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# Shape sensitivity analysis and optimal design of physically nonlinear plates

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A UNIFORM variational formulation of sensitivity analysis for physically nonlinear plates is presented in terms of generalized stresses and strains. Both the external and internal boundary shape modifications are treated within this formulation. Next, optimal design problems for stress and deflection constraints are formulated and the relevant optimality conditions are derived using the concept of a linear adjoint plate. Finally, some illustrative examples of sensitivity analysis and optimal design problems are presented.

Wariacyjne sformułowanie analizy wrażliwości dla płyt fizycznie nieliniowych w zakresie teorii małych ugięć jest przedstawione przy zastosowaniu uogólnionych naprężeń i odkształceń. Rozpatrzono wariacje zewnętrznego kształtu płyty i wariacje kształtu wewnętrznych powierzchni. Optymalne projektowanie dla warunków naprężeniowych i odkształceniowych zostało następnie rozpatrzone i warunki optymalności zostały wyrażone jako warunki ekstremum odpowiednich funkcjonałów. Przykłady ilustrujące zastosowanie ogólnego podejścia wariacyjnego do analizy wrażliwości zostały przedstawione w końcowej części pracy.

Вариационное сформулирование анализа чувствительности для физически нелинейных плит в области теории малых прогибов представлено при применении обобщенных напряжений и деформаций. Рассмотрены внешние вариации формы плиты и вариации формы внутренних поверхностей. Затем рассмотрено оптимальное проектирование для условий напряжения и деформации и условия оптимальности выражены как условия экстремума соответствующих функционалов. Примеры, иллюстрирующие применение общего вариационного подхода к анализу чувствительности, представлены в окончательной части работы.

## 1. Introduction

THE PRESENT PAPER is devoted to a variational formulation of sensitivity analysis and optimal design of plates subjected to flexure within small deflection and strain theory. However, a nonlinear relation is assumed between generalized stresses and strains. Such a situation corresponds, for instance, to fiber-reinforced composite plates which exhibit nonlinearity even within small strain and deflection ranges, as a result of progressing damage and inelasticity within fibers or matrix. Thus the assumption of nonlinearity provides a more accurate description of the deformation of composite structures subjected to flexure.

In optimal design problems of such structures, local or global constraints are usually set on displacements and stresses. The objective function then corresponds to a minimum of weight or cost of material of a structure. In order to derive the relevant optimality conditions, explicit expressions for variations of constraint equations and objective functions in terms of the variations of design functions are to be determined (sensitivity analysis). For linear elastic structures such variations were derived in Refs. [1–5] for any stress, strain or displacement functionals assuming both material parameters and shape variations. The general case of sensitivity analysis in the case of physical nonlinearity was discussed in [8, 9] and the case of both physically and geometrically nonlinear beams and plates was considered in [10, 12]. The present work supplements the results of [10, 12] by considering the derivation of variations of functionals for plates with respect to external and internal boundary modifications. In spite of the fact that our analysis will be limited to the geometrically linear case, extension to the geometrically nonlinear theory can easily be obtained by following the present analysis and the one presented in [10, 12]. The concept of an adjoint structure and its mechanical interpretation discussed in [1–4] remains valid in the nonlinear case. However, the stiffness matrix for the adjoint structure is specified as the tangent stiffness matrix of the primary structure. Hence the adjoint stiffness matrix is not constant but depends on the strain or displacement fields of the primary structure. This renders the iterative solution of the optimal design problem more complicated since the tangent stiffness matrix should be updated after each redesign step.

In Sect. 2, the sensitivity analysis of an arbitrary functional with respect to variation of an external plate boundary will be discussed and in Sect. 3 the case of an interface variation within a plate will be considered. Variations of potential and complementary energies associated with shape variations will be derived in Sect. 4. In Sect. 5, the optimal design problem will be formulated and the relevant optimality conditions will be derived. Some illustrative examples will be presented in Sect. 6.

## 2. Sensitivity analysis of an arbitrary functional with respect to external boundary variation

Consider a plate occupying the domain $A$ with the boundary $S$, Fig. 1. The plate is subjected to transverse load $p$, whereas either generalized tractions or displacements are specified on $S$. Denote the generalized stresses (i.e., bending and twisting moments within plate domain) by $\mathbf{M}$, the associated strains (i.e., curvatures and torsion) by $\boldsymbol{\varkappa}$, and the

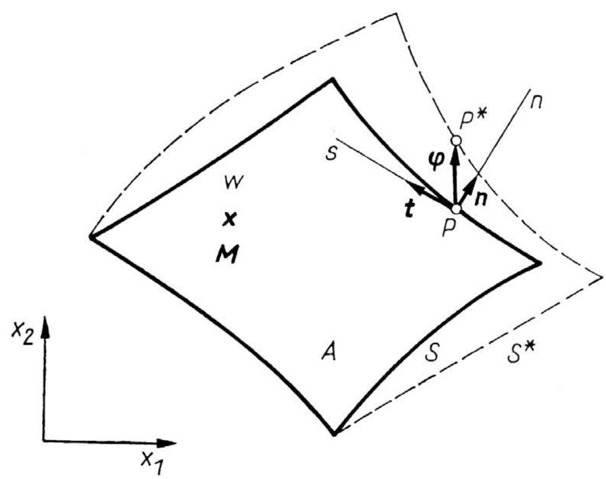


FIG. 1. Plate occupying domain $A$ with boundary $S$.

lateral deflection by $w$. It is assumed that the nonlinear stress-strain relations are generated by strain and stress potentials such that

$$\mathbf{M}(\boldsymbol{\varkappa}) = \frac{\partial U(\boldsymbol{\varkappa})}{\partial \boldsymbol{\varkappa}}, \quad \boldsymbol{\varkappa}(\mathbf{M}) = \frac{\partial W(\mathbf{M})}{\partial \mathbf{M}}, \tag{2.1}$$

where

$$U(\boldsymbol{\varkappa}) = \int_0^{\varkappa} \mathbf{M} \cdot d\boldsymbol{\varkappa}, \quad W(\mathbf{M}) = \int_0^{M} \boldsymbol{\varkappa} \cdot d\mathbf{M} \tag{2.2}$$

and the dot between two symbols denotes the scalar product or the summation with respect to indices of lower order tensors. The incremental form of Eq. (2.1) is expressed as follows:

$$d\mathbf{M} = \frac{\partial^2 U}{\partial \boldsymbol{\varkappa} \partial \boldsymbol{\varkappa}} \cdot d\boldsymbol{\varkappa} = \mathbf{D} \cdot d\boldsymbol{\varkappa}, \quad d\boldsymbol{\varkappa} = \frac{\partial^2 W}{\partial \mathbf{M} \partial \mathbf{M}} \cdot d\mathbf{M} = \mathbf{C} \cdot d\mathbf{M}, \tag{2.3}$$

where

$$\mathbf{D} = \frac{\partial^2 U}{\partial \boldsymbol{\varkappa} \partial \boldsymbol{\varkappa}} = \frac{\partial \mathbf{M}}{\partial \boldsymbol{\varkappa}}, \quad \mathbf{C} = \frac{\partial^2 W}{\partial \mathbf{M} \partial \mathbf{M}} = \frac{\partial \boldsymbol{\varkappa}}{\partial \mathbf{M}}. \tag{2.4}$$

For a stable elastic material, $\mathbf{D}$ is a symmetric and positive definite tangent stiffness matrix, whereas $\mathbf{C}$ is a compliance matrix.

Under applied loads the plate passes from its initial configuration to a deformed one specified by the deflection field $w$. In addition to the deformation process, let us consider a transformation process which modifies the plate domain, $\mathbf{x}^t = \mathbf{x} + \boldsymbol{\varphi}$, with the imposed transformation field $\boldsymbol{\varphi}(\mathbf{x})$ specified within $A$, Fig. 1. Obviously this transformation field modifies the shape of the external boundary of a plate or its internal interfaces between different materials and affects deflection, strain and stress fields within plate domain.

