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Viscoelastic boundary layer: the integral momentum procedure for layers of the "elastic-type"

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THE INTEGRAL momentum equations with the corresponding boundary conditions and the additional relations at the wall have been derived for the "elastic-type" boundary layer in stagnation point flows treated as flows with dominating extensions. Some numerical results for the velocity profile are presented and certain formulae for the boundary layer thickness are discussed in greater detail.

Wyprowadzono równania całkowego pędu wraz z odpowiednimi warunkami brzegowymi i dodatkowymi zależnościami na ściance dla warstwy przyściennej typu "sprężystego" w przepływach stagnacyjnych, traktowanych jako przepływy z dominującymi efektami rozciągania. Przedstawiono pewne wyniki numeryczne dla profilów prędkości oraz bardziej szczegółowo przedyskutowano niektóre wzory na grubość warstwy przyściennej.

Выведены уравнения интегрального импульса совместно с соответствующими граничными условиями и дополнительными зависимостями на стенке для пограничного слоя ,,упругого'' типа в застойных течениях, трактованных как течения с доминирующими эффектами растяжения. Представлены некоторые численные результаты для профилей скорости и более подробно обсуждены некоторые формулы для толщины пограничного слоя.

1. Introduction

BASIC problems connected with viscoelastic boundary layers in stagnation point flows treated as flows with dominating extensions were extensively discussed in our previous paper [1]. To this end the notion of the so-called boundary layer of the "elastic-type" (cf. [2, 3]) was used and the corresponding scaling procedure valid for small values of the elasticity number, i.e. the ratio of Weissenberg to Reynolds number, respectively, was developed in greater detail. It is noteworthy that only for purely stagnation point flows the exact similarity transformations could be performed and some approximate solutions of the governing equations presented (cf. [1]).

In the present paper we consider the integral momentum procedure and the modified von Kármán–Pohlhausen method as applied to boundary layers of the "elastic-type". This approximate method may be very useful in flows near the leading edge of blunt bodies and profiles where the extensional effects are of greater importance as compared with shearing effects, i.e. for higher values of Weissenberg numbers.

At the beginning of the paper we briefly repeat some main results for layers of the "elastic-type" as well as for the flows with dominating extensions (cf. [1, 4]). Next, we derive the corresponding integral momentum equations together with the boundary conditions and the additional relations satisfied at the wall. In what follows, some numerical results describing the velocity profiles are presented in a graphical form and the formulae for the boundary layer thickness are discussed in certain particular cases. As in previous considerations, the emphasis is placed on the role of the extensional viscosity function and its variability with respect to the extension rate (gradient).

2. Preliminary results and notations

In our previous paper [1] we discussed the Prandtl-type equations for the "elastic-type" boundary layers in the following form:

$$\begin{gathered}\varrho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dp^*}{dx}+\frac{\partial T^{12}}{\partial y}+\frac{\partial}{\partial x}(T^{11}-T^{22}),\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\end{gathered}\tag{2.1}$$

where u and v denote velocity components, ϱ is a density, and T^{ij} $(i,j=1,2)$ are stress components in Cartesian coordinates. The modified pressure p^* is connected with an external flow by the Euler-type equation:

$$U\frac{dU}{dx}=-\frac{1}{\varrho}\frac{dp^*}{dx},\tag{2.2}$$

where $U(x)$ denotes velocity of an inviscid solution at the wall. The relevant boundary conditions are as follows:

$$u(0)=v(0)=0,\quad \lim_{y\to\infty}u(y)=U.\tag{2.3}$$

Let us remind that the concept of a viscoelastic boundary layer is based rather on the intuitive than physical assumption that in many situations there exists some thin layer close to the wall in which not only viscous but also viscoelastic (normal-stress) effects are meaningful and the external flow is exactly an inviscid one, governed by the Euler equation (cf. [1, 3]).

