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A uniqueness theorem in anisotropic viscothermoelasticity of integral type

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THIS PAPER is concerned with a uniqueness theorem for the mixed problem within the framework of a linearized theory of materials with memory predicting finite speeds of propagation.

W pracy zajęto się twierdzeniem o jednoznaczności rozwiązań dla problemów mieszanych zlinearyzowanej teorii materiałów z pamięcią, prowadzących do skończonych prędkości propagacji zaburzeń.

В работе занимаются теоремой об единственности решений для смешанных задач линеаризованной теории материалов с памятью, приводящих к конечным скоростям распространения возмущений.

1. Introduction

A MATHEMATICAL theory which is able to account for memory effects and for propagation of finite thermal discontinuities with finite speeds was proposed by GURTIN and PIPKIN [1]. Their theory is confined to rigid materials. In [2] MCCARTHY removes this restriction by formulating a theory of thermomechanical materials with the same essential features as those from [1]. The linearized version of the constitutive equations considered in [2] is given in [3].

There exist some investigations concerning uniqueness theorems for history-value problems appropriate to the linearized theory from [1] see [4–7]. In this work we give a uniqueness theorem concerning the anisotropic viscothermoelastic material defined in [3] and occupying a bounded region in space. Its proof makes use of an argument given by Edelstein and Gurtin [8] for a similar result in the case of anisotropic viscoelastic solids.

2. Notations

Statement of the mixed problem. Let D be a regular bounded region (in the sense of KELLOG [9]) of a three-dimensional Euclidean space E. Denote by $\bar{D}$, ∂D and $\mathbf{n}$ its closure, boundary and unit outward normal, respectively. D_u, D_σ and D_θ, D_2 stand for complementary subsets of ∂D i.e., $\partial D = D_u \cup D_\sigma = D_\theta \cup D_2$, $D_u \cap D_\sigma = D_\theta \cap D_2 = \Phi$. $\mathbf{x}$ is a point of E and t stands for time. The initial value $f(\mathbf{x}, 0)$ (or $f(0)$) of a function $f(\mathbf{x}, t)$ (or $f(t)$) is denoted by $f_0(\mathbf{x})$ (or f_0).

The constitutive equations of a visco-thermoelastic material of integral type are (see [3])

$$
\begin{aligned}
\mathbf{T}(\mathbf{x}, t) &= \mathbf{G}_0(\mathbf{x})\mathbf{E}(\mathbf{x}, t) + \int_0^\infty \dot{\mathbf{G}}(\mathbf{x}, s)\mathbf{E}(\mathbf{x}, t-s)\,ds - \mathbf{N}_0(\mathbf{x})T(\mathbf{x}, t) \\
&\quad - \int_0^\infty \dot{\mathbf{N}}(\mathbf{x}, s)T(\mathbf{x}, t-s)\,ds + \int_0^\infty \mathbf{R}(\mathbf{x}, s)\mathbf{g}(\mathbf{x}, t-s)\,ds, \\
\varrho_0\eta(\mathbf{x}, t) &= \mathbf{N}_0(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}, t) + \int_0^\infty \dot{\mathbf{P}}(\mathbf{x}, s)\cdot\mathbf{E}(\mathbf{x}, t-s)\,ds + g_0(\mathbf{x})T(\mathbf{x}, t) \\
&\quad + \int_0^\infty \dot{g}(\mathbf{x}, s)T(\mathbf{x}, t-s)\,ds + \int_0^\infty \mathbf{r}(\mathbf{x}, s)\cdot\mathbf{g}(\mathbf{x}, t-s)\,ds, \\
\frac{1}{\vartheta_0}\mathbf{q}(\mathbf{x}, t) &= \mathbf{J}_0(\mathbf{x})\mathbf{E}(\mathbf{x}, t) + \int_0^\infty \dot{\mathbf{J}}(\mathbf{x}, s)\mathbf{E}(\mathbf{x}, t-s)\,ds + \mathbf{h}_0(\mathbf{x})T(\mathbf{x}, t) \\
&\quad + \int_0^\infty \dot{\mathbf{h}}(\mathbf{x}, s)T(\mathbf{x}, t-s)\,ds + \int_0^\infty \mathbf{K}(\mathbf{x}, s)\mathbf{g}(\mathbf{x}, t-s)\,ds.
\end{aligned}
\tag{2.1}
$$

