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Can the finite memory of a simple material be nontrivial?

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DIRECT implementation of the COLEMAN–MIZEL theory [1–3] to continuous response functionals of materials with finite memory implies the trivial (vanishing) memory of the materials, which is characteristic of an elastic material. To overcome this drawback, another procedure is proposed in which most of the convenient features of the original theory are preserved and, moreover, on the history space every continuous functional represents a material with finite memory.

Bezpośrednie zastosowanie teorii COLEMANA–MIZELA [1–3] do ciągłych funkcjonałów reakcji materiałów ze skończoną pamięcia prowadzi do trywialnej (znikającej) pamięci, charakterystycznej dla materiałów sprężystych. Dla uniknięcią tej wady zastosowano odmienną procedurę zachowującą większość własności teorii oryginalnej, a ponadto każdy ciągły funkcjonał na przestrzeni historii reprezentuje materiał ze skończoną pamięcią.

Непосредственное применение теории КОЛЕМАНА–МИЗЕЛЯ [1–3] к непрерывным функционалам реакции материалов с конечной памятью приводит к тривиальной (исчезающей) памяти, характеристической для упругих материалов. Для устранения этого недостатка применена другая процедура, сохраняющая большинство свойств оригинальной теории, и кроме этого каждый непрерывный функционал на пространстве истории представляет материал с конечной памятью.

1. Introduction

A MATERIAL for which in the case of mechanical processes the present response, i.e. stress, is a functional of the entire past history of the configuration, i.e. deformation, is called a simple material with memory (cf. [4]). On the other hand it is clear that not every functional of the entire past history will represent a real material. Here we understand that a functional represents a real material if there exists a material in real life such that modelling its physical properties yields just the dependence described by the given functional. Searching for necessary conditions that would be satisfied for each functional having a real counterpart, we obtain a minimal set of requirements to a mathematical theory of materials. Among others, such a minimal set of the requirements should provide a continuity of the dependence of the present response on the past events (i.e. on the history of deformation, in the case of a mechanical theory, cf. [3–5]). Moreover, some empirical inequalities (cf. [4]) and thermodynamic restrictions (cf. [5–7]) impose further requirements on the response functional.

Our special interest here is to inspect the consequences of the continuity requirement in the case of modelling a class of materials possessing certain observed physical behaviour, namely the so-called finite memory.

It is obvious that the concept of continuity is strongly related to the topology. It will, however, introduce no restrictions on a given response functional if the topology is just constructed to make it continuous. It was the case considered by W. NOLL in his fundamental paper [6]. If, on contrary, the topology is assumed as a primary notion, one can obtain a framework for the description of material classes with some desired property. This way was proposed by B. D. COLEMAN and V. J. MIZEL in their series of papers [1–3]. For the case of materials with fading memory this seems to be the mathematically most attractive way. Hence there were many attempts to apply their idea to obtain other classes of materials (cf. [7–9]).

There has recently been an increasing interest in studying materials with so-called finite memory due to the possibility of numerical identification and carrying out simulations by computational method [15, 16]. Unfortunately, a direct implementation of the Coleman–Mizel theory together with the assumption that each continuous functional on the history space should model materials with finite memory confines the possible framework for description of elastic materials, only. This procedure is contrasted with the present approach of describing materials with finite memory in which most of the convenient features of the Coleman–Mizel theory are preserved, and still each continuous functional on the history space represents a material with finite (nontrivial) memory, as well. This has been done by changing the main strategy of forming the framework for description: the domain of constitutive (response) functionals to be considered is restricted by a finite memory assumption additionally to the postulates appearing in the Coleman–Mizel theory.

The present paper brings a positive answer to the question posed in the title, provided the classical procedure of the Coleman–Mizel approach is replaced by a more physical one described further in the paper. Section 2 brings some of the most important facts from the COLEMAN–MIZEL theory [1–3] together with the proof that in the original theory simple materials with finite memory are trivial. Section 3 contains the discussion of possible improvements of that theory. In Sect. 4 another approach is proposed in which instead of the whole history space the domain of definition of a constitutive functional is restricted by a finite memory assumption additionally to the postulates identical with those appearing in the Coleman–Mizel approach. An example of a constitutive functional of a material with finite memory is given together with some concluding remarks in this section.

