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# Nonlinear scalar parabolic equation describing the temperature and flow of a heat conducting gas

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WE ARE CONCERNED with the quasi-static flow of a heat conducting gas under some additional assumptions allowing for reduction of the system of five flow and energy equations to a certain nonlinear parabolic equation. Some properties of this equation are considered; the uniqueness and existence of global solutions is proved. The elliptic (stationary) case is also considered.

Zajmujemy się quasi-statycznym przepływem gazu przewodzącego ciepło przy pewnych dodatkowych założeniach umożliwiających sprowadzenie pięciu równań przepływu i energii do nieliniowego równania parabolicznego. Rozważono pewne własności tego równania; udowodniono twierdzenia o istnieniu i jednoznaczności rozwiązań globalnych. Rozważono również przypadek eliptyczny (stacjonarny).

Займемся квазистатическим течением теплопроводящего газа при некоторых дополнительных предположениях дающих возможность сведения пяти уравнений течения и энергии к нелинейному параболическому уравнению. Обсуждены некоторые свойства этого уравнения; доказана теорема существования и единственности глобальных решений. Обсужден тоже эллиптический (стационарный) случай.

## 1. Introduction

IN MANY circumstances, when considering the flow of heat conducting gas, the flow itself is relatively slow whereas the temperature and hence the density of the gas may widely vary a few times or even more. In such cases when the flow takes place under constant exterior pressure, the influence of local variation of pressure due to dynamical effects on the local density can be neglected. Indeed, the dynamical pressure is of the order of $\varrho\,\dfrac{(\delta v)^2}{2}$ which, compared to $p = RT\varrho$, leads to $\dfrac{\delta p}{2} = \dfrac{(\delta v)^2}{2RT} = (\delta v)^2/2c_0^2$, where $c_0^2 = RT$ is the isothermal sound velocity. Therefore, if the variations of the velocity in the flow are not too large, when compared to the velocity of sound, the pressure in the equation of state can be assumed to be constant. In this way the density $\varrho$ becomes the function of the temperature only. Thus the fluid is considered to be incompressible but temperature extendable. On the other hand, the gradient of the pressure cannot be removed from the momentum equation since it is of the same order as other terms in the equation. Thus the pressure in the momentum equation becomes a free variable not related to density as it always takes place in the theory of incompressible fluids.

When the flow is realized in the vessel of a finite volume, the total pressure may change due to the heat supply. In that case one may still preserve the splitting of the pressure into the independent dynamical part $p$, appearing in the momentum equation and the "average pressure" $P$ which appears in the state equation. This can be done by assuming that the average pressure $P$ can only depend on time. Integrating the state equation over the whole volume

$$P(t)\int\limits_{\Omega}\frac{1}{T}dx = R\int\limits_{\Omega}\varrho\, dx = Rm,$$

the dependence of $P$ on $t$ can be determined. Here $m$ denotes the total mass of the gas. In such a case, however, we have a functional dependence of $\varrho$ on the temperature

$$\varrho = \frac{m}{T}\left(\int\limits_{\Omega}\frac{1}{T}dx\right)^{-1}.$$

Although further considerations can be carried out, in this case we will not deal with them because they lead to some additional complications for the boundary conditions.

## 1. Derivation of the basic equation

We start from the following system of equations in $\mathbf{R}^3$:

$$\begin{aligned}
&\varrho\frac{dv}{dt}+\nabla p = \eta\Delta v+\nabla(\zeta\,\mathrm{div}v),\\
(1.1)\qquad &\frac{d\varrho}{dt}+\varrho\,\mathrm{div}v = 0,\\
&\varrho c_p\frac{dT}{dt} = \mathrm{div}(\varkappa\nabla T)+Q,
\end{aligned}$$

where $c_p = c_p(T) > 0$, $\varkappa = \varkappa(T) > 0$, $Q(x, t) \geqslant 0$. Here $\frac{d}{dt} = \frac{\partial}{\partial t}+v\cdot\nabla$; $v = v(x, t)$ — velocity field.