Equations (2.1) have been derived under the assumption that the parameter

$$\varepsilon=\frac{\delta}{L}=O\left(\sqrt{\mathrm{El}}\right)=O\left(\sqrt{\frac{\nu_{10}}{L^2}}\right)\tag{2.4}$$

is small enough, and the elasticity number El (cf. [2]) is defined through

$$\mathrm{El}=\mathrm{Ws}/\mathrm{Re},\qquad \mathrm{Re}=\frac{U_0L}{\nu_0},\qquad \mathrm{Ws}=\frac{\nu_{10}U_0}{\nu_0L}.\tag{2.5}$$

In the above definitions of the Reynolds and Weissenberg numbers, U_0 and L denote some characteristic velocity and dimension, respectively, and ν_0, ν_{10} are the kinematic viscosity and normal-stress coefficients. As a consequence of Eq. (2.4) we obtain

$$\delta=O\left(\sqrt{\nu_{10}}\right),\tag{2.6}$$

this means that the thickness of the "elastic-type" boundary layer is independent of U_0 and L.

Thus the concept of the "elastic-type" boundary layer may be used at some distance from the stagnation point where the corresponding assumptions are satisfied.

It was also shown in our paper [1] that for plane flows with dominating extensions, the constitutive equations of an incompressible simple fluid (cf. [5]) can be used in a form perturbed with respect to the additional velocity field distinguishing the flow under consideration from the purely extensional one. This assumption leads to the following stress components (increments):

$$T^{11} = -p+\frac{1}{4}\frac{d\beta}{dc}\left(\frac{\partial u}{\partial y}\right)^2,$$

(2.7)
$$T^{22} = -p-\frac{1}{4}\frac{d\beta}{dc}\left(\frac{\partial u}{\partial y}\right)^2,$$

$$T^{12} = \beta\frac{\partial u}{\partial y},$$

where c denotes the extension gradient (constant or depending on x), and the material function $\beta(c)$ is simply related to the elongational viscosity function η^*, viz.

(2.8)
$$\eta^* = \frac{T^{11}-T^{22}}{c} = 4\beta(c)$$

if the stresses are taken for purely extensional flow.

For stagnation point flows of an inviscid fluid for which

(2.9)
$$U = cx, \quad V = -cy, \quad c = U_0/L = \text{const},$$

we have, in particular,

(2.10)
$$\delta = O\left(\sqrt{\frac{\nu_0}{c}\,\mathrm{Ws}}\right), \quad \mathrm{Ws} = \frac{1}{2}k = \frac{1}{2}\frac{db}{dc}c,$$

where k is defined as the Weissenberg number multiplied by factor 2, and $b = \beta/\nu_0$ denotes the dimensionless extensional viscosity function. It should be emphasized, however, that for $\mathrm{Ws} \to 0$ the notion of the "elastic-type" boundary layer loses its physical sense. Then, the concept of the boundary layer of the "viscous-type" must be used (cf. [2, 3]).

3. Integral momentum equations

Equation $(2.1)_1$ can be written in the following equivalent form:

(3.1)
$$\frac{\partial}{\partial x}[u(U-u)]+\frac{\partial}{\partial y}[v(U-u)]+(U-u)\frac{dU}{dx} = -\frac{1}{\varrho}\frac{\partial T^{12}}{\partial y}-\frac{1}{\varrho}\frac{\partial}{\partial x}(T^{11}-T^{22}).$$

Integration of the above equation over the finite thickness δ (it may correspond to the distance where 99% of the velocity $U(x)$ is reached, cf. [6]) leads to

(3.2)
$$\int_0^\delta \frac{\partial}{\partial x}[u(U-u)]dy+[v(U-u)]_0^\delta+\frac{dU}{dx}\int_0^\delta (U-u)dy$$
$$= -\frac{1}{\varrho}[T^{12}]_0^\delta-\frac{1}{\varrho}\int_0^\delta \frac{\partial}{\partial x}(T^{11}-T^{22})dy.$$

Assuming that the velocity profiles u are similar along the whole boundary layer, viz.

