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Integrity conditions for elastic-plastic damaged solids subjected to cyclic loading

B. DRUYANOV and I. ROMAN

Graduate School of Applied Science, The Hebrew University of Jerusalem, Givat Ram Campus 91904 Jerusalem, Israel

INTEGRITY CONDITIONS for elastic-plastic, isotropically damaged solids with isotropic and kinematic strain hardening as subjected to cyclic loading, are in question. It is assumed that the damage process is coupled with the process of plastic deformation. The shakedown conditions are assumed to be satisfied. A new sufficient condition for shakedown accounting for a mixed isotropic and kinematic hardening is developed. The problem of evaluating limit yield-condition arguments is reduced to a min-max problem. In the case of plain strain, the problem is equivalent to a hyperbolic equation in partial derivatives of the second order. A method for computing the purely elastic damaged response of the solid to the prescribed loading program is proposed. The limit yield condition with specified arguments makes it possible to obtain upper and lower estimates for the local actual limit values of the damage parameter admitted by the yield condition for the given loading program. The estimates lead to necessary and sufficient conditions of integrity. The proposed method is illustrated by an example.

1. Introduction

UNDER CERTAIN CONDITIONS, irreversible changes in the material of elastic-plastic damaged solids subjected to cyclic loading vanish after a period of adaptation, and the solid experiences only elastic deformation starting from some time on. One says in this case that the solid adapted (shook down) to the prescribed loading program; in other words, the deformation process reached a stationary (post-adaptation) stage. If the damage and plastic deformation processes are coupled, then the values of damage and internal parameters at this stage do not change. These values and the corresponding yield surface will be referred to as the limiting ones.

The shakedown theory provides us with the means to predict directly, i.e. without a detailed computation of the deformation path, whether the solid will adapt itself to the prescribed cyclic loading program or not. This gives us a chance to establish the estimates of the damage parameters limiting value. Reviews of the modern achievements in the shakedown theory are available in [1–4]. Presently the question of extension of the shakedown theory to damaged elastic-plastic solids is topical.

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HACHEMI and WEICHERT extended the Melan theorem to elastic-plastic isotropically damaged solids with unlimited [5] and limited [6] linear kinematic strain-hardening. SIEMASZKO [7] developed a method of step-by-step inadaptation analysis for elastic-plastic structures subjected to repeated loading, which accounts for nonlinear isotropic strain hardening, developing of damage, and nonlinear geometrical effects. POLIZZOTTO, BORINO and FUSCHI [8] included the damage variable into a set of internal variables, and developed an elastic-plastic damaged material model with associated constitutive relations and non-linear elasticity. Employing the D-stability principle introduced by them, they extended the static Melan shakedown theorem to this model. The theorem was also extended to the elastic damaged material model as well.

DRUYANOV and ROMAN [9] extended the Melan theorem to the model of damaged elastic-plastic solids with isotropic and isotropic-like strain hardening.

All known extensions of the Melan theorem to elastic plastic solids with isotropic damage can be formulated as follows: if there exists a time-independent field of effective residual stress tensor $\hat{\bar{\rho}}(\mathbf{x})$, which satisfies the yield inequality $\Phi(\bar{\sigma}^{E}(\mathbf{x},\mathbf{t}),+\hat{\bar{\rho}}(\mathbf{x}),\chi(\mathbf{x},\mathbf{t}))<\mathbf{0}$ from some time on, then the total plastic dissipation and damage parameter are bounded. Here $\Phi(\bar{\sigma},\chi)$ is the yield function, $\bar{\sigma}$ is the effective stress tensor, χ denotes the strain hardening parameter, and $\bar{\sigma}^{E}$ is the effective stress tensor representing the current, purely elastic response of the solid to the prescribed loading program.

Obviously, even if the conditions of the extended Melan theorem are satisfied, a damaged elastic plastic solid can fail due to accumulation of damage because the conditions of local integrity may be violated. For example, in the case of isotropic damage, the limit value of the damage parameter may exceed its critical value, and the solid will collapse before the plastic deformation process ceases.

To establish the conditions of integrity, FENG and YU [10, 11] supposed that the state of damage is described by a scalar quantity connected with the plastic strain tensor. They introduced a damage factor as a local average of this quantity, and assumed that the safety of structures subjected to cyclic loading is guaranteed, if the damage factor is less than its critical value. Assuming a piece-wise linear yield condition, they reduced the computation of an upper bound for the damage factor to a problem of mathematical programming. Besides, a method of obtaining a lower bound was developed.

HACHEMI and WEICHERT [6, 12], WEICHERT and HACHEMI [13] employed the model of elastic plastic damaged material by Lemaitre [14] and the Generalized Standard Material Model approach (HALPHEN and NGUYEN [15]) to derive upper bounds for the accumulated plastic strain and damage parameter. As a result, they reduced the problem of determining of the safety factor to a problem of mathematical programming. A numerical method capable of controlling the current value of the damage parameter was also developed.

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A method to find lower estimates of the limit value of isotropic damage parameter was developed in DRUYANOV and ROMAN [16].

Thus, the fulfillment of the shakedown conditions is only a necessary condition for the safety of solids subjected to cyclic loading. To assure their safety, the condition of local integrity at every point of the solids has to be satisfied.

Hence, a problem appears: for the prescribed loading program, to derive the conditions of shakedown and upper and lower estimates to the limit value of damage parameter at every point of the solid, and based on them to set a priori conditions of integrity. Below, a method to solve this problem is proposed.

In the main body of the paper, the method is developed for elastic plastic solids with isotropic damage and isotropic strain hardening. Then the method is extended to solids with both kinematic and isotropic strain hardening.

The method is based on the assumption that the shakedown conditions are fulfilled. Therefore, for the sake of completeness, a novel sufficient shakedown condition accounting for kinematic strain hardening additionally to the isotropic one is proposed in Appendix 1.

A shakedown condition for structures of elastic-perfectly plastic materials with linear kinematic hardening was formulated by MELAN, as early as in 1938 [17]. A mixed linear kinematic-isotropic hardening was considered by MAIER and NOVATI [18]. Other early extensions are available in the book by KÖNIG [19]. STEIN *et al.* [20] showed how the shakedown theorems can be extended to material models with nonlinear kinematic hardening. POLIZZOTTO *et al.* extended the shakedown theory to a material model with dual internal variables and a thermodynamic potential [21]. Static and kinematic approaches to shakedown conditions for the generalized standard material model with limited kinematic/isotropic hardening was considered by NGUYEN and PHAM [22].

The quality of the obtained estimation depends on the degree of strain hardening. The less is the strain hardening, the better is the quality.

An example of application of the developed method is given.

2. Constitutive relations. Extended Melan theorem

The elastic-plastic isotropic damage material models by LEMAITRE [14] and JU [23] are employed as a basis for the formulation of constitutive relations. The formulation is incomplete: only the relations which are employed in the subsequent argumentation are formulated.

The damage and strain hardening are taken as isotropic. Thermal fluxes and inertia forces are neglected. The components of the total strain tensor ($\boldsymbol{\varepsilon}$) are assumed small, so that $\boldsymbol{\varepsilon}$ can be decomposed into plastic ($\boldsymbol{\varepsilon}^e$) and elastic ($\boldsymbol{\varepsilon}^p$) parts: $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^e+\boldsymbol{\varepsilon}^p$.

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The Hooke law is expressed by the equation $\varepsilon^e = \mathbf{C}_0^{-1} : \bar{\sigma}$, $\bar{\sigma} = \sigma/(1-\Delta)$ where Δ is the damage parameter, $\mathbf{C}_0$ is the initial fourth-rank elastic stiffness tensor,$\bar{\sigma}$and σ are the tensors of effective and nominal stresses respectively, and $(\mathbf{C}\!: \varepsilon)_{ij} = \mathbf{C}_{ijkl}\varepsilon_{kl}$. The current (damaged) value of the elastic stiffness tensor is $\mathbf{C}=\mathbf{C}_0\ (1-\Delta)$.

It is assumed that $0\leq \Delta \leq \Delta_c<1$ where Δ_c is the critical value of the damage parameter, i. e. the material preserves its local integrity until $\Delta<\Delta_c$.

The plastic strain rate tensor is supposed to obey to the associated flow rule: $\dot{\varepsilon}^p = \dot{\lambda}\partial\Phi/\partial\bar{\sigma}$, $\dot{\lambda}\Phi(\bar{\sigma},\chi)=0$, $\dot{\lambda} \geq 0$, $\Phi(\bar{\sigma},\chi) \leq 0$ where $\Phi(\bar{\sigma},\chi) \equiv \zeta$ is the yield function, $\dot{\lambda}$ is the plastic multiplier, and χ is the strain-hardening parameter.

The yield function is assumed to be strictly convex in the argument $\bar{\sigma}$, and the inequality $\Phi(\bar{\sigma},\chi)<0$ corresponds to the interior of the yield surface $\Phi(\bar{\sigma},\chi)=0$ in the effective stress space $\bar{\sigma}$. Consequently, if $\Phi(\bar{\sigma},\chi)=0$ and $\Phi(\hat{\bar{\sigma}},\chi)<0$ where $\hat{\bar{\sigma}}$ is an effective virtual stress, then $(\bar{\sigma}-\hat{\bar{\sigma}}) : \dot{\varepsilon}^{\mathrm{p}} > 0$.

Let $\hat{\bar{\sigma}} = \dfrac{\hat{\sigma}}{1-\Delta(\mathbf{x},t)}$ where $\hat{\sigma}$ is the nominal virtual stress, and $\Delta(\mathbf{x},t)$ is the actual value of the damage parameter. Then the last inequality becomes

$$(\sigma - \hat{\sigma}) : \dot{\varepsilon}^{\mathrm{p}} \geq 0. \tag{2.1}$$

The unloading process is assumed purely elastic with the current damage value of the elastic stiffness tensor $\mathbf{C}=\mathbf{C}_0(1-\Delta)$.

The damage process is assumed to be coupled with the process of plastic deformation, i.e. the damage can develop only if the plastic deformation process is in progress.

It is supposed that the hardening is limited, i.e. there exists a material constant χ^* such that $0\leq \chi \leq \chi*$. The constant χ^* corresponds to the state of hardening saturation. See also [6].

There are two concurring growing parameters in elastic-plastic damaged solids: the parameter of isotropic strain hardening χ, and the parameter of softening Δughe first increases the yield surface, the second diminishes it. It is assumed that in the interval $0\leq \chi \leq \chi^*$ the material is stable, i.e. all the subsequent yield surfaces corresponding to increasing values of χ and Δ comprise the previous ones. In other words, this means that in the interval $0\leq \chi \leq \chi^*$the rate of strain hardening surpasses the rate of damage growth.

Thus, if there exists a field of virtual stress $\hat{\sigma}(\mathbf{x},t)$ that satisfies the inequality

$$\Phi\left(\frac{\hat{\sigma}(\mathbf{x},t)}{1-\Delta_0}, \chi_0\right) < 0 \tag{2.2}$$

for any $t \geq t_0$ where $\Delta_0 = \chi_0 = 0$ are the initial values of Δ and χ, then the inequality

$$\Phi\left(\frac{\hat{\sigma}(\mathbf{x},t)}{1-\Delta(\mathbf{x},t)}, \chi(\mathbf{x},t)\right) < 0 \tag{2.3}$$

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is valid for any $t_0 \geq 0$ until $\chi \leq \chi^*$, where $\Delta(\mathbf{x},t)$ and $\chi(\mathbf{x},t)$ are the actual values of Δ and χ. This inequality shows that the stress $\hat{\sigma}(t) = \dfrac{\hat{\sigma}(\mathbf{x},t)}{1-\Delta(\mathbf{x},t)}$ is in the interior of the current yield surface for $t \geq 0$. Consequently, due to the assumption of material stability, inequality (2.1) preserves its format for $t \geq 0$ and $\chi_0 \leq \chi \leq \chi^*$.

Now with inequality (2.1) in hand, it is possible to extend the Melan theorem to the chosen material model [24]. The extended theorem can be formulated as follows.

If there exists a virtual stationary stress field $\hat{\mathbf{r}}(\mathbf{x})$ such that the fictitious virtual decomposition $\hat{\sigma} = \mathbf{s}^E(\mathbf{x},t) + \hat{\mathbf{r}}(\mathbf{x})$ satisfies inequality (2.2) at a time t_o, then the structure under consideration shakes down, i.e. the plastic strain rate tensor tends to zero: $\dot{\varepsilon}^p \to 0$, and the total plastic dissipation is bounded: $W < w^* < \infty$, where W is the total plastic dissipation, and w^* is a number.

This condition is not only sufficient, but also necessary for shakedown.

Here $\mathbf{s}^{\mathrm{E}}(\mathbf{x},t)$ represents the fictitious actual purely elastic response of the structure to the actual value of the load and for the time-independent value of the damage parameter $\Delta_0(\mathbf{x})$ equal to its initial value (the value corresponding to the time t_0), and $\hat{\mathbf{r}}(\mathbf{x})$ is a fictitious virtual time-independent stress.

More specifically, $\mathbf{s}^{\mathrm{E}}(\mathbf{x},t)$ is computed for the initial (given) value of the elastic stiffness tensor $\mathbf{C}=\mathbf{C}_0(1-\Delta_0(\mathbf{x}))$, whereas, in contrast to it, the actual elastic response $\sigma^{\mathrm{E}}(\mathbf{x},t)$ is computed for the current (damaged) value of the elastic stiffness tensor $\mathbf{C}=\mathbf{C}_0(1-\Delta)$ where $\Delta(\mathbf{x},t)$ is the current value of the damage parameter.

As the initial values of the damage and hardening parameters (Δ_0, χ_0) should be given, the computation of $\mathbf{s}^{\mathrm{E}}$ does not cause any principal difficulties.

3. Features of the post-adaptation stage under cyclic loading

In conditions of cyclic loading, the features of the post-adaptation stage provide us with the possibility to obtain directly, i.e. without detailed investigation of the loading path, a relation between limit values of the residual stress tensor and the damage parameter.

If the condition of the extended Melan theorem is satisfied, i.e. if there exists a stationary residual stress tensor $\hat{\rho}(\mathbf{x})$ such that the stress $\hat{\sigma} = \sigma^{\mathrm{E}}(\mathbf{x},t) + \hat{\rho}(\mathbf{x})$ satisfies inequality (2.2) for any $t \geq 0$, then the structure shakes down, i.e. eventually the deformation process reaches the post-adaptation stage.

Time-independent values of the residual stress tensor ρ_s, the damage parameter Δ_s and the hardening parameter χ_s are characteristic for the post-adaptation stage of the deformation process, if it exists. These values and the corresponding yield surface will be called hereafter the limit ones.

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At the post-adaptation stage, the representative actual stress point in the effective stress space $\bar{\sigma}$ reaches the limit yield surface repeatedly, but the stress does not cause plastic deformation and damage, and the limit yield surface does not change. This is possible, if either some parts of the stress path $\bar{\sigma}$ are placed on the yield surface (neutral loading), or the stress path touches it at some isolated points. In particular, this is valid at the time instants t^* corresponding to the beginning of unloading. These time points will be named the departure instants.

At the departure instants the effective stress satisfies the equation of the yield surface, therefore the following relation is valid:

$$\zeta = \zeta^* = \Phi\Big(\frac{\sigma(t^*)}{1-\Delta_s}, \chi_s\Big)\Big) = 0. \tag{3.1}$$

The stress point cannot escape from the yield surface. According to the assumption, $\zeta<0$ for the stress points situated in the interior of the yield surface, and $\zeta=0$ for the points of it. Hence, the departure points are the points of absolute maximum of the yield function $\zeta = \Phi\,(\bar{\sigma}(t),\chi)$ with respect to t [25].

However, due to cyclic nature of loading, the yield function can have several points of local maxima in the elastic region of the deformation process. These points are situated in the interior of the yield surface.

This statement is valid for the departure instants during the whole deformation process. At the post-adaptation stage, the quantities $\boldsymbol{\rho}$, $\boldsymbol{\chi}$ and Δ do not change in time. To account for this property, it is necessary to return to the nominal stress tensor, and to employ the known decomposition $\boldsymbol{\sigma}(\mathbf{x},t)=\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t)+\boldsymbol{\rho}$.

Consequently, at the departure instants equation (3. 1) can be represented as

$$\zeta(\mathbf{x}) = \zeta^*(\mathbf{x}) = \Phi\left(\frac{\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t^*)}{1-\Delta_{\mathrm{s}}(\mathbf{x})} - \boldsymbol{\eta}_{\mathrm{s}}(\mathbf{x},\boldsymbol{\eta}_{\mathrm{s}}(\mathbf{x}))\right) = 0. \tag{3.2}$$

$$\boldsymbol{\eta}_s(\mathbf{x}) = -\frac{\boldsymbol{\rho}_{\mathrm{s}}(\mathbf{x})}{1-\Delta_{\mathrm{s}}(\mathbf{x})}. \tag{3.3}$$

The function

$$\bar{\boldsymbol{\sigma}}(\mathbf{x}) = \frac{\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t)}{1-\Delta_{\mathrm{s}}(\mathbf{x})} - \boldsymbol{\eta}_{\mathrm{s}}(\mathbf{x}). \tag{3.4}$$

determines the the stress path at the point $\mathbf{x}$ of the solid, in the stress space $\bar{\boldsymbol{\sigma}}$.