Considering a simultaneous variation of transformation and state fields (cf. [4]), any point $P$ within plate domain, initially placed at $\mathbf{x}$, is transformed to the actual position $\mathbf{x}^*$ according to the rule

$$P \to P^*: \ x_k^* = x_k + \delta\varphi_k(\mathbf{x}), \quad k = 1, 2, \tag{2.5}$$

whereas the state fields for the actual configuration of plate are

$$\begin{aligned} &w^*(\mathbf{x}^*) = w(\mathbf{x}) + \mathring{\delta} w(\mathbf{x}), \quad \boldsymbol{\varkappa}^*(\mathbf{x}^*) = \boldsymbol{\varkappa}(\mathbf{x}) + \mathring{\delta}\boldsymbol{\varkappa}(\mathbf{x}), \\ &\mathbf{M}^*(\mathbf{x}^*) = \mathbf{M}(\mathbf{x}) + \mathring{\delta}\mathbf{M}(\mathbf{x}) \end{aligned} \tag{2.6}$$

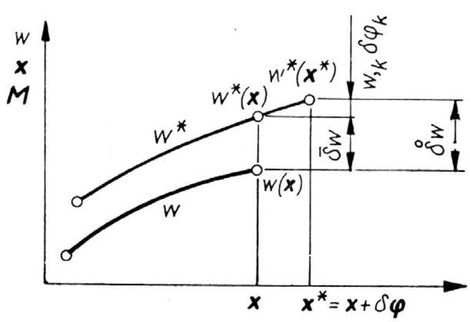


FIG. 2. Variations of state field within plate domain.

where $\mathring{\delta}(\cdot)$ denotes the total variation of enclosed quantity with respect to a fixed Cartesian reference system. From Eq. (2.5) and (2.6) it follows that, Fig. 2,

$$\text{(2.7)} \qquad \mathring{\delta} w = \bar{\delta} w + w_{,k}\delta\varphi_k, \qquad \mathring{\delta}\boldsymbol{\varkappa} = \bar{\delta}\boldsymbol{\varkappa} + \boldsymbol{\varkappa}_{,k}\delta\varphi_k, \qquad \mathring{\delta}\mathbf{M} = \bar{\delta}\mathbf{M} + \mathbf{M}_{,k}\delta\varphi_k,$$

where the comma denotes partial differentiation and $\bar{\delta}(\cdot) = (\cdot)^*(\mathbf{x}) - (\cdot)(\mathbf{x})$ is the local state variation for a fixed configuration of the plate. Furthermore, the following transformation rules occur (cf. [4]):

$$\text{(2.8)} \qquad \begin{gathered} \mathring{\delta}(dA) = \delta\varphi_{k,k}dA, \qquad \mathring{\delta}(dS) = (\delta\varphi_{k,k} - n_k\delta\varphi_{k,n})dS, \\ \mathring{\delta} n_j = n_j^* - n_j = (n_j n_l - \delta_{jl})n_k\delta\varphi_{k,l}, \\ \mathring{\delta} t_j = t_j^* - t_j = (\delta_{jk} - t_j t_k)t_l\delta\varphi_{k,l}, \end{gathered}$$

where $dA$, $dS$ denote the area and boundary length elements, $\mathbf{n}$, $\mathbf{t}$ are the unit normal and tangential vectors to $S$, respectively, and $\delta_{jl}$ denotes Kronecker's symbol. Note that the vectors $\mathbf{n}$, $\mathbf{t}$ form the local right-hand orthogonal reference system $(n, s)$ along the external boundary of a plate.

Consider now any kinematically admissible deflection field $w^k$ and any statically admissible stress field $\mathbf{M}^s$ within a plate of fixed configuration. For the small strain theory, the generalized strain $\boldsymbol{\varkappa}^k$ is obtained from $w^k$ by a linear equation

$$\text{(2.9)} \qquad \varkappa_{ij}^k = -w_{,ij}^k,$$

where $w_{,ij}^k$ is the in-plane second-order gradient of the deflection field. Thus the equilibrium condition for a plate can be expressed in terms of the virtual work equation, namely,

$$\text{(2.10)} \qquad \int \mathbf{M}^s \cdot \boldsymbol{\varkappa}^k dA - \int p^s w^k dA + \int (M_{in}^s w_{,i}^k - V^s w^k)dS = 0,$$

where

$$\text{(2.11)} \qquad M_{in}^s = M_{ij}^s n_j$$

are the boundary moment components with respect to a fixed Cartesian coordinate system and $V^s$ denotes the shear force along the boundary $S$, that equals ([11])

$$\text{(2.12)} \qquad V^s = M_{ij,j}^s n_i.$$

In view of Eqs. (2.7) and (2.8), the total variations of $M_{in}^s$ and $V^s$ can be expressed as follows

$$\text{(2.13)} \qquad \begin{aligned} \mathring{\delta} M_{in}^s &= \mathring{\delta} M_{ij}^s n_j + M_{ij}^s \mathring{\delta} n_j = \bar{\delta} M_{in}^s + M_{ij,k}\delta\varphi_k n_j + M_{ij}^s(n_j n_l - \delta_{jl})n_k\delta\varphi_{k,l}, \\ \mathring{\delta} V^s &= \mathring{\delta}(M_{ij,j}^s)n_i + M_{ij,j}^s\mathring{\delta} n_i = \bar{\delta} V^s + M_{ij,jk}^s\delta\varphi_k n_i + M_{ij,j}^s(n_i n_l - \delta_{il})n_k\delta\varphi_{k,l}, \end{aligned}$$

where $\bar{\delta} M_{in}^s$ and $\bar{\delta} V^s$ denote the local variations for a fixed plate configuration.

Consider now the following functional:

$$\text{(2.14)} \qquad G = \int_A \psi(\mathbf{M}, \boldsymbol{\varkappa}, p, w)dA$$

depending on generalized stress and strain fields, transverse load and deflection within plate domain. The major question now posed is how the value of this functional is modified as a result of transformation of plate domain. Thus it is our goal to determine the first variation of $G$ with respect to the variation of plate shape. Assuming $\psi$ to be a continuous and differentiable function of its arguments, the first variation of $G$ equals

$$(2.15)\qquad \delta G = \int (\psi_{,\mathbf{M}}\cdot \overset{\circ}{\delta}\mathbf{M}+\psi_{,\varkappa}\cdot \overset{\circ}{\delta}\boldsymbol{\varkappa}+\psi_{,p}\overset{\circ}{\delta}p+\psi_{,w}\overset{\circ}{\delta}w+\psi\delta\varphi_{k,k})\,dA$$
$$= \int (\psi_{,\mathbf{M}}\cdot \bar{\delta}\mathbf{M}+\psi_{,\varkappa}\cdot \bar{\delta}\boldsymbol{\varkappa}+\psi_{,p}\bar{\delta}p+\psi_{,w}\bar{\delta}w)\,dA+\int \psi\delta\varphi_n\,dS,$$

where $\delta\varphi_n = \mathbf{n}\cdot\delta\boldsymbol{\varphi}$ denotes the normal component of boundary variation on $S$.

To eliminate $\bar{\delta}\mathbf{M}$, $\bar{\delta}\boldsymbol{\varkappa}$ and $\bar{\delta}w$ from Eq. (2.15), let us introduce an adjoint, physically linear plate of the same shape as the primary one, but subjected to the imposed fields of initial stresses and strains specified by

$$(2.16)\qquad \mathbf{M}^{ai} = \psi_{,\varkappa},\qquad \boldsymbol{\varkappa}^{ai} = \psi_{,\mathbf{M}}\qquad \text{within } A$$

and loaded by

$$(2.17)\qquad p^a = \psi_{,w}\qquad \text{within } A.$$

Furthermore we assume that on the boundary $S$ of the adjoint plate either generalized tractions or generalized displacements vanish and the adjoint plate is supported in the same way as the primary one. The stress field $\mathbf{M}^a$ within the adjoint plate is related to its strain field $\boldsymbol{\varkappa}^a$ by the relation

$$(2.18)\qquad \mathbf{M}^a = \mathbf{D}^T\cdot(\boldsymbol{\varkappa}^a-\boldsymbol{\varkappa}^{ai})-\mathbf{M}^{ai}$$

with the stiffness matrix $\mathbf{D}$ specified by Eq. (2.4). Obviously $\mathbf{M}^a$ satisfies the equilibrium conditions for the adjoint plate and $\boldsymbol{\varkappa}^a$ is the associated strain field that follows from the deflection field $w^a$. Using now Eqs. (2.16), (2.17) and noting that in view of Eq. (2.3) we have

$$(2.19)\qquad \bar{\delta}\mathbf{M} = \mathbf{D}\cdot\bar{\delta}\boldsymbol{\varkappa},$$

Eq. (2.15) can be rewritten in the form

$$(2.20)\qquad \delta G = \int [(\mathbf{D}^T\cdot\boldsymbol{\varkappa}^{ai}+\mathbf{M}^{ai})\cdot\bar{\delta}\boldsymbol{\varkappa}+\psi_{,p}\bar{\delta}p+p^a\bar{\delta}w]\,dA+\int\psi\delta\varphi_n\,dS$$
$$= \int (\boldsymbol{\varkappa}^a\cdot\bar{\delta}\mathbf{M}-\mathbf{M}^a\cdot\bar{\delta}\boldsymbol{\varkappa}+\psi_{,p}\bar{\delta}p+p^a\bar{\delta}w)\,dA+\int\psi\delta\varphi_n\,dS.$$

Identifying now $\mathbf{M}^s$, $w^k$ and $\boldsymbol{\varkappa}^k$ with $\mathbf{M}^a$, $\bar{\delta}w$ and $\bar{\delta}\boldsymbol{\varkappa}$, respectively, the virtual work equation (2.10) can be written in the form

$$(2.21)\qquad \int\mathbf{M}^a\cdot\bar{\delta}\boldsymbol{\varkappa}\,dA-\int p^a\bar{\delta}w\,dA+\int (M^a_{in}\bar{\delta}w_{,i}-V^a\bar{\delta}w)\,dS = 0.$$

On the other hand, setting $\mathbf{M}^s = \bar{\delta}\mathbf{M}$, $w^k = w^a$ and $\boldsymbol{\varkappa}^k = \boldsymbol{\varkappa}^a$, it follows from Eq. (2.10) that

$$(2.22)\qquad \int\bar{\delta}\mathbf{M}\cdot\boldsymbol{\varkappa}^a\,dA-\int\bar{\delta}p\,w^a\,dA+\int (\bar{\delta}M_{in}w^a_{,i}-\bar{\delta}V w^a)\,dS = 0.$$

Then, in view of Eqs. (2.21) and (2.22), Eq. (2.20) can be transformed as follows:

$$(2.23)\qquad \delta G = \int (w^a+\psi_{,p})\bar{\delta}p\,dA+\int (\bar{\delta}V w^a-\bar{\delta}M_{in}w^a_{,i}-V^a\bar{\delta}w+M^a_{in}\bar{\delta}w_{,i}+\psi\delta\varphi_n)\,dS.$$

Thus the first variation of $G$ is expressed in terms of local variations of boundary moment, shear force and deflection of primary plate along its boundary.