$$u = \phi(\eta)\,U(x), \qquad \eta = y/\delta(x), \tag{3.3}$$

and taking into account the following boundary conditions (cf. (2.3)):

$$\begin{aligned} &\phi(0) = 0, \qquad \phi(1) = 1, \\ &\phi'(1) = 0, \quad \phi''(1) = 0, \quad \phi'''(1) = \text{ etc.} \end{aligned} \tag{3.4}$$

(the number of conditions depends on the order of smoothness assumptions imposed on ϕ for $\eta = 1$) and the relations:

$$\begin{aligned} &\int_0^\delta \frac{\partial}{\partial x}[u(U-u)]dy = \frac{d}{dx}\int_0^\delta u(U-u)\,dy - u(U-u)|_\delta \frac{d\delta}{dx}, \\ &\int_0^\delta \frac{\partial}{\partial x}(T^{11}-T^{22})dy = \frac{d}{dx}\int_0^\delta (T^{11}-T^{22})dy - (T^{11}-T^{22})|_\delta \frac{d\delta}{dx}, \end{aligned} \tag{3.5}$$

we arrive at

$$\left(1-2\frac{\beta_1}{\alpha}\frac{\nu_{10}}{\delta^2}\right)\delta\frac{d\delta}{dx} = \frac{\nu_0 b(c)\phi'(0)}{\alpha U(x)} - \frac{U'}{U}\left[\delta^2\left(2+\frac{\gamma}{\alpha}\right)+2\frac{\beta_1}{\alpha}\nu_{10}\right], \tag{3.6}$$

where we have denoted

$$\alpha = \int_0^1 (1-\phi)\phi\,d\eta, \quad \gamma = \int_0^1 (1-\phi)\,d\eta, \quad \beta_1 = \int_0^1 \phi'^2 d\eta. \tag{3.7}$$

Introducing the Pohlhausen parameter λ:

$$\lambda = \frac{\delta^2 U'}{\nu_0}, \quad \nu_{10} = \frac{1}{2}k\frac{\delta^2}{\lambda} = \frac{\delta^2}{\bar{\lambda}}, \tag{3.8}$$

we obtain, instead of Eq. (3.6),

$$(\alpha\bar{\lambda}-2\beta_1)\,\delta\frac{d\delta}{ax} = \frac{\nu_0 b(c)\phi'(0)\bar{\lambda}}{U} - \delta^2\frac{U'}{U}\left[(2\alpha+\gamma)\bar{\lambda}+2\beta_1\right], \tag{3.9}$$

where we have taken into account that $\lambda = \frac{1}{2}k\bar{\lambda} = \mathrm{Ws}\bar{\lambda}$, of course, under the assumption that $k \neq 0$ (Ws $\neq 0$).

For the case of purely stagnation point flows, for which the velocity field takes the form (2.9), λ as well as δ are constants and

$$\lambda = \frac{\delta^2 c}{\nu_0}, \quad \nu_{10} = \frac{1}{2}k\frac{\nu_0}{c} = \frac{\delta^2}{\bar{\lambda}}. \tag{3.10}$$

Then Eq. (3.9) simplifies to the form

$$a_0\frac{b}{k} = \frac{1}{2}\left[(2\alpha+\gamma)\bar{\lambda}+2\beta_1\right], \tag{3.11}$$

where by $a_0 \equiv \phi'(0)$ we have denoted the inclination of the velocity profile at $\eta = 0$.