2. Materials with finite memory in the Coleman–Mizel theory

In order to be able to answer the main question put in the title, let us recall in this section some of the most important facts from the Coleman–Mizel theory(1).

Let us first consider the set ϱ of all nonnegative functions defined for all nonnegative real numbers, and measurable with respect to a nontrivial, σ — finite, positive, regular Borel measure μ, called further an influence measure. Let ν be a nontrivial function norm with the sequential Fatou property (cf. ZAANEN [10] or [2, 8]). Now, if V is a nontrivial, separable Banach space with the norm $|\ |$ and by $\mathscr{V}^0$ we denote the set of V-valued μ-measurable functions of the half line $R^+ = [0, \infty)$, then on the set

$$\mathscr{V} := \{\phi \in \mathscr{V}^0 | \nu(|\phi|) < \infty\}$$

(1) For brevity we shall write C–M theory.

the function norm ν will define the semi-norm $|| \; ||$ by

$$||\phi|| := \nu(|\phi|).$$

Identifying two functions whenever the semi-norm of their difference vanishes, we obtain a normed function space $(\mathscr{B}, || \; ||)$, which is complete because the sequential Fatou property implies the Riesz–Fischer property for ν (cf. [10]).

The functions ϕ in $\mathscr{V}$ are called histories(²) with $\phi(0)$ as the present value, while the past values $\phi(s)$ are those for which $s > 0$. Further, to every ϕ in $\mathscr{V}$ we relate its restriction to the open interval $(0, \infty)$ denoted further by ϕ_r and called the past history of ϕ. The collection of all those ϕ_r forms the Banach space $\mathscr{B}_r$, in the natural way, with the norm

$$||\phi_r||_r := ||\phi\chi_{(0,\infty)}||,$$

where $\chi_{(0,\infty)}$ denotes the characteristic function of the interval $(0, \infty)$.

Now, the concept of a material with memory consists in introducing a continuous constitutive operator r defined on a cone $\mathscr{D} \subset \mathscr{B}$ with its values in a finite dimensional vector space S.

Let us cite the first three postulates admitted in [3] and some of their consequences. They form the structure of the C–M theory. If ϕ is in $\mathscr{V}^0$ and $\sigma \geqslant 0$, then we define

$$\phi^{(\sigma)}(s) = \begin{cases} \phi(0), & s \in [0, \sigma), \\ \phi(s-\sigma), & s \in [\sigma, \infty), \end{cases}$$

$$\phi_{(\sigma)}(s) = \phi(s+\sigma), \quad s \in [0, \infty).$$

In terms of that two families of transformations (the first is called a *static continuation*, the latter — the *section* or the right translation), the three postulates read as follows.

POSTULATE CM1. If ϕ is in $\mathscr{V}$, then $\phi^{(\sigma)}$ is in $\mathscr{V}$ for all $\sigma \geqslant 0$, and the conditions $\phi, \psi \in \mathscr{V}$ with $||\phi-\psi|| = 0$ imply $||\phi^{(\sigma)}-\psi^{(\sigma)}|| = 0$ for all $\sigma \geqslant 0$.

POSTULATE CM2. If ϕ is in $\mathscr{V}$, then so are all functions $\phi_{(\sigma)}$ for all $\sigma \geqslant 0$.

POSTULATE CM3. If $\alpha \in V$, then so is the constant function $\alpha^\dagger$, where $\alpha^\dagger(s) = \alpha$ for any $s \geqslant 0$

As consequences of CM1–CM3 one can receive among others the following results [1–3]:

LEMMA 0. A) If we put $E^\sigma\phi := \phi^{(\sigma)}$ for any $\sigma \geqslant 0$, then E^σ is a well defined operator on $\mathscr{B}$ with values in $\mathscr{B}$. (We call it static continuation by the amount σ).