Here $\mathbf{T}$ is the stress tensor, $\mathbf{E} = \hat{\nabla}\mathbf{u}$, where $\mathbf{u}$ is the displacement and $\hat{\nabla}$ is the symmetric gradient, η — the specific entropy, $\mathbf{q}$ — the heat flux, ϱ_0, θ_0 — the uniform(¹) density and absolute temperature, respectively, in the reference configuration, $T = \theta - \theta_0$ — the temperature difference, $\mathbf{g}$ — the temperature gradient and tr — the trace operator. The constitutive functions appearing in Eq. (2.1) are defined on $\overline{\mathbf{D}} \times [0, \infty)$ and have the following values: $\mathbf{G}(\mathbf{x}, s)$ is a fourth-order tensor, $\mathbf{N}(\mathbf{x}, s)$, $\mathbf{P}(\mathbf{x}, s)$ are symmetric tensors of order 2, $\mathbf{K}_0(\mathbf{x}, s)$ is a tensor of order 2, $\mathbf{R}(\mathbf{x}, s)$, $\mathbf{J}(\mathbf{x}, s)$ are third-order tensors, $\mathbf{r}(\mathbf{x}, s)$, $\mathbf{h}(\mathbf{x}, s)$ are vectors and $\mathbf{g}(\mathbf{x}, s)$ is the scalar. The superposed dot stands for time differentiation. In [3] it is shown that the Clausius–Duhem inequality implies

(a) $\mathbf{G}_0(\mathbf{x})$ symmetric,

(b) $\mathbf{N}_0(\mathbf{x}) = \mathbf{P}_0(\mathbf{x})$,

(c) $\mathbf{K}(\mathbf{x})$ symmetric and negative semi-definite.

Other symmetry properties have been proved using additional assumptions, logically independent of the entropy inequality. Thus it is found that the heat-work done on every closed path starting from the virgin state is invariant under time reversal if and only if

$$
\begin{aligned}
&\mathbf{G}(\mathbf{x}, s) \text{ is symmetric,} \\
&\mathbf{N}(\mathbf{x}, s) = \mathbf{P}(\mathbf{x}, s), \\
&\mathbf{R}(\mathbf{x}, s) = \mathbf{J}^T(\mathbf{x}, s) + \text{constant}, \\
&\mathbf{r}(\mathbf{x}, s) = -\mathbf{h}(\mathbf{x}, s) + \text{constant}, \qquad \text{for every } (\mathbf{x}, s) \in \overline{D} \times [0, \infty).
\end{aligned}
\tag{2.2}
$$

In the theorem some of these properties are used.

The balance equations which are to be satisfied on $D \times (-\infty, \infty)$ are

$$
\varrho_0 \ddot{\mathbf{u}} = \operatorname{div}\mathbf{T} + \varrho_0 \mathbf{b}, \qquad \varrho_0 \theta_0 \dot{\eta} = -\operatorname{div}\mathbf{q} + \varrho_0 r,
\tag{2.3}
$$

(¹) In fact ϱ_0 may depend upon $\mathbf{x}$.

where $\mathbf{b}$ is the body force and r is the heat supply. Suppose that

(i) $\mathbf{b}(\mathbf{x}, t) = \mathbf{0}, \quad r(\mathbf{x}, t) = 0 \quad \text{on } \bar{D}\times(-\infty, 0);$

(ii) $\mathbf{b}(\mathbf{x}, t)$ and $r(\mathbf{x}, t)$ are of class $\mathscr{C}^1$ on $D\times(-\infty, \infty)$;

(iii) $\mathbf{G}(\mathbf{x}, s)$, $\mathbf{N}(\mathbf{x}, s)$, $\mathbf{R}(\mathbf{x}, s)$, $\mathbf{P}(\mathbf{x}, s)$, $g(\mathbf{x}, s)$, $\mathbf{r}(\mathbf{x}, s)$, $\mathbf{J}(\mathbf{x}, s)$, $\mathbf{h}(\mathbf{x}, s)$, $\mathbf{K}(\mathbf{x}, s)$

are of class $\mathscr{C}^2$ on $\bar{D}\times[0, \infty)$.