For the case of plane flows past blunt profiles, for which the velocity field $U(x)$ depends on the distance x from the leading edge, we introduce instead of Eqs. (2.5) the following definitions:

$$\mathrm{El}_x = \mathrm{Ws}_x/\mathrm{Re}_x, \quad \mathrm{Re}_x = \frac{Ux}{\nu_0}, \quad \mathrm{Ws}_x = \frac{\nu_{10} U}{\nu_0 x}, \tag{3.12}$$

where the subscript x reminds that Re_x as well as Ws_x both depend on x, i.e., on the shape of a profile. As a consequence of the above assumptions, we have

$$\delta^2 = \frac{1}{2} k\bar{\lambda} \frac{\nu_0}{U'}, \quad k = \frac{db}{d\bar{c}} \bar{c}, \tag{3.13}$$

where the corresponding extension rate $c = U'(x)$ is not constant along the boundary layer. The question arises whether the parameter λ may be treated as a constant independent of x in the vicinity of the leading edge ($x \equiv 0$). The answer seems to be positive, at least for constant ratios b/k, i.e., for a power-law dependence of b on c (cf. results of Sect., 5).

Under such an assumption, Eq. (3.9) leads to

$$\frac{b}{k} a_0 = \frac{1}{2} [(2\alpha+\gamma)\bar{\lambda}+2\beta_1] + \frac{1}{4}\left(\frac{k'}{k} U' - U''\right) \frac{U}{U'^2} (2\beta_1 - \alpha\bar{\lambda}), \tag{3.14}$$

where

$$k' \equiv \frac{dk}{dx} = \frac{d^2 b}{d\bar{c}^2} \bar{c} + \frac{db}{d\bar{c}} \frac{d\bar{c}}{dx}. \tag{3.15}$$

It can easily be seen that for stagnation point flows: $k' = 0$ and $U'' = 0$; thus Eq. (3.14) simplifies to the form (3.11).

If, on the other hand, the profile considered for small x differs only slightly from a flat wall (purely stagnation point flow), we may assume that

$$\delta^2 = \delta_0^2 + \Delta\delta^2, \tag{3.16}$$

where

$$\delta_0^2 = \frac{\nu_0 \phi'(0) b(c) \bar{\lambda}}{c[(2\alpha+\gamma)\bar{\lambda}+2\beta_1]}, \tag{3.17}$$

denotes the constant thickness valid for a stagnation point flow and $\Delta\delta^2$ is the corresponding linear increment. Substituting the above result into Eq. (3.9), we arrive at the following differential equation:

$$\frac{d\Delta\delta^2}{dx} + 2\frac{U'}{U} \frac{[(2\alpha+\gamma)\bar{\lambda}+2\beta_1]}{(\alpha\bar{\lambda}-2\beta_1)} \Delta\delta^2 = 0, \tag{3.18}$$

the solution of which is

$$\Delta\delta^2 = C\exp\left(2\int \frac{U'[(2\alpha+\gamma)\bar{\lambda}+2\beta_1]}{U(2\beta_1-\alpha\bar{\lambda})} dx\right), \tag{3.19}$$

where C denotes an integration constant.

It can be verified in a straightforward way that the results (3.17) and (3.19) introduced into Eq. (3.9) lead again to the integral momentum equation in the form (3.11).

It is also noteworthy that the constant C in Eq. (3.19) cannot be determined from the condition that for $x = 0$, $\delta^2 = \delta_0^2$.

4. Additional relations at the wall

The Prandtl-type equation $(2.1)_1$ should be satisfied at the wall, i.e., for $y = 0$. Introducing Eqs. (2.2) and (3.3) into Eq. $(2.1)_1$ and putting $y = 0$, we arrive at the relation

$$\phi''(0) = -\frac{k}{b}\phi'^2(0) - \frac{\lambda}{b}, \tag{4.1}$$

where λ denotes the Pohlhausen parameter already defined by Eq. $(3.8)_1$. It turns out that for the case of Newtonian fluids ($k = 0$, $b = 1$), Eq. (4.1) takes the form well known in the classical von Kármán–Pohlhausen method (cf. [6, 7]).

In a similar way, differentiating Eq. $(2.1)_1$ with respect to y and next putting $y = 0$, we obtain the second relation (subsequent differentiations do not lead to any new results):

$$\phi'''(0) = -2\frac{k}{b}\phi'(0)\phi''(0). \tag{4.1}$$

For the case of Newtonian fluids ($k = 0$, $b = 1$) we rediscover the well-known result: $\phi'''(0) = 0$ (cf. [6, 7]).