B) The measure μ must have an atom at $s = 0$ and be absolutely continuous on $R^{++} = (0, \infty)$ with respect to the Lebesgue measure λ. Furthermore, either $\mu(R^{++}) = 0$ or λ is absolutely continuous on R^{++} with respect to μ.

C) The space $\mathscr{B}$ is algebraically and topologically the direct sum of V and $\mathscr{B}_{r'}$, i.e. $\mathscr{B} = V \oplus \mathscr{B}_r$ and the norm $|| \; ||$ is equivalent to $|| \; ||'$ defined by

$$||\phi||' := |\phi(0)| + ||\phi_r||_r. \; \blacksquare$$

The points of Lemma 0 state that each element of $\mathscr{V}$ has its trace at $s = 0$; the value of ϕ at 0 contributes in the same degree to the norm at $s = 0$ as the whole past history. This observation will be crucial in getting the next result.

(²) In the mechanical theory of continua V is usually the space of symmetric second order Euclidean tensors in which the positive cone represents all possible values of the Cauchy–Green strain tensor. Then ϕ is a strain history.

As it is well known, the concept of fading memory was put in the precise mathematical — functional analysis setup for the first time in [11] (cf. also [12]). In the C–M theory it appears in the form of two postulates, namely the separability of $\mathscr{B}$ and the relaxation property of the norm. Not introducing those postulates, we would like to check whether the finite memory is possible in the framework of the general theory of material with memory restricted by Postulates CM1–CM3, only.

The finiteness of the memory will be introduced by the following

POSTULATE 4′. There exists a finite positive ω such that for arbitrary $\phi_1, \phi_2 \in \mathscr{B}$, the condition

$$\phi_1|_{[0,\omega]} = \phi_2|_{[0,\omega]} \quad \text{implies} \quad \phi_1 = \phi_2. \tag{2.1}$$

For further discussion we introduce

DEFINITION 1. *A material with memory represented by the operator* $\mathfrak{r}$ *defined in the space* $\mathscr{B}$ *satisfying Postulates* CM1, CM2, CM3 and 4′ *will be called a material with finite memory.*

We see that for a material with finite memory two histories do not differ (they are equivalent) if they are the same on the (final) time interval of duration ω. The minimal amount ω the same for all histories and for which Eq. (2.1) is true, if existing, can be treated as an intrinsic (constitutive) value.

The next observation has a fundamental meaning.

REMARK 1. The material with finite memory has the relaxation property for its norm and, consequently, it has fading memory(3).

P r o o f. Let ϕ be in $\mathscr{V}$ and consider its static continuation by the amount ω. We obtain

$$(E^\sigma\phi)|_{[0,\omega]} = \phi(0)\chi_{[0,\omega]}|_{[0,\omega]} = \phi(0)^\dagger|_{[0,\omega]}.$$

Hence, by Eq. (2.1), we have $||E^\omega\phi - \phi(0)^\dagger|| = 0$ and, consequently,

$$\lim_{\sigma\to\infty} ||E^\sigma\phi - \phi(0)^\dagger|| = 0.$$

In view of the arbitrariness of ϕ, the last relation expresses the relaxation property and hence the proof is complete. ■

Since in $\mathscr{B}$ the norm is introduced through the function norm ν, the latter being based on the measure μ, the following observation is essential

REMARK 2. The requirement formed by Postulate 4′ implies that

$$\mu((\omega, \infty)) = 0, \tag{2.2}$$

and in view of B) in Lemma 0 the finite memory is trivially short.

P r o o f. Let c_1 and c_2 be two different elements of V, then

$$c_1\chi_{(\omega,\infty)}|_{[0,\omega]} = c_2\chi_{(\omega,\infty)}|_{[0,\omega]}.$$

From the requirement (2.1) of Postulate 4′, we infer

$$c_1\chi_{(\omega,\infty)} = c_2\chi_{(\omega,\infty)}.$$

This, however, in view of the fact that $c_1 \neq c_2$, is equivalent to

$$\mu(\{s \in R^+ | c_1\chi_{(\omega,\infty)}(s) \neq c_2\chi_{(\omega,\infty)}(s)\}) \equiv \mu((\omega, \infty)) = 0.$$

(3) According to the classical definition [C–M 2] $\mathscr{B}$ should be separable.