Due to cyclic nature of the loading, stress path (3.4) has a number of apexes, which are specified by the loading program. It is supposed, as it usually is, that ζ is a non-decreasing function of effective stress tensor. Therefore the local and

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absolute extrema of the yield function correspond to the apexes of the stress path (3.4).

Hereafter only the post-adaptation stage is considered, so the subscript "s" is omitted.

At the post-adaptation stage, the stress path reaches the yield surface at one, two or more points. The event, when the stress path reaches the yield surface at a single point corresponds to one-sided loading and a deformation of the same sign. This is the event of ratcheting. This paper is, however, devoted to Low Cycle Fatigue. In this case the stress path reaches the yield surface at least at two points. At these points the yield function reaches its absolute maximum value equal to zero, i.e. these values of the yield function are equal to each other. To model this situation, it is necessary to require that the absolute maximums of the yield function should be minimal. As a result, the following specification of equation (3.2) is arrived at:

$$\zeta_m(\mathbf{x}) \equiv \min_{\boldsymbol{\eta}} \max_{t} \zeta(\mathbf{x}) \equiv \min_{\boldsymbol{\eta}} \max_{t} \Phi\Big(\frac{\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x}, t(\mathbf{x}))}{1-\Delta} - \boldsymbol{\eta}(\mathbf{x}), \chi\Big) = 0. \tag{3.5}$$

The above min-max problem should be solved for fixed values of $\mathbf{x}, \Delta$ and χ.

The variables $\boldsymbol{\eta}$, Δ and $\boldsymbol{\rho}$ are connected by the relation $\boldsymbol{\rho} = (1-\Delta)\boldsymbol{\eta}$. Since Δ is fixed, the minimum of the function max ζ should be found with respect to $\boldsymbol{\rho}$. The residual stress tensor $\boldsymbol{\rho}$ should satisfy the equilibrium equations $\nabla \cdot \boldsymbol{\rho}=0$, and the boundary conditions $\boldsymbol{\rho}\cdot\boldsymbol{\nu}=0$ at the part of the solid surface S_p, where tractions are prescribed. Here, ∇ is the vector with components $\partial/\partial x_i$, $\boldsymbol{\nu}$ is the unit vector of the external normal to S_p, and $\mathbf{a}\cdot\mathbf{b} = a_i b_i$.

REMARK 1. The equilibrium equations can be satisfied by means of introducing the Airy functions, which are defined by the min-max problem at the left-hand side of (3.5). The application of these functions to the problem of plane strain/stress of linear elasticity is widely known [26]. In the case of plane strain and the von Mises yield condition, the application of the Airy functions reduces the min-max problem (3.5) to a boundary-value problem for a hyperbolic equation in partial derivatives of the second order (Appendix 2).

The solution of the min-max problem provides us with the values $\zeta_m(\mathbf{x})$, $\boldsymbol{\eta}_m(\mathbf{x})=\boldsymbol{\rho}_m/(1-\Delta)$ and $t^*(\mathbf{x})$ at every point of the solid under consideration as a function of Δ and χ. It exists if the yield function $\zeta = \Phi(\bar{\boldsymbol{\sigma}}, \chi)$ is convex in $\bar{\boldsymbol{\sigma}}$ for any admissible values of Δ and χ.

There could be more than one solution of equation (3.5) with different values of $\boldsymbol{\eta}_m$ corresponding to different shifts of the stress path in the case, when the stress path has more than two apexes. This situation is shown in Fig. 1 where a triangular stress path ABC is considered, as an example. Three solutions of the

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min-max problem can be obtained in this case by such shifts of the triangle ABC which result in adjoining one of the sides AB, BC or CA to the yield surface. Consequently, in the case under consideration three solutions to Eq. (3.5) are possible.

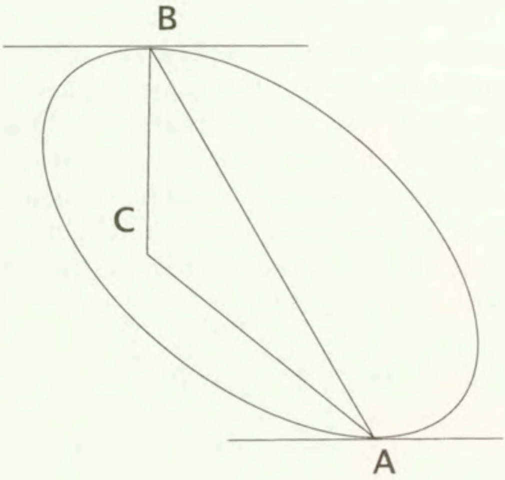

FIG. 1.

Equation (3.5) establishes a dependence between χ and Δ. This dependence defines the diameter of yield surface because changes in both χ and Δ transform the yield surfaces similarly. Notice that different solutions of (3.5) lead to different dependences, i.e. to the yield surfaces of different diameters.

Consider the von Mises yield condition for example. Its equation can have the form as $\Phi = f(\sigma) - (1-\Delta)\,\kappa(\chi) = 0$ where $\kappa(\chi)$ is the yield stress. The quantity $(1-\Delta)\kappa(\chi)$ defines the radius of the von Mises cylinder. If $f(\sigma)$ is fixed, then the radius is fixed as well.

Let us fix a pair (Δ,χ) satisfying (3.5). The quantity η_m specifies a certain position of the stress path (3.4) with respect to the yield surface defined by the pair (Δ, χ). Simultaneously η_m provides a minimal value to the function $\max\zeta$. As $\min\max\zeta$ is equal to zero, a change in η_m provides a positive value to $\max\zeta$, i.e. it shifts the stress path in such a way that at least one of its apexes falls outside the limit yield surface.

In the case when the stress path has only two apexes placed at the limit yield surface, Eq. (3.5) leads to such a position of the stress path that these apexes coincide with the opposite ends of the chord of the maximal length, whose direction coincides with the direction of the corresponding chord of the stress path (Fig. 1).

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Hence, in the classical case, when the applied load ranges between two values, the solution to equation (3.5) is unique.

If there are more than one solution to equation (3.5), the solution resulting in the best estimates, i.e. in the maximal lower estimate, and in the minimal upper estimate, should be chosen.

4. Estimating the limit value of the damage parameter and conditions of integrity

It is assumed in the subsequent argumentation that the conditions of shakedown are satisfied.

Equation (3. 5) depends on the function $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t^*)$ which represents the current damaged, purely elastic response of the solid to the prescribed loading program. This function satisfies the system of the linear elasticity equations with Hook's law $\boldsymbol{\varepsilon}^{\mathrm{E}} = \mathbf{C}^{-1} : \boldsymbol{\sigma}^{\mathrm{E}}$ where $\mathbf{C} = \mathbf{C}_0\,(1-\Delta)$, for the boundary conditions corresponding to the departure instants t^*. Obviously, the value of Δ at $t = t^*$ is unknown in advance.

To overcome this deficiency, the following method is proposed. At the departure instant t^* the function $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t^*) + \boldsymbol{\rho}_m\mathbf{x}$ satisfies the yield condition. Here $\boldsymbol{\rho}_m = \boldsymbol{\eta}_m(1-\Delta)$. This property gives us the possibility of computing $\boldsymbol{\sigma}^{E}$ at $t = t^*$ directly, i.e. without a detailed investigation of the deformation process, by means of resolving the boundary-value problem for the system of the elasticity equations supplemented with the equations $\mathbf{C} = \mathbf{C}_0\,(1-\Delta)$ and (3. 5) for the corresponding boundary conditions. The solution of this system for fixed values of Δ and $\boldsymbol{\chi}$ provides us with the values of $t^*(\mathbf{x})$ and $\boldsymbol{\eta}_m\,(\mathbf{x},t^*)$, aside from $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t^*)$.

For known $t^*(\mathbf{x})$, $\boldsymbol{\eta}_m\,(\mathbf{x},t^*)$ and $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t^*)$, equation (3.5) defines Δ as a function of χ at every point of the solid at the time instant t^*. Because Δ is a non-decreasing function of t, its values at $t = t^*$ are the limit values of Δ admitted by the yield function under the prescribed loading program.

Suppose for definiteness that ζ_m is a monotonic function of Δ. Then the extremal values of χ (χ_0 and χt^*) define the extremal limit values (bounds) of Δ admitted by the yield function and possible under the prescribed loading program: $\Delta_{\min}$ and $\Delta_{\max}$.

Hence, to determine the bounds for Δ, it is necessary to set subsequently $\chi = \chi_0$, and $\chi = \chi^*$. Then the above-mentioned system of equations becomes definite, and defines the unknown variables $t^*(\mathbf{x})$, $\boldsymbol{\eta}_m\,(\mathbf{x},t^*)$ and $\boldsymbol{\sigma} t^*\mathrm{E}(\mathbf{x},t^*)$ and the bounds $\Delta_{\min}$ and $\Delta_{\max}$..

As a result, the following estimate for the limit values of Δ at every point of the solid is obtained.

$$\Delta_{\min}(\mathbf{x}) \leq \Delta\mathbf{x} \leq \Delta_{\max}(\mathbf{x}). \tag{4.1}$$

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This estimate provides us with the necessary and sufficient conditions of local integrity for the given loading program: $\Delta_{\min} < \Delta_c$ and $\Delta_{\max} < \Delta_c$, respectively. If $\Delta_{\max} < \Delta_c$, the local integrity is not violated. On the other hand, if $\Delta_{\min} \geq \Delta_c$, then the solid looses its local integrity.

The quality of the estimation depends on the difference $\delta = \Delta_{\max} - \Delta_{\min} \geq 0$. The less is δ the better is the quality; otherwise, the lower is the degree of strain hardening, the better is the quality.

The condition of overall integrity could be formulated as follows: the solid saves its overall integrity, if the maximal value of the upper estimate over the solid is less than the critical value of the damage parameter. It is supposed that the necessary condition of overall integrity coincides with that of local integrity: if the necessary condition of local integrity is violated, then the condition of overall integrity is violated as well.

5. Accounting for kinematic strain hardening.

In this section, the developed method is extended to material models with kinematic strain hardening, and additionally to the isotropic one.

Let $\boldsymbol{\beta}\boldsymbol{\varepsilon}^{\mathrm{p}}$ denote the back-stress tensor. It is assumed that the state of saturation exists for the kinematic strain hardening, i.e. the values of back-stress components are bounded by a material constant β^*: $|\beta_{ij}| \leq \beta^*$.

The yield surface equation is written as $\Phi(\bar{\sigma} - \boldsymbol{\beta}\chi) = 0$. The variables $\boldsymbol{\beta}(\boldsymbol{\varepsilon}^{\mathrm{p}})$ and $\chi(\boldsymbol{\varepsilon}^{\mathrm{p}})$ define the position and size of the yield surface.

Suppose that the solid under consideration shakes down to the prescribed loading program. All the arguments given in Secs. 3, 4 remain valid. The only difference is in the definition of $\boldsymbol{\eta}$ which becomes (see (3.3) for comparison)

$$\boldsymbol{\eta} = -\frac{\boldsymbol{\rho}}{1-\Delta} + (1-\Delta)\boldsymbol{\beta}. \tag{5.1}$$

The method developed above proceeds from the assumption that the conditions for shakedown are fulfilled. To extend it to kinematic strain hardening, a new sufficient shakedown condition accounting for both the isotropic and kinematic strain hardening is proposed. The condition is based on the notion of the depository surface.

The yield surface $\Phi(\bar{\sigma}, \chi) = 0$ is assumed regular with the principal radii of curvature R_{i} greater than β^*. Under this condition, the surface $\breve{\Phi}(\bar{\sigma}, \chi) = 0$ "parallel" to $\Phi(\bar{\sigma}, \chi) = 0$ exists with the principal radii of curvature equal to $R_{\mathrm{i}} - \beta^*$. This surface will be named the depository one. The depository surface $\breve{\Phi}(\bar{\sigma}, \chi) = 0$ is in the interior of the yield surface $\Phi(\bar{\sigma}, \chi) = 0$, and separated from it by a layer of constant thickness equal to β^*.

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In certain respect, the depository is similar to the "reduced elastic domain" [27], and the "sanctuary" [28].

Consider for example the von Mises yield surface. In the principal stress space it is a circular cylinder of the radius $\sigma_s\sqrt{2/3}$ where $\boldsymbol{\sigma}_s$ is the yield stress of the material in tension. The corresponding depository surface is also a cylinder of the radius $\boldsymbol{\sigma}_s\sqrt{2/3} - \beta^*$.

Assume that the initial (at $t=0$) values of the damage and strain hardening parameters are $\Delta_0 = \chi_0 = 0$. Suppose that there exists a stationary residual stress field $\hat{\boldsymbol{\rho}}$ such that the following inequality is valid for $t \geq 0$:

$$\breve{\Phi}\left(\frac{\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t) + \hat{\boldsymbol{\rho}}(\mathbf{x})}{1-\Delta_0}\right) < 0 \tag{5.2}$$

at every point of the solid under consideration.

Under the above assumptions, the proposed theorem can be formulated as follows: the total plastic dissipation is bounded from above, if there exists a stationary field of residual stress $\hat{\boldsymbol{\rho}}(\mathbf{x})$, such that the stress $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t) + \hat{\boldsymbol{\rho}}(\mathbf{x})$ satisfies inequality (5.2) for any $t \geq t_0$.

The detailed proof of this theorem is presented in the Appendix. An example of application of the theorem is given in the next section.

6. Example

Let us consider the structure shown in Fig. 2. The structure consists of three rods of the same cross-sectional area S, and the same material. The rods 2, 3 are twice as long as that of rod 1: $l_3 = l_2 = 2l_1$. The structure is loaded by a variable force $P(t)$ ranging in the interval $-P_1 \leq P \leq P_2$, $P_1 \leq P_2$, where $P(t)$ is a given function of time. The rods can bear only uniaxial tensile/compressive deformation.

Due to the symmetry, the strains and stresses in rods 2 and 3 are identical: $\varepsilon_2 = \varepsilon_3$, $\sigma_2 = \sigma_3$. The strains of rods 1 and 2 are connected by the relation: $\varepsilon_1 = 2\varepsilon_2$. The stresses in the rods satisfy the equilibrium equation: $\sigma_1 + 2\sigma_2 = \mathrm{p}$, where $p_1 \leq p(t) \leq p_2$, $p = P(t)/S$, $p_1 = P_1/S$, $p_2 = P_2/S$.

It is assumed that the damage process is coupled with the process of plastic deformation, i.e. the damage can develop only if the plastic deformation process is in progress. It is assumed also that the damage process starts simultaneously with the process of plastic deformation, i.e. the damage threshold is small enough.

In the elastic undamaged state $\sigma_1^E = 2\sigma_2^E = p/2$. Assume that rods 2, 3 remain elastic, whereas rod 1 experiences plastic deformation accompanied by damage. The yield condition of the rod 1 material is taken in the form: $\Phi = |\bar{\sigma} - \beta(\varepsilon^p)| - \kappa(\chi) = 0$, or

$$\Phi = |\boldsymbol{\sigma} - (1-\Delta)\beta(\varepsilon^p)| - (1-\Delta)\kappa(\chi) = 0 \tag{6.1}$$

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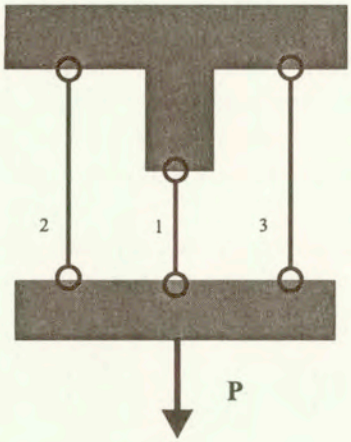

FIG. 2.

where Δ is the current value of damage parameter in rod 1, σ and ε^p are the nominal stress and plastic deformation, $\beta(\varepsilon^p)$ is the back stress, χ is the hardening parameter, and $\kappa(\chi)$ is the yield stress of undamaged material. The functions $\kappa(\chi)$ is assumed to be limited: $\kappa(\chi)$ is a known increasing function of χ for $\chi<\chi^*$, and $\kappa = \kappa(\chi^*)$=const for $\chi \geq \chi^*$. It is supposed that for $-p_1 \leq p \leq p_2$ the material is stable, i.e. the effective yield stress $\bar{\kappa} = (1-\Delta)\kappa(\chi)$ is an increasing function of time up to $\chi = \chi^*$.

The equation of the depository surface is: $\breve{\Phi} = |\boldsymbol{\sigma}| - (1-\Delta)(\kappa(\chi) - \beta^*) = 0$. Assume that $\varepsilon^p = \beta = \chi = \Delta = 0$ at t= 0. The purely elastic response of rod 1 is equal to $\sigma_1^{\mathrm{E}} = p(t)/2$. Inequality (5. 2) takes the form: $\breve{\Phi} = \left|\dfrac{p}{2} + \hat{\rho}\right| < \kappa(0) - \beta^*$. Consequently, the shakedown occurs, if $\kappa(0) - \beta^* - 0.5p_2 > \hat{\rho} - \kappa(0) + \beta^* + 0.5p_1$. This inequality leads to the requirement $(p_1 + p_2) < 4(\kappa(0) - \beta^*)$. If the last inequality is valid, then it is possible to find such $\hat{\rho}$ that (5. 2) would be satisfied. Consequently, the last inequality defines the constraints to values of p_1 and p_2 for which shakedown occurs.