One can now express these local variations by means of total variations. Making use of Eqs. (2.7) and (2.13) and noting the following identity that holds on the plate boundary $S$

$$(2.24)\quad \int (M_{il,k} w^a_{,i} - M_{lj,jk} w^a) n_l \delta\varphi_k dS = \int [(V w^a_{,k} - M_{in} w^a_{,ik}) \delta\varphi_k - (\mathbf{M} \cdot \boldsymbol{\varkappa}^a - p w^a) \delta\varphi_n + (V w^a - M_{in} w^a_{,i}) \delta\varphi_{k,k} + (M_{il} w^a_{,i} - M_{lj,j} w^a) n_k \delta\varphi_{k,l}] dS,$$

Eq. (2.23) can be rewritten in the form

$$(2.25)\quad \delta G = \int (w^a + \psi_{,p}) \bar{\delta} p\, dA + \int [(\psi - \mathbf{M} \cdot \boldsymbol{\varkappa}^a + p w^a) \delta\varphi_n + (V w^a - M_{in} w^a_{,i}) (\delta_{s,s} - K \delta\varphi_n) + (V w^a_{,k} - M_{in} w^a_{,ik} + V^a w_{,k} - M^a_{in} w_{,ik}) \delta\varphi_k + \mathring{\delta} V w^a - \mathring{\delta} M_{in} w^a_{,i} - V \mathring{\delta} w + M^a_{in} \mathring{\delta} w_{,i}] dS,$$

where $\delta\varphi_n = \delta\boldsymbol{\varphi} \cdot \mathbf{n}$ and $\delta\varphi_s = \delta\boldsymbol{\varphi} \cdot \mathbf{t}$ are the normal and tangential components of the transformation field along plate boundary and $K$ denotes the curvature of $S$. Equation (2.25) expresses the first variation of any functional $G$ in terms of components of boundary tractions and deflection and their derivatives of both primary and adjoint plates with respect to the fixed Cartesian reference system as well as the total variations of primary state fields. Specifying boundary conditions, it is generally more convenient to specify them in a local coordinate system $(\mathbf{n}, \mathbf{t}, \mathbf{b})$ associated with plate boundary, see Fig. 3a.

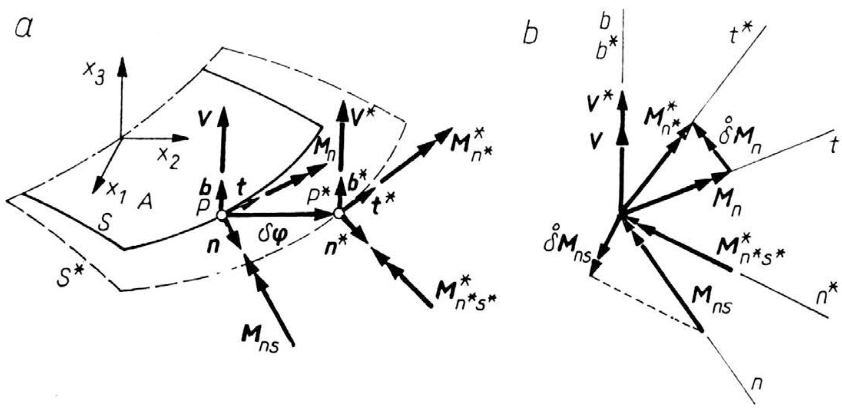


FIG. 3. Boundary conditions along $S$; (a) components of generalized surface tractions, (b) total variations of traction components.

The generalized tractions along plate boundary are then the bending moment $M_n$ and generalized shear force $Q$ which can be expressed in terms of $M_{in}$ and $V$ as follows

$$(2.26)\quad M_n = M_{in} n_i, \quad Q = V + M_{ns,s},$$

where the twisting moment $M_{ns}$ along plate edge equals

$$(2.27)\quad M_{ns} = M_{in} t_i.$$

During the infinitesimal transformation of plate boundary to its actual configuration $S^*$, a typical point $P$ passes to $P^*$ and the unit vectors $\mathbf{n}, \mathbf{t}$ are translated and rotated to their actual orientation $\mathbf{n}^*, \mathbf{t}^*$, whereas the vector $\mathbf{b}$ is translated to $\mathbf{b}^*$, Fig. 3a, specifying the new local coordinate system $(\mathbf{n}^*, \mathbf{t}^*, \mathbf{b}^*)$. The change in orientation of $\mathbf{n}$ and $\mathbf{t}$ in a global fixed Cartesian coordinate system during this transformation is expressed by the last two equations of the set (2.8) that can be now rewritten in the form

$$\mathring{\delta} n_j = -t_j(K\delta\varphi_s + \delta\varphi_{n,s}), \qquad \mathring{\delta} t_j = n_j(K\delta\varphi_s + \delta\varphi_{n,s}), \tag{2.28}$$

whereas $\mathring{\delta}\mathbf{b} = 0$ and variation of the boundary length element described by the second line the set (2.8) can be expressed in the equivalent form

$$\delta(dS) = (\delta\varphi_{s,s} - K\delta\varphi_n)dS. \tag{2.29}$$

Consider now any vector field $\mathbf{f}(s)$ specified along plate boundary $S$, whose components in the global fixed Cartesian coordinate system are denoted by $f_j$ ($j = 1, 2, 3$) and in the local coordinate system $(\mathbf{n}, \mathbf{t}, \mathbf{b})$ by $f_n$, $f_s$ and $f_b$, respectively. During the transformation process of plate boundary shape, $\mathbf{f}$ changes to $\mathbf{f}^*$ with components $f^*_{n*}$, $f^*_{s*}$ and $f^*_{b*}$ with respect to the axes $n^*$, $t^*$, $b^*$. The total variation of $\mathbf{f}$ with respect to a fixed coordinate system is defined as $\mathring{\delta}\mathbf{f} = \mathbf{f}^* - \mathbf{f}$ with components $\mathring{\delta} f_j = f^*_j - f_j$. For purposes of our subsequent analysis, besides considering the total variations $\mathring{\delta} f_j$, let us introduce the corotational variations of components of $\mathbf{f}$ with respect to the local reference system, which do not take into account the rotation of this system during the transformation process. Since the components of $\mathbf{f}$ and $\mathbf{f}^*$ are denoted by $(f_n, f_s, f_b)$ and $(f^*_{n*}, f^*_{s*}, f^*_{b*})$ in configurations $S$ and $S^*$, respectively, then the corotational variations of $\mathbf{f}$ are defined as follows:

$$\begin{aligned} \delta f_n &= f^*_{n*} - f_n = f^*_j n^*_j - f_j n_j = \mathring{\delta} f_j n_j + f_j \mathring{\delta} n_j, \\ \delta f_s &= f^*_{s*} - f_s = f^*_j t^*_j - f_j t_j = \mathring{\delta} f_j t_j + f_j \mathring{\delta} t_j, \\ \delta f_b &= f^*_{b*} - f_b = f^*_j b^*_j - f_j b_j = \mathring{\delta} f_j b_j. \end{aligned} \tag{2.30}$$

The solution of Eqs. (2.30) with respect to $\mathring{\delta} f_j$ provides the relations between the total variations $\mathring{\delta} f_j$ of any vector field $\mathbf{f}$ and its corotational variations $\delta f_n$, $\delta f_s$, $\delta f_b$. Noting that $n_3 = \mathring{\delta} n_3 = t_3 = \mathring{\delta} t_3 = 0$ and $b_1 = b_2 = 0$, $b_3 = 1$ and taking into account Eqs. (2.28), it follows from Eqs. (2.30) that

$$\begin{aligned} \mathring{\delta} f_j &= n_j \delta f_n + t_j \delta f_s + (n_i f_s - t_i f_n)(K\delta\varphi_s + \delta\varphi_{n,s}), \qquad j = 1, 2, \\ \mathring{\delta} f_3 &= \delta f_b. \end{aligned} \tag{2.31}$$