To simplify the system (1.1) we assume the following hypothesis:

H1. density $\varrho$ is a function of the temperature only, i.e., $\varrho = \varrho(T)$;

H2. $\eta$ — the first viscosity coefficient is constant;

H3. a) the inertial force $\varrho\frac{dv}{dt}$ is negligible, or

b) $\varrho\frac{dv}{dt}$ may be replaced by $\tilde{\varrho}\frac{dv}{dt}$, where (average) $\tilde{\varrho}$ is constant;

H4. the ratio $\frac{\varrho_{,T}}{\varrho^2 c_p} := -\alpha$ is constant, $\alpha \geqslant 0$.

The last assumption is satisfied, for example, in the case of an ideal gas $P = RT\varrho$ under constant pressure. In this case one gets $\alpha = R/Pc_p$.

In general H4 implies that the specific enthalpy, under constant pressure, is given (or may be approximated) by the formula

$$h(T) = h_0+\frac{1}{\alpha\varrho(T)}$$

resulting from the integration of the formula in H4. The assumption H1 is well justified when the flow velocity is small compared to the velocity of sound. In that case the influence of small changes in pressure on the density may be neglected. Multiplying the continuity equation $(1.1)_2$ by $\varrho c_p$ and the energy equaton $(1.1)_3$ by $\varrho_{,T}$, one arrives, after subtracting one from the other, to the following equation:

$$\varrho^2 c_p \operatorname{div} v + \varrho_{,T}[\operatorname{div}(\varkappa \nabla T) + Q] = 0,$$

which, by the assumption H4, is reduced to

$$\operatorname{div}(v - \alpha\varkappa\nabla T) = \alpha Q. \tag{1.2}$$

The general solution to this equation can be written as

$$v = \alpha(\varkappa\nabla T + \nabla\psi) + v_s, \tag{1.3}$$

where $\Delta\psi = Q$ and $\operatorname{div} v_s = 0$.

Now, according to the assumption H3, we consider two cases:

a. When H3. a) holds, then $v_s$ must satisfy the Stokes equations

$$\zeta\nabla \operatorname{div} v_s + \eta\Delta v_s = \nabla p, \qquad \operatorname{div} v_s = 0. \tag{1.4}$$

Since the inertial terms are not present in the case H3. a, the initial condition for $v$ must be relaxed. As this case can be formally obtained by assuming $\varrho \to 0$, we have the following physical picture — in the short time scale, of order $1/\varrho$, the flow parameters are adjusted in such a way that $v_s$ satisfies Eq. (1.4).

The natural boundary condition for $v_s$ in $\mathbf{R}^3$ is that $v_s$ tends to a constant flow $v_0$, for $|x| \to \infty$. In this case the only solution of Eq. (1.4) for $v_s$ is $v_s \equiv v_0$. However, one can admit also unbounded solutions for $v_s$, assuming, for example, that $v_s$ is a shear flow.

b. When H3. b) holds, we assume that $v_s$ is a gradient of a potential $\varphi$ satisfying $\Delta\varphi = 0$ or even somewhat less for non-simply connected domains, that $v_s$ is rotation and divergence-free

$$\operatorname{rot} v_s = 0, \qquad \operatorname{div} v_s = 0. \tag{1.5}$$

Let us notice that if $v_s$ satisfies Eq. (1.5), then it satisfies also the Navier–Stokes equations with constant density.

Inserting the expression (1.3) into the momentum equation $(1.1)_1$, we see that in both cases a) and b) of the hypothesis H3 the equation is fulfilled since every term in the equation has the form of a gradient. Therefore this equation serves for the determination of the pressure.

Applying now the expression (1.3) in the energy equation $(1.1)_3$, we arrive at the following nonlinear parabolic equation:

$$\varrho c_p \left\{ \frac{\partial T}{\partial t} + \alpha\varkappa(\nabla T)^2 + q \cdot \nabla T \right\} = \operatorname{div}(\varkappa\nabla T) + Q, \tag{1.6}$$

where $q(x, t) = \alpha\nabla\psi + v_s$. At this moment it is convenient to introduce a new variable, heat potential $s = \int \varkappa(T)dT$. This reduces our equation to

$$\frac{\varrho c_p}{\varkappa} \left\{ \frac{\partial u}{\partial t} + \alpha(\nabla u)^2 + q \cdot \nabla u \right\} = \Delta u + Q, \tag{1.7}$$

or to another divergent form

$$\frac{\partial u}{\partial t} = \operatorname{div}(\alpha \nabla u) - \beta (\nabla u)^2 - q \cdot \nabla u + \alpha Q, \tag{1.8}$$

where

$$\alpha = \frac{\varkappa}{\varrho c_p}, \quad \beta = 1 + \frac{d\alpha}{ds}.$$