By a solution to the mixed problem we mean a pair $(\mathbf{u}, T)$ having the properties:

(iv) $\mathbf{u}$ is thrice continuously differentiable, while T is twice continuously differentiable on $\bar{D}\times(-\infty, \infty)$;

(v) $(\mathbf{u}, T)$ satisfies the field equations (2.3) and the constitutive equations (2.1) on $D\times(-\infty, \infty)$;

(vi) $(\mathbf{u}, T)$ satisfies the initial conditions

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad T(\mathbf{x}, t) = 0 \quad \text{on} \quad \bar{D}\times(-\infty, 0]; \tag{2.4}$$

(vii) on ∂D the following boundary conditions are to be satisfied:

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \hat{\mathbf{u}}(\mathbf{x}, t) \quad \text{on} \quad \partial D_u\times[0, \infty),\\ \mathbf{T}(\mathbf{x}, t)\mathbf{n}(\mathbf{x}) \equiv \mathbf{t}(\mathbf{x}, t) &= \hat{\mathbf{t}}(\mathbf{x}, t) \quad \text{on} \quad \partial D_\sigma\times[0, \infty),\\ \mathbf{T}(\mathbf{x}, t) &= \hat{T}(\mathbf{x}, t) \quad \text{on} \quad \partial D_\theta\times[0, \infty),\\ \mathbf{q}(\mathbf{x}, t)\cdot\mathbf{n}(\mathbf{x}) &= \hat{\mathbf{q}}(\mathbf{x}, t) \quad \text{on} \quad \partial D_2\times[0, \infty). \end{aligned} \tag{2.5}$$

In view of the initial conditions (2.4), the constitutive equations (2.1) become

$$\begin{aligned} \mathbf{T}(\mathbf{x}, t) &= \int_0^t \mathbf{G}(\mathbf{x}, t-s)\dot{\mathbf{E}}(\mathbf{x}, s)\,ds - \int_0^t \mathbf{N}(\mathbf{x}, t-s)\dot{T}(\mathbf{x}, s)\,ds + \int_0^t \mathbf{R}(\mathbf{x}, t-s)\mathbf{g}(x, s)\,ds,\\ \varrho_0\eta(\mathbf{x}, t) &= \int_0^t \mathbf{P}(\mathbf{x}, t-s)\cdot\dot{\mathbf{E}}(\mathbf{x}, s)\,ds + \int_0^t \mathbf{g}(\mathbf{x}, t-s)\dot{T}(\mathbf{x}, s)\,ds + \int_0^t \mathbf{r}(\mathbf{x}, t-s)\cdot\mathbf{g}(\mathbf{x}, s)\,ds,\\ \frac{1}{\theta_0}\mathbf{q}(\mathbf{x}, t) &= \int_0^t \mathbf{J}(\mathbf{x}, t-s)\dot{\mathbf{E}}(\mathbf{x}, s)\,ds + \int_0^t \mathbf{h}(\mathbf{x}, t-s)\dot{T}(\mathbf{x}, s)\,ds + \int_0^t \mathbf{k}(\mathbf{x}, t-s)\mathbf{g}(\mathbf{x},s)\,ds. \end{aligned} \tag{2.6}$$

3. Uniqueness theorem

The proof of the theorem uses the following lemma.

LEMMA 3.1. Let $(\mathbf{u}, T)$ be a solution to the mixed problem corresponding to null data. Then

$$\begin{aligned} 0 = \frac{1}{2}\int_D \{\varrho_0\ddot{\mathbf{u}}(\mathbf{x}, t)\cdot\ddot{\mathbf{u}}(\mathbf{x}, t) + \dot{\mathbf{E}}(\mathbf{x}, t)\cdot\mathbf{G}_0(\mathbf{x})\dot{\mathbf{E}}(\mathbf{x}, t) + g_0(\mathbf{x})\dot{T}^2(\mathbf{x}, t)\\ - \mathbf{g}(\mathbf{x}, t)\cdot\mathbf{K}_0(\mathbf{x})\mathbf{g}(\mathbf{x}, t)\}\,d\mathbf{x}\\ + \int_D\int_0^t \{[-\dot{\mathbf{G}}_0(\mathbf{x})\dot{\mathbf{E}}(\mathbf{x}, \tau) + \dot{\mathbf{N}}_0(\mathbf{x})\dot{T}(\mathbf{x}, \tau) - \dot{\mathbf{R}}_0(\mathbf{x})\mathbf{g}(\mathbf{x}, \tau)]\cdot\dot{\mathbf{E}}(\mathbf{x}, \tau) \end{aligned} \tag{3.1}$$

