paper presents one way of resolving the discrepancy.

1. Introduction

THE REISSNER-SAGOCI problem [1] in a homogeneous elastic medium has received much attention in the literature, e.g. [2–10], just to mention a few. Some including [2, 10] and several others have obtained the exact solution to the corresponding static problem. For the dynamic problem approximate solutions have been obtained in [3, 4, 9] when the frequency factor is low and in [5, 6, 7, 8] when the frequency factor is high. The above list of references is by no means exhaustive; for a fuller list see [7]. Attempts have also been made in [1] and [7], for example, to obtain a general solution that is valid for all values of the frequency factor. It is observed, however, that such general solutions are usually in the form of a very complicated infinite series of rather complicated higher transcendental functions which are not very useful for practical purposes. Probably this is one reason why interest in the problem continues to be sustained long after Reissner and Sagoci have given the general solution of the problem in terms of series of spheroidal wave functions. Thus in cases where the frequency of oscillations is deemed high or low, the corresponding appropriate high or low frequency approximate solutions which yield simpler results are considered more useful provided, of course, such approximate solutions are sufficiently accurate. The results in [9] and [7, 8] are examples of such solutions in the low and high frequency regimes respectively.

One important aspect of the problem to which inadequate attention has been devoted in the high-frequency factor studies of the problem, is the nature of shear stress, $\sigma_{\theta z}$, just above the disc and close to the disc edge from within. In the static case and also for the low frequency factor case, it is well documented that the stress is unbounded in the sense that it has the usual spatial square-root singularity in the region. In the high-frequency factor range some attention has been devoted to the stress singularity situation in [6]; approximate results show that the stress varies linearly from the centre of the disc and exhibits no spatial singularity at all at the disc edge, quite contrary to results for the low frequency and static cases.

It is clear, however, that the solution of the governing field equations must be a continuous function of the frequency factor ω, for all ω in $[0, \infty)$. Hence if the high frequency solution in [6] and the low frequency solution are to be acceptable as part of the genera solution in their respective regions of validity, then, intuitively, one way in which the discrepancy of the existence and non-existence of the spatial stress singularity at low and high frequencies respectively could be resolved, is to have a high frequency factor solution which possesses an additional expression, containing terms with the spatial square-root singularity, whose terms vanish as the frequency factor gets very large or tends to infinity. Such an additional expression could be obtained only through a sufficiently accurate approximate analysis of the problem. In this paper we present the quantitative result we have found and which supports the above assertion.

In [8] we considered the usual Reissner-Sagoci Problem at high frequencies but more generally in an ([1]) inhomogeneous medium occupying the region $0 \leqslant r < \infty$, $0 < z < \infty$, $0 \leqslant \theta \leqslant 2\pi$, and whose shear modulus, μ, and density, ϱ vary radially as $(\mu, \varrho) = (\mu_0, \varrho_0)r^{-\varepsilon}$; (r, θ, z) are cylindrical coordinates, $\mu_0, \varrho_0, \varepsilon$ are constants with $-\infty < \mathrm{Re}(\varepsilon) \leqslant 0$. The rigid disc occupies the region $0 \leqslant r < 1$, $z = 0$, distances having been nondimensionalised by dividing by the radius of the disc. We assumed a time dependence of $e^{-i\omega t}$. An asymptotic method was used to reduce the resulting integral equation to the Wiener-Hopf type, which was then solved exactly. Physical quantities like the moment of the applied forces necessary to oscillate the disc were calculated as functions of the frequency factor and the inhomogeneity parameter, ε. By setting $\varepsilon = 0$ (homogeneous medium), the results were found to agree with those in [7] whose results have been found to agree well with experimental ones.

2. Shear stress just above the disc

Employing the results in [8] we have

$$\frac{\partial u}{\partial z}(r<1,0) = \frac{2}{\pi}\sigma_0\frac{\partial}{\partial z}\int_0^\infty d\xi\left[\frac{\xi r}{\beta^2} + r^{\varepsilon/2}\left(\frac{-\xi}{\beta}K_{\nu+1}(\beta) + \xi\int_1^\infty d\lambda\,\lambda^\nu\psi(\lambda)K_\nu(\beta)\right)\times I_\nu(\beta r)\right]\sin\xi z, \tag{2.1}$$

evaluated at $z = 0, r < 1$, where

$$\beta = \begin{cases} \sqrt{\xi^2-\omega^2}, & \xi > \omega, \\ -i\sqrt{\omega^2-\xi^2}, & \xi < \omega, \end{cases}$$