From the condition B) in Lemma 0 follows $\mu((0, \infty)) = 0$. This, however, means that whenever $\phi_1(0) = \phi_2(0)$ the histories ϕ_1 and ϕ_2 are equivalent, i.e. $||\phi_1 - \phi_2|| = 0$. ■

In terms of the constitutive properties from the above follows that for a given constitutive operator $\mathfrak{r}$ there exists the function $r: V \to S$ which realizes $\mathfrak{r}$, i.e.,

$$\mathfrak{r}(\phi) = r(\phi(0)).$$

Since just elastic materials (in the sense of Cauchy) are characterized by trivial (instantaneous) memory, we can formulate the final result of this section as follows:

THEOREM 1. *Every material with finite memory in the sense of Definition 1 is elastic.*

3. Searching for a material with finite memory

Theorem 1 of the above section was obtained as the result of strengthening the relaxation property. This was done in order to get the finiteness of the memory. Theorem 1 cannot be, however, treated as a satisfactory solution. In this section we shall try to find another formulation of Postulate 4′ that could help us to give a weaker condition than Eq. (2.1).

Searching for a gap between the relaxation property and the trivial outcome of Theorem 1, let us note that the relaxation property for the norm is equivalent to the following condition: for every continuous functional $\mathfrak{r}: \mathscr{B} \to R$ and any $\phi \in \mathfrak{B}$

$$\lim_{\sigma \to \infty} \mathfrak{r}(E^{\sigma}\phi) = \mathfrak{r}\phi^{\dagger}(0),$$

which is in fact expressed in terms of constitutive operators(4). Now we formulate the following

POSTULATE 4. There exists a positive ω such that for every continuous functional $\mathfrak{r}: \mathscr{B} \to R$ and every pair $\phi_1, \phi_2 \in \mathfrak{B}$, the condition

$$\{\phi_1 \chi_{[0,\omega]} = \phi_2 \chi_{[0,\omega]}\} \quad \text{implies} \quad \{\mathfrak{r}(\phi_1) = \mathfrak{r}(\phi_2)\}.$$

LEMMA 1. Postulate 4′ is equivalent to Postulate 4.

Proof. The proof of the implication "⇒" is obvious. To show the opposite implication, let us assume that there exists a pair $\phi_1, \phi_2 \in \mathfrak{B}$, such that $\phi_1 \neq \phi_2$ and $\phi_1 \chi_{[0,\omega]} = \phi_2 \chi_{[0,\omega]}$. Then there exists(5) a continuous functional (even linear) $\mathfrak{r}: \mathfrak{B} \to R$, such that $\mathfrak{r}(\phi_1) \neq \mathfrak{r}(\phi_2)$, which contradicts Postulate 4′. ■

The equivalent formulation of the requirement (2.1) gives some hints for the direction in which a weaker postulate could be looked for. Namely, by exchanging quantificators the uniform existence condition P4 could be replaced by the pointwise existence. To this end we propose

POSTULATE 4a. For every continuous functional $\mathfrak{r}: \mathscr{B} \to R$ there exists a positive ω such that for every pair $\phi_1, \phi_2 \in \mathscr{B}$, the condition

$$\{\phi_1 \chi_{[0,\omega]} = \phi_2 \chi_{[0,\omega]}\} \quad \text{implies} \quad \{\mathfrak{r}(\phi_1) = \mathfrak{r}(\phi_2)\}. \tag{3.1}$$

(4) Compare COLEMAN and MIZEL [1, p. 109] and the weaker version of the relaxation property in the form of the constitutive asymptotic stability property in KOSIŃSKI and VALANIS [9, p. 544].