During the damage process, the current value of the unloading Young's modulus of rod 1 is: $E = E_0(1-\Delta)$, where E_0 is its undamaged value. At the same time, according to the assumption, Young's moduli of rods 2, 3 preserve their initial value E_0. Thus the purely elastic response of the structure to the current value of the load $p(t)$, after the plastic-damage process in rod 1 has started, is $\sigma_1^{\mathrm{E}} = p\left(1 - \dfrac{1}{2-\Delta}\right)$, $\sigma_2^{\mathrm{E}} = \sigma_3^{\mathrm{E}} = \dfrac{p}{2}\dfrac{1}{2-\Delta}$. Thus, the current values of nominal stresses can be represented as: $\sigma_1 = p\left(1 - \dfrac{1}{2-\Delta}\right) + \rho_1$, $\sigma_2 = \sigma_3 = \dfrac{p}{2}\dfrac{1}{2-\Delta}\rho_2$ where ρ_1 and ρ_2 are the actual residual stresses in rods 1 and 2, 3, respectively.

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Since the residual stresses should be self-equilibrated, $2\rho_2 = \rho_1 = \rho$ where ρ is a new notation for ρ_1.

Now the yield function of rod 1 can be rephrased as $\Phi = |p\left(1 - \frac{1}{2-\Delta}\right) + \theta| - (1-\Delta)\kappa(\chi)$ with $\theta = \rho - (1-\Delta)\beta$, where $\rho\Delta\beta$ are actual values, and with θ instead for η.

At the post-adaptation stage ε^p, β, χ, Δ do not vary. Under fixed values of θ and χ, the function $\zeta = \Phi$ reaches its absolute maximum value under $p = p_2$, if $p\left(1 - \frac{1}{2-\Delta}\right) + \theta \geq 0$:$\max\Phi = \Phi_2 = p_2\left(1 - \frac{1}{2-\Delta}\right) + \theta - (1-\Delta)\kappa(\chi)$. However, if $p\left(1 - \frac{1}{2-\Delta}\right) + \theta \leq 0$, the yield function reaches its absolute maximum value under $p = -p_1$: $\max\,\Phi = \Phi_1 = p_1\left(1 - \frac{1}{2-\Delta}\right) - \theta - (1-\Delta)\kappa(\chi)$. The function $\max\,\Phi$ is minimum, if $\Phi_1 = \Phi_2$. This equation yields $\theta = -\frac{1}{2}(p_2 - p_1)\left(1 - \frac{1}{2-\Delta}\right)$. The corresponding value of the yield function is equaled to $\zeta_m \equiv \min\max\zeta = \frac{p_1+p_2}{2}\left(1 - \frac{1}{2-\Delta}\right) - (1-\Delta)\kappa(\chi)$.

The equation $\zeta_m = 0$ determines Δ as a function of χ: $\Delta = 2 - \frac{p_1+p_2}{2\kappa(\chi)}$.

Because χ is not greater than χ^*, and $\kappa(\chi)$ is an increasing function of χ, then

$$\Delta_{\max} = 2 - \frac{p_1+p_2}{2\kappa(\chi^*)}. \tag{6.2}$$

This equality establishes the upper bound for Δ-variation admitted by the yield condition (6.1) under the given amplitude of the force P. One can see from (6.1) that the length of the yield segment at the σ-axis is in the inverse relation to the value of Δ. That is why the value of Δ is also in the inverse relation to the quantity $p_1 + p_2$ equaled to the amplitude of p: the larger is the amplitude, the smaller is the range of admissible values of Δ.

On the other hand, the lower estimate for Δ is obtained for χ=0

$$\Delta_{\min} = 2 - \frac{p_1+p_2}{2\kappa(0)}. \tag{6.3}$$

If $\Delta_{\max} \leq \Delta_c$, then rod 1 preserves its integrity. If $\Delta_{\min} \geq \Delta_c$, the rod fails.

7. Concluding remarks

7.1. According to the developed method, the general algorithm of computing the damage parameter estimates (bounds) can be sketched as follows. First of all,

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the shakedown conditions for the solid under consideration have to be verified. The boundary-value problem for the system of elasticity equations supplemented with equations (3.5) and $\mathbf{C}=\mathbf{C}_0(1-\Delta)$ should be resolved for the extremal values of the strain hardening parameter and for the given boundary conditions. The solution of this system provides the bounds to the limit value of the damage parameter at every point of the solid under consideration. With the bounds in hand, it is possible to examine fulfilling of the local conditions of integrity.

In order to derive the conditions for overall integrity, the maximal value of the local upper bounds over the solid should be found and compared with the damage parameter critical value.

7.2. Under certain conditions, the proposed method can be extended to events, when the loading program is unknown as a function of time, and only the apexes of the load trajectory are given. For example, let us consider a solid subjected to a few repeated loads such that at any time point only one of the loads is active, i.e. the loads are applied in turn. The frequency of the application of loads, and the laws of their changing in time are unknown. It is possible that the loads are applied accidentally. The developed method is applicable to such situations, if only the maximal values of the loads are known.

Actually, at the time instant corresponding to the maximal value of a load, the stress tensor components reach their extremal values because they are proportional to the value of the load. Hence, for every separately taken load, the yield function reaches its maximal value at the instances corresponding to the load maximal values. Therefore the maximal values of the load can be taken as the boundary conditions for the boundary-value problem outlined in Sec. 5.

8. Summary

The method for estimating the local limit value of damage parameter was developed. The method is based on the relation between the damage and strain hardening parameters resulting from the limit yield condition, in which the rest of arguments is specified.

It was assumed that the material is linear-elastic during unloading, and the damage process was coupled with the process of plastic deformation: the first can be in progress, only if the second develops. The current stress tensor was decomposed into purely elastic and residual parts. The dependence of the current values of the elastic moduli on damage was taken into account.

The shakedown conditions were assumed to be fulfilled so that the post-adaptation (limit) stage of deformation existed. Although the limit values of damage and strain hardening parameters depend on the deformation path, and thus they are unknown in advance; nevertheless it is known that they satisfy the

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equation expressing the limit yield condition at the time points when the stress path reaches it. This equation was utilized to obtain upper and lower estimates of the limit value of damage parameter admitted by the given yield condition under the prescribed loading program. The problem was in a proper evaluating the residual stress that is connected with a parameter η.

The consideration was restricted by the requirement that the stress path reached the limit yield surface at least at two apexes. This situation is characteristic for the phenomenon of Low Cycle Fatigue. To model this situation, the parameter h is defined by the solution of min-max problem (3.5).

A system of equations was set which enables direct evaluation of the purely elastic, damaged response (σ^E) of the solid under consideration to the prescribed loading program at the departure instants.

Once the parameter η and stress σ^E have been specified, the limit yield condition issues in a relation between the admissible limit values of the damage and the strain hardening parameters. Because the strain hardening parameter is assumed bounded, this relation makes it possible to obtain the minimal and maximal limit values of the damage parameter. These values provide a priori bounds for the local limit value of the damage parameter admitted by the yield condition for the prescribed loading program.

The quality of the obtained estimate depends on strain hardening. The lower is the strain hardening, the better is the quality.

If the solid under investigation shakes down, and the upper bound at a solid point is less than the critical value of damage parameter (which is a material parameter), then the solid preserves its integrity at the point under consideration. This is a sufficient condition of local integrity. On the other hand, if the lower bound is greater than the damage critical value, then the local integrity is violated. This is a sufficient condition for local failure. It can be rephrased as a necessary condition of local integrity.

If a solid has to preserve its overall integrity, the condition of the local integrity has to be fulfilled at every point of the solid. An alternative formulation is as follows: a solid preserves its overall integrity, if the maximal value of the upper bound for damage parameter over the solid is smaller than the critical value of the damage parameter.

Appendix A. Sufficient shakedown condition accounting for kinematic strain hardening

The proposed condition is valid for classical material models with the only restriction: the kinematic strain hardening is bounded. Let $\beta(\varepsilon^p)$ denote the back-stress tensor. It is assumed that the state of saturation exists for the kine-

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matic strain hardening, i.e. the values of back-stress components are bounded by a material constant $\beta^* : |\beta_{ij}| \leq \beta^*$.

The yield surface equation is written as $\Phi(\bar{\sigma} - \beta, \chi) = 0$ where χ is the parameter of isotropic strain hardening. The parameters $\beta\varepsilon^P$ and $\chi\varepsilon^P$ define position and size of the yield surface.

It is assumed that the yield function is strictly convex in the first argument $s = \bar{\sigma} - \beta$, and the inequality $\Phi(s,\chi)<0$ corresponds to the interior of the yield surface $\Phi(s,\chi)=0$. Consequently, if $\Phi(s,\chi)=0$, and $\Phi(\hat{s},\chi)<0$ where $\hat{s} = \hat{\bar{\sigma}} - \beta$, then $(s - \hat{s}) : \dot{\varepsilon}^P \geq 0$. This inequality results in inequality (2.1).

As previously, it is assumed that in the interval $0 \leq \chi \leq \chi^*$ the material is stable, i.e. inequalities (2.2), (2.3) are valid. However, if the kinematic strain hardening is taken into account, inequality (2.3) can be violated during the deformation process because the stress $\hat{\sigma}$ can fall out of the current yield surface due to its shift caused by the back-stress.

In order to make inequality (2.1) valid in the case where the back-stress is taken into account, it is necessary to modify condition (2.2). To that end, a "depository" surface $\breve{\Phi}(\bar{\sigma}, \chi)=0$ is introduced. Its interior will be named the depository. This surface possesses the property that the requirement $\Phi(\bar{\sigma} - \beta\chi)<0$ is satisfied, if the inequality $\breve{\Phi})(\bar{\sigma}, \chi) < 0$ is valid.

According to the assumption, the values of the back-stress components are bounded by a constant β^*. Let use consider the case when the yield surface $\Phi(\bar{\sigma},\chi)=0$ is regular with the principal radii of curvature R_i greater than β^*. Under these conditions, the surface $\breve{\Phi}(\bar{\sigma}, \chi) = 0$ "parallel" to $\Phi(\bar{\sigma}, \chi)=0$ can be constructed with the principal radii of curvature equal to $R_i - \beta^*$. This surface is the depository one.

The surface $\Phi(\bar{\sigma} - \beta\chi) = 0$ results from $\Phi(\bar{\sigma}\chi) = 0$ by the shift equal to β. As $|\beta_{ij}| \leq \beta^*$, the depositary surface $\Phi(\bar{\sigma}, \chi) = 0$ is in the interior of the surface $\Phi(\bar{\sigma} - \beta\chi) = 0$, or touches it. Therefore, if $\Phi(\hat{\bar{\sigma}}, \chi) < 0$, then $\Phi(\hat{\bar{\sigma}} - \beta\chi) < 0$ as well.

Assume that the initial (at $t = 0$) values of the damage and hardening parameters are zero: $\Delta_0 = \chi_0 = 0$. Suppose that there exists a stationary residual stress field $\hat{\rho}$ such that the following inequality is valid for $t \geq 0$:

$$\text{(A.1)} \qquad \breve{\Phi}\left(\frac{\sigma^E(\mathbf{x}, t) + \hat{\rho}(\mathbf{x})}{1 - \Delta_0}, \chi_0\right) < 0,$$

at every point of the volume V of the solid under consideration.

According to the assumed material stability, a current yield surface comprises the previous ones. Analogously, a current depository surface also comprises the previous ones. Consequently, the following inequality holds at the every point of the volume V for any $t \geq t_0$

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$$\breve{\Phi}\left(\frac{\sigma^{E}(\mathbf{x},t)+\hat{\rho}(\mathbf{x},t)}{1-\Delta(\mathbf{x},t)},\chi(\mathbf{x},t)\right)<0, \tag{A.2}$$

Since the depository surfaces are in the interior of the corresponding yield surfaces, inequality (A.2) results in the inequality

$$\breve{\Phi}\left(\frac{\sigma^{E}(\mathbf{x},t)+\hat{\rho}(\mathbf{x},t)}{1-\Delta(\mathbf{x},t)},\beta(\mathbf{x},t),\chi(\mathbf{x},t)\right)<0, \tag{A.3}$$

Combining the above arguments it is possible to conclude that, if the stress $\hat{\sigma}(\mathbf{x},t)=\sigma^{E}(\mathbf{x},t)+\hat{\rho}(\mathbf{x})$ satisfies inequality (A.1) everywhere in V for $\Delta_0=\chi_0=0$, and for any $t\geq 0$, then inequality (A. 3) is valid for any $t\geq 0$ where $\Delta(\mathbf{x},t)$, $\chi(\mathbf{x},t)$ and $\beta(\mathbf{x},t)$ are actual values, and $\chi_0\leq\chi(t)\leq\chi^{*}$.

Hence, if condition (A.1) is valid, then inequality (2. 3) holds in the case where the back-stress is taken into account.

Inequalities (A.1), (A.3) are the extensions of inequalities (2. 2), (2. 3) accounting for kinematic strain hardening.

Notice that although the virtual stress path $\sigma=\hat{\sigma}(\mathbf{x},t)$ is in the interior of the depository surface, the actual stress path $\sigma=\sigma(t,\mathbf{x})$ exits out of it.

Thus, if there exists a stationary field of residual stress $\hat{\rho}(\mathbf{x})$ such that the virtual stress $\hat{\sigma}=\sigma^{E}(\mathbf{x},t)+\hat{\rho}(\mathbf{x})$ satisfies inequality (A.1), then inequality (2. 3) is valid.

Now repeating again the arguments developed in [22], it is possible to show that in the case under consideration, the rate of plastic strain tends to zero, as well as that the total plastic dissipation is bounded.

Under the above assumptions the proposed theorem can be formulated as follows: the total plastic dissipation is bounded from above and the plastic rate strain tensor tends to zero, if there exists a stationary field of residual stress $\hat{\rho}(\mathbf{x})$, such that the stress $\hat{\sigma}=\sigma^{E}(\mathbf{x},t)+\hat{\rho}$ satisfies inequality (5. 2) for any $t\geq t_0$ where $\chi_0(\mathbf{x})$ is the initial (at $t=t_0$) value of the hardening parameter.

Obviously this condition for shakedown is not necessary, it is only sufficient.

Appendix B. Reduction of min-max problem (3.5) in the case of plane strain.

Assume that Δ and χ are fixed. Min-max problem (2.5) posses an additional condition on the residual stress field at every point of the solid. It is shown below that in the case of plane strain, min-max problem (3. 5) is equivalent to a boundary-value problem for a hyperbolic equation in partial derivatives.

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The tensor $\boldsymbol{\eta}$ is in proportion to $\boldsymbol{\rho}$: $\boldsymbol{\eta}(\mathbf{x})=\dfrac{\boldsymbol{\rho}(\mathbf{x})}{1-\Delta}$. As Δ is fixed, the minimum of function max ζ should be found with respect of $\boldsymbol{\rho}$. Consequently, (2.5) can be reshaped in the form

$$\zeta_m \equiv \min_{\boldsymbol{\rho}} \max_{t} \zeta = \min_{\boldsymbol{\rho}} \max_{t} \hat{\Phi}\Big(\frac{\boldsymbol{\sigma}^{\mathrm{E}}(\mathrm{x},t)+\boldsymbol{\rho}(\mathrm{x})}{1-\Delta}, \chi(\mathrm{x})\Big) = 0. \tag{B.1}$$

The minimum of the function max ζ with respect to $\boldsymbol{\rho}$ should be found under the constraints: $\nabla\cdot\boldsymbol{\rho}=0$ and $\boldsymbol{\rho}\cdot\boldsymbol{\nu}=0$ at the part of the structure surface S_p where tractions are prescribed.

In the case of plane strain, the components of the stress tensor can be represented as: $\sigma_{11} = -\mathrm{Y}_{,22}$, $\sigma_{22} = \mathrm{Y}_{,11}$, $\sigma_{12} = \mathrm{Y}_{,12}$ where $\mathrm{Y}(\mathrm{x}_1,\mathrm{x}_2)$ is the stress function, and comma denotes partial derivative [24].

Suppose for simplicity that the surface tractions are prescribed as the product of a periodic function of time $\boldsymbol{\varphi}(t)$ and a function of boundary coordinate $\boldsymbol{\theta}(\mathbf{x})$. Then the tensor $\boldsymbol{\sigma}^{\mathrm{E}}$ is also represented as the product: $\boldsymbol{\sigma}^{\mathrm{E}} = \boldsymbol{\varphi}(t)\mathbf{r}(\mathrm{x}_1,\mathrm{x}_2)$. The components of tensor $\mathbf{r}(\mathbf{x})$ are determined by the solution of the elastic boundary-value problem under the surface traction $\boldsymbol{\theta}(\mathbf{x})$. Below r is considered as known.