$$
\begin{aligned}
(3.1)\quad [\text{const.}] \qquad & + [\dot{\mathbf{P}}_0(\mathbf{x}) \cdot \dot{\mathbf{E}}(\mathbf{x}, \tau) + \dot{g}_0(\mathbf{x}) \dot{T}(\mathbf{x}, \tau) + \dot{\mathbf{r}}_0(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}, \tau)] \dot{T}(\mathbf{x}, \tau) \\
& + [\dot{\mathbf{J}}_0(\mathbf{x}) \dot{\mathbf{E}}(\mathbf{x}, \tau) + \dot{\mathbf{h}}_0(\mathbf{x}) \dot{T}(\mathbf{x}, \tau) + \dot{\mathbf{K}}_0(\mathbf{x}) \mathbf{g}(\mathbf{x}, \tau)] \cdot \mathbf{g}(\mathbf{x}, \tau)\} d\tau\, d\mathbf{x} \\
& + \int_D \int_0^t \{[\dot{\mathbf{G}}(\mathbf{x}, t-\tau) \dot{\mathbf{E}}(\mathbf{x}, \tau) - \dot{\mathbf{N}}(\mathbf{x}, t-\tau) \dot{T}(\mathbf{x}, \tau) + \dot{\mathbf{R}}(\mathbf{x}, t-\tau) \mathbf{g}(\mathbf{x}, \tau)] \cdot \dot{\mathbf{E}}(\mathbf{x}, t) \\
& + [-\dot{\mathbf{J}}(\mathbf{x}, t-\tau) \dot{\mathbf{E}}(\mathbf{x}, \tau) - \dot{\mathbf{h}}(\mathbf{x}, t-\tau) \dot{T}(\mathbf{x}, \tau) - \dot{\mathbf{K}}(\mathbf{x}, t-\tau) \mathbf{g}(\mathbf{x}, \tau)] \cdot \mathbf{g}(\mathbf{x}, t)\} d\tau\, d\mathbf{x} \\
& + \int_D \Big\{ \dot{\mathbf{E}}(\mathbf{x}, t) \cdot \mathbf{R}_0(\mathbf{x}) \mathbf{g}(\mathbf{x}, t) - \int_0^t \left(\mathbf{R}_0(\mathbf{x}) + \mathbf{J}_0^T(\mathbf{x})\right) \dot{\mathbf{g}}(\mathbf{x}, \tau) \cdot \dot{\mathbf{E}}(\mathbf{x}, \tau) d\tau \\
& + \int_0^t \left(\mathbf{r}_0(\mathbf{x}) - \mathbf{h}_0(\mathbf{x})\right) \cdot \dot{\mathbf{g}}(\mathbf{x}, \tau) \dot{T}(\mathbf{x}, \tau) d\tau \Big\} d\mathbf{x} \\
& + \int_D \int_0^t \int_0^\tau \{[-\ddot{\mathbf{G}}(\mathbf{x}, \tau-s) \dot{\mathbf{E}}(\mathbf{x}, s) + \ddot{\mathbf{N}}(\mathbf{x}, \tau-s) \dot{T}(\mathbf{x}, s) - \ddot{\mathbf{R}}(\mathbf{x}, \tau-s) \mathbf{g}(\mathbf{x}, s)] \cdot \dot{\mathbf{E}}(\mathbf{x}, \tau) \\
& + [\ddot{\mathbf{P}}(\mathbf{x}, \tau-s) \cdot \dot{\mathbf{E}}(\mathbf{x}, s) + \ddot{g}(\mathbf{x}, \tau-s) \dot{T}(\mathbf{x}, s) + \ddot{\mathbf{r}}(\mathbf{x}, \tau-s) \cdot \mathbf{g}(\mathbf{x}, s)] \dot{T}(\mathbf{x}, \tau) \\
& + [\ddot{\mathbf{J}}(\mathbf{x}, \tau-s) \dot{\mathbf{E}}(\mathbf{x}, s) + \ddot{\mathbf{h}}(\mathbf{x}, \tau-s) \dot{T}(\mathbf{x}, s) + \ddot{\mathbf{K}}(\mathbf{x}, \tau-s) \mathbf{g}(\mathbf{x}, s)] \cdot \mathbf{g}(\mathbf{x}, \tau)\} ds\, d\tau\, d\mathbf{x}.
\end{aligned}
$$