([1]) In the references [8, 10, 11] the inhomogeneous behaviour should be taken to be given by the power law $(\mu, \varrho) = (\mu_0, \varrho_0)r^{-\varepsilon}, -\infty < \mathrm{Re}(\varepsilon) < 0$, rather than the exponential law erroneously written therein.

and $u(r, z)$ is the circumferential displacement,

$$\lambda^{\nu}\psi(\lambda) = \lambda^{1/2}\phi(\omega(\lambda-1)), \quad \nu = 1-\varepsilon/2,$$

$$(2.1') \quad \phi(x) = \operatorname{erfc}\sqrt{-ix} - \frac{(\nu+1/2)}{\omega}\left(\sqrt{\frac{-ix}{\pi}}\, e^{ix} + ix\operatorname{erfc}\sqrt{-ix}\right)$$
$$+\frac{e^{i(2\omega-\nu\pi+\pi/4)}}{\sqrt{2\pi}\,(2\omega+x)^{3/2}}\left(\sqrt{\frac{-2ix}{\pi}}\, e^{ix} - 1/2 e^{-ix}\operatorname{erf}\sqrt{-2ix}\right) + 0\left(\frac{1}{\omega^{5/2}}\right),$$

ω is the frequency factor, erf, erfc are the error and complementary error functions given by

$$\operatorname{erf}x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \quad \operatorname{erfc}x = 1-\operatorname{erf}x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt,$$

I_ν, K_ν are the modified Bessel functions of order ν. Eq. (2.1) may be written as

$$(2.2) \quad \frac{\partial u}{\partial z}(r < 1, 0) = \sigma_0\left[i\omega r + \frac{2}{\pi} r^{\varepsilon/2}\int_0^\infty \xi^2 F(\beta)\, I_\nu(\beta r)\, d\xi\right]$$
$$= \sigma_0\left[i\omega r + \frac{2}{\pi} r^{\varepsilon/2}\int_{-i\omega}^\infty \beta\sqrt{\beta^2+\omega^2}\, F(\beta) I_\nu(\beta r) d\beta\right],$$

σ_0 is a constant, the amplitude of oscillation applied on the rigid disc given by $u = \sigma_0 r$, $0 \leqslant r \leqslant 1$, $z = 0$

$$F(\beta) = -K_{\nu+1}(\beta) - \int_\infty^1 \lambda^\nu\psi(\lambda) K_\nu(\beta\lambda) d\lambda,$$

where the path of the β-integration is from $-i\omega$ to the origin along the imaginary axis, and from the origin to infinity along the real axis. Thus following the analysis in [8], the ξ-integration in Eq. (2.1) is converted to a β-integration whose path of integration can be deformed to a path on which $\operatorname{Re}(\beta) > 0$, $|\beta|$ is large throughout the range of integration, so long as ω is large.

Using the formula

$$(2.3) \quad -\frac{1}{\beta} K_{\nu+1}(\beta) + \int_1^\infty \lambda^{1/2} K_\nu(\beta\lambda) d\lambda = \frac{-(\nu+1/2)}{\beta}\int_1^\infty \lambda^{-1/2} K_{\nu+1}(\beta\lambda) d\lambda$$

to simplify and then replacing the Bessel functions occurring by their asymptotic forms for the large argument, noting that interest is in the field in the region $r \to 1^-$, we find that

$$(2.4) \quad \frac{1}{\sigma_0}\frac{\partial u}{\partial z}(r \to 1^-, 0) \sim i\omega r + r^{\frac{\varepsilon-1}{2}}\int_{-i\omega}^\infty \beta d\beta(\beta^2+\omega^2)^{1/2} e^{-\eta\beta}$$
$$\times\left[\frac{-(\nu+1/2)}{\beta^2}\left(1 - \frac{\mu_1}{\beta}\left(\int_0^\infty \frac{e^{-x\beta}}{x+1}\, dx + \left(1 - \frac{\mu_1}{\beta}\right)\frac{\mu_1}{\beta}\int_0^\infty \frac{e^{-x\beta}}{(1+x)^2}\, dx\right.\right.\right.$$