Furthermore, for any quantity defined along plate boundary length, the following identity can be written:

$$\delta[(\cdot)_{,s}] = \delta\left[\frac{d(\cdot)}{ds}\right] = \frac{d[\delta(\cdot)]}{ds} - \frac{d(\cdot)}{ds}\frac{\delta[dS]}{ds}. \tag{2.32}$$

In view of Eq. (2.29) it follows from Eq. (2.32) that

$$\delta[(\cdot)_{,s}] = [\delta(\cdot)]_{,s} - (\cdot)_{s}(\delta\varphi_{s,s} - K\delta\varphi_n). \tag{2.33}$$

Using Eq. (2.31) we can now express the total variations $M_{in}$ ($i = 1, 2$) in terms of corotational variations of the boundary bending moment $M_n$ and twisting moment $M_{ns}$, namely,

$$\mathring{\delta} M_{in} = n_i \delta M_n + t_i \delta M_{ns} + (n_i M_{ns} - t_i M_n)(K\delta\varphi_s + \delta\varphi_{n,s}). \tag{2.34}$$

Furthermore, we have $\mathring{\delta} V = \delta V$, and then in view of Eqs. (2.26) and (2.33) we can write

$$\mathring{\delta} V = \delta Q - \delta(M_{ns,s}) = \delta Q - (\delta M_{ns})_{,s} + M_{ns,s}(\delta\varphi_{s,s} - K\delta\varphi_n). \tag{2.35}$$

The relation between total and corotational variations of the deflection field $w$ and its gradient along the plate boundary, in view of Eq. (2.31) and (2.33) has the form

$$
(2.36)\quad \begin{aligned} \overset{\circ}{\delta} w &= \delta w, \\ \overset{\circ}{\delta} w_{,i} &= n_i \delta(w_{,n}) + t_i(\delta w)_{,s} - t_i w_{,s}(\delta\varphi_{s,s} - K\delta\varphi_n) + (n_i w_{,s} - t_i w_{,n})(K\delta\varphi_s + \delta\varphi_{n,s}). \end{aligned}
$$

Moreover, we can write the two following equalities along plate boundary:

$$
(2.37)\quad \begin{aligned} w_{,k}\delta\varphi_k &= w_{,s}\delta\varphi_s + w_{,n}\delta\varphi_n, \\ w_{,ik}\delta\varphi_k &= -(n_i\varkappa_n + t_i\varkappa_{sn})\delta\varphi_n - (n_i\varkappa_{sn} + t_i\varkappa_s)\delta\varphi_s, \end{aligned}
$$

where

$$
(2.38)\quad \varkappa_n = -w_{,nn}, \quad \varkappa_s = -(w_{,ss} - Kw_{,n}), \quad \varkappa_{sn} = -(w_{,ns} + Kw_{,s})
$$

denote the curvatures and torsion of the deformated plate which are expressed in the local coordinate system $(\mathbf{n}, \mathbf{t})$.

Using now Eqs. (2.34)–(2.38) in Eq. (2.25) after some transformations and integrations by parts along the plate boundary, the first variation of the functional $G$ can be expressed in the following form:

$$
\begin{aligned} (2.39)\quad \delta G = &\int (w^a + \psi_{,p})\overline{\delta p}\, dA + \int [\psi + pw^a - M_s\varkappa_s^a - 2M_{ns}\varkappa_{ns}^a + Qw^a_{,n} + Q^a w_{,n} \\ &+ M_n^a\varkappa_n - (Qw^a - M_n w^a_{,n})K - (M_n w^a_{,s} + M_n^a w_{,s})_{,s}]\,\delta\varphi_n dS + \int (M_{n,s} w^a_{,n} - Q_{,s} w^a \\ &- M_n^a w_{,ns} + Q^a w_{,s})\,\delta\varphi_s dS + \int [\delta Q w^a - \delta M_n w^a_{,n} - Q^a\delta w + M_n^a \delta(w_{,n})]\, dS \\ &+ \int [M_{ns}^a \delta w - \delta M_{ns} w^a + (Qw^a - M_n w^a_{,n} - M_{ns} w^a_{,s} - M_{ns}^a w_{,s})\,\delta\varphi_s \\ &+ (M_n w^a_{,s} - M_{ns} w^a_{,n} + M_n^a w_{,s} - M_{ns}^a w_{,n})\,\delta\varphi_n]_{,s} dS. \end{aligned}
$$

The last integral on the right-hand side of Eq. (2.39) vanishes when the plate boundary is smooth and all terms of this integral are continuous functions of the boundary parameter $s$. On the other hand, when there exist some singular points $S_i$ along the plate boundary at which either the plate boundary is not smooth or some terms of the last integral of Eq. (2.39) suffer discontinuities, it is reduced to the form

$$
\begin{aligned} (2.40)\quad \int [\ldots]_{,s} dS = \sum_i \{&[\![M_{ns}^a]\!]\delta w - [\![\delta M_{ns}]\!] w^a + [\![(Qw^a - M_n w^a_{,n} - M_{ns} w^a_{,s} \\ &- M_{ns}^a w_{,s})\,\delta\varphi_s]\!] + [\![(M_n w^a_{,s} - M_{ns} w^a_{,n} + M_n^a w_{,s} - M_{ns}^a w_{,n})\,\delta\varphi_n]\!]\}, \end{aligned}
$$

where $[\![f]\!] = f(S_i^-) - f(S_i^+)$ denotes the jump of proper quantity calculated as a difference of its values on both sides of the singular point $S_i$.

Equation (2.39) expresses the first variation of any functional $G$ defined over the plate domain $A$ in terms of its integrand $\psi$, deflections, generalized stresses and strains of both primary and adjoint plates as well as in terms of normal and tangential components of plate boundary shape variation. Note, furthermore, that since along the boundary $w$ or $Q$ and $w_{,n}$ or $M_n$ are specified in advance, then their corotational variations are also known and can be expressed in terms of $\delta\boldsymbol{\varphi}$. Similarly, the variations $[\![\delta M_{ns}]\!]$ which are equal to the variations of concentrated forces at the boundary singular points can be calculated from the specified boundary conditions in terms of $\delta\boldsymbol{\varphi}$.

The formula (2.39) derived for the first variation of an arbitrary functional may appear to be rather complex. This complexity results from a general formulation of the problem and a general form of nonhomogeneous boundary conditions along plate edges. Even in the most general case, all terms occurring in Eq. (2.39) are computable and $\delta G$ may be calculated analytically or numerically. However, in most applications many of terms that appear in Eq. (2.39) will vanish and many others will have a simple form. Assume, for instance, the homogeneous boundary conditions along plate edges. Thus $w$ or $Q$ and $w_{,n}$ or $M_n$ are equal to zero along the boundary of the primary plate. Since, in addition, the generalized tractions and/or displacements vanish along the edges of the adjoint plate, then Eq. (2.39) is simplified to the form

$$\delta G = \int (w^a + \psi_{,p}) \delta p \, dA + \int (\psi + p w^a - M_s \varkappa_s^a - 2M_{ns} \varkappa_{ns}^a + Q w^a_{,n} + Q^a w_{,n} + M_n^a \varkappa_n) \delta \varphi_n \, dS. \tag{2.41}$$

## 3. Sensitivity analysis for interface shape variation

Consider now a two-phase elastic plate contained in a domain $A$ and bounded by the boundary $S$, Fig. 4. Assume the plate to be composed of two materials occupying the subdomains $A_1$ and $A_2$ and separated by the interface $\Gamma$, that is $A = A_1 \cup A_2$. The interface

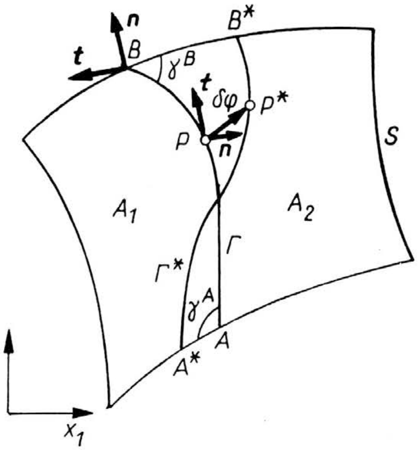


FIG. 4. Two-phase plate with interface.

can separate either domains of different material properties or domains of different thickness in a plate. Moreover, it is assumed that the interface $\Gamma$ does not contain any singular points. Regardless of the finite jump of material properties or plate ridigity on $\Gamma$, the deflection field within the plate is continuous and smooth and the generalized internal tractions on $\Gamma$ are continuous. Thus this assumption yields

$$\begin{aligned} [\![w]\!] = 0, \quad & [\![w_{,n}]\!] = 0, \\ [\![M_n]\!] = 0, \quad & [\![Q]\!] = 0, \quad \text{on } \Gamma, \end{aligned} \tag{3.1}$$

where $[\![\ ]\!]$ denotes the jump of the enclosed quantity on $\Gamma$ calculated as a difference of the respective values in the domains $A_1$ and $A_2$. Furthermore the continuity and smoothness of the deflection field $w$ assure the continuity of tangential curvature and torsion along $\Gamma$, that is

$$(3.2) \qquad [\![\varkappa_s]\!] = 0, \quad [\![\varkappa_{ns}]\!] = 0 \quad \text{on } \Gamma.$$

The case when the internal tractions suffer discontinuity across $\Gamma$ will be treated in a separate paper.