Take, for example, the Mises yield function: $\Phi(\boldsymbol{\sigma}\chi)=f(\boldsymbol{\sigma}\chi)-4\mathrm{k}(\chi)^2$ where $\mathrm{k}(\chi)$ is the yield stress, and $f(\boldsymbol{\sigma})=(\sigma_{11}-\sigma_{22})^2+4\sigma_{12}^2 \geq 0$. Then

$$f(\bar{\boldsymbol{\sigma}}^{\mathrm{E}}+\bar{\boldsymbol{\rho}})(1-\Delta) = \boldsymbol{\varphi}^2(t)f(\mathbf{r})+f(\boldsymbol{\rho})-2\boldsymbol{\varphi}(t)[(r_{11}-r_{22})(\rho_{11}-\rho_{22}+4r_{12}\boldsymbol{\rho}_{12})].$$

Suppose that the function $\boldsymbol{\varphi}(t)$ ranges between $\varphi_1 = \varphi(t_1) \leq 0$ and $\varphi_2 = \varphi(t_2) > 0$. The function $f(\bar{\boldsymbol{\sigma}}^{\mathrm{E}}+\bar{\boldsymbol{\rho}})$ considered as a function of $\boldsymbol{\varphi}$ is assumed to be convex below. Therefore it reaches its absolute maximal value either at $\boldsymbol{\varphi}=\varphi_1$, or $\boldsymbol{\varphi}=\varphi_2$, depending on the relation between the values of quantities $f(\mathbf{r})$, $f(\boldsymbol{\rho})$, $2\left[(r_{11}-r_{22})(\sigma_{11}-\sigma_{22})+4r_{12}\sigma_{12}\right]$, which in turn depend on the values of residual stress tensor components $(\rho_{11}-\rho_{22})$ and ρ_{12}. These values of $f(\bar{\boldsymbol{\sigma}}^{\mathrm{E}}+\bar{\boldsymbol{\rho}})$ are denoted by f_1 and f_2 correspondingly. The absolute maximum value of $f(\bar{\boldsymbol{\sigma}}^{\mathrm{E}}+\bar{\boldsymbol{\rho}})$ is minimal, if $f_1 = f_2$. This condition leads to the equation $(r_{11}-r_{22})(\rho_{11}-\rho_{22})+2r_{12}\rho_{12} = (\boldsymbol{\varphi}(t_1)+\boldsymbol{\varphi}(t_2))f(\mathbf{r})$. Expressing the components of the residual stress tensor through the stress function, we arrive at the following equation in partial derivatives with respect of the stress function Y:

$$(r_{11}-r_{22})(Y_{,11}-Y_{,22})+2r_{12}Y_{,12} = (\boldsymbol{\varphi}(t_1)+\boldsymbol{\varphi}(t_2))f(\mathbf{r}).$$

This equation is of hyperbolic type. It has two orthogonal families of characteristic lines, which coincide with the trajectories of shear stress of the tensor $\mathbf{r}$. The characteristic relations are:

$$p_{,\xi}\sin\alpha - q_{,\xi}\cos\alpha = -\frac{\phi_1+\phi_2}{4}\sin 2\alpha, \quad p_{,\eta}\sin\alpha = \frac{\phi_1+\phi_2}{4}\sin 2\alpha$$

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where ξ, η are the coordinates along the characteristic lines, α is the angle between the abscissa axis and the ξ-lines, and $p = Y_{,1}$, $q = Y_{,2}$.

Along the solid border S_p where the traction is prescribed $F_x = -dq/ds$, $F_y = dp/ds$ where F_x and F_y are the components of the traction corresponding to the x, y axes, and s is a coordinate along the border. Because $F_x = F_y = 0$ along S_p then at S_p of boundary p, q = const.

The solution of this boundary-value problem determines the $\boldsymbol{\rho}$ - field in the region of influence of the boundary conditions that is bordered by the S_p - segment of the solid boundary, and the characteristic lines originating from the ends of the S_p - segment.

The $\boldsymbol{\rho}$ - field in the rest of the solid is defined by the solution of the elastic boundary-value problem under the condition of the contact stress continuity at the interfacial boundary between the elastic and plastic parts, and the conditions at the part of solid boundary where the displacement prescribed.

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On couple-stress fluid permeated with suspended particles heated from below

R. C. SHARMA(1), SUNIL(2), Y. D. SHARMA(2), RAJENDER SINGH CHANDEL(3)

(1) *Department of Mathematics, Himachal Pradesh University, Summer Hill, Shimla-171 005, India*

(2) *Department of Applied Sciences, National Institute of Technology, Hamirpur, (H. P.) – 177 005, India.*

(3) *Department of Mathematics, Government Degree College, Palampur, (H.P.) – 176 061, India.*

A LAYER of a couple-stress fluid heated from below and permeated with suspended particles is considered. For the case of stationary convection, the couple-stress has a stabilizing effect whereas suspended particles have a destabilizing effect on the couple-stress fluid permeated with suspended particles, heated from below. Graphs have been plotted by giving numerical values to the parameters, to depict the stability characteristics. The principle of exchange of stabilities is found to hold true for the couple-stress fluid in the presence of suspended particles, heated from below.

Key words: couple-stress fluid, heated from below, suspended particles.

1. Introduction

THE THEORY of Bénard convection in a viscous, Newtonian fluid layer heated from below has been given by CHANDRASEKHAR [1]. CHANDRA [2] observed that in an air layer, convection occurred at much lower gradients than those predicted if the layer depth was less than 7mm, and called this motion, "Columnar instability". However, for layers deeper than 10mm, a Bénard-type cellular convection was observed. Thus there is a contradiction between the theory and the experiment. SCANLON and SEGEL [3] have considered the effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure fluid was supplemented by that of the particles.

The study of a layer of fluid heated from below is motivated theoretically and by its practical applications in engineering. Among the applications in engineering disciplines one can find the food process industry, chemical process industry, solidification and centrifugal casting of metals.

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With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. Stokes [4] formulated the theory of couple-stress fluid. One of the applications of couple-stress fluid is its use to the study of the mechanisms of lubrication of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing which has articular cartilage as the bearing and synovial fluid as the lubricant. When a fluid film is generated, squeeze-film action is capable of providing considerable protection to the cartilage surface. The shoulder, hip, knee and ankle joints are the loaded-bearing synovial joints of the human body and these joints have a low friction coefficient and negligible wear.

Normal synovial fluid is clear or yellowish and is a viscous, non-Newtonian fluid. According to the theory of Stokes [4], couple-stresses are found to appear in noticeable magnitudes in fluids with very large molecules. Since the long chain hyaluronic acid molecules are found as additives in synovial fluid, Walicki and Walicka [5] modeled synovial fluids as a couple-stress fluid in human joints. Environmental pollution is the main cause of dust to enter into the human body. The metal dust, which filters into the blood stream of those working near the furnace, causes extensive damage to the chromosomes and the genetic mutations so observed are likely to breed cancer or malformations in the coming progeny. Therefore it is very essential to study the blood flow with dust particles. Considering blood as a couple-stress fluid and dust particles as microorganisms, Rathod and Thippeswamy [6] have studied the gravity flow of pulsatile blood through closed rectangular inclined channel with micro-organisms.

Keeping in mind the importance of non-Newtonian fluids and convection in fluid layer heated from below; the present paper attempts to study the couple-stress fluid, permeated with suspended particles, heated from below.

2. Formulation of the problem and perturbation equations

Here we consider an infinite, horizontal, incompressible couple-stress fluid layer of thickness d, heated from below so that the temperatures and densities at the bottom surface $z = 0$ are T_0 and ρ_0, and at the upper surface $z = d$ are T_d and ρ_d respectively, and that uniform temperature gradient $\beta\left(= |dT/dz|\right)$ is maintained. This layer is acted on by the gravity field $\mathbf{g}=(0, 0, -\mathrm{g})$ pervading the system.

Let ρ, p, T and $\mathbf{v}=(u,\ v,\ w)$ denote respectively the fluid density, pressure, temperature, and filter velocity, $\mathbf{v}\left(\bar{x}, t\right)$ and $N\left(\bar{x}, t\right)$ denote the velocity and number density of the particles, respectively. Then the equations of motion and

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continuity of a couple-stress fluid (STOKES [4], JOSEPH [8]) are

$$\left[\frac{\partial \mathbf{v}}{\partial t}+\left(\mathbf{v}.\nabla\right)\mathbf{v}\right]=-\nabla\left(\frac{p}{\rho_0}\right)+\mathbf{g}\left(1+\frac{\delta\rho}{\rho_0}\right)+\left(\nu-\frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2\mathbf{v}+\frac{KN}{\rho_0}\left(\mathbf{v}_d-\mathbf{v}\right), \tag{2.1}$$

$$\nabla.\mathbf{v}=0, \tag{2.2}$$

where the suffix zero refers to values at the references level $z = 0$ and in writing (2.1), use has been made of the Boussinesq approximation which states that the density variations are ignored in all terms in the equation of motion except the external force term; $\mathbf{g}$ is acceleration due to gravity, $\bar{\mathbf{x}} = (x, y, z)$ and $K = 6\pi\mu\eta', \eta'$ being particle radius, is the Stoke's drag coefficient. The kinematic viscosity ν, couple-stress viscosity μ', thermal diffusivity κ and coefficient of thermal expansion α are all assumed to be constants.

Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion for the particles. The buoyancy force and pressure force acting on the particles are neglected. Inter-particle reactions are also not considered for we assume that distances between particles are quite large compared with their diameters. If mN is the mass of the particles per unit volume, then the equations of motion and continuity for the particles, under the above assumptions, are

$$mN\left[\frac{\partial \mathbf{v}_d}{\partial \mathrm{t}}+\left(\mathbf{v}_d.\nabla\right)\mathbf{v}_d\right]=KN\left(\mathbf{v}-\mathbf{v}_d\right), \tag{2.3}$$

$$\frac{\partial N}{\partial \mathrm{t}}+\nabla.\left(N\,\mathbf{v}_d\right)=0. \tag{2.4}$$

Let c_v, c_p, c_{pt}, T, and q denote respectively, the heat capacity of fluid at constant volume, heat capacity of fluid at constant pressure, heat capacity of particles, temperature, and "effective thermal conductivity" of the clean fluid. Since the volume fraction of the particles is assumed to be small, the effective properties of the suspension are taken to be those of the clean fluid. If we assume that the particles and the fluid are in thermal equilibrium, the equation of heat gives

$$\rho c_v\frac{\partial T}{\partial \mathrm{t}}+\rho\, c_v\left(\mathbf{v}.\nabla\right)T+mNc_{pt}\left(\frac{\partial}{\partial\,\mathrm{t}}+\mathbf{v}_d.\nabla\right)T=q\nabla^2T. \tag{2.5}$$

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The equation of state for the fluid is given by

(2.6) $$\rho = \rho_0\Big[1 - \alpha(T - T_0)\Big].$$

The basic motionless solution is

(2.7) $$\mathbf{v} = \Big(0,0,0\Big), \quad \mathbf{v}_d = \Big(0,0,0\Big), \quad T = -\beta z + T_0,$$
$$\rho = \rho_0\Big(1 + \alpha\beta z\Big), \quad N = N_0, \quad \text{a constant.}$$

Assume small perturbations around the basic solution, and let $\delta\rho$, δp, θ, $\mathbf{v} = (u, v, w)$, $\mathbf{v}_d = (\ell, r, s)$ and N denote respectively the perturbations in fluid density ρ_0, pressure p_0, temperature T, couple-stress fluid velocity (0, 0, 0), suspended particles velocity (0, 0, 0) and suspended particles number density N_0. The change in density $\delta\rho$, caused mainly by the perturbation θ in temperature, is given by

(2.8) $$\delta\rho = -\alpha\rho_0\theta.$$

Then the linearized perturbation equations of the couple-stress fluid and particles are

(2.9) $$\frac{\partial \mathbf{v}}{\partial \mathrm{t}} = \frac{1}{\rho_0}(\nabla\delta p) - \mathbf{g}\alpha\theta + \left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2\mathbf{v} + \frac{KN_0}{\rho_0}\Big(\mathbf{v}_d - \mathbf{v}\Big),$$

(2.10) $$\nabla.\mathbf{v} = 0,$$

(2.11) $$\Big(1 + h\Big)\frac{\partial\theta}{\partial \mathrm{t}} = \beta(w + hs) + \kappa\nabla^2\theta.$$

(2.12) $$mN_0\frac{\partial \mathbf{v}_d}{\partial \mathrm{t}} = KN_0\big(\mathbf{v} - \mathbf{v}_d\big),$$

(2.13) $$\frac{\partial N}{\partial \mathrm{t}} + \nabla.\Big(N_0\mathbf{v}_d\Big) = 0.$$

Here $h = f\dfrac{c_{pt}}{c_v}$, $f = \dfrac{mN_0}{\rho_0}$, $\kappa = \dfrac{q}{\rho_0 c_v}$.

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Eliminating $\mathbf{v}_d$ in (2.9) with the help of (2.12), writing the scalar components of Eq. (2.9) and eliminating $u, v, \delta p$ between them, by using (2.10), we obtain

$$\left[\left\{\left(\frac{m}{K}\frac{\partial}{\partial t}+1\right)+\frac{mN_0}{\rho_0}\right\}\frac{\partial}{\partial t}-\left\{\left(\frac{m}{K}\frac{\partial}{\partial t}+1\right)\left(\nu-\frac{\mu'}{\rho_0}\nabla^2\right)\right\}\nabla^2\right]\nabla^2 w - \left(\frac{m}{K}\frac{\partial}{\partial t}+1\right)\left\{g\alpha\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\theta\right\}=0. \tag{2.14}$$

Eliminating s with the help of (2.12), equation (2.11) yields

$$\left(\frac{m}{K}\frac{\partial}{\partial t}+1\right)\left[(1+h)\frac{\partial}{\partial t}-\kappa\nabla^2\right]\theta=\beta\left(\frac{m}{K}\frac{\partial}{\partial t}+1+h\right)w. \tag{2.15}$$

3. The dispersion relation

Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$\left[w,\theta\right]=\left[W(z),\Theta(z)\right]\exp\left(ik_x x+ik_y y+nt\right), \tag{3.1}$$

where k_x, k_y are the wave numbers along the x- and y- directions respectively, $k=\sqrt{\left(k_x^2+k_y^2\right)}$ is the resultant wave number and n is the growth rate which is, in general, a complex constant.

Expressing the coordinates x, y, z in the new unit of length d, time t in the new unit of length $\frac{d^2}{\kappa}$ and let $a=kd$, $\sigma=\frac{nd^2}{\nu}$, $p_1=\frac{\nu}{\kappa}$, $F=\frac{1}{\nu}\frac{\mu'}{(\rho_0 d^2)}$, $\sigma'=\frac{n'd^2}{\nu}$, H=h+1, $\tau=\frac{m\kappa}{KD^2}$, $n'=n\left[1+\frac{mN_0K/\rho_0}{mn+K}\right]$ and $D=\frac{d}{dz}$; Eqs. (2.14), and (2.15), using (3.1), yield

$$\left[\sigma'-\left\{1-F(D^2-a^2)\right\}(D^2-a^2)\right](D^2-a^2)W=-\frac{g\alpha d^2}{\nu}a^2\Theta, \tag{3.2}$$

$$\left(\frac{\tau\nu}{d^2}\sigma+1\right)\left\{D^2-a^2-(1+h)p_1\sigma\right\}\Theta=-\frac{\beta d^2}{\kappa}\left(H+\frac{\tau\nu}{d^2}\sigma\right)W. \tag{3.3}$$

Eliminating Θ between Eqs. (3.2) - (3.3), we obtain

$$\left(1+\frac{\nu\tau}{d^2}\sigma\right)\left[\sigma'-\left\{1-F\left(D^2-a^2\right)\right\}\left(D^2-a^2\right)\right]\left(D^2-a^2\right) \tag{3.4}$$

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(3.4) [cont.]
$$\left\{D^2 - a^2 - (1+h)p_1\sigma\right\}W = Ra^2\left(H + \frac{\nu\tau}{d^2}\sigma\right)W,$$

where $R = \dfrac{g\alpha\beta d^4}{\nu\kappa}$ is the Rayleigh number.