$$-\frac{1}{\beta}\int_0^\infty dx\left(1+\frac{\mu_2}{\beta(x+1)}\right)\operatorname{erf}\sqrt{-i\omega x}\,e^{-x\beta}\left(1-\frac{\mu_1}{\beta}\right)$$

$$+\frac{i}{\beta}\left(1-\frac{\mu_1}{\beta}\right)\int_0^\infty dx\left(\frac{\mu_1}{\omega}-\frac{\mu_2}{\omega(x+1)}\right)\left(\sqrt{\frac{-i\omega x}{\pi}}\,e^{i\omega x}+i\omega x\operatorname{erfc}\sqrt{-i\omega x}\right)e^{-x\beta}\Bigg],$$

where

$$\eta = 1-r,\quad \mu_1=\frac{4(\nu+1)^2-1}{8},\quad \mu_2=\frac{4\nu^2-1}{8},\quad \omega\gg 1,\quad r\sim 1^-.$$

Doing the calculations, it is finally found that for $\omega \gg 1$, we have

$$\sigma_{\theta z}(r\sim 1^-,0)\sim\sigma_0\mu_0 r^{-\varepsilon}\left[i\omega r+\frac{r^{\frac{\varepsilon-1}{2}}}{\pi}R(\alpha;\omega)\right],\quad \omega\gg 1,\quad r\sim 1^-, \tag{2.5}$$

where

$$R(\alpha;\omega)=i\omega f_1+f_2+\frac{1}{\sqrt{\omega}}f_3+0\left(\frac{1}{\omega}\right),$$

$$f_1=\sqrt{\frac{\pi}{i\alpha}}\,e^{-i\alpha}-\pi\operatorname{erfc}\sqrt{i\alpha},$$

$$f_2=(\nu+1/2)\left[\left(\pi i\alpha-\sqrt{\pi i\alpha}\right)e^{-i\alpha}+\frac{\pi}{2}\operatorname{erfc}\sqrt{i\alpha}\right]$$

$$+(\nu+1/2)\left(\frac{e^{i\alpha}}{\sqrt{2}}\left[\sqrt{\frac{\pi}{2i\alpha}}\,e^{-2i\alpha}-\pi\operatorname{erfc}\sqrt{2i\alpha}\right]\right.$$

$$\left.+\pi i\left[H_0^{(2)}(\alpha)+2\alpha\left(iH_0^{(2)}(\alpha)-H_1^{(2)}(\alpha)\right)\right]\right),$$

$$f_3=\frac{i}{2}\sqrt{\frac{\pi}{2}}\,e^{i\left(\alpha+2\omega-\frac{3\pi}{4}+\frac{\varepsilon\pi}{2}\right)}\operatorname{erfc}\sqrt{2i\alpha},$$

$$\alpha=\omega(r-1),\quad H_0^{(2)},H_1^{(2)},$$

are, respectively, the zero-th and first-order Hankel functions of the second kind.

It is thus seen that if the first term on the right hand side of the expression (2.5) were the only term obtained for the stress, the stress would exhibit no spatial singularity at all and would vary linearly as some earlier researchers have previously obtained. However, it is seen that the additional terms $R(\alpha;\omega)$ that are obtained in this paper are also important especially when ω is large but finite, because for any large but finite ω, $R(\alpha;\omega)$ exhibits a spatial singularity whose highest order is of the usual square-root type as $r\to 1^-$ (i.e. $\alpha\to 0^-$).

The spatial stress singularity as $r\to 1^-$ will not be there if

(i) the repeated limits

$$\operatorname*{Lim}_{r\to 1^-}\operatorname*{Lim}_{\omega\to\infty} R(\alpha;\omega),$$

or (ii) the double limit

$$\operatorname*{Lim}_{(\omega,r)\to(\infty,1^-)} R(\alpha;\omega)$$

are taken.

The cases (i) and (ii) are not physically very meaningful, since one insists that ω though large, should be finite—in which case the spatial square-root singularity exists via $R(\alpha; \omega)$. Our conclusion is that a spatial stress singularity whose dominant part is of the square--root type does exist at the disc edge ($r \to 1^-$) in the stress, $\sigma_{\theta z}$, even when the frequency factor is high but kept finite. But as the frequency factor gets too large or tends to infinity, the first term of the expression for $\sigma_{\theta z}$ as given by Eq. (2.5) may be deemed to be the dominant part of the stress.

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