Consider now an infinitesimal variation of plate configuration prescribed by a continuous and differentiable transformation vector field $\delta\boldsymbol{\varphi}(\mathbf{x})$. The domains $A_1$ and $A_2$ are then transformed into domains $A_1^*$ and $A_2^*$, with the interface transformed into $\Gamma^*$. We assume that when the interface $\Gamma$ does not penetrate the external boundary $S$ then the function $\delta\boldsymbol{\varphi}(\mathbf{x})$ vanishes on $S$ so that the external shape of the plate is not changed. On the other hand, when the interface penetrates the external boundary at points $A$ and $B$, see Fig. 4, then $S$ can undergo the tangential transformation only, so that $\delta\boldsymbol{\varphi}\cdot\mathbf{n} = 0$ on $S$.

Similarly as in the previous Section, we introduce now an arbitrary functional $G$ expressed in the form

$$(3.3) \qquad G = G_1 + G_2 = \int_{A_1} \psi_1(\mathbf{M}, \boldsymbol{\varkappa}, p, w)\,dA_1 + \int_{A_2} \psi_2(\mathbf{M}, \boldsymbol{\varkappa}, p, w)\,dA_2$$
$$= \int_{A_1\cup A_2} \psi(\mathbf{M}, \boldsymbol{\varkappa}, p, w)\,dA,$$

and derive its first variation with respect to the shape variation of the interface $\Gamma$. To do this, we shall utilize the results obtained in Section 2. First of all, introduce the adjoint plate that is defined by Eqs. (2.16)–(2.18). It is obvious that the adjoint solutions $w^a$, $\boldsymbol{\varkappa}^a$, $\mathbf{M}^a$ satisfy the continuity conditions along $\Gamma$, expressed in a form similar to Eqs. (3.1) and (3.2). Next, to write the expression for the first variation of Eq. (3.3), we apply Eq. (2.39) to both subdomains $A_1$ and $A_2$ of the plate domain $A$. Keeping in mind the conditions (3.1), (3.2) and those similar for adjoint fields, we then obtain

$$(3.4) \qquad \delta G = \delta G_1 + \delta G_2 = \int (w^a + \psi_{,p})\,\delta p\,dA + \int ([\![\psi]\!] + [\![p]\!]w^a - [\![M_s]\!]\varkappa_s^a$$
$$-2[\![M_{ns}]\!]\varkappa_{ns}^a + M_n^a[\![\varkappa_n]\!])\,\delta\varphi_n\,d\Gamma + \int (M_{n,s}w^a_{,n} - Q_{,s}w^a - M_n^a w_{,ns} + Q^a w_{,s})\,\delta\varphi_s\,dS$$
$$+\int [\delta Q w^a - \delta M_n w^a_{,n} - Q^a\delta w + M_n^a\,\delta(w_{,n})]\,dS + \int [M_{ns}^a\,\delta w - \delta M_{ns} w^a$$
$$+(Qw^a - M_n w^a_{,n} - M_{ns}w^a_{,s} - M_{ns}^a w_{,s})\,\delta\varphi_s]_{,s}\,dS + \{[\![M_{ns}^a]\!]\,\delta w - [\![\delta M_{ns}]\!]\,w^a$$
$$-([\![M_{ns}]\!]w^a_{,s} + [\![M_{ns}^a]\!]w_{,s})\,\delta\varphi_s - ([\![M_{ns}]\!]w^a_{,n} + [\![M_{ns}^a]\!]w_{,n})\,\delta\varphi_n\}\Big|_{B\Gamma}^{A\Gamma},$$

where $\{\ldots\}\big|_{B\Gamma}^{A\Gamma}$ denotes the difference of enclosed quantities at points $B$ and $A$ calculated along $\Gamma$.

When the interface $\Gamma$ is a closed curve within the plate domain, then in Eq. (3.4) all integrals along the external boundary $S$ and the last term in square brackets vanish. On the other hand, when $\Gamma$ penetrates $S$, we assume that the points $A$ and $B$ are placed on the smooth parts $S_a$ and $S_b$ of the boundary $S$, so that $S = S_a\cup S_b\cup S_0$, and the tangential transformation $\delta\varphi_s$ of the external boundary influences these parts only. Moreover, we assume that on the remaining boundary portion $S_0$ there exist no such singular points at which concentrated forces are specified if this portion is unsupported. In view of such assumptions, and noting the following identities which hold at points $A$ and $B$

$$(w_{,s}\,\delta\varphi_s + w_{,n}\,\delta\varphi_n)^{\Gamma} = (w_{,s}\,\delta\varphi_s)^S, \tag{3.5}$$

where the symbols $(\ )^{\Gamma}$ and $(\ )^S$ denote that the enclosed quantities are calculated along $\Gamma$ and $S$, respectively, Eq. (3.4) is reduced to its final form

$$\delta G = \int (w^a + \psi_{,p})\,\delta p\,dA + \int_{\Gamma} ([\![\psi]\!] + [\![p]\!]\,w^a - [\![\mathbf{M}\cdot\boldsymbol{\varkappa}^a]\!] + M_n[\![\boldsymbol{\varkappa}_n^a]\!] + M_n^a[\![\varkappa_n]\!])\,\delta\varphi_n\,d\Gamma + \int_{S_a\cup S_b} (M_{n,s}\,w^a_{,n} - Q_{,s}\,w^a - M_n^a\,w_{,ns} + Q^a w_{,s})\,\delta\varphi_s\,dS + \int_{S_a\cup S_b} [\delta Q\,w^a - \delta M_n\,w^a_{,n} - Q^a\delta w + M_n^a\delta(w_{,n})]\,dS. \tag{3.6}$$

When the primary plate is subjected to the set of homogeneous boundary conditions, then Eq. (3.6) is much simplified since the last two integrals on the right-hand side vanish.

The assumption that the points $A$ and $B$ move along the boundary $S$ during shape transformation of the interface $\Gamma$ yields, in addition, the following relationship between normal variation of $\Gamma$ and tangential variation of $S$ at $A$ and $B$:

$$\delta\varphi_s\Big|^S = -\frac{1}{\sin\gamma}\,\delta\varphi_n\Big|^{\Gamma}, \tag{3.7}$$

where $\gamma$ denotes the angle between $S$ and $\Gamma$, see Fig. 4. The change of this angle during the transformation process is expressed as follows:

$$\delta\gamma = \left(K^S\,\frac{1-2\sin^2\gamma}{\sin\gamma} - K^{\Gamma}\right)\delta\varphi_n - \delta\varphi_{n,s}, \tag{3.8}$$

where $\delta\varphi_n$ denotes the normal component of the shape variation of $\Gamma$ at $A$ or $B$ and $K^S$, $K^{\Gamma}$ are the curvatures of $S$ and $\Gamma$ at $A$ or $B$, respectively. If we assume no change of angle $\gamma$ during the transformation process, then the following constraint has to be set down on the rate of $\delta\varphi_n$ at $A$ or $B$:

$$\delta\varphi_{n,s} = \left(K^S\,\frac{1-2\sin^2\gamma}{\sin\gamma} - K^{\Gamma}\right)\delta\varphi_n \quad \text{at} \quad A \text{ or } B. \tag{3.9}$$

Up to now, we consider the problem of variation of an arbitrary functional $G$ defined over the whole domain of a primary plate. Thus the functional $G$ has been treated as the global structural response of a plate. However, the same approach can be applied to a closely related class of problems associated with variation of local generalized stress and strain components or deflection at a typical point $\mathbf{x}_0$ of a plate domain, or associated with variation of any quantity $f(\mathbf{x}_0)$ depending on state fields at $\mathbf{x}_0$. Using the well-known property of the Dirac delta function $\delta(\mathbf{x}-\mathbf{x}_0)$, any local quantity $f[\mathbf{M}(\mathbf{x}_0), \boldsymbol{\varkappa}(\mathbf{x}_0), w(\mathbf{x}_0)]$ can be converted to the global one by the following relationship:

$$f(\mathbf{x}_0) = G = \int f[\mathbf{M}(\mathbf{x}), \boldsymbol{\varkappa}(\mathbf{x}), w(\mathbf{x})]\,\delta(\mathbf{x}-\mathbf{x}_0)\,dA. \tag{3.10}$$

Comparing Eq. (3.10) with Eqs. (2.14) or (3.3), it can be easily noted that

$$\psi(\mathbf{M}, \boldsymbol{\varkappa}, w) = f[\mathbf{M}(\mathbf{x}), \boldsymbol{\varkappa}(\mathbf{x}), w(\mathbf{x})]\,\delta(\mathbf{x}-\mathbf{x}_0) \tag{3.11}$$

and then the discussed approach can be used in order to determine $\delta f(\mathbf{x}_0)$, with proper qualification of the adjoint plate.