Proof. Taking into account the null data, the divergence theorem, Fubini's theorem and the balance equations (2.3), one arrives at

$$
\begin{aligned}
0 = & \int_0^t \int_{\partial D} \dot{\mathbf{t}}(\mathbf{x}, \tau) \cdot \ddot{\mathbf{u}}(\mathbf{x}, \tau) d\mathbf{x}\, d\tau + \int_0^t \int_D \varrho_0 \dot{\mathbf{b}}(\mathbf{x}, \tau) \cdot \ddot{\mathbf{u}}(\mathbf{x}, \tau) d\mathbf{x}\, d\tau \\
& - \int_0^t \int_{\partial D} \frac{1}{\theta_0} \dot{T}(\mathbf{x}, \tau) \dot{\mathbf{q}}(\mathbf{x}, \tau) \cdot \mathbf{n}(\mathbf{x}) d\mathbf{x}\, d\tau + \int_0^t \int_D \frac{\varrho_0}{\theta_0} \dot{T}(\mathbf{x}, \tau) \dot{\mathbf{r}}(\mathbf{x}, \tau) d\mathbf{x}\, d\tau \\
& = \frac{1}{2} \varrho_0 \ddot{\mathbf{u}}(\mathbf{x}, t) \cdot \ddot{\mathbf{u}}(\mathbf{x}, t) d\mathbf{x} \\
& + \int_0^t \int_D \left\{ \dot{\mathbf{T}}(\mathbf{x}, \tau) \cdot \ddot{\mathbf{E}}(\mathbf{x}, \tau) + \varrho_0 \ddot{\eta}(\mathbf{x}, \tau) \dot{T}(\mathbf{x}, \tau) - \frac{1}{\theta_0} \dot{\mathbf{q}}(\mathbf{x}, \tau) \cdot \dot{\mathbf{g}}(\mathbf{x}, \tau) \right\} d\mathbf{x}\, d\tau.
\end{aligned}
$$

Integrating by parts and using the initial conditions (2.4), the foregoing relation becomes

$$
\begin{aligned}
(3.2)\qquad 0 = & \frac{1}{2} \int_D \varrho_0 \ddot{\mathbf{u}}(\mathbf{x}, t) \cdot \ddot{\mathbf{u}}(\mathbf{x}, t) d\mathbf{x} + \int_D \left\{ \dot{\mathbf{T}}(\mathbf{x}, t) \cdot \dot{\mathbf{E}}(\mathbf{x}, t) - \frac{1}{\theta_0} \dot{\mathbf{q}}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x}, t) \right\} d\mathbf{x} \\
& + \int_0^t \int_D \left\{ -\ddot{\mathbf{T}}(\mathbf{x}, \tau) \cdot \dot{\mathbf{E}}(\mathbf{x}, \tau) + \varrho_0 \ddot{\eta}(\mathbf{x}, \tau) \dot{T}(\mathbf{x}, \tau) + \frac{1}{\theta_0} \ddot{\mathbf{q}}(\mathbf{x}, \tau) \cdot \mathbf{g}(\mathbf{x}, \tau) \right\} d\mathbf{x}\, d\tau.
\end{aligned}
$$

Substituting in Eq. (3.2) the constitutive relations (2.6) and using the symmetry properties (a)–(c), after a long routine calculus one arrives at the theorem (3.1).

Now we give the uniqueness theorem.

THEOREM 3.1. *If the following properties*

$$\begin{aligned} &\mathbf{G}_0(\mathbf{x}) \quad \text{is positive definite,}\\ &\mathbf{g}_0(\mathbf{x}) > 0,\\ &\mathbf{K}_0(\mathbf{x}) \quad \text{is negative definite,}\\ &\mathbf{R}_0(\mathbf{x}) = \mathbf{J}_0^T(\mathbf{x}) = \mathbf{0},\\ &\mathbf{r}_0(\mathbf{x}) = \mathbf{h}_0(\mathbf{x}) \end{aligned} \tag{3.3}$$

hold, then there is at the most one solution to the mixed problem.

Proof. Let $(\mathbf{u}_1, T_1)$ and $(\mathbf{u}_2, T_2)$ be two solutions to the mixed problem. The uniqueness is proved if we may show that $(\mathbf{u}, T) = 0$ on $\overline{D}\times(-\infty, \infty)$ where $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $T = T_1 - T_2$.