(5) Note that $\mathscr{B}$ is a Banach space.

Unfortunately, in the new version the requirement of finiteness of the memory leads to the same result as that of Remark 2. To make this obvious, we take for $\mathfrak{r}$ the norm, i.e.,

$$\mathfrak{r}(\phi) = ||\phi|| \quad \text{for every} \quad \phi \in \mathscr{B}.$$

Then, in view of Postulate 4a, there should exist an $\omega > 0$ for which Eq. (3.1) becomes true. In particular, if $\phi|_{[0,\omega]} = 0^{\dagger}|_{[0,\omega]}$, then $\mathfrak{r}(\phi) = \mathfrak{r}(0^{\dagger}) = 0$, and, consequently, Eq. (2.2) holds.

According to the authors, the last possible improvement of the finiteness requirement in Postulate 4 can be done in the following form:

POSTULATE 4b. For every continuous functional $\mathfrak{r}: \mathscr{B} \to R$ and for every $\phi_1 \in \mathscr{B}$ there exists an $\omega > 0$, such that for every $\phi_2 \in \mathscr{B}$ the condition

$$\{\phi_1 \chi_{[0,\omega]} = \phi_2 \chi_{[0,\omega]}\} \quad \text{implies} \quad \{\mathfrak{r}(\phi_1) = \mathfrak{r}(\phi_2)\} \tag{3.2}$$

holds.

In this case, however, the outcome is not alluring, either. Namely, for a fixed ϕ^* from $\mathscr{B}$ we can define the continuous functional $\mathfrak{r}$ by

$$\mathfrak{r}(\phi) = ||\phi^* - \phi||. \tag{3.3}$$

If we take $\phi_1 = \phi^*$, then from the last formulation we can conclude the existence of a positive ω with the property

$$\phi_1|_{[0,\omega]} = \phi_2|_{[0,\omega]} \tag{3.4}$$

which leads to $\mathfrak{r}(\phi_1) = \mathfrak{r}(\phi_2)$. However, in view of Eq. (3.3) $\mathfrak{r}(\phi_1) = 0$; it means that the condition (3.4) with fixed ϕ_1 and an arbitrary ϕ_2, implies $\phi_1 = \phi_2$. As in the previous case, the proposition of Remark 2 remains unaffected by our efforts to weaken it. It means that our efforts do not lead to the desired result, mainly due to our way of the improvement preserving the first three postulates of the C–M theory. We are forced to choose another way and to neglect some of the primitive notions of the classical theory. It will be done in the next section.

We conclude this section with a remark of general nature. It concerns the concept of a state space, for the first time introduced by NOLL [6] in the theory of simple materials and subsequently discussed by PERZYNA and KOSIŃSKI [13], and the present authors in [8, 14]. The starting point for the domain of definition of an arbitrary constitutive function of a simple material with memory in the C–M theory is the space $\mathscr{B}$. In terms of the concept of a state we can say that the space $\mathscr{B}$ restricted by Postulates CM1–CM3 differs, in general, from the state space of the material in the sense of NOLL [6] and others [9, 13, 14]. The difference can even appear for the case of a viscoelastic material (in the sense of Boltzmann), not mentioning a material with finite memory. The latter case is certain.

4. The finite history space

The procedure which we are attempting to formulate now has to modify the framework of the C–M theory. It will be done by allowing the representation of histories by functions defined on a finite time interval and preserving the convenient features of the C–M theory.

In this way the procedure becomes alluring through its direct applicability to computer simulations, especially in the description of non-elastic material behaviour.

We are not going to reject the main concept of the C–M theory which requires that each continuous functional on a given history space ought to exhibit the demanded features (i.e. the finiteness of memory), since it could lead to great difficulties. One of them could be observed in the identification problems, for in the space $\mathscr{B}^*$, regarded as a space of all continuous (linear and nonlinear) functionals on $\mathscr{B}$, there would be functionals of materials with infinite memory in each neighbourhood of a functional of a material with finite memory.