## 4. Variation of potential and complementary energies

Consider now a particular case when the functional $G$ coincides with potential or complementary energies of a plate and derive their first variations associated with the shape variation of external or internal boundaries. The analysis of such a case is simpler than in the general case since the solutions of the adjoint plate can be expressed in terms of solutions of the primary plate.

Assuming the homogeneous boundary conditions along plate edges, consider first the potential plate energy that equals

$$\Pi_u = \int [U(\boldsymbol{\varkappa}) - pw]\,dA, \tag{4.1}$$

where $U$ denotes the specific strain energy per unit area of a plate. Comparing Eq. (4.1) with (2.14) or (3.3) we easily observe that

$$\psi = U - pw \tag{4.2}$$

and then, according to the relations (2.16) and (2.17) the adjoint plate is loaded by a transverse load $p^a$

$$p^a = \psi_{,w} = -p \tag{4.3}$$

with the imposed field of initial stresses

$$\mathbf{M}^{ai} = \psi_{,\varkappa} = \mathbf{M} \tag{4.4}$$

and vanishing generalized tractions or displacements on $S$. Moreover, we should note that

$$\psi_{,p} = -w. \tag{4.5}$$

Thus the state fields within the adjoint plate are

$$w^a = 0, \quad \boldsymbol{\varkappa}^a = 0, \quad \mathbf{M}^a = -\mathbf{M} \tag{4.6}$$

and the first variation of $\Pi_u$ can be obtained from the general expressions (2.41) and (3.6). When the external boundary is subjected to shape variation, then from Eq. (2.41) we obtain

$$\delta\Pi_u = \int (U - pw - M_n\varkappa_n - Qw_{,n})\,\delta\varphi_n\,dS - \int \overline{\delta p}\,w\,dA, \tag{4.7}$$

whereas for interface shape variation Eq. (3.6) yields

$$\delta\Pi_u = \int ([\![U]\!] - [\![p]\!]w - M_n[\![\varkappa_n]\!])\,\delta\varphi_n\,d\Gamma - \int \overline{\delta p}\,w\,dA. \tag{4.8}$$

Assume now that the functional $G$ coincides with the complementary energy of a plate, that is

(4.9) $$\Pi_\sigma = \int W(\mathbf{M})\, dA,$$

where $W$ denotes the specific stress energy per unit area. Comparing Eq. (4.9) with Eqs. (2.14) or (3.3), we have $\psi = W$ and the adjoint plate is subjected to the imposed field of initial strains

(4.10) $$\boldsymbol{\varkappa}^{ai} = \psi_{,M} = \boldsymbol{\varkappa}$$

with vanishing external loading and homogeneous boundary conditions along plate edges. Thus the state fields within the adjoint plate are

(4.11) $$w^a = w, \quad \boldsymbol{\varkappa}^a = \boldsymbol{\varkappa}, \quad \mathbf{M}^a = 0$$

and the first variation of $\Pi_\sigma$, with respect to the shape variation of the external boundary equals

(4.12) $$\delta\Pi_\sigma = \int (W + pw - M_s \varkappa_s - 2M_{ns}\varkappa_{ns} + Qw_{,n})\, \delta\varphi_n\, dS + \int w\bar{\delta}p\, dA.$$

When the interface $\Gamma$ undergoes shape variation, then in view of Eq. (3.6) we have

(4.13) $$\delta\Pi_\sigma = \int ([\![W]\!] + [\![p]\!]w - [\![M_s]\!]\varkappa_s - 2[\![M_{ns}]\!]\varkappa_{ns})\, \delta\varphi_n\, d\Gamma + \int w\bar{\delta}p\, dA.$$

Noting that $U + W = \mathbf{M} \cdot \boldsymbol{\varkappa}$, it is easy to prove that $\delta\Pi_\sigma = -\delta\Pi_u$.

## 5. Optimal shape design for specified displacement and stress constraints

The typical optimal design problem involves minimization of the cost function

(5.1) $$C = \int c\, dA \to \min_\varphi,$$

where $c$ is a specific material cost subject to the global constraint imposed on generalized stresses, strains or deflection, i.e.,

(5.2) $$G = \int \psi(\mathbf{M}, \boldsymbol{\varkappa}, p, w)\, dA - G_0 \leqslant 0$$

or constraint on local or maximum values of stresses, strains or deflection, and other geometrical constraints which will not be considered here. Note that the constraint imposed on local values of stress, strain or deflection can be easily converted to the global form (5.2) by using Eq. (3.11) Similarly, any constraint imposed on maximum values of the stress or strain component or deflection can be also expressed in global form. The maximum local deflection, for instance, can be represented by the functional

(5.3) $$G = \left[\int |w|^p dA\right]^{1/p}$$

since for $p \to \infty$, $w \to w_{\max}$. The maximum local stress component or generalized stress intensity can be obtained by considering the functional

(5.4) $$G = \left[\int \psi^p(\mathbf{M})\, dA\right]^{1/p},$$

where $p$ is even and $\psi$ is assumed to be a homogeneous function of generalized stresses of order one. For $p \to \infty$, $G \to \sup\psi$, that is the functional tends to the maximum value of its integrand. Another approach to impose a constraint on maximum stress is to apply the penalty approach. Namely, introducing the acceptable stress intensity level $\psi_0$, we can consider the functional

$$G = \int \left| \frac{\psi(\mathbf{M})}{\psi_0} \right|^p dA. \tag{5.5}$$

For $p \to \infty$ the integrand $(\psi/\psi_0)^p$ of Eq. (5.5) tends to zero for $\psi/\psi_0 < 1$ and tends to infinity for $\psi/\psi_0 > 1$. This provides a proper penalty functional which for large $p$ takes very small values when $\psi < \psi_0$ and very large ones when $\psi > \psi_0$.

Introducing the functional

$$C' = C + \lambda(G - G_0 + \alpha^2), \tag{5.6}$$

where $\lambda$ denotes the Lagrange multiplier and $\alpha$ is a slack function, its stationarity condition yields the optimality condition

$$\delta C = -\lambda \delta G \tag{5.7}$$

with the switching and constraint conditions of the form

$$\lambda\alpha = 0, \quad \delta\lambda(G - G_0 + \alpha^2) = 0. \tag{5.8}$$

The variation of the constraint (5.2) is expressed here by Eqs. (2.39) or (3.6), whereas the variation of structural cost equals

$$\delta C = c \int \delta\varphi_n dS \tag{5.9}$$

for the case of external boundary variation or is expressed by

$$\delta C = [\![c]\!] \int \delta\varphi_n d\Gamma \tag{5.10}$$

for interface shape variation.

An alternative formulation of the optimal design problem would require the minimization (or maximization) of $G$ with the upper bound set on the structural cost, thus

$$\min G \quad \text{subject to} \quad C - C_0 \leqslant 0. \tag{5.11}$$

Introducing now the functional

$$G' = G + \lambda(C - C_0 + \beta^2), \tag{5.12}$$

where $\beta$ is a slack function, we can obtain the following set of conditions:

$$\delta G = -\lambda \delta C, \quad \lambda\beta = 0, \quad \delta\lambda(C - C_0 + \beta^2) = 0 \tag{5.13}$$

which are equivalent to Eqs. (5.7), (5.8).

## 6. Examples

In this Section, let us consider three simple examples which should illustrate the analysis presented in the previous Sections.