Let us note that having any solution of Eq. (1.6) (let it satisfy $T \to$ const at infinity), one can determine from Eq. (1.3) the corresponding solution of the whole set of flow equations (with $\varrho = 0$ or $\varrho \simeq \tilde{\varrho}$ according to H3. a or b).

## 2. Existence theorems

In order to prove the existence of solutions, we refer to suitable theorems proved in [2]. Chapter 5. We are searching, however, nonnegative solutions of Eq. (1.8) and need additional estimates. These estimates are based on the notion of sub and supersolutions. Let us consider the initial boundary value problem (IBVP)

$$\begin{aligned} \mathscr{L}u &= u_t - \operatorname{div}(\alpha(u)\nabla u) + b(x, t, u, \nabla u), \\ u|_{\partial' \Omega_T} &= \chi, \quad \Omega_T = \Omega \times [0, T], \end{aligned} \tag{2.1}$$

where $\partial' \Omega_T$ denotes $\bar{\Omega} \times \{0\} \cup \partial\Omega \times [0, T]$ and where $\Omega \times \mathbf{R}^n$ is a bounded domain with $C^{2+\beta}$ boundary $\partial\Omega$. We assume that $\chi(x, t)$ in the relation (2.1) is a restriction of a $C^{2+\beta, 1+\beta/2}$ function on $\mathbf{R}^n \times \mathbf{R}$ to $\partial' \Omega_T$.

DEFINITION. *A function $\underset{\sim}{u}(x, t) \in C^{2,1}(\Omega_T)$ (or $\tilde{u} \in C^{2,1}(\Omega_T)$) is called a subsolution (a supersolution) for IBVP* (2.1) *if the following inequalities are true*:

$$\mathscr{L}\underset{\sim}{u} \leqslant 0 \quad (\mathscr{L}\tilde{u} \geqslant 0) \quad \text{in } \Omega_T \quad \text{and} \quad \underset{\sim}{u}|_{\partial' \Omega_T} \leqslant \chi \, (\tilde{u}|_{\partial' \Omega_T} \geqslant \chi).$$

Sub and supersolution are useful in finding *a priori* estimates for the supremum norm of the solution. We recall:

THEOREM 1. *Let $\underset{\sim}{u}(x, t) \in C^{2,1}(\Omega_T)$ (or $\tilde{u}(x, t)$) be a subsolution (supersolution) of the IBVP* (2.1) *and let $u(x, t)$ be a solution of IBVP* (2.1) *of class $C^{2,1}$, then*

$$\underset{\sim}{u}(x, t) \leqslant u(x, t) \quad \big(u(x, t) \leqslant \tilde{u}(x, t)\big) \quad \text{in } \Omega_T.$$

In order to prove this, let us notice that denoting by $\omega$ the difference $u - \underset{\sim}{u}$, we have

$$\mathscr{L}u - \mathscr{L}\underset{\sim}{u} \equiv \omega_t - \operatorname{div}(\alpha_1 \nabla \omega) - \operatorname{div}(\xi \omega) + d\nabla \omega + c\omega \geqslant 0 \tag{2.2}$$

with

$$\omega|_{\partial' \Omega_T} \geqslant 0, \text{ where } \alpha_1 > 0, \quad \xi = (\xi^1, \ldots, \xi^n), \quad d = (d_1, \ldots, d_n)$$

and $c$ are functions depending only on $x$ and $t$. In the derivation of the inequality (2.2) we made use of the following formula:

$$f(y) - f(y_0) = \sum_{i=1}^{n} (y^i - y_0^i) \int_0^1 \frac{\partial f}{\partial y^i} \big(y_0 + s(y - y_0)\big) \, ds,$$