The function $\phi(\eta)$ used in the modified von Kármán–Pohlhausen method may be any function satisfying the boundary conditions (3.4) and the additional relations (4.1), (4.2). Frequently, various polynomials of the sufficiently high order can be applied. Bearing in mind monotonicity of the velocity profiles as well as smoothness assumptions with respect to η for sufficiently small values of the ratio b/k (less than 1/2), we have finally chosen the seventh order polynomials, the derivatives of which with respect to η are of the form

$$\phi'(\eta) = (1-\eta)^4(a_0 + a_1\eta + a_2\eta^2). \tag{4.3}$$

After integrating the above equations with respect to η, and taking into account the boundary conditions (3.4), we obtain

$$\begin{aligned}\phi(\eta) = a_0\eta + \frac{1}{2}(a_1 - 4a_0)\eta^2 + \frac{1}{3}(a_2 - 4a_1 + 6a_0)\eta^3 \\ + \frac{1}{4}(6a_1 - 4a_2 - 4a_0)\eta^4 + \frac{1}{5}(6a_2 - 4a_1 + a_0)\eta^5 + \frac{1}{6}(a_1 - 4a_2)\eta^6 + \frac{1}{7}a_2\eta^7.\end{aligned} \tag{4.4}$$

Next, eliminating the quantities a_1 and a_2 by means of the additional relations (4.1), (4.2), we arrive at the following third order algebraic equation for $a_0 - \frac{5}{2}\frac{b}{k}$:

$$\begin{aligned}\left(a_0 - \frac{5}{2}\frac{b}{k}\right)^3 + \left(\frac{105}{4}\frac{b^2}{k^2} + \frac{\bar{\lambda}}{2}\right)\left(a_0 - \frac{5}{2}\frac{b}{k}\right) \\ + \frac{b}{k}\left(\frac{325}{4}\frac{b^2}{k^2} - 105\frac{b}{k} - \frac{5}{2}\bar{\lambda}\right) = 0,\end{aligned} \tag{4.5}$$

where $\bar{\lambda}$ results from the relation: $\lambda = \frac{1}{2} k\bar{\lambda}$, for $k \neq 0$. The polynomial (4.4) can also be written as

$$(4.6) \quad \phi(\eta) = a_0\eta - \frac{1}{2}\frac{k}{b}\left(a_0^2 + \frac{\bar{\lambda}}{2}\right)\eta^2 + \left(35 - 15a_0 + \frac{5}{2}\frac{k}{b}\left(a_0^2 + \frac{\bar{\lambda}}{2}\right)\right)\eta^3 + \left(40a_0 - 105 - 5\frac{k}{b}\left(a_0^2 + \frac{\bar{\lambda}}{2}\right)\right)\eta^4 + \left(126 - 45a_0 + 5\frac{k}{b}\left(a_0^2 + \frac{\bar{\lambda}}{2}\right)\right)\eta^5 + \left(\frac{71}{3}a_0 - 70 - \frac{5}{2}\frac{k}{b}\left(a_0^2 + \frac{\bar{\lambda}}{2}\right)\right)\eta^6 + \left(15 - 5a_0 + \frac{1}{2}\frac{k}{b}\left(a_0^2 + \frac{\bar{\lambda}}{2}\right)\right)\eta^7.$$

Equations (3.11) and (4.5) contain, either explicitly or by means of the functions α, γ and β_1 (cf. (3.7)), the following three quantities: the ratio b/k, the parameter $\bar{\lambda}$ and the inclination of the velocity profile at the wall $a_0 = \phi'(0)$. Therefore, for a given value of b/k, the mentioned equations enable, in principle, determination of the remaining quantities $\bar{\lambda}$ and a_0. This can be done exclusively in a numerical way.

In what follows, we use a slightly modified approach to numerical calculations which is presented in the next Section.