EXAMPLE 1. Consider a circular plate of radius $r_e$ with a central hole of radius $r_i$, simply supported on the outer edge. The plate is loaded uniformly by bending moments $M_e$ and $M_i$, Fig. 5a. Consider the mean compliance design for which both radii $r_e$ and $r_i$

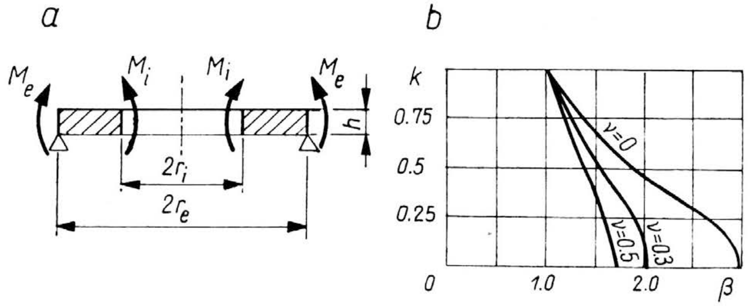


FIG. 5. Circular plate with a hole; (a) boundary conditions, (b) ratio $r_i/r_e$ versus ratio $M_i/M_e$.

are to be determined so that the complementary energy $\Pi_\sigma$ attains a minimum. The plate is subject to the condition of constant structural cost

$$r_e^2 - r_i^2 = \frac{C_0}{c\pi} = C. \tag{6.1}$$

The optimality conditions in this case follows from (5.13) where the variation of the objective functional $G = \Pi_\sigma$ is expressed by Eq. (4.12) whereas the variation of structural cost, in view of Eq. (5.9) equals

$$\delta C = 2r_e\,\delta r_e - 2r_i\,\delta r_i. \tag{6.2}$$

Thus, in view of Eq. (4.12), (5.13) and (6.2), the optimality conditions can be expressed as follows:

$$2\pi\left(\frac{1}{2}M_r\varkappa_r - \frac{1}{2}M_s\varkappa_s - \frac{1}{r}M_r w_{,r}\right)\Bigg|_{r=r_e} r_e\delta r_e - 2\pi\left(\frac{1}{2}M_r\varkappa_r - \frac{1}{2}M_s\varkappa_s - \frac{1}{r}M_r w_{,r}\right)\Bigg|_{r=r_i} r_i\delta r_i = -2\lambda r_e\,\delta r_e + 2\lambda r_i\,\delta r_i,$$
$$r_e^2 - r_i^2 = C. \tag{6.3}$$

Expressing the radial and circumferential bending moments and curvatures in terms of the deflection field $w$, Eqs. (6.3) yield the following set of optimality equations:

$$\left(w_{,rr} + \frac{1}{r}w_{,r}\right)^2 - 2(1-\nu)\frac{1}{r^2}w_{,r}^2 = -\frac{4\lambda}{D} \quad \text{for} \quad r = r_e \quad \text{and} \quad r = r_i,$$
$$r_e^2 - r_i^2 = C. \tag{6.4}$$

The deflection field within the plate is expressed as follows:

$$w = -C_2 \ln\frac{r}{a} + \frac{1}{4}C_1(r_e^2 - r^2), \tag{6.5}$$

where

$$C_1 = \frac{2(r_e^2 M_e - r_i^2 M_i)}{(1+\nu)D(r_e^2 - r_i^2)}, \qquad C_2 = \frac{r_e^2 r_i^2 (M_e - M_i)}{(1-\nu)D(r_e^2 - r_i^2)}. \tag{6.6}$$

Introducing now the nondimensional quantities

$$\beta = \frac{M_i}{M_e}, \quad k = \frac{r_i}{r_e} \tag{6.7}$$

and using Eq. (6.5) the optimality conditions (6.4) result in the following optimality equation with respect to $k$:

$$\begin{aligned}[(1-\nu)(1-\beta k^2)+(1+\nu)(1-\beta)k^2](1-k^2)-2(1-\beta k^2)k^2 \\ = [(1-\nu)(1-\beta k^2)+(1+\nu)(1-\beta)](1-k^2)\beta-2(1-\beta)(1-\beta k^2).\end{aligned} \tag{6.8}$$

Solution of this equation yields the optimal value of the ratio of $r_i$ and $r_e$, namely,

$$k_{\text{opt}} = \sqrt{\frac{\frac{3-\nu}{1+\nu}-\beta}{\frac{3-\nu}{1+\nu}-1}} \tag{6.9}$$

valid for

$$1 \leqslant \beta \leqslant \frac{3-\nu}{1+\nu}. \tag{6.10}$$

Figure 5b presents the variation of the ratio $r_i/r_e$ in function of $M_i/M_e$ for different values of $\nu$. It is seen that for $\beta < 1$ the optimal solution corresponds to the plate with vanishing hole, whereas for $\beta$ varying within the range corresponding to the condition (6.10) the plate is gradually transformed from a thin ring into a circular plate without the hole.

The relative compliance of the plate is expressed by

$$\frac{(1-\nu^2)D\Pi_\sigma}{M_e^2 C} = \frac{(1-\nu)(1-\beta k^2)^2+(1+\nu)(1-\beta)^2k^2}{(1-k^2)^2}. \tag{6.11}$$

Figure 6 shows the variation of the relative compliance as a function of $k$ for $\beta = 2$ and $\nu = 0.3$. It is easy to see that the value of $k$ satisfying the optimality condition (6.9) corresponds to a global minimum of the mean plate compliance.

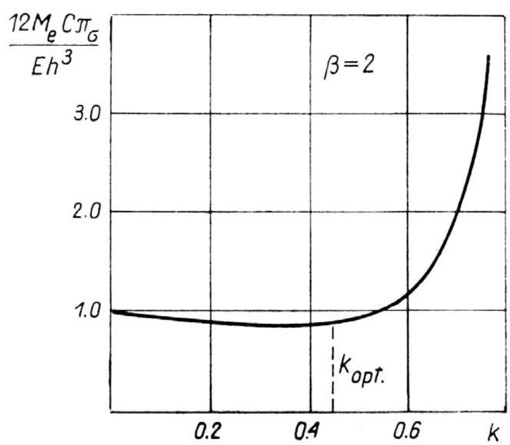


FIG. 6. Variation of relative compliance of a circular plate versus ratio of $r_i/r_e$.

EXAMPLE 2. As the second example, consider a rectangular plate of dimensions $a \times b$, simply supported on its boundary, Fig. 7, and loaded by a transverse load of the form

$$p = p_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}. \tag{6.12}$$

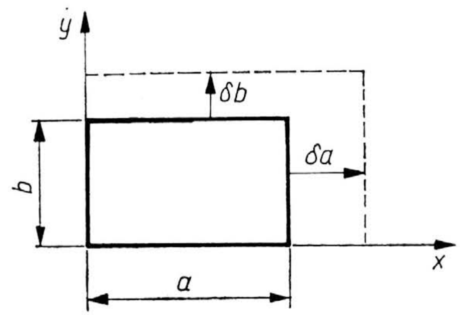


FIG. 7. Shape variation of rectangular plate.

Assume now the material of the plate to be linear orthotropic, with principal axes of orthotropy parallel to the plate boundaries, so that the equilibrium equation for the plate can be written in the form

$$D_x w_{,xxxx} + 2\sqrt{D_x D_y}\, w_{,xxyy} + D_y w_{,yyyy} = p, \tag{6.13}$$

where $D_x$ and $D_y$ denote the bending stiffness moduli of the plate with respect to the principal directions of orthotropy. The deflection field that satisfies Eq. (6.13) together with the proper set of boundary conditions can be written in the form

$$w = \frac{p_0}{\pi^4 k D_y \left(\frac{1}{a^2} + \frac{1}{\sqrt{k} b^2}\right)^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}, \tag{6.14}$$

where $k$ denotes the ratio of the bending stiffness moduli $D_x/D_y$. Assuming the constant area of a plate, we are looking for its optimal dimensions $a$ and $b$ for which the global measure of the deflection field is minimized, that is

$$G = \int |w|^n dA \to \min \quad \text{for} \quad ab = C_0 = \text{const}. \tag{6.15}$$

Thus this example can be related to the general theory in Sect. 2, by introducing the adjoint plate of the same shape as the primary one, simply supported on its edges and subjected to the transverse load of the form

$$p^a = \frac{\partial[|w|^n]}{\partial w} = \operatorname{sgn}(w) n |w|^{n-1}, \tag{6.16}$$

where

$$\operatorname{sgn}(w) = \begin{cases} -1 & w < 0, \\ 0 \quad \text{for} & w = 0, \\ 1 & w > 0. \end{cases} \tag{6.17}$$

Assume now the the plate boundaries $x = 0$ and $y = 0$ are fixed and the two remaining boundaries are allowed to translate in $x$ and $y$ directions, respectively, see Fig. 7. Thus the transformation field within the plate domain can be assumed in the form $\delta\varphi_x = \frac{x}{a}\delta a$, $\delta\varphi_y = \frac{y}{b}\delta b$ and, in view of Eq. (2.39), the general optimality conditions (5.13) for the functional (6.15) can be expressed in the form

$$
\begin{gathered}
-\int_0^b\int_0^a p_{,x} w^a \frac{x}{a}\,dx\,dy + \int_0^b (Qw^a_{,x} - M_y\varkappa^a_y - 2M_{xy}\varkappa^a_{xy} + Q^a w_{,x})\,dy|_{x=a} = -\lambda b,\\
-\int_0^b\int_0^a p_{,y} w^a \frac{y}{b}\,dx\,dy - \int_0^b (Qw^a_{,y} - M_x\varkappa^a_x - 2M_{xy}\varkappa^a_{xy} + Q^a w_{,y})\,dx|_{y=b} = -\lambda a,\\
ab - C_0 = 0,
\end{gathered}
\tag{6.18}
$$

where the primary fields are generated by the deflection field (6.14) and the adjoint fields are the generalized stresses, strains and deflection of adjoint plate subjected to transverse load (6.16).