which is valid for the $C^1$ function $f(y^1, \ldots, y^n)$ defined in a convex domain of $\mathbf{R}^k$. Taking $u$, $\nabla u$ for $y$ and $\underset{\sim}{u}$, $\nabla \underset{\sim}{u}$ for $y_0$ and then substituting their values at $(x, t) \in \Omega_T$ under the integral, one obtains appropriate terms in Eq. (2.2). Hence $\omega$ satisfies a homogeneous linear equation with non-negative initial and boundary conditions; therefore by the maximum principle [3], $\omega \geqslant 0$ in $\Omega_T$. This proves the theorem in the case of subsolution. In a similar way taking $\mathscr{L}\tilde{u} - \mathscr{L}u$, one arrives at its counterpart for a supersolution. Basically the maximum principle is valid for bounded domains. It may be extended, however, to the unbounded domains imposing some restrictions on the growth of $u$ for $|x| \to \infty$ (the Phragmen–Lindelöff principle [3]) as for example assuming that $u \to 0$ for $|x| \to \infty$.

Having this, we can apply the theorem (6.1) in [2], Chapter 5, which we specify here in the simpler form, more suited to the case of Eq. (1.8).

THEOREM 2. *Let us assume that* $\underset{\sim}{u}(x, t)$, $\tilde{u}(x, t)$ *are bounded sub and supersolutions, respectively, of class* $C^{2,1}$ *of IBVP* (2.1) *in* $\Omega_T$ ($\Omega_T$ *of class* $C^{2+\beta}$) *and let the following conditions be satisfied*:

1) $\alpha(u) > 0$ *for* $u \in I := [\inf_{\Omega_T} \underset{\sim}{u}, \sup_{\Omega_T} \tilde{u}]$ *and* $\alpha \in C^{1+\beta}(I)$;

2) $b(x, t, u, p)$ *is Hölder continuous in* $\Omega \times [0, T] \times \mathbf{R}_+ \times \mathbf{R}^n$ *with the exponents* $\beta, \beta/2, \beta, \beta$;

*there exists a constant* $\mu(I)$ *such that for every* $(x, t) \in \bar{\Omega}_T$, $u \in I$

$$|b(x, t, u, p)| < \mu(1+|p|^2);$$

3) *for every* $\varrho > 0$ *there exists a function* $\varphi_\varrho(x, t)$ *such that for every* $(x, t) \in \Omega_T$, $u \in I$ *and* $|p| \leqslant \varrho$

$$|b_{,p^i}, b_{,u}, b_{,t}| \leqslant \varphi_\varrho(x, t)$$

*and*

$$\|\varphi_\varrho\|_{q,r,\Omega_T} := \Big[\int_0^T dt \Big(\int_\Omega |\varphi_\varrho|^q dx\Big)^{r/q}\Big]^{1/r} \leqslant \mu(\varrho) \leqslant \infty,$$

*where*

$$\frac{1}{r} + \frac{n}{2q} = 1 - \varepsilon, \quad q \in \left[\frac{n}{2(1-\varepsilon)}, \infty\right], \quad r \in \left[\frac{1}{1-\varepsilon}, \infty\right], \quad 0 < \varepsilon < 1.$$

*Then there exists a unique solution of IBVP* (2.1) *in* $C^{1+\beta, 1+\beta/2}(\bar{\Omega}_T)$ *and, moreover, its mixed derivatives* $u_{x_i}$ *are in* $L^2(\Omega_T)$.

We assume the following additional conditions:

H.5. The function $\alpha(u)$ in Eq. (1.8) defined on $\mathbf{R}_+ = (0, \infty)$ is positive for positive $u$ and is locally of class $C^{1+\beta}$. The vector field $v(x, t)$ in Eq. (1.8) is of class $C^{\beta, \beta/2}(\Omega_T)$ and its time derivative $v_t$ has a finite $\|\cdot\|_{q,r}$ norm in $\Omega_T$ where $r$, $q$ is given in theorem 1, $\alpha(u)$ grows not faster than linearly, i.e.,

$$\alpha(s) < c_1 s + c_2 \quad \text{for} \quad s \in \mathbf{R}_+$$

and $Q(x, t, s) \in C^{\beta, \beta/2, \beta}$ satisfies 1) in Theorem 2. There exists a positive constant $\mu$ (or positive subsolution) such that

$$Q(x, t, \mu) \geqslant 0 \quad \text{in} \quad Q_T.$$

(Denote by $\boldsymbol{\mu}$ the infimum of all such constants $\mu$). $Q$ is bounded from above;

$$Q \leqslant C \quad \text{for} \quad (x, t) \in \Omega_T, \quad u \in \mathbf{R}_+$$

and has bounded (weak) derivatives with respect to $t$ and $u$.