5. Approximate procedure of numerical calculations

It was shown in our previous paper [1] that the inclinations of the velocity profile at the wall, i.e., for $y = \eta = 0$, are exactly the same for Newtonian fluids and for viscoelastic fluids for which $\Delta b/k = 1.52$ ($\Delta b = b-1$). This fact results from the general properties of the differential equations describing the stagnation point flows treated as flows with dominating extensions (cf. [1, 4]).

For Newtonian fluids we have $a_0 = \phi'(0) \simeq 2.96$ assuming that $\delta = 2.4\sqrt{\nu_0/c}$, i.e., that the finite boundary layer thickness corresponds to 99% of the velocity $U(x)$, (cf. [6]). On the other hand, from the classical von Kármán–Pohlhausen method we obtain (cf. [6, 7])

$$(5.1) \qquad a_0 = 2 + \frac{\lambda}{6},$$

what, after putting $\lambda = 7.052$, i.e. the value valid for stagnation point flows, gives $a_0 = 3.17$.

Now, we assume that $\Delta b/k \simeq 1.5$, what, for Ws = 1 ($k = 2$), leads to $b/k \simeq 2.0$. It is noteworthy that Ws = 1 ($k = 2$) corresponds exactly to the case in which $\bar{\lambda} = \lambda$. Moreover, Ws = 1 is the limit value of the Weissenberg number for which El = 1/Re (cf. Sect. 2).

For $b/k = 2$ the inclinations $a_0 = 2.96$ or 3.17 introduced into Eq. (3.11) (integral momentum equation) give $\bar{\lambda} = 4.6$ or 4.0, respectively. These latter values of $\bar{\lambda}$ used in Eq. (4.5) lead to $a_0 = 3.13$ or 3.10. The above quantities can be introduced again into Eq. (3.11) and the whole iteration process is repeated. The procedure applied is of quite fast convergence with respect to $\bar{\lambda}$ as well as a_0. Similar approaches can be applied for various moderate ratios b/k and $k = 2\mathrm{Ws}$.

We should bear in mind, however, that for very small Ws (large ratios b/k) the notion of the boundary layer of the "elastic-type" should be replaced by that of the "viscous-type" (cf. [2]). Similarly, for very large Ws (small ratios b/k) the basic assumptions of the boundary layer theory may not be satisifed.

By way of illustration, the velocity profiles for $\Delta b/k \simeq 1.5$ and for the indicated values of $\mathrm{Ws} = \dfrac{1}{2} k$ are shown in Fig. 1. It should be noted that the boundary layer

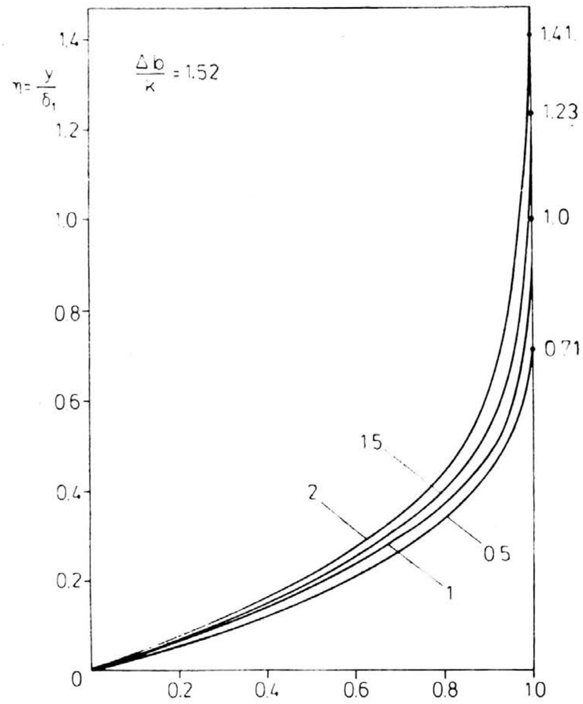

FIG. 1.

thickness is referred to δ_1, i.e., to the thickness corresponding to $\mathrm{Ws} = 1$. On the other hand, the velocity profiles for $\mathrm{Ws} = 1$ ($k = 2$) and for the indicated values of parameter b/k are shown in Fig. 2. It is seen from the latter graph that the velocity profiles may be crossed over for certain values of b/k. In the same figure the limit profile for $b/k \to \infty$ has been marked with a broken line.