For $n = 2$, the adjoint deflection field is expressed in the form

$$
w^a = \frac{2w}{\pi^4 k D_y \left(\frac{1}{a^2} + \frac{1}{\sqrt{k}b^2}\right)^2}
\tag{6.19}
$$

and two first equations of the set (6.18) yield the optimal ratio of the plate dimensions $a$ and $b$; that equals

$$
\frac{a}{b} = \sqrt[4]{k}\,.
\tag{6.20}
$$

EXAMPLE 3. The third example is related to the optimal design of interfaces within plate domain, treated previously in [2, 13]. Consider now a circular sandwich plate which is simply supported at the outer edge and uniformly loaded by the lateral pressure $p$. The plate is made of a linearly elastic material with sheet thickness $t_k$ constant over annular subdomains defined by the radii $r_k$ $(k = 1, 2, 3)$, Fig. 8. For the prescribed sheet thicknesses

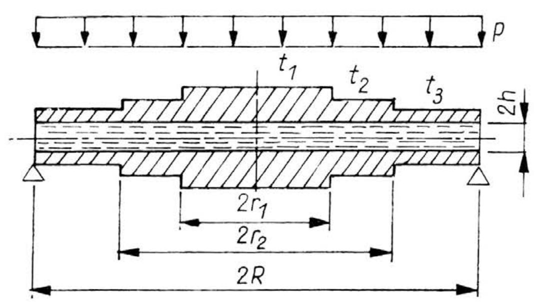


FIG. 8. Circular sandwich plate uniformly loaded by lateral pressure.

$t_k$, the optimization problem is reduced to determining the radii $r_1$ and $r_2$, for which the global plate compliance attains minimum, within the class of plates of constant structural cost that is proportional to

$$C = 2\pi \sum_{k=1}^{3} t_k(r_k^2 - r_{k-1}^2) = C_0 \tag{6.21}$$

with $r_0 = 0$ and $r_3 = R$. Assume as the measure of global compliance the complementary energy of the plate, equal to

$$\Pi_\sigma = \frac{\pi}{Eh^2} \sum_{k=1}^{3} \frac{1}{t_k} \int_{r_{k-1}}^{r_k} (M_r^2 - 2\nu M_r M_s + M_s^2) r\, dr, \tag{6.22}$$

where $M_r$, $M_s$ are the radial and circumferential bending moments, $2h$ is the core thickness, and $E$, $\nu$ are elastic constants. Thus, applying the stationarity conditions (5.13) and Eq. (3.6) and using the continuity conditions (3.1)–(3.2) for $r = r_k$, we obtain

$$\begin{gathered} \left.\frac{M_r^2 - M_s^2}{t_k}\right|_{r_k^-} - \left.\frac{M_r^2 - M_s^2}{t_{k+1}}\right|_{r_k^+} = 4\lambda Eh^2(t_{k+1} - t_k), \\ \sum_{k=1}^{3} t_k(r_k^2 - r_{k-1}^2) = qR^2 t_0, \qquad k = 1, 2, \end{gathered} \tag{6.23}$$

where $r_k^-$ and $r_k^+$ denote the values of $M_r$ and $M_s$ on the respective sides of the interface $r = r_k$, whereas $q > 1$ and $t_0$ are prescribed quantities. The quantity $q$ can be termed the relative cost of the design. The bending moments $M_r$, $M_s$ can now be expressed in the form (cf. [2])

$$\begin{gathered} M_r = A_k + \frac{B_k}{r^2} - \frac{pr^2}{16}(3+\nu), \qquad M_s = A_k - \frac{B_k}{r^2} - \frac{pr^2}{16}(1+3\nu), \\ r_{k-1} \leqslant r \leqslant r_k, \qquad k = 1, 2, 3, \end{gathered} \tag{6.24}$$

where the constants $A_k$, $B_k$ have to satisfy the conditions

$$\begin{gathered} B_1 = 0, \qquad A_3 + \frac{B_3}{R^2} - \frac{pR^2}{16}(3+\nu) = 0, \\ A_{k+1} + \frac{B_{k+1}}{r_k^2} = A_k + \frac{B_k}{r_k^2}, \qquad k = 1, 2, \\ 2t_k A_{k+1} = A_k[(1+\nu)t_k + (1-\nu)t_{k+1}] + \left[\frac{B_k}{r_k^2}(1+\nu) + \frac{pr_k^2}{16}(1-\nu^2)\right](t_k - t_{k+1}). \end{gathered} \tag{6.25}$$

Using the form (6.24) in Eq. (6.23) the optimality conditions take the form

$$\begin{gathered} (1-\nu^2)\left[A_k + \frac{B_k}{r_k^2} - \frac{pr_k^2}{16}(3+\nu)\right]^2 + \frac{t_{k+1}}{t_k}\left[A_k(1-\nu) - \frac{B_k}{r_k^2}(1+\nu) \right. \\ \left. - \frac{pr_k^2}{16}(1-\nu^2)\right]^2 = 4\lambda Eh^2 t_k t_{k+1}, \qquad k = 1, 2, \\ \sum_{k=1}^{3} t_k(r_k^2 - r_{k-1}^2) = qR^2 t_0. \end{gathered} \tag{6.26}$$

Equations (6.25) and (6.26) constitute a set of equations with 9 unknowns $A_k$, $B_k$ ($k = 1, 2, 3$), $r_1$, $r_2$ and $\lambda$, from which the optimal values of $r_1$ and $r_2$ can be determined.

In order to simplify the solutions of Eqs. (6.25), (6.26) assume that the plate thickness varies according to the relationship

$$t_k = (4-k)t_0 \tag{6.27}$$

which implies that the structure is made up of plate elements of given thickness $t_0$. The solution for the case $\nu = 0$ is illustrated in Fig. 9a which shows the dependence of the

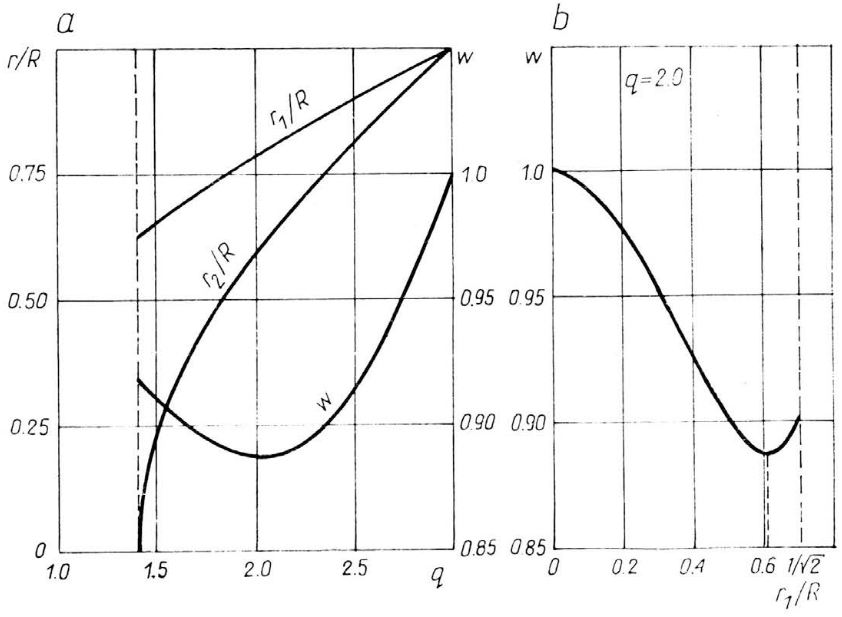


FIG. 9. Optimal solutions for a circular sandwich plate; (a) optimal radii versus relative cost of design, (b) relative compliance of a plate versus radius of first subdomain.

optimal radii $r_1$ and $r_2$, as well as the relative compliance (i.e. the ratio of the mean compliance of the optimal plate to the mean compliance of a plate of constant thickness and the same cost) on the relative cost $q$. It is seen that for decreasing relative cost $q$, the optimal solution corresponds to the vanishing interface $r = r_1$, whereas for $q$ tending to 3, the optimal plate is gradually transformed into the plate of uniform thickness. Figure 9b shows the variation of relative compliance for $q = 2.0$ as a function of $r_1$ defining the interface position between the first and the second subdomains. It is easy to see that the values of $r_1$ and $r_2$ satisfying the optimality conditions (6.26) correspond to a global minimum of the mean plate compliance.

## 7. Concluding remarks

The present paper supplements the results of previous works [3–5, 9, 10] and provides a systematic variational approach to sensitivity analysis and optimal design for plates with varying external boundaries and interfaces. The analysis is limited to geometrical linear and physical nonlinear plates for which the concept of adjoint plate provides an effective tool in generating the first variation of any functional prescribed over plate domain. The extension to the geometrical nonlinear plate can also be obtained by following the present analysis and that presented in [10].

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