From the relations (3.1) and $(3.3)_4$ it follows that

$$\begin{aligned} 0 = \frac{1}{2}\int_D \{&\varrho_0\ddot{\mathbf{u}}(\mathbf{x},t)\cdot\ddot{\mathbf{u}}(\mathbf{x},t)+\dot{\mathbf{E}}(\mathbf{x},t)\cdot\mathbf{G}_0(\mathbf{x})\dot{\mathbf{E}}(\mathbf{x},t)+g_0(\mathbf{x})\dot{T}^2(\mathbf{x},t)\\ &-\mathbf{g}(\mathbf{x},t)\cdot\mathbf{K}_0(\mathbf{x})\mathbf{g}(\mathbf{x},t)\}d\mathbf{x}\\ +\int_D\int_0^t \{&[-\dot{\mathbf{G}}_0(\mathbf{x})\dot{\mathbf{E}}(\mathbf{x},\tau)+\dot{\mathbf{N}}_0(\mathbf{x})\dot{T}(\mathbf{x},\tau)-\dot{\mathbf{R}}_0(\mathbf{x})\mathbf{g}(\mathbf{x},\tau)]\cdot\dot{\mathbf{E}}(\mathbf{x},\tau)\\ &+[\dot{\mathbf{P}}_0(\mathbf{x})\cdot\dot{\mathbf{E}}(\mathbf{x},\tau)+\dot{g}_0(\mathbf{x})\dot{T}(\mathbf{x},\tau)+\dot{\mathbf{r}}_0(\mathbf{x})\cdot\mathbf{g}(\mathbf{x},\tau)]\dot{T}(\mathbf{x},\tau)\\ &+[\dot{\mathbf{J}}_0(\mathbf{x})\dot{\mathbf{E}}(\mathbf{x},\tau)+\dot{\mathbf{h}}_0(\mathbf{x})\dot{T}(\mathbf{x},\tau)+\dot{\mathbf{K}}_0(\mathbf{x})\mathbf{g}(\mathbf{x},\tau)]\cdot\mathbf{g}(\mathbf{x},\tau)\}d\tau d\mathbf{x}\\ +\int_D\int_0^t \{&[\dot{\mathbf{G}}(\mathbf{x},t-\tau)\dot{\mathbf{E}}(\mathbf{x},\tau)-\dot{\mathbf{N}}(\mathbf{x},t-\tau)\dot{T}(\mathbf{x},\tau)+\dot{\mathbf{R}}(\mathbf{x},t-\tau)\mathbf{g}(\mathbf{x},\tau)]\cdot\dot{\mathbf{E}}(\mathbf{x},t)\\ &+[-\dot{\mathbf{J}}(\mathbf{x},t-\tau)\dot{\mathbf{E}}(\mathbf{x},\tau)-\dot{\mathbf{h}}(\mathbf{x},t-\tau)\dot{T}(\mathbf{x},\tau)-\dot{\mathbf{K}}(\mathbf{x},t-\tau)\mathbf{g}(\mathbf{x},\tau)]\cdot\mathbf{g}(\mathbf{x},t)\}d\tau d\mathbf{x}\\ +\int_D\int_0^t\int_0^\tau \{&[-\ddot{\mathbf{G}}(\mathbf{x},\tau-s)\dot{\mathbf{E}}(\mathbf{x},s)+\ddot{\mathbf{N}}(\mathbf{x},\tau-s)\dot{T}(\mathbf{x},s)-\ddot{\mathbf{R}}(\mathbf{x},\tau-s)\mathbf{g}(\mathbf{x},s)]\cdot\dot{\mathbf{E}}(\mathbf{x},\tau)\\ &+[\ddot{\mathbf{P}}(\mathbf{x},\tau-s)\cdot\dot{\mathbf{E}}(\mathbf{x},s)+\ddot{g}(\mathbf{x},\tau-s)\dot{T}(\mathbf{x},s)+\ddot{\mathbf{r}}(\mathbf{x},\tau-s)\cdot\mathbf{g}(\mathbf{x},s)]\dot{T}(\mathbf{x},\tau)\\ &+[\ddot{\mathbf{J}}(\mathbf{x},\tau-s)\dot{\mathbf{E}}(\mathbf{x},s)+\ddot{\mathbf{h}}(\mathbf{x},\tau-s)\dot{T}(\mathbf{x},s)+\ddot{\mathbf{K}}(\mathbf{x},\tau-s)\mathbf{g}(\mathbf{x},s)]\cdot\mathbf{g}(\mathbf{x},\tau)\}ds\,d\tau\,dx. \end{aligned} \tag{3.4}$$

Now identify the tensor-valued constitutive functions with their representative matrices in a rectangular Cartesian frame as follows (see [8]): $\mathbf{G}(\mathbf{x}, s)$ denotes a 6×6 matrix, $\mathbf{N}(\mathbf{x}, s)$ — a 6×1 column vector, $\mathbf{P}(\mathbf{x}, s)$ — a 1×6 line vector, $\mathbf{R}(\mathbf{x}, s)$ — a 6×3 matrix, $\mathbf{J}(\mathbf{x}, s)$ — a 3×6 matrix, $\mathbf{r}(\mathbf{x}, s)$ — a 1×3 line vector, $\mathbf{h}(\mathbf{x}, s)$ — a 3×1 column vector and $\mathbf{K}(\mathbf{x}, s)$ — a 3×3 matrix. Also let I be the 3×3 identity matrix, $\sqrt{\varrho_0}\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{g}(\mathbf{x}, t)$ —

Substituting in Eq. (3.2) the constitutive relations (2.6) and using the symmetry properties (a)–(c), after a long routine calculus one arrives at the theorem (3.1).