Consider the case where both boundaries are free as well as perfect conductors of heat, while the adjoining medium is perfectly conducting. The case of two free boundaries is somewhat artificial but relevant for stellar atmospheres (SPIEGEL [7]). However, it enables us to find analytical solutions and to make some qualitative conclusions. The boundary conditions, with respect to which Eq. (3.4) must be solved, are

$$W = D^2W = 0, \quad \Theta = 0, \quad \text{at } z = 0 \text{ and } 1. \tag{3.5}$$

The constitutive equation for the couple-stress fluid are

$$\tau_{ij} = \left(2\mu - 2\mu'\nabla^2\right)e_{ij}, \quad e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{3.6}$$

The condition on a free surface are

$$\tau_{xz} = \tau_{yz} = 0, \tag{3.7}$$

which yield

$$(\mu - \mu'\nabla^2)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0, \tag{3.8}$$

$$(\mu - \mu'\nabla^2)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0. \tag{3.9}$$

Since w vanishes (for all x and y) on the bounding surface, it follows from Eqs. (3.8) and (3.9) that

$$\left(\mu - \mu'\nabla^2\right)\left(\frac{\partial u}{\partial z}\right) = 0, \quad \left(\mu - \mu'\nabla^2\right)\left(\frac{\partial v}{\partial z}\right) = 0. \tag{3.10}$$

From the equation of continuity (2.2) differentiated with respect of z, we conclude that

$$\left[\mu - \mu'\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right]\frac{\partial^2 w}{\partial z^2} = 0, \tag{3.11}$$

which implies that

$$\frac{\partial^2 w}{\partial z^2} = 0, \quad \frac{\partial^4 w}{\partial z^4} = 0, \quad \text{on } z = 0 \text{ and } z = d. \tag{3.12}$$

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Using expression (3.1), the boundary conditions (3.12), in non-dimensional form, yield additional boundary condition

$$D^4 w = 0 \text{ on } z = 0 \text{ and } z = 1. \tag{3.13}$$

Equations (3.2) and (3.3), using boundary conditions (3.5) and (3.13), yield

$$D^6 w = 0 \text{ on } z = 0 \text{ and } z = 1. \tag{3.14}$$

Using (3.5), (3.13) and (3.14), Eq. (3.4), yields

$$D^8 w = 0 \text{ on } z = 0 \text{ and } z = 1. \tag{3.15}$$

Differentiating Eq. (3.4) twice, four times,.....w.r.t. z and using the preceding boundary conditions (29), it can be shown that all the even order derivatives of W must vanish for $z = 0$ and $z = 1$, and hence the proper solution of W characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{3.16}$$

where W_0 is a constant. Substituting the proper solution (3.16) in Eq. (3.4), we obtain the dispersion relation

$$R_1 = \frac{\mathbf{A}}{\mathbf{B}} \tag{3.17}$$

where

$$\mathbf{A} = (1+x_1)(1+x_1+iHp_1\sigma_1)\left(1+i\frac{\nu\tau\pi^2}{d^2}\sigma_1\right)\left[i\sigma_1' + \left\{1+\pi^2 F(1+x_1)\right\}(1+x_1)\right],$$

$$\mathbf{B} = x_1\left(H + i\frac{\nu\tau\pi^2}{d^2}\sigma_1\right),$$

and, where $x_1 = \dfrac{a^2}{\pi^2}$, $i\sigma_1 = \dfrac{\sigma}{\pi^2}$ and $R_1 = \dfrac{R}{\pi^4}$.

4. The stationary convection

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relation (3.17) reduces to

$$R_1 = \frac{(1+x_1)^3\{1+\pi^2 F(1+x_1)\}}{x_1 H}. \tag{4.1}$$

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To study the effects of couple-stress parameter and suspended particles, we examine the natures of $\frac{dR_1}{dF}$ and $\frac{dR_1}{dH}$ analytically. Equation (4. 1) yields

(4.2) $$\frac{dR_1}{dF} = \frac{\pi^2 (1+x_1)^4}{x_1 H^2},$$

(4.3) $$\frac{dR_1}{dH} = -\frac{(1+x_1)^3 \{1 + \pi^2 F (1+x_1)\}}{x_1 H^2},$$

which imply that the couple-stress a stabilizing effect whereas suspended particles have a destabilizing effects on the onset of convection in couple-stress fluid permeated with suspended particles heated from below.

The dispersion relation (4. 1) is analysed numerically. In Fig. 1, R_1 is plotted against x_1 for $F = 1, 2, 3$ and $H = 10$. It is clear that the couple-stress has a stabilizing effect on the onset of convection as the Rayleigh number increases with the increase in couple-stress parameter. In Fig. 2, R_1 is plotted against x_1 for $H = 10, 20, 30$ and $F = 2$. Here we find the suspended particles have a destabilizing effect, as the Rayleigh number decreases with the increase in suspended particles parameter, on couple-stress fluid permeated with suspended particles heated from below.

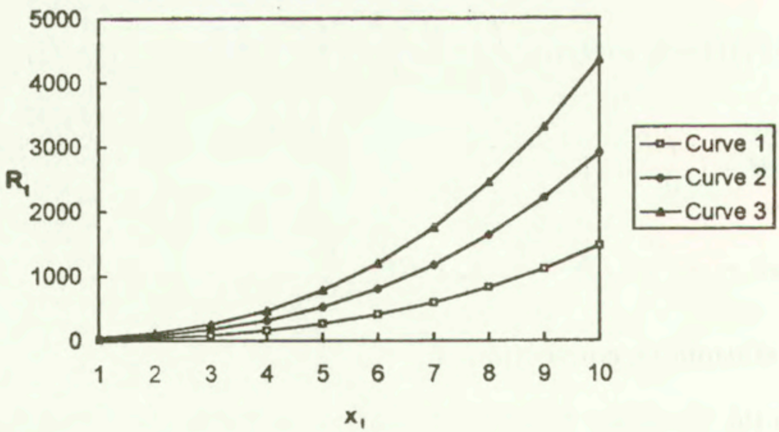

FIG. 1. The variation of Rayleigh number (R_1) with wave number (x_1) for H = 10, F=1 for Curve 1, F=2 for Curve 2 and F=3 for Curve 3.

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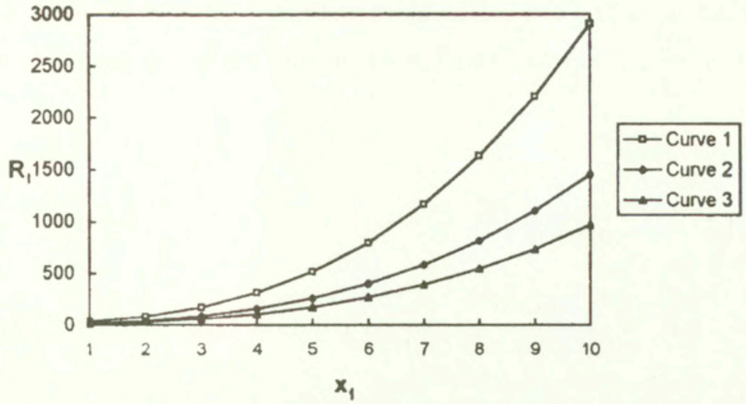

FIG. 2. The variation of Rayleigh number (R_1) with wave number (x_1) for F = 2, H=10 for Curve 1, H=20 for Curve 2 and H=30 for Curve 3.

5. Principle of exchange of stabilities

Multiplying (3.2) by W^*, the complex conjugate of W, and using (3.3) together with the boundary conditions (3.5) and (3.13), we obtain

$$FI_1 + I_2 + \sigma' I_3 - \left(\frac{g\alpha\kappa a^2}{\nu\beta}\right)\left(\frac{d^2+\nu\tau\sigma^*}{Hd^2+\nu\tau\sigma^*}\right)\Big[I_4 + Hp_1\sigma^* I_5\Big] = 0, \tag{5.1}$$

where

$$I_1 = \int_0^1 \left(|D^3W|^2 + 3a^2|D^2W|^2 + 3a^4|DW|^2 + a^6|W|^2\right)dz,$$

$$I_2 = \int_0^1 \left(|D^2W|^2 + 2a^2|DW|^2 + +a^4|W|^2\right)dz,$$

$$I_3 = \int_0^1 \left(|DW|^2 + a^2|W|^2\right)dz, \tag{5.2}$$

$$I_4 = \int_0^1 \left(|D\Theta|^2 + a^2|\Theta|^2\right)dz$$

$$I_5 = \int_0^1 \left|\Theta|^2\right)dz$$

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The integrals $I_1, ..., I_5$ are all positive definite. Since $\sigma = \sigma_r + i\sigma_i$, putting $\sigma = i\sigma_i$, $f = \dfrac{mN_0}{\rho_0}$ where σ_i is real and equating the imaginary parts of Eq. (5. 1), we obtain

$$\text{(5.3)} \quad \sigma_i\left[\left(1+\frac{f}{1+p_1^2\tau^2\sigma_1^2}\right)I_3 + \frac{g\alpha\kappa a^2}{\nu\beta\left(H^2d^2+\nu^2\tau^2\sigma^2\right)}\left\{d^2\nu\tau h I_4 + \left(Hd^4+\nu^2\tau^2\sigma^2\right)Hp_1I_5\right\}\right] = 0$$

But the quantity inside the brackets is positive definite. Hence

$$\text{(5.4)} \qquad \sigma_i = 0.$$

This shows that whenever $\sigma_r = 0$ implies that $\sigma_i = 0$, then the stationary (cellular) pattern of flow prevails on the onset of instability. In other words, the principle of exchange of stabilities is valid for the couple-stress fluid permeated with suspended particles, heated from below.

6. Conclusion

The presence of small amounts of additives in a lubricant can improve the bearing performance by increasing the lubricant viscosity and thus producing an increase in the load capacity. These additives in a lubricant also reduce the coefficient of friction and increase the temperature range in which the bearing can operate. A number of theories of the microcontinuum have been postulated and applied (STOKES [4], LAI *et al.* [9], WALICKA [10]). The theory due to STOKES [4] allows for polar effects such as the presence of couple stresses and body couples. STOKES [4] theory has been applied to the study of some simple lubrication problems (see e.g. SINHA *et al.* [11], BUJURKE and JAYARAMAN [12], LIN [13]).

A layer of a couple-stress fluid heated from below and permeated with suspended particles is considered. Here we use linearized stability theory and normal mode analysis method. We consider the case where both boundaries are free as well as perfect conductors of heat, while the adjoining medium is perfectly conducting. For the case of stationary convection, the couple-stress has a stabilizing effect whereas suspended particles have a destabilizing effect on the couple-stress fluid, permeated with suspended particles, heated from below. Graphs have been plotted by giving numerical values to the parameters, to depict the stability

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characteristics. It is clear that the couple-stress has a stabilizing effect on the onset of convection as the Rayleigh number increases with the increase in the couple-stress parameter. Also here we find that the suspended particles have destabilizing effect as the Rayleigh number decreases with the increase in suspended particles parameter on the couple-stress fluid, permeated with suspended particles, heated from below. We also see that oscillatory modes are not allowed due to the presence of kinematic viscoelasticity and suspended particles, since whenever $\sigma_r = 0$ implies that $\sigma_i = 0$, then the stationary (cellular) pattern of flow prevails on the onset of instability. In other words, the principle of exchange of stabilities is valid for the couple-stress fluid permeated with suspended particles, heated from below.

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Inverse analysis of the heat conduction process induced by impinging jet

M. WOELKE [1], L. BOGUSŁAWSKI

Poznań University of Technology, Chair of Thermal Engineering, ul. Piotrowo 3, 60 -965 Poznań,
e-mail: woelkem@sol.put.poznan.pl

THIS PAPER PRESENTS an analysis of the cooling process of a solid, induced by the impingement of an air jet. Solutions of the inverse heat conduction problem were obtained by applying the heat functions to formulate the base functions of the Finite Element Method. The applied heat functions identically satisfy the heat conduction equation in dimensionless co-ordinates. The minimisation of the functional, presented in this paper, leads to the solutions of the analysed problem. The temperature distribution of the analysed solid was determined by solving the inverse heat conduction problem by means of the temperature measurements taken inside the solid. Properties of the heat function were applied to reconstruct the distribution of the Bi number on the heat exchange surface; this in turn enables to determine the heat transfer coefficient on the analysed surface. The results of the analysis were compared with the data found in the literature.

Notations

a	temperature compensatory coefficient,
$Bi = \dfrac{hZ}{k}$	Biot's number,
c	velocity,
g	temperature gradient,
h	heat transfer coefficient,
i	number of the nodes,
I	direct functional,
J	inverse functional,
j	number of measurement points,
k	heat conduction coefficient,
K	number of finite elements,
l	number of the heat functions,
L	final number of finite elements,,
m	final number of measurement points,
n	time step,
N	final number of the nodes in each finite element,

[1] Graduate from study for doctor degree at PUT

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p	parameter,
q	density of the heat flux,
t	time,
T	temperature surplus,
w	generating function,
z	characteristic dimension for the heat conduction,
Z	length of the analysed solid in the heat conduction direction,
Greek letters	
δ	standard deviation,
θ	approximate value of the temperature surplus,
θ^n	approximate value of the temperature surplus from n time step,
$\xi = \frac{z}{Z}$	dimensionless co-ordinates,
$Fo = \tau = \frac{a}{Z^2}t$	Fourier's number,
v	heat function,
$\varphi_i(\xi, \tau)$	base function of FEM,
Ω	surface of the finite element.

1. Introduction

The concept of solution of the inverse heat conduction problem implies the determination of the unknown boundary condition on the basis of the temperature measurements taken inside the solid. The inverse heat conduction problems are ill-posed; it means that small changes of the temperature inside the solid could correspond to larger changes of the temperature on the heat exchange surface. As a consequence, small uncertainty in temperature measurements taken inside the solid causes significant errors in temperature on the heat exchange surface determined by the solution of the inverse problem. The inverse heat conduction problems are analysed in the papers [2, 3, 4, 5, 8, 9, 10, 12, 13, 14].

The determination of the heat exchange boundary conditions, on the surface cooled by impinging air jet, is based on solving the inverse heat conduction problem.

The temperature field satisfies the equation

$$\frac{\partial^2 T(\xi,\tau)}{\partial \xi^2} = \frac{\partial T(\xi,\tau)}{\partial \tau}, \qquad \tau > 0,\ \xi \in (0,\, 1) \tag{1.1}$$

and is subjected to the initial condition

$$T(\xi, 0) = T_p(\xi, 0), \qquad \xi \in <0, 1> \tag{1.2}$$

and to the boundary condition

$$\frac{\partial T(0,\tau)}{\partial \xi} = 0 \quad \text{for} \quad \xi = 0, \tau > 0, \tag{1.3}$$

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(1.4) $$T(1,\tau) = T_B(1,\tau) \quad \text{for} \quad \xi = 1, \tau > 0.$$

The boundary condition (1.4), in the inverse problem, was investigated by experimental studies.

The problem was analysed in dimensionless co-ordinates $Bi = \dfrac{hZ}{k}$ (Biot's number), $\xi = \dfrac{z}{Z}$ and $Fo = \tau = \dfrac{a}{Z^2}t$ (Fourier's number).

The determination of the boundary condition for ξ=1 leads, by means of the heat functions properties, to the determination of the heat transfer coefficient h on the heat exchange surface.

Due to complexity of the geometry conditions of the heat machines, the determination of the heat transfer coefficient h is complicated, because the mounting of the measurement sensor might be very difficult.

2. Numerical analysis of the heat transfer

The solution of the inverse heat conduction problem consists in the solution of the direct problem with the unknown boundary conditions as a parameter.

The unknown boundary condition is determined by minimisation of the difference between the temperature calculated and the experimental temperature measurements [5].

2.1. Determination of the direct problem

The temperature field in a flat layer is analysed in order to solve the direct heat conduction problem. The temperature field satisfies the heat Eq.(1.1) with boundary conditions (1.3), (1.4) and initial condition (1.2). The area of the solution of the analysed problem is shown in Fig. 1.

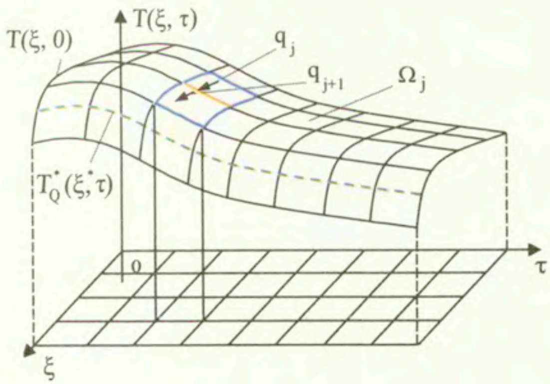

FIG. 1. The discreet area of the solution of the analysed problem.

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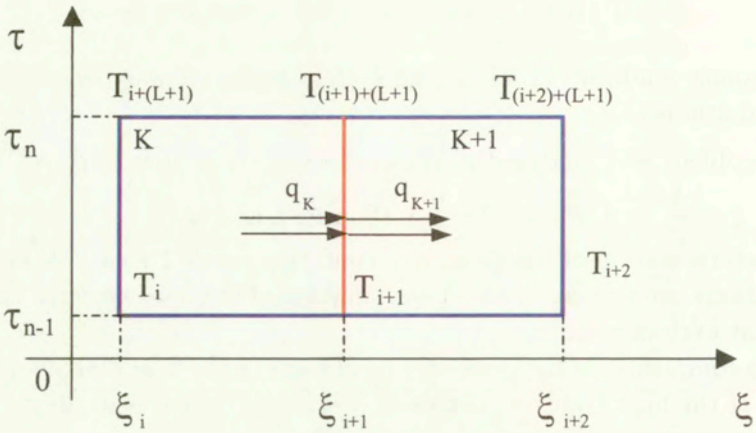

FIG. 2. Two finite elements of the discreet area of the solution of the analysed problem.