As we know in the approach given by C–M, the domain of a constitutive functional of a material with memory is defined in a formal way in terms of μ-measurable functions. This attempt leaves only one object unspecified for the further analysis, namely the influence measure μ. Consequently all postulates impose restrictions on this measure. In the procedure we are now suggesting the same general postulates to hold, but they should restrict the domain rather than the measure.

Before the new definition of the domain is given, we would like to point out one more problem. Namely, if ϕ is a given history, then in the case of a material with finite memory, characterized by a positive number (amount) ω, the whole information contained in $\phi|_{(\omega,\infty)}$ has no influence on the response of the material. This observation, expressed by the property (2.2) of the influence measure μ, unables to satisfy Postulate CM2 by an arbitrary μ-measurable function, because the function

$$\psi(s) = \begin{cases} 0 & \text{on } [0,\omega], \\ \infty & \text{on } (\omega,\infty) \end{cases}$$

is in $\mathscr{B}$ and no σ-section $\psi_{(\sigma)}$ of ψ is in $\mathscr{B}$, if σ is greater than 0. This fact implies the question: why, in the case of a material with finite memory, do physically non-admissible histories have to be regarded as proper objects for the derivation of the restrictions on the influence measure if the domain of definition of the corresponding constitutive functional does not contain them? Now we are well prepared for the following:

DEFINITION 2. *The domain of definition of a constitutive functional of a material with finite memory is a cone* $\mathfrak{G}^\mu$

$$\mathfrak{G}^\mu := \{\phi \in \mathscr{D} \mid \underset{\sigma \geqslant 0}{\forall}\ \nu(|\phi_{(\sigma)}|) < \infty\}, \tag{4.1}$$

in the Banach space $\mathscr{B}$ *defined*[6] *in Sect.* 2 (*together with its cone* $\mathscr{D} \subset \mathscr{B}$), *the space* $\mathscr{B}$ *has to be restricted by the postulate* CM1 *and* CM3. *Furthermore, the measure* μ *has to have the following splitting in the Dirac measure at* $s = 0$ *and an absolutely continuous with respect to the Lebesgue measure* λ *on* $(0,\infty)$

$$\mu = c_1\,\delta(0) + \lambda', \tag{4.2}$$

where c_1 *is a positive constant and* $d\lambda/(d\lambda') \not\equiv 0$ *on* $(0,\infty)$.

Using the results of [3] concerning the consequences of Postulates CM1–CM3, we get the point A) of the Lemma 0, as well as the point C). The second part of the point B)

(6) Another interesting case occurs when one changes the norm of $\mathscr{B}$ into the alternative one

$$\|\Phi\| := \sup\{\nu(|\Phi_{(\sigma)}|) : \sigma \geqslant 0\}.$$

is no longer true, for Postulate CM2 has not been assumed, and, moreover, Eq. (4.1) introduces the restriction on the domain rather, than on the whole space $\mathscr{B}$. Now, if we identify μ — a.e. equal histories and choose representatives vanishing on (ω, ∞), we obtain now a new space of "histories" of finite duration. The new space will be exemplified in what follows.

Let $\mathscr{F}$ be a Banach function space, the elements of which are μ' — measurable functions on $[0, \omega]$ where the measure μ' is the restriction of μ to $[0, \omega]$. If $\| \|'$ denotes the norm in $\mathscr{F}$ then the first property of the finite history space is given by the following:

POSTULATE F1. If $\varphi_{1'}, \varphi_2 \in \mathscr{F}$ and $\|\varphi_1 - \varphi_2\| = 0$, then $\varphi_1(0) = \varphi_2(0)$.