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\end{aligned} \tag{3.3}$$

hold, then there is at the most one solution to the mixed problem.

Proof. Let $(\mathbf{u}_1, T_1)$ and $(\mathbf{u}_2, T_2)$ be two solutions to the mixed problem. The uniqueness is proved if we may show that $(\mathbf{u}, T) = 0$ on $\overline{D} \times (-\infty, \infty)$ where $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $T = T_1 - T_2$.

From the relations (3.1) and $(3.3)_4$ it follows that

$$\begin{aligned}
0 = \frac{1}{2} \int_D \{\varrho_0 \ddot{\mathbf{u}}(\mathbf{x}, t) \cdot \ddot{\mathbf{u}}(\mathbf{x}, t) + \dot{\mathbf{E}}(\mathbf{x}, t) \cdot \mathbf{G}_0(\mathbf{x}) \dot{\mathbf{E}}(\mathbf{x}, t) + g_0(\mathbf{x}) \dot{T}^2(\mathbf{x}, t) \\
- \mathbf{g}(\mathbf{x}, t) \cdot \mathbf{K}_0(\mathbf{x}) \mathbf{g}(\mathbf{x}, t)\} d\mathbf{x} \\
+ \int_D \int_0^t \{[-\dot{\mathbf{G}}_0(\mathbf{x}) \dot{\mathbf{E}}(\mathbf{x}, \tau) + \dot{\mathbf{N}}_0(\mathbf{x}) \dot{T}(\mathbf{x}, \tau) - \dot{\mathbf{R}}_0(\mathbf{x}) \mathbf{g}(\mathbf{x}, \tau)] \cdot \dot{\mathbf{E}}(\mathbf{x}, \tau) \\
+ [\dot{\mathbf{P}}_0(\mathbf{x}) \cdot \dot{\mathbf{E}}(\mathbf{x}, \tau) + \dot{g}_0(\mathbf{x}) \dot{T}(\mathbf{x}, \tau) + \dot{\mathbf{r}}_0(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}, \tau)] \dot{T}(\mathbf{x}, \tau) \\
+ [\dot{\mathbf{J}}_0(\mathbf{x}) \dot{\mathbf{E}}(\mathbf{x}, \tau) + \dot{\mathbf{h}}_0(\mathbf{x}) \dot{T}(\mathbf{x}, \tau) + \dot{\mathbf{K}}_0(\mathbf{x}) \mathbf{g}(\mathbf{x}, \tau)] \cdot \mathbf{g}(\mathbf{x}, \tau)\} d\tau d\mathbf{x} \\
+ \int_D \int_0^t \{[\dot{\mathbf{G}}(\mathbf{x}, t-\tau) \dot{\mathbf{E}}(\mathbf{x}, \tau) - \dot{\mathbf{N}}(\mathbf{x}, t-\tau) \dot{T}(\mathbf{x}, \tau) + \dot{\mathbf{R}}(\mathbf{x}, t-\tau) \mathbf{g}(\mathbf{x}, \tau)] \cdot \dot{\mathbf{E}}(\mathbf{x}, t) \\
+ [-\dot{\mathbf{J}}(\mathbf{x}, t-\tau) \dot{\mathbf{E}}(\mathbf{x}, \tau) - \dot{\mathbf{h}}(\mathbf{x}, t-\tau) \dot{T}(\mathbf{x}, \tau) - \dot{\mathbf{K}}(\mathbf{x}, t-\tau) \mathbf{g}(\mathbf{x}, \tau)] \cdot \mathbf{g}(\mathbf{x}, t)\} d\tau d\mathbf{x} \\
+ \int_D \int_0^t \int_0^\tau \{[-\ddot{\mathbf{G}}(\mathbf{x}, \tau-s) \dot{\mathbf{E}}(\mathbf{x}, s) + \ddot{\mathbf{N}}(\mathbf{x}, \tau-s) \dot{T}(\mathbf{x}, s) - \ddot{\mathbf{R}}(\mathbf{x}, \tau-s) \mathbf{g}(\mathbf{x}, s)] \cdot \dot{\mathbf{E}}(\mathbf{x}, \tau) \\
+ [\ddot{\mathbf{P}}(\mathbf{x}, \tau-s) \cdot \dot{\mathbf{E}}(\mathbf{x}, s) + \ddot{g}(\mathbf{x}, \tau-s) \dot{T}(\mathbf{x}, s) + \ddot{\mathbf{r}}(\mathbf{x}, \tau-s) \cdot \mathbf{g}(\mathbf{x}, s)] \dot{T}(\mathbf{x}, \tau) \\
+ [\ddot{\mathbf{J}}(\mathbf{x}, \tau-s) \dot{\mathbf{E}}(\mathbf{x}, s) + \ddot{\mathbf{h}}(\mathbf{x}, \tau-s) \dot{T}(\mathbf{x}, s) + \ddot{\mathbf{K}}(\mathbf{x}, \tau-s) \mathbf{g}(\mathbf{x}, s)] \cdot \mathbf{g}(\mathbf{x}, \tau)\} ds d\tau dx.
\end{aligned} \tag{3.4}$$