In this paper, the results of the analyses are presented for the finite element consisted of four nodes, but the number of nodes in each finite element could be N=4, 6, 8.

Approximation of the temperature in the element K of the analysed area may be presented in the form

$$\theta_K(\xi,\tau) = \sum_{i=1}^{N} \varphi_i^K(\xi,\tau)\,T_i \qquad \text{for } K = 1,\ldots,L. \tag{2.1}$$

The heat functions were used to determine the base function $\varphi_i(\xi,\tau)$ of Finite Element Method. The heat functions are obtained by expansion in series of generating function $\exp\left(p\xi + p^2\tau\right)$ [4]

$$w(\xi,\tau,p) = e^{p\xi+p^2\tau} = \sum_{l=1}^{\infty} \nu_l(\xi,\tau)\cdot\frac{p^l}{l!} \qquad \tau \ge 0, \quad \xi \in (0,1) \tag{2.2}$$

where coefficients of the series (2.2) are found from the formulae [4]

$$\begin{gathered} v_{l+1}(\xi,\tau) = \frac{\xi}{l}\cdot v_l(\xi,\tau) + \frac{2\tau}{l}\cdot v_{l-1}(\xi,\tau),\ l \ge 2, \\ v_0 = 0, \quad v_1(\xi,\tau) = 1, \quad v_2(\xi,\tau) = \xi, \\ \frac{\partial v_l(\xi,\tau)}{\partial \xi} = v_{l-1}(\xi,\tau), \qquad \frac{\partial v_l(\xi,\tau)}{\partial \tau} = v_{l-2}(\xi,\tau), \quad l = 1, 2, 3, \ldots. \end{gathered} \tag{2.3}$$

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The heat functions determined above, identically satisfy Eq. (1.1). The base functions of the Finite Element Method are formed as linear combinations of the heat functions. Therefore the basic functions automatically satisfy the heat conduction equation (1.1)[4].

For the purpose of determination of unknown temperatures at the nodes of finite elements mesh, the following functional was minimised:

$$I = \int_{\tau}^{\tau+\Delta\tau} \left(\frac{\partial \theta_1(0,t)}{\partial \xi} - 0 \right)^2 dt + \int_{\tau}^{\tau+\Delta\tau} (\theta_{K=L}(1,t) - T_B(1,t))^2 dt + \sum_{K=1}^{L} \int_{\xi_K}^{\xi_{K+1}} (\theta_K(\xi,0) - T_p(\xi,0))^2 d\xi + \sum_{K=1}^{L} \int_{\tau}^{\tau+\Delta\tau} \left[-\frac{\partial \theta_K(\xi_g,t)}{\partial \xi} + \frac{\partial \theta_{K+1}(\xi_g,t)}{\partial \xi} \right]^2 dt, \tag{2.4}$$

Minimisation of the functional (2.4) with respect to unknown temperatures at the nodes of finite elements mesh leads to solutions of the direct heat conduction problem. The solution of the direct problem demonstrates the relation between the temperature of the previous time step and the boundary condition

$$\{\theta^n\} = [DD]\,\{\theta^{n-1}\} + [GB]\,T_B^n + [GA]\,T_B^{n-1}. \tag{2.5}$$

Most significant is the relation between the solution of the direct heat conduction problem (2.5) and the boundary condition (1.4) for ξ=1.

2.2. Determination of the inverse problem

The temperature $T_B^n(\tau_n)$ is sought by solving the inverse heat conduction problem. The temperature inside the solid should be measured for the purpose of solving the inverse heat conduction problem. The measurements of the $T_Q^*(\tau_n)$ temperature were performed to build the functional, the minimisation of which leads to the determination of the temperature distribution of the solid

$$J = \left\|\theta_K(\xi^*,\tau_n) - T_Q^*(\tau_n)\right\|^2 = \sum_{j=1}^{m} \left(\theta_K(\xi^*,\tau_n) - T_Q^*(\tau_n)\right)_j^2. \tag{2.6}$$

Here j – the number of measurement points (ξ^*,τ_n) of the $T_Q^*(\tau_n)$ temperature (here the analysis was accomplished for m=1).

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Minimisation of the functional (2.6) with respect to the unknown temperature $T_B^n(\tau_n)$ leads to the solution of the inverse heat conduction problem

$$\{\theta^n\} = [GDD]\,\{\theta^{n-1}\} + \sum_{j=1}^{m}[GNN]_j \cdot T^*_{Q,j} + [GT]\cdot T_B^{n-1}. \tag{2.7}$$

In the paper [14] were published the results of the solution of the inverse problem, calculated by means of the numerically generated temperature measurements. The size of this stability area of solution of the inverse problem was determined as a function of localisation of the temperature measurement sensor. The obtained results confirm the applicability of the heat functions to analyse the heat conduction equation.

3. Experimental analysis

The heat transfer process analysed in this paper consists in conducting heat by the solid through the heat exchange surface and taking over the heat by the impinging cooling air jet. The experimental research was aimed at validation of the applicability of the inverse analyses solved by means of the heat function to examine the heat exchange intensification on the analysed surface.

For the purpose of ensuring one-dimensional heat conduction, heat transfer through the thickness of the flat plate was replaced by the heat transfer along the length of the cylinder.

The change of the shape of the analysed solid is negligible for the solution validity. It simplifies the fulfilment of the assumed heat transfer boundary conditions. Due to this change, the shape of the analysed solid was adjusted to the shape of the impinging air jet.

In the experimental research, the temperature was measured along the symmetry axis of the cylinder in five measurement points. The temperature measurements were taken for solving the inverse heat conduction problem and for the assessment of validity of the obtained solution. The locations of the measurement points are shown in Table 1 and Fig. 3.

Table 1. The locations of temperature measurement points

Location of temperature	z=1mm	z=2mm	z=3mm	z=20mm	z=40mm
measurements	$\xi = 0.983$	$\xi = 0.978$	$\xi = 0.950$	$\xi = 0.667$	$\xi = 0.334$

The analysed solid was made of 0H18N9 steel ($\lambda = 14,65\,W/mK$), with length Z=0,6m chosen as a characteristic dimension.

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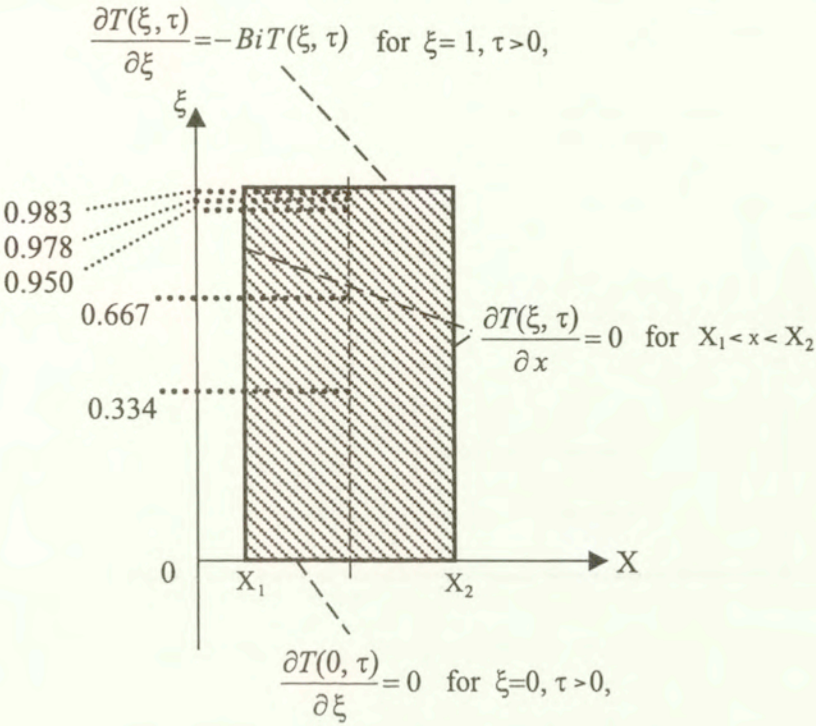

FIG. 3. The cross section of analysed solid with boundary conditions and measurement points locations.

4. Experimental apparatus

Schematic diagram of the test set-up with the most important working sections like: fan inducing air flow, open wind tunnel generating axisymmetrical jet, measurement sensors and record circuit, is shown in Fig. 3.

The diameter of the outlet nozzle of the wind tunnel was selected to emit impinging cooling jet where the diameter was larger than the one of the analysed specimen, in order to ensure smaller change of the temperature along the radius of the cylinder with respect to the axis temperature of the cylinder. The surface of the analysed specimen was placed in the axis of the cooling jet, at the distance of 2D (D-diameter of the nozzle) from the outlet nozzle section. The jet impinges the plate where the analysed solid was placed. During experimental studies, the diameter of the nozzle as well as the distance between the analysed solid and the outlet nozzle section were constant.

At the first stage of the experiment, a plate made of polystyrene isolated the heat exchange surface. After heating the specimen to the temperature of 100^{o}C,

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the plate was rapidly removed starting the cooling process with impinging air jet (the remaining heat exchange surfaces were insulated).

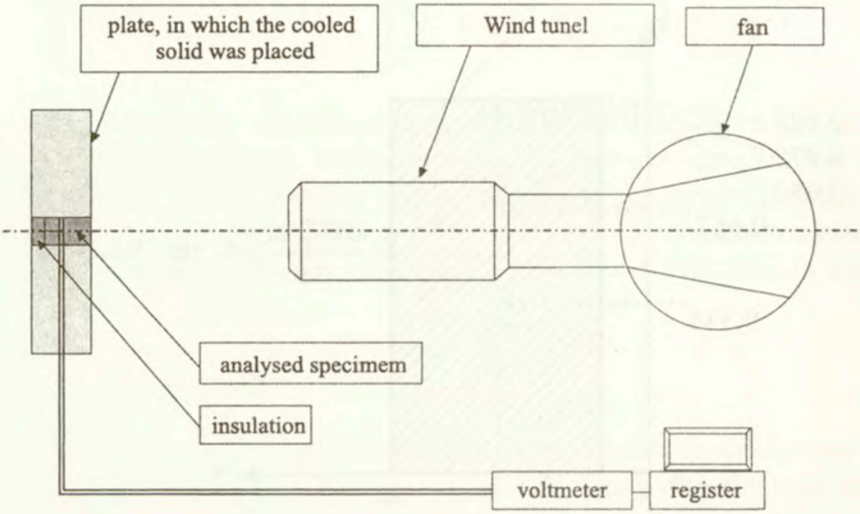

FIG. 4. Schematic diagram of the test rig.

During the experimental analyses, variation of the temperature inside the solid was monitored with Cu-constantan thermocouple. The measurements of the average velocity of the coolant flow were accomplished by the measuring nozzle and the pressure differences processor Furness Control FCO-14. The measurement data were recorded by applying voltmeter Keithley 2000 under the supervision of the LabView computer application.

5. Sensitivity of the solution of the inverse heat conduction problem to the random temperature measurement errors

During intensive heating/cooling processes the changes of the temperature inside the solid are smothered and delayed in time to the changes of the temperature on the heat exchange surface. As a consequence, small errors in temperature measured inside the solid are enlarged. They appear as large oscillations in determined values of the boundary conditions on the heat exchange surface.

For the purpose of showing the sensitivity of the solution of the inverse heat conduction problem to the random temperature measurement errors, the temperature field satisfying stationary equation in cylindrical co-ordinate system

$$\frac{\partial}{\partial \xi}\left(\xi \frac{\partial T}{\partial \xi}\right) = 0 \tag{5.1}$$

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was analysed. In this case, the temperature measurements were disturbed in 0.5%. The temperature field fulfilling stationary Eq. (5.1) has the solution

$$T = T_2 + \frac{T_1 - T_2}{\ln \dfrac{\xi_1}{\xi_2}} \ln \frac{\xi}{\xi_2} = T_2 + g \ln \frac{\xi}{\xi_2}. \tag{5.2}$$

The value of the temperature gradient containing the random temperature measurement errors was determined as

$$g_b = \frac{T_1 - (T_2 + \delta T_2)}{\ln \dfrac{\xi_1}{\xi_2}}. \tag{5.3}$$

The deviation between the temperature gradient containing the random temperature measurement errors and the exact temperature gradient was determined as

$$\delta_G = \sqrt{\frac{(g_b - g_{\text{exact}})^2}{g_{\text{exact}}^2}}. \tag{5.4}$$

The schematic distribution of the temperature determined by means of the undisturbed temperature measurement and the schematic distribution of the temperature determined by means of the disturbed temperature measurement are shown in Fig. 5.

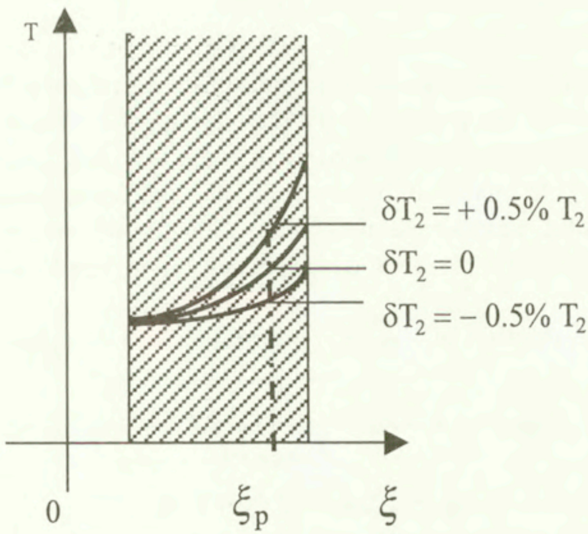

FIG. 5. Temperature distribution.

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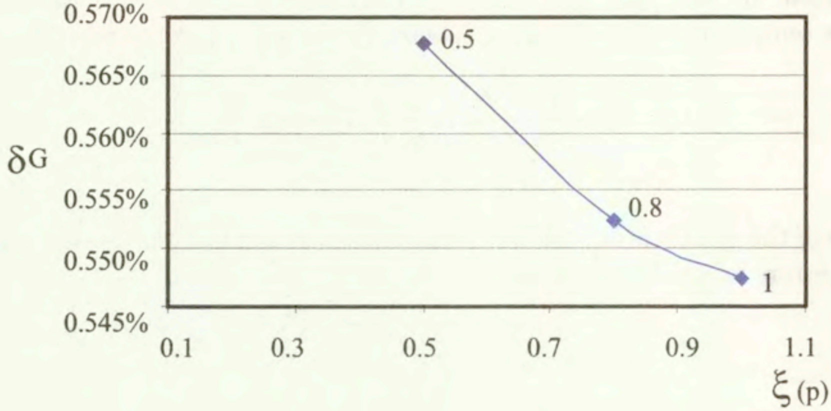

FIG. 6. The distribution of δ_G.

The increasing value of deviation of the temperature gradient as the increase of the distance of the measurement point location from the heat exchange surface is shown in Fig. 6.

The reconstruction of the boundary conditions on the basis of distorted measurements is a very complicated subject and susceptible to random temperature measurement errors [13].

6. Temperature distribution of the analysed solid

The results of the analysis of the inverse heat conduction problem were obtained by applying the temperature data measured at one point inside the solid, 1mm from the heat exchange surface. By introducing the temperature measurements to the computer application written in programming language – Fortran 77, solving the inverse heat conduction problem, the temperature distribution in the analysed solid was determined. The solution of the inverse problem was compared with temperature measurements taken inside the solid in five points Fig. 7.

Accuracy of the solution of the inverse problem was determine by means of

$$\delta_\theta = \sqrt{\frac{\left(T_{\text{measured}} - \theta_{\text{inverse solution}}\right)^2}{T^2_{\text{measured}}}}. \tag{6.1}$$

The standard deviation δ_θ varies from 0.10% to 1.25%.

The temperature measurements applied to solve the inverse heat conduction problem significantly influence the determination of the heat flux transferred from the analysed solid to the cooling air jet.

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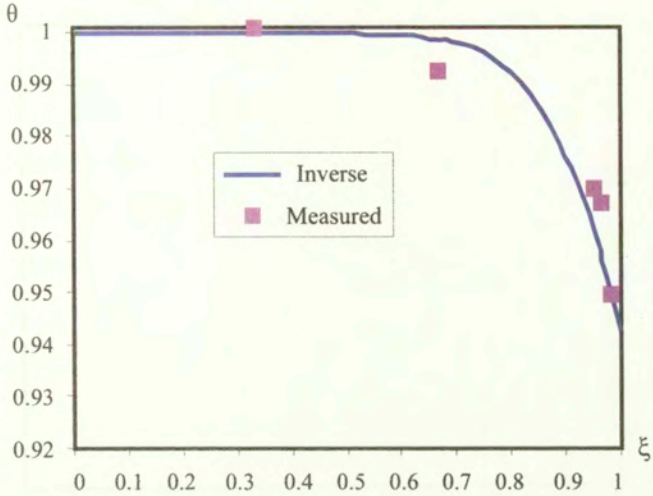

FIG. 7. Temperature distribution inside the analysed solid after 15s from the beginning of cooling.