Now we can formulate the existence theorem.

THEOREM 3. *Assuming that $\Omega$ is of class $C^{2+\beta}$ and the hypothesis* H.5 *is satisfied, then the IBVP* (2.1) *for Eq.* (1.8) *has a unique $C^{2\beta, 1+\beta/2}(\bar{\Omega}_T)$ solution for any $T > 0$ provided that* $\min\limits_{\partial'\Omega_T}(\chi - \boldsymbol{\mu}) > 0$ (*it may reach zero when $\boldsymbol{\mu}$ is positive*).

In order to prove this, lest us notice that there exists a positive constant $\underset{\sim}{s}$ such that $\min\limits_{\partial'\Omega_T} \chi \geqslant \underset{\sim}{s} \geqslant \boldsymbol{\mu}$. One easily verifies that $\underset{\sim}{s}$ is a positive subsolution. On the other hand, the solution $\bar{s}(t)$ to the ordinary differential equation

$$\frac{d}{dt}\tilde{s} = C(c_1 \tilde{s} + c_2), \qquad \tilde{s}(0) = \max_{\partial'\Omega_T} \chi$$

is a supersolution of the IBVP for Eq. (1.8). Hence we have the bounds for the solution $u(x, t)$

$$\underset{\sim}{s} \leqslant u(x, t) \leqslant \bar{s}(t).$$

Applying Theorem 2 we conclude the validity of Theorem 3.

Let us also notice that whenever $\alpha_{,u} \geqslant 0$ the solution of IBVP for Eq. (1.8) with removed quadratic term $\alpha(\nabla u)^2$ is also a supersolution for the full equation (1.8). In the case of unbounded domain we assume that $u \to u_0 \geqslant \underset{\sim}{s}$ for $|x| \to \infty$.

## 3. Elliptic problem

When searching for stationary solutions, one can again use sub and supersolutions which are defined in a similar way, the only difference being that there is no $u_t$ in $\mathscr{L}$ and instead of $\Omega_T$ we have $\Omega$. Again one easily checks that $\underset{\sim}{s}$ satisfying $\inf u|_{\partial\Omega} \geqslant \underset{\sim}{s} \geqslant \boldsymbol{\mu}$ is a subsolution and $\tilde{s} = \max|_{\partial\Omega}$ is a supersolution of BVP. Hence, using the results of the paper [1] and the form of Eq. (1.7), one easily deduces the existence of a solution $u(x)$ satisfying $\underset{\sim}{s} \leqslant u(x) \leqslant \bar{s}$.

REMARKS. Solutions of Eq. (1.8) in a domain with boundary can describe a real flow if appropriate boundary conditions for $T$ and $v$ are satisfied. Imposing Dirichlet conditions for the temperature, we are not, in general, able to control $\nabla T$ at $\partial\Omega$ and hence, according to Eq. (1.3) velocity cannot be prescribed there — the fluid is leaking through the boundary (perforated boundary). One does not encounter these difficulties in the case of whole $\mathbf{R}^3$ when the flow is given at infinity.

## References

1. J. L. KAZDAN, R. J. KRAMER, *Invariant criteria for existence of solutions to second order quasi-linear elliptic equations*, Comm. Pure Appl. Math., **31**, 619–645, 1978.
2. O. A. LADYZHENSKA, V. A. SOLONNIKOV, N. N. URALTSEVA, *Linear and quasi-linear equations of parabolic type* [in Russian], Nauka, Moskva 1967.

3. M. H. PROTTER, H. F. WEINBERGER, *Maximum principles in differential equations*, Prentice Hall, Englewood Cliffs, New Jersey 1967.

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