Now identify the tensor-valued constitutive functions with their representative matrices in a rectangular Cartesian frame as follows (see [8]): $\mathbf{G}(\mathbf{x}, s)$ denotes a 6×6 matrix, $\mathbf{N}(\mathbf{x}, s)$ — a 6×1 column vector, $\mathbf{P}(\mathbf{x}, s)$ — a 1×6 line vector, $\mathbf{R}(\mathbf{x}, s)$ — a 6×3 matrix, $\mathbf{J}(\mathbf{x}, s)$ — a 3×6 matrix, $\mathbf{r}(\mathbf{x}, s)$ — a 1×3 line vector, $\mathbf{h}(\mathbf{x}, s)$ — a 3×1 column vector and $\mathbf{K}(\mathbf{x}, s)$ — a 3×3 matrix. Also let I be the 3×3 identity matrix, $\sqrt{\varrho_0}\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{g}(\mathbf{x}, t)$ —

hence

$$\zeta(\mathbf{x}, t) = \mathbf{0} \quad \text{on} \quad \bar{D} \times [0, \infty).$$

This implies

$$\ddot{\mathbf{u}}(\mathbf{x}, t) = \mathbf{0}, \quad \dot{T}(\mathbf{x}, t) = 0 \quad \text{on} \quad \bar{D} \times [0, \infty).$$

Since $\dot{\mathbf{u}}(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$ and $T(\mathbf{x}, t)$ are continuous and because of the initial conditions (2.3), one obtains the desired conclusion

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad T(\mathbf{x}, t) = 0 \quad \text{on} \quad \bar{D} \times (-\infty, \infty).$$

Note that the conditions $(3.3)_1$–$(3.3)_3$ become in the isotropic case

$$\lambda_0 > 0, \quad \mu_0 > 0, \quad c_0 > 0, \quad \varkappa_0 > 0, \text{(}^2\text{)}$$

and make possible the propagation with finite velocities of the thermomechanical disturbances (see [10]). But the restrictions $(3.3)_{4,5}$ have no theoretical or experimental grounds. They are only sufficient to have a unique solution to the mixed problem. Also, if the heat-work done every closed path starting from the virgin state is invariant under time-reversal, then the assumptions $(3.3)_{4,5}$ determine the constants from the conditions $(2.2)_{3,4}$. However, if the material is centrosymmetric or isotropic, then

$$\mathbf{R}(\mathbf{x}, s) = \mathbf{J}^T(\mathbf{x}, s) = 0, \quad \mathbf{r}(\mathbf{x}, s) = \mathbf{h}(\mathbf{x}, s) = 0,$$

hence the conditions $(3.3)_{4,5}$ are automatically satisfied.

It is worth mentioning that the thermoelastic material of the Cattaneo type for which $\mathbf{q}(\mathbf{x}, 0) = \mathbf{0}$ is a particular kind of visco-thermoelastic material (see [11]). In view of the null data required by the proof of the uniqueness theorem, it follows that for this particular material the uniqueness problem is equivalent to that considered within the framework of the theory of the Cattaneo type. Thus our Theorem 3.1 provides the uniqueness to the mixed problem corresponding to the mentioned theory.

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(²) For notations see [10] or [11].

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