7. The temporary Bi number values

On the basis of determined temperature distributions, applying the heat functions properties, the Bi number distributions during the cooling process were reconstructed

$$Bi = \frac{\left.\dfrac{\partial\theta(1,\tau)}{\partial\xi}\right|_{\xi=1}}{\theta(1,\tau)} = \frac{\left.\sum\limits_{i=1}^{N}\left(\sum\limits_{n=1}^{N} U_{ni}\, v_{n-1}(1,\tau)\right)\theta(\tau)\right|_{\xi=1}}{\theta(1,\tau)}. \tag{7.1}$$

The temporary Bi number values, shown in Fig. 8, slowly rise until the heat flux conducted on the surface of analysed solid achieves the maximum value which can be taken over by cooling air jet.

The fluctuations in the Bi number distribution are caused by sensitivity of the solution of the inverse heat conduction problem to the random temperature measurement errors. For that reason, the elimination of the random temperature measurements errors by appropriate approximation of the measured temperature distribution is essential.

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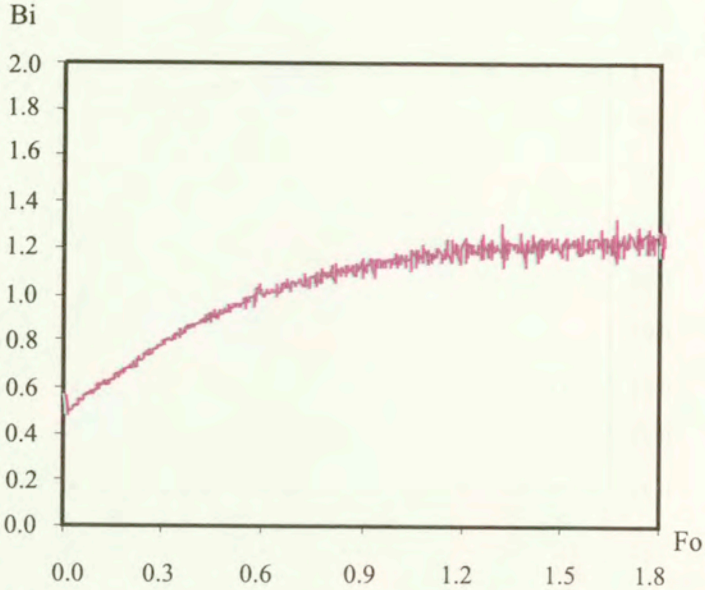

FIG. 8. The temporary Bi number values calculating on the basis of distorted temperature date.

8. Elimination of the random temperature measurement errors

For the purpose of obtaining the Bi number distribution devoid of fluctuation, the measured temperature distribution is approximated by the second-degree polynomials and by the $e^{\gamma\tau}$ function. The results of the Bi number distribution calculated on the basis of the approximate date were shown with the Bi number distribution calculated on the basis of the measured data, to demonstrate the effectiveness of the applied approximation.

The results of the Bi number distribution calculated on the basis of approximation of the temperature distribution by the second-degree polynomials, shown in Fig. 9, are not satisfying. The approximate Bi number values after achieving maximal value are decreasing, causing distortion in the Bi number distribution. In order to avoid unfavourable properties of the approximation of the second degree polynomials, the approximation of temperature distribution by the $e^{\gamma\tau}$ function was applied. On the basis of the results of Bi number distribution, shown in Fig. 10, it was confirmed that the approximation of the temperature distribution by the $e^{\gamma\tau}$ function gives a satisfactory solution.

Averaging of the temperature measurements taken inside the solid is another method of decreasing the influence of the random temperature measurement errors on Bi number distribution.

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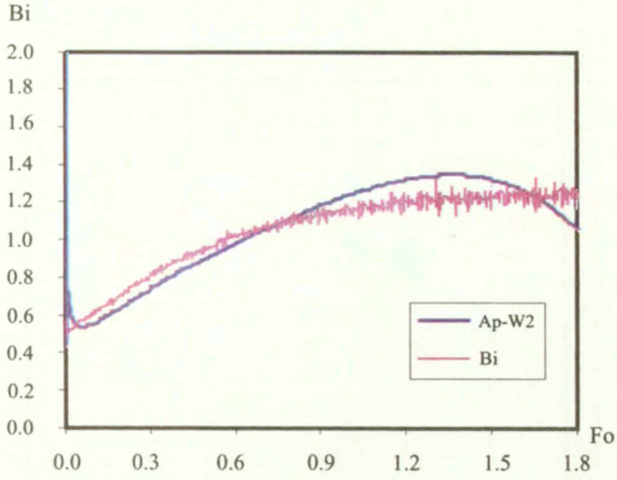

FIG. 9. The Bi number distribution approximated by second-degree polynomials.

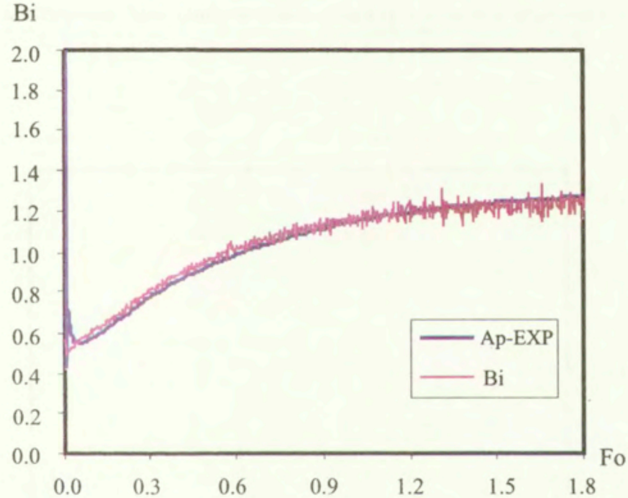

FIG. 10. The Bi number distribution approximated by $e^{\gamma\tau}$ function.

The fluctuation in the Bi number distribution calculated from the averaged temperature measurements, demonstrated in Fig. 12, is significantly smaller than the fluctuation in the solution calculated from the single temperature measurement, shown in Fig. 11. The application of averaged temperature measurements taken inside the solid to calculate the Bi number distribution reduces the influence of the random temperature measurement errors on the Bi number distribution.

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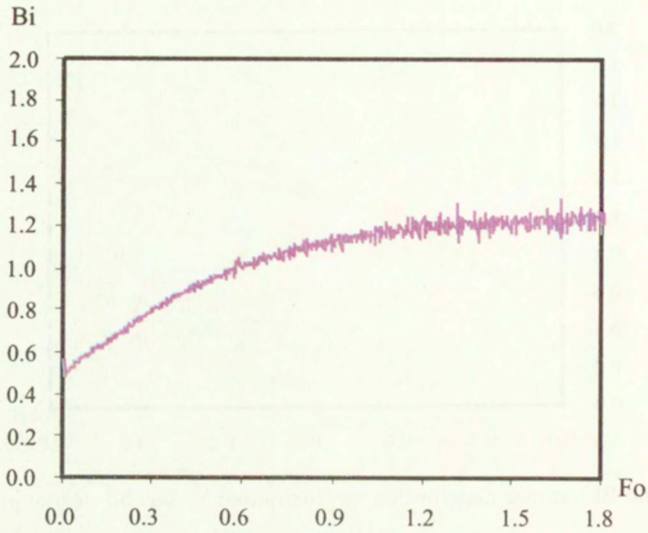

FIG. 11. The Bi number distribution calculated on the single temperature measurement.

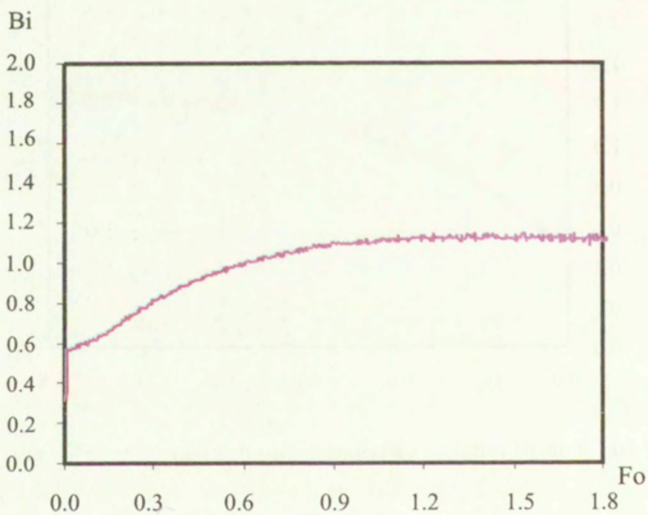

FIG. 12. The Bi number distribution calculated on the averaged temperature measurements.

9. The distribution of the heat transfer coefficient

The definition of the Bi number was applied to determine the heat transfer coefficient

$$h = \frac{Bi_{av}\, k_s}{Z}. \tag{9.1}$$

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The comparison of the heat transfer coefficient h determined from the solution of the inverse problem (the temporary Bi-number values) with the data given in the literature is shown in Fig. 13. The heat transfer coefficient h from the solution of the inverse problem was determined by averaging the Bi number, after the maximum value is reached by the heat flux which is taken over by the cooling air jet.

Literature data were applied to determine the accuracy of the presented method. The discrepancies in the distribution of heat transfer coefficients known from the literature are shown in Fig. 13. The author of the paper [1] claims in his works that the discrepancies in heat transfer coefficients distributions in the literature are caused by the errors which appeared during experiments, and by the variability of the structure of the jets generated by the differently shaped emitters and nozzles, the details of which are not mentioned in the publications. In consequence, there are discrepancies in heat transfer coefficients distributions in the literature data shown in Fig. 13

The values of heat transfer coefficients following from the solution of the inverse problem were compared with the data published in paper [1]. These data [1] were determined on the basis of the experimental research led in the Chair of Thermal Engineering, accomplished by applying the same test set-up which was used to obtain the data to solve the inverse problem. The air impinging jet was generated by applying the same nozzle. The accuracy of determination of the heat transfer coefficient h from the solution of the inverse heat conduction problem compared to the data from paper [1] was determined by

$$\delta_h = \sqrt{\frac{\left(h_{[1]} - h_{\text{inverse solution}}\right)^2}{h_{[1]}^2}}. \tag{9.2}$$

The standard deviation δ_h varies from 0.37% to 4.07%.

Comparison of the heat transfer coefficient h, determined by the solution of the inverse problem – calculated by applying average of nine temperature measurements taken inside the solid, with the date known from literature, is shown in Fig. 14.

The standard deviation of the solution of the inverse heat conduction problem – calculated by applying average temperature measurements with relation to data given in paper [1] was determine by

$$\delta_h = \sqrt{\frac{\left(h_{[1]} - h_{\text{inverse solution}}\right)^2}{h_{[1]}^2}} = 2.08\%. \tag{9.3}$$

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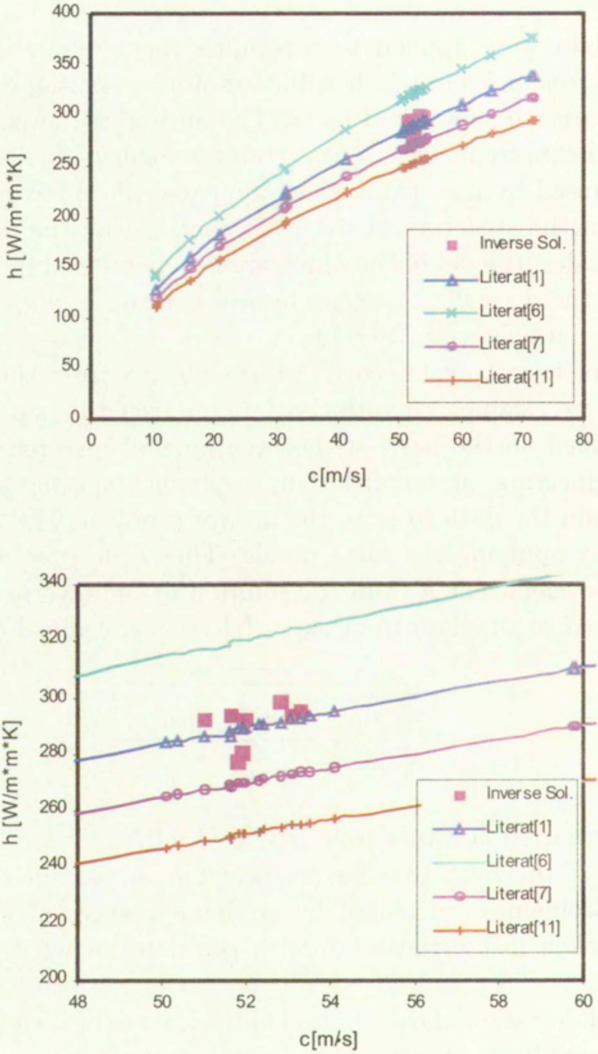

FIG. 13. Values of the heat transfer coefficient.

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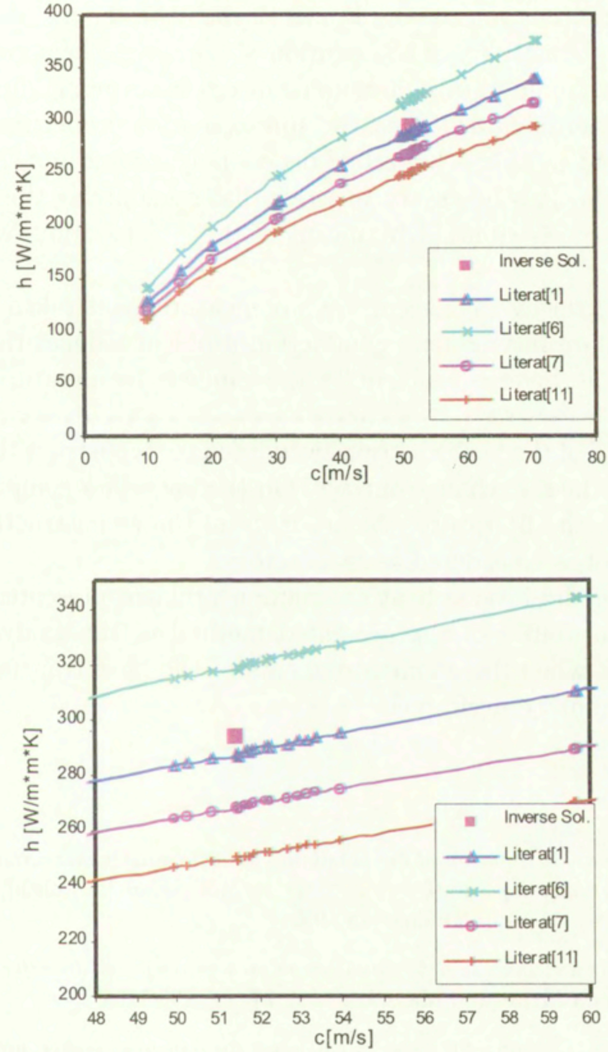

FIG. 14. Values of the heat transfer coefficient for the averaged data.

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10. Conclusions

The investigation was aimed at determination of the properties of the solution of the inverse heat conduction problem occurring in real conditions. The inverse heat conduction problem is solved by using the heat functions applied to formulate the base functions of the Finite Element Method.

Because of the sensitivity of the solution of the inverse heat conduction problem to the random temperature measurement errors occurring during laboratory and industrial experimental studies, the approximation properties of the temperature measurement by using the second degree polynomials and the $e^{\gamma\tau}$ function were demonstrated. It is necessary to stress that by applying the approximation of the temperature distribution by means of the $e^{\gamma\tau}$ function, we obtain satisfactory results.

Application of the averaged temperature measurements taken inside the solid in order to calculate inverse heat conduction problem reduces the sensitivity of the solution of the inverse problem to the random temperature measurement errors.

The properties of the heat functions were used to reconstruct the heat transfer coefficient on the heat exchange surface. On the basis of a comparison with the data known from the literature, the accuracy of the reconstruction of the heat transfer coefficient is considered as satisfactory.

The results of the inverse heat conduction problem presented in this paper confirm the applicability of the presented method to the analysis of the heat exchange process when direct measurements of the heat transfer coefficient or the heat flux cannot be performed.

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Stationary thermoelasticity and stochastic homogenization

B. GAMBIN[(1)], J. J. TELEGA[(1)] and L. V. NAZARENKO[(2)]

[(1)] *Institute of Fundamental Technological Research,*
Polish Academy of Sciences,
Świętokrzyska 21, 00-049 Warsaw, Poland
e-mail: bgambin@ippt.gov.pl, jtelega@ippt.gov.pl

[(2)] *Institute of Hydromechanics of NAS of Ukraine, Kiev, Ukraine*

THE AIM OF THE PAPER is twofold: first, the stochastic homogenization theorem formulated by DAL MASO and MODICA [9, 10] is extended to the case applicable to a class of nonlinear problems of mechanics. Second, this new theorem is applied to determine the effective thermoelastic response of the material body with stochastically periodic microstructure. As a result, one obtains the closed form of effective (homogenized) stored energy function. As a specific case, one-dimensional problem is solved analytically.