From this requirement it follows that the measure μ' must possess an atom at $s = 0$. To formulate the next property, let us now notice that the function φ (or more precisely — an equivalent class) from $\mathscr{F}$ can be regarded as a history of finite duration and, consequently, the static continuation map T^σ defined by

$$(4.3) \qquad (T^\sigma \varphi)(s) = \begin{cases} \varphi(0), & \text{if} \quad s \leqslant \min(\sigma, \omega), \\ \varphi(s-\sigma), & \text{if} \quad \sigma < s \leqslant \omega, \end{cases}$$

and reflecting the properties of the map E^σ given in Sect. 2 for the case of infinite memory does not have any sense for $\sigma > \omega$. However, for any $\sigma \leqslant \omega$ this map should be well defined in $\mathscr{F}$, and should be continuous, as E^σ was (cf. [8]). Hence the next requirement will be

POSTULATE F2. For any $\sigma \in [0, \omega]$ the map T^σ defined by (4.1) is continuous as a map from $\mathscr{F}$ into $\mathscr{F}$

The image of φ under the map T^σ can be regarded as the result of a composition of an element from $\mathscr{F}$ with a constant function $\varphi(0)^\dagger$ on $[0, \sigma]$. We would like, however, to be able to compose elements from $\mathscr{F}$ with nonconstant functions, as well. The latter can be called processes. To this end we introduce the following requirements:

POSTULATE F3. There exists a class Π of S-valued functions defined on the closed intervals of the type $[0, d]$, $d \geqslant 0$, such that:

a) for every $\varphi \in \mathscr{F}$ and $P \in \Pi$ the superposition $\varphi * P$, called continuation of φ by the process P, and defined by

$$(4.4) \qquad (\varphi * P)(s) := \begin{cases} P(d_P - s) & \text{if} \quad s \leqslant \min\{d_P, \omega\}, \\ \varphi(s - d_P) & \text{if} \quad d_P < s \leqslant \omega, \end{cases}$$

belongs to $\mathscr{F}$, where $\operatorname{dom} P = [0, d_p]$;

b) for every $\varphi \in \mathscr{F}$ and $\sigma \in [0, \omega]$, $\tilde{\varphi}|_{[0,\omega]} \cap \Pi_\sigma \neq \emptyset$, where

$$\tilde{\varphi}(s) := \varphi(\omega - s) \quad \text{and} \quad \Pi_\sigma := \{P \in \Pi \mid \operatorname{dur} P = \sigma\};$$

c) for every $P \in \Pi$ and each pair (t_1, t_2) such that $0 \leqslant t_1 < t_2 < d_p$ and $t_2 - t_1 < \omega$, there exists an element $\varphi \in \mathscr{F}$ such that(7) $P|_{[t_1, t_2]} \in \varphi|_{[0, t_2 - t_1]}$.

We can see that this representation of the finite history space $\mathscr{F}$ contains an additional object in the description, namely the class of processes Π. That class is introduced in a way, which makes it possible to lengthen a finite "history" by such a process to obtain a new finite history. The properties of the prolongation are natural for the model of material

(7) Note that processes are functions while histories are classes of equivalent functions.

with finite memory. Thanks to this, the properties of class Π are similar to that required by NOLL in his framework of "a new mathematical theory of materials" [6].

At the end, let us notice that the space $\mathscr{F}$ serves as a complete example for the domain of definition of a material with finite memory if we put $\omega = 1$ and $d\nu'/d\lambda(\tau) = 1-\tau^2$, $\tau \in [0, 1]$, and $\mu' = \delta(0)+\nu'$. With this measure at hand we can put

$$\mathscr{F} = L^2_{\mu'}([0, 1]) \quad \text{and} \quad \Pi = \bigcup_{t \geqslant 0} \Pi_t,$$

with

$$\Pi_t = \left\{P:[0, t] \to S \,|\, \int_0^1 (1-\tau^2) P^2(\tau) d\tau < \infty\right\}.$$

A particular form of the constitutive functional could be

$$\mathfrak{r}(\varphi) = E\left((\varphi(0) + \int_0^1 (1-\tau^2)\varphi(\tau) d\tau\right).$$

The functional will describe a viscoelastic material if we identify the values of $\mathfrak{r}$ with the stress tensor and of φ — with the strain. In case of a beam made of such a viscoelastic material, further identification is necessary. It will be done in the next paper [15] together with the investigation of a vibration problem.

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