The question arises whether the velocity profiles shown in Figs. 1 and 2 can be compared, at least qualitatively, with those quoted in our previous paper [1]. The answer is positive and such comparisons lead to a fairly good agreement between the corresponding profiles. When making any comparisons we should remember, however, that the coordinates used now and those in [1] are entirely different, that $\phi'(0) \neq f''(0)$, etc. More quantitative comparisons are possible only after performing necessary calculations for the governing equation derived in the previous paper.

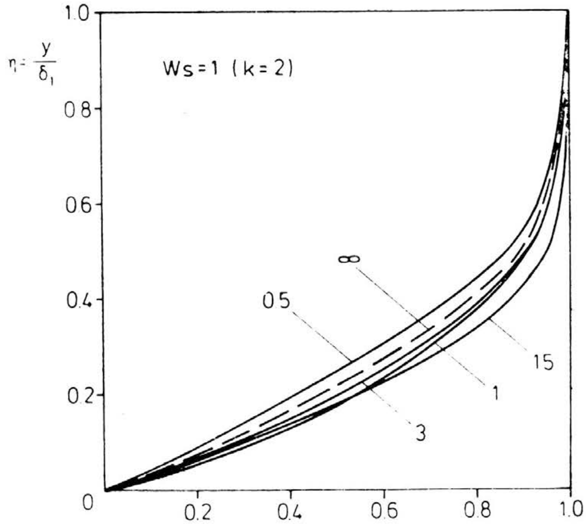

FIG. 2.

6. Boundary layer thickness for various blunt profiles

The integral momentum equation in the form (3.9) is valid for any plane flows past blunt profiles if the basic assumptions of the "elastic-type" boundary layer remain satisfied. This equation, being the first order ordinary differential equation for δ^2, can be integrated in a closed form. To this end, we write Eq. (3.9) as

$$\frac{d\delta^2}{dx} + 2\frac{U'}{U}\frac{[(2\alpha+\gamma)\bar{\lambda}+2\beta_1]}{(\alpha\bar{\lambda}-2\beta_1)}\delta^2 = \frac{2\nu_0\phi'(0)b(\bar{c})\bar{\lambda}}{(\alpha\bar{\lambda}-2\beta_1)U}, \tag{6.1}$$

where $\bar{c} = U'(x)$, and its solution can be presented in the form

$$\delta^2 = \left[\int \frac{2\nu_0\phi'(0)b(\bar{c})\bar{\lambda}}{(\alpha\bar{\lambda}-2\beta_1)U}\exp\left(2\int\frac{U'[(2\alpha+\gamma)\bar{\lambda}+2\beta_1]}{U(\alpha\bar{\lambda}-2\beta_1)}dx\right)dx + D\right]\exp\left(-2\int\frac{U'[(2\alpha+\gamma)\bar{\lambda}+2\beta_1]}{U(\alpha\bar{\lambda}-2\beta_1)}dx\right), \tag{6.2}$$

where D denotes an integration constant.

For stagnation point flows ($x = 0$, $U = cx$, $U' = c$), we rediscover the previous results (3.16). In general, the constant D cannot be determined from the condition that $\delta^2 = \delta_0^2$ for $x = 0$. To this end, another condition valid for any finite value $x \neq 0$ should be taken into account. In what follows we shall briefly discuss the case of flow past a circular cylinder as well as the case of flow past a blunt wedge.

In the case of flow past a circular cylinder of radius R, we can assume, for example, that the thickness of the "elastic-type" boundary layer and that of the "viscous-type" are exactly the same for some finite value x_0. Since the extensional effects are the most important near the leading edge of a profile, while the shearing effects become more significant closer to the upper (lower) point of a cylindrical surface, i.e., $x = \pi R/2$, we may assume that there exists some intermediate point on a cylinder where both kinds of effects are approximately of the same significance. For real pressure and stress distributions, the maximum of shearing stresses at the wall corresponds approximately to $x = \pi R/3$ and this point may be taken into consideration.

Therefore, the condition that $\delta_v^2 = \delta_{el}^2$ for $x = \pi R/3$, where the subscripts v and el denote "viscous" and "elastic", respectively, leads to the following value of D:

$$D = \frac{\nu_0 R\bar{\lambda}_{el}}{2U_\infty}\left(\frac{\pi}{3}\frac{\bar{\lambda}_v}{\bar{\lambda}_{el}} - \frac{\phi'(0)b(\bar{c})}{[(2\alpha+\gamma)\bar{\lambda}_{el}+2\beta_1]}\right)\left(\frac{\pi}{3}\right)^{-\frac{2[(2\alpha+\gamma)\bar{\lambda}_{el}+2\beta_1]}{(2\beta_1-\alpha\bar{\lambda}_{el})}}, \tag{6.3}$$

where we have taken into account the following

$$\delta_v^2 = \bar{\lambda}_v \frac{\nu_0 x}{U(x)}, \qquad U(x) = 2U_\infty \sin\frac{x}{R}. \tag{6.4}$$

In the case of flow past a blunt wedge, Eq. (6.2) gives

$$\delta^2 = \frac{\nu_0 b(c)\phi'(0)\bar{\lambda}_{el}}{c\left\{m[(2\alpha+\gamma)\bar{\lambda}_{el}+2\beta_1]+\frac{1}{2}(m-1)(\alpha\bar{\lambda}_{el}-2\beta_1)\right\}}x^{1-m} + Ex^{-\frac{2m[(2\alpha+\gamma)\bar{\lambda}_{el}+2\beta_1]}{(\alpha\bar{\lambda}_{el}-2\beta_1)}}, \tag{6.5}$$

where E is an integration constant. If we assume that $\delta_v^2 = \delta_{el}^2$ for some $x = x_0$, we arrive at

$$E = \left[\bar{\lambda}_v - \frac{b(c)\phi'(0)\bar{\lambda}_{el}}{m[(2\alpha+\gamma)\bar{\lambda}_{el}+2\beta_1]+\frac{1}{2}(m-1)(\alpha\bar{\lambda}_{el}-2\beta_1)}\right] \times \frac{\nu_0}{c} x_0^{\frac{2\{m[(2\alpha+\gamma)\bar{\lambda}_{el}+2\beta_1]+\frac{1}{2}(m-1)(\alpha\bar{\lambda}_{el}-2\beta_1)\}}{(\alpha\bar{\lambda}_{el}-2\beta_1)}}. \tag{6.6}$$

For other profiles of more complex geometry, the boundary layer thickness can be determined in a similar way after performing the corresponding integrations in Eq. (6.2).

7. Remarks

On the basis of the present considerations as well as the previous results obtained in [1], the following final remarks can be formulated:

1) the thickness of the "elastic-type" boundary layer may be much larger as compared with that of the "viscous-type", especially for higher values of Weissenberg numbers Ws;

2) the velocity profiles in the "elastic-type" boundary layer much weaker depend on the parameter b/k, i.e., the ratio of the extensional viscosity b to the double Weissenberg number $k = 2\mathrm{Ws}$;

3) for any moderate Weissenberg number Ws there exists the ratio b/k for which the inclination of the velocity profile is the largest (e.g., for $\mathrm{Ws} = 1$, such $b/k \simeq 1.5$);

4) the variable thickness of the "elastic-type" boundary layer in the neighbourhood of the edge may increase or decrease depending on the shape of a blunt profile and the additional boundary condition imposed on δ at some distance x_0.

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