1. Introduction

EFFECTIVE MATERIAL moduli of nonhomogeneous, linear, thermoelastic solids were derived by various authors. FRANCFORT [11] solved the problem of homogenization of a thermoelastic solid with microperiodic structure. The method of two-scale asymptotic expansions was used to obtain effective thermoelastic constants, cf. [22]. The idea was further developed by GAŁKA *at al.* [12] in the case of diffusion in a thermoelastic body. Thermopiezoelectric composites were investigated in [13].

In the case of random microstructure various, rather engineering-type, stochastic approaches were used, cf. [23]. The method of conditional moments due to KHOROSHUN [16] was applied to predict the effective properties of stochastic composites. Particularly, the effective thermoelastic moduli of anisotropic composites with ellipsoidal inclusions were determined in [17]. Thermoelastic properties of porous anisotropic materials were investigated in [20]. The micromechanical approach based on the Green function technique, as well as the interfacial Hill operators, was applied in [6] to the analysis of thermoelastostatic behaviour of composites with coated randomly distributed inclusions. The local effective thermoelastic properties of graded random structure matrix composites were considered in [5] under the hypothesis of effective field homogeneity near the inclusions.

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Papers dealing with the application of rigorous homogenization methods to randomly inhomogeneous materials are not numerous [9, 10], cf. also [8, 14]. The aim our paper is to perform homogenization of randomly inhomogeneous thermoelastic media in the case of stochastically periodic microstructure. To this end we apply the method of stochastic Γ-convergence. As a specific case, one-dimensional problem is solved analytically.

We observe that the Γ-convergence method is applicable only to stationary thermoelasticity, the case investigated in this paper.

2. Stochastic homogenization theorem

The aim of the present paper is to determine the global thermoelastic response of the material body with stochastically periodic microstructure. To this end the method of stochastic Γ-convergence is used. As a result, one obtains the closed form of effective (homogenized) stored energy function. To find this function explicitly, provided that a stochastic microstructure is prescribed, one has to solve a counterpart of a so-called cell problem. Unfortunately, this can be done only in specific cases, Sec. 5. The microstructure is understood here as a real heterogeneous thermoelastic body whose properties vary rapidly and are stochastically periodic in space, see below. The real dimension of a single cell of periodicity is large enough to permit the application of the concept of continuum, but the number of cells is too large to apply any numerical procedure for solving the proper system of partial differential equations. To cope with such a difficulty, a passage to the limit with suitably defined small parameter is performed. The limit procedure is nothing else but smearing out the microheterogeneities, i.e. the number of cells goes to infinity and at the same time their characteristic dimension becomes infinitely small.

Our considerations are based on employing the notion of stochastic Γ - convergence. An alternative approach would consist in applying stochastic G - convergence [15, 18] or stochastic two-scale convergence in the mean [2, 3]. Suitable comments will be provided at the end of Sec. 3. In the present section we are going to formulate a general stochastic homogenization theorem. In Sec. 3 we provide the proof and comments.

Let us pass to the formulation of general stochastic homogenization theorem applicable to performinig homogenization of equations of stationary thermoelesticity. We denote by $\mathcal{A}_0$ the family of all bounded open subsets of $\mathbb{R}^N$. Obviously, from the physical point of view $N = 1, 2$, or 3. Nevertheless no such restriction on the space dimension is needed in Secs. 2 or 3. For every $A \in \mathcal{A}_0$ we denote by $W^{1,\alpha}(A)$ the Sobolev space of functions of $L^\alpha(A)$ whose first-order weak derivatives belong to $L^\alpha(A)$.

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Let us fix $\alpha > 1$, $\beta > 1$, $c_1 \ge c_0 > 0$. We denote by $\mathcal{F} = \mathcal{F}(c_0, c_1, \alpha, \beta)$ the class of all functionals

$$F : (L^\alpha_{loc}(\mathbb{R}^N)^N \times L^\beta_{loc}(\mathbb{R}^N) \times \mathcal{A}_0 \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$$

such that

$$F(\mathbf{u}, T, A) \tag{2.1}$$

$$= \begin{cases} \int\limits_A f[\mathbf{x}, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla T(\mathbf{x})] d\mathbf{x} & \text{if } \begin{cases} \mathbf{u}_{|A} \in W^{1,\alpha}(A)^N, \\ T_{|A} \in W^{1,\beta}(A), \end{cases} \\ +\infty, & \text{otherwise} \end{cases}$$

Here $f : \mathbb{R}^N \times \mathbb{E}^N_s \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is any function satisfying the following conditions:

(i) $f(\mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q})$ is Lebesgue measurable in $\mathbf{x}$ and convex in $\boldsymbol{\epsilon}$, ξ and $\mathbf{q}$;

(ii) $c_0(|\boldsymbol{\epsilon}|^\alpha + |\xi|^\beta + |\mathbf{q}|^\beta \le f(\mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q}) \le c_1(|\boldsymbol{\epsilon}|^\alpha + |\xi|^\beta + |\mathbf{q}|^{\beta+1})$ for each $(\mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q}) \in \mathbb{R}^N \times \mathbb{E}^N_s \times \mathbb{R} \times \mathbb{R}^N$.

We denote by $\mathbb{E}^N_s$ the space of symmetric $N \times N$ matrices; in the case of linear thermoelasticity $\alpha = \beta = 2$. Moreover $\mathbf{e}(\mathbf{u})$ denotes the small strain tensor

$$e_{ij}(\mathbf{u}) = u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.2}$$

where, as usual, $u_{(i,j)} = \dfrac{\partial u_i}{\partial x_j}$.

We observe that DAL MASO and MODICA [9, 10] studied only the integrands of the form $f(\mathbf{x}, \nabla u(\mathbf{x}))$, cf. also SAB [21].

In order to perform stochastic homogenization of equation of stationary thermoelasticity, a more general approach is obviously needed.

In Sec. 3 we shall consider a possibility of weakening the assumptions (i), (ii). After DAL MASO and MODICA [9, 10], we equip $\mathcal{F}$ with the metric d so that the $\mathcal{F}$ is a compact metric space. To define the metric d, we first introduce the ϵ-Yosida ($\epsilon > 0$) transform of $F \in \mathcal{F}$:

$$T_\varepsilon F(\mathbf{u}, T, A) = \inf\Big\{F(\mathbf{v}, R, A) + \frac{1}{\varepsilon}\int\limits_A |\mathbf{v} - \mathbf{u}|^\alpha d\mathbf{x} + \frac{1}{\varepsilon}\int\limits_A |R - T|^\beta d\mathbf{x} \Big| \mathbf{v} \in L^\alpha_{loc}(\mathbb{R}^N)^N, \ \ R \in L^\beta_{loc}(\mathbb{R}^N)\Big\}. \tag{2.3}$$

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Now we are in a position to define a distance on $\mathcal{F}$. Let us choose a countable dense subset $\mathcal{W} = \{\mathbf{w}_j | \, j \in \mathbb{N}\} \times \{g_j | \, j \in \mathbb{N}\}$ of $W^{1,\alpha}(\mathbb{R}^N)^N \times W^{1,\beta}(\mathbb{R}^N)$ and a countable subfamily $\mathcal{B} = \{B_k | k \in \mathbb{N}\}$ of $\mathcal{A}_0$. Here $\mathbb{N}$ denotes the set of natural numbers. For instance, $\mathcal{B}$ could be chosen as the family of all bounded open subsets of $\mathbb{R}^N$ which are finite unions of rectangles with rational vertices. Let us define for $F, G \in \mathcal{F}$

$$d(F,G) = \sum_{i,j,k=1}^{+\infty} \frac{1}{2^{i+j+k}} |\phi(T_{1/i}F(\mathbf{w}_j, g_j, B_k)) - \phi(T_{1/i}G(\mathbf{w}_j, g_j, B_k))|. \tag{2.4}$$

Here $\phi : \overline{\mathbb{R}} \to \mathbb{R}$ is any increasing, continuous bounded function. For instance, we may take $\phi = \arctan$ [9].

To prove that d is a distance on $\mathcal{F}$ it suffices to show that if $d(F,G) = 0$ then $F = G$. Indeed, Proposition 1.11 and Corollary 1.6 due to DAL MASO and MODICA [9], now extended to our more general case, are formulated as follows.

PROPOSITION 1.

(a) Let $F \in \mathcal{F}$, $\mathbf{u} \in L^{\alpha}_{loc}(\mathbb{R}^N)^N$, $T \in L^{\beta}_{loc}(\mathbb{R}^N)$, $A \in \mathcal{A}_0$.

Then

$$\lim_{\varepsilon \to 0^+} T_\varepsilon F(\mathbf{u}, T, A) = \sup_{\varepsilon > 0} T_\varepsilon F(\mathbf{u}, T, A) = F(\mathbf{u}, T, A).$$

(b) Let $\mathcal{W}$ be a dense subset of $W^{1,\alpha}(\mathbb{R}^N)^N \times W^{1,\beta}(\mathbb{R}^N)$ and $\mathcal{B}$ a dense subfamily of $\mathcal{A}_0$. If $F, G \in \mathcal{F}$ and $F(\mathbf{w}, g, B) = g(\mathbf{w}, g, B)$ $\forall (\mathbf{w}, g) \in \mathcal{W}$, $\forall B \in \mathcal{B}$ then $F = G$. □

Now we have to show that the metric space $(\mathcal{F}, d)$ is compact, hence complete and separable. To this end we have to introduce the notion of Γ-convergence. For more details the reader is referred to [7,10] and the relevant references cited therein. We observe that this type of variational convergence was introduced by E. De Giorgi and profoundly developed by the Italian School of the Calculus of Variations.

Let X be a metric space and let $\{F_\delta\}_{\delta>0}$ be a sequence of functions defined on X with values in $\overline{\mathbb{R}}$. For instance, in our case $X = L^{\alpha}(A)^N \times L^{\beta}(A)$. We say that $\{F_\delta\}$ $\Gamma(X)$-converges at a point $z_\infty \in X$ to $\lambda \in \overline{\mathbb{R}}$ if the following two conditions are satisfied:

$(\mathrm{A})_1$ $\lambda \le \liminf_{\delta \to 0^+} F_\delta(z_\delta)$ for any sequence $\{z_\delta\}_{\delta>0}$ converging in X to z_∞;

$(\mathrm{A})_2$ there exists a sequence $\{z_\delta\}_{\delta>0}$ converging in X to z_∞ such that $\limsup_{\delta\to 0^+} F_\delta(z_\delta) \le \lambda$.

In such a case we write $\lambda = \Gamma(X) \lim_{\delta \to 0^+} F_\delta(z_\infty)$. More precisely, we should write $\Gamma(X^-)$ instead of $\Gamma(X)$, cf. [7,9]. Since only the above notion of Γ-convergence is used in this paper, therefore we prefer to use our simpler notation.

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If there exists $F_\infty : X \to \overline{\mathbb{R}}$ such that

$$F_\infty(z) = \Gamma(X) \lim_{\delta\to 0^+} f_\delta(z), \quad \forall z \in X$$

we say that $\{F_\delta\}$ $\Gamma(X)$-converges to F_∞. Then from (A_1) and (A_2) we conclude that

$$(2.5) \qquad F_\infty(z_\infty) = \min\{\liminf_{\delta\to 0^+} F_\delta(z_\delta) | z_\delta \text{ converges in } X \text{to } z_\infty\}$$

for every $z_\infty \in X$. Consequently, the $\Gamma(X)$-limit F_∞ is determined uniquelly.

Let now $\{F_\delta\}$ be a sequence in $\mathcal{F}$. Then we write

$$(2.6) \qquad \Gamma(L^\alpha(A)^N \times L^\beta(A)) \lim_{\delta\to 0^+} F_\delta(\mathbf{u}, T) = F_\infty(\mathbf{u}, T)$$

$\forall(\mathbf{u}, T) \in L^\alpha(A)^N \times L^\beta(A)$, whenever $A \in \mathcal{A}_0$. More precisely, in (2.6) we should write $(F_\delta)_A$ and $(F_\infty)_A$ instead of F_δ and F_∞, cf. [9]. Indeed, each $F \in \mathcal{F}$ defines, for every $A \in \mathcal{A}_0$, a functional $F_A: \ L^\alpha(A)^N \times L^\beta(A) \to \overline{\mathbb{R}}$, cf. [9]. It suffices to extend $(\mathbf{u}, T) \in L^\alpha(A)^N \times L^\beta(A)$ to an element $(\widetilde{\mathbf{u}}, \widetilde{T})$ of $L^\alpha_{loc}(\mathbb{R}^N)^N \times L^\beta_{loc}(\mathbb{R}^N)$. We observe that the value of $F(\widetilde{\mathbf{u}}, \widetilde{T}, A)$ does not depend on the extension $(\widetilde{\mathbf{u}}, \widetilde{T})$ of $(\mathbf{u}, T)$.

As we shall see, the distance d on $\mathcal{F}$ has been chosen to be defined by (2.4) since then there is a link between d and Γ-convergence. Primarily, however, we formulate a compactness result.

PROPOSITION 2. The class $\mathcal{F}$ is compact for the $\Gamma(L^\alpha \times L^\beta)$-convergence, i.e. every sequence $\{F_\delta\}_{\delta>0}$ in $\mathcal{F}$ contains a subsequence that $\Gamma(L^\alpha \times L^\beta)$-converges to a functional $F_\infty \in \mathcal{F}$.

Proof. Let $\{F_\delta\}_{\delta>0}$ be a sequence in $\mathcal{F}$. By Theorems 2.4 and 4.3 of [7] there exists a subsequence $\{F_{\delta'}\}$ and a function $F_\infty: \ \mathbb{R}^N \times \mathbb{E}^N_s \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, non-negative, Lebesgue measurable in $\mathbf{x}$ and convex in the remaining variables, such that

$$(2.7) \quad \Gamma(L^\alpha(A)^N \times L^\beta(A)) \lim_{\delta'\to 0} (F_{\delta'})_A(\mathbf{u}, T) = \int_A f_\infty(\mathbf{x}, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla T(\mathbf{x})) d\mathbf{x}$$

for every $A \in \mathcal{A}_0$ and $(\mathbf{u}, T) \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$. If $(\mathbf{u}, T) \in (L^\alpha(A)^N \setminus W^{1,\alpha}(A)^N) \times (L^\beta(A) \setminus W^{1,\beta}(A))$ and $\{\mathbf{u}_\delta, T_\delta\}_{\delta>0}$ is a sequence converging in $L^\alpha(A)^N \times L^\beta(A)$ to $(\mathbf{u}, T)$, then $\{\mathbf{u}_\delta, T_\delta\}$ cannot have bounded subsequences in $W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$. Indeed, extending slightly Corollary 1.4 of DAL MASO and MODICA [9] we conclude that if $A \in \mathcal{A}_0$ then any bounded sequence in

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$W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$ contains a subsequence that converges in $L^\alpha_{loc}(A)^N \times L^\beta_{loc}(A)$, weakly in $W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$ and pointwise almost everywhere in A.

Consequently, either $\{\mathbf{u}_\delta, T_\delta\}_{\delta>0} \notin W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$, or

$$\lim_{\delta''\to 0} \int_A (|\mathbf{e}(\mathbf{u}_{\delta''})|^\alpha + |T_{\delta''}|^\beta + |\nabla T_{\delta''}|^\beta) d\mathbf{x} = +\infty$$

for each subsequence $\{\mathbf{u}_{\delta''}, T_{\delta''}\}_{\delta''>0}$ contained in $W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$.

In both cases we get

$$\liminf_{\delta\to 0^+} (F_{\delta'})_A(u_\delta) = +\infty.$$

We recall that $\{\delta'\}$ is a subsequence of $\{\delta\}$. For instance, $\{\delta\} = \left\{\dfrac{1}{n}\right\}$, $n \in \mathbb{N}$, $\{\delta'\} = \left\{\dfrac{1}{n_k}\right\}$.

Thus we arrive at

$$\Gamma(L^\alpha(A)^N \times L^\beta(A)) \lim_{\delta'\to 0^+} (F_{\delta'})_A(\mathbf{u}, T)$$
$$= \begin{cases} \displaystyle\int_A f_\infty(\mathbf{x}, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla T(\mathbf{x})) d\mathbf{x} & \text{if } (\mathbf{u}, T) \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A) \\ +\infty & \text{if } (\mathbf{u}, T) \in (L^\alpha(A)^N \setminus W^{1,\alpha}(A)^N) \times (L^\beta(A) \setminus W^{1,\beta}(A)) \end{cases}$$

for every $A \in \mathcal{A}_0$.

The r.h.s. of the last equality defines a functional $F_\infty : L^\alpha_{loc}(\mathbb{R}^N)^N \times L^\beta_{loc}(A) \times \mathcal{A}_0 \to \overline{\mathbb{R}}$ which is the $\Gamma(L^\alpha \times L^\beta)$-limit of $\{F_{\delta'}\}_{\delta'>0}$. It now remains to prove that $F_\infty \in \mathcal{F}$, i.e. that condition (ii) following formula (2.1) is satisfied. Indeed, we have

$$c_0 \int_A (|\mathbf{e}(\mathbf{u})|^\alpha + |T|^\beta + |\nabla T|^\beta) d\mathbf{x} \le F_{\delta'}(\mathbf{u}, T, A)$$
$$\le c_1 \int_A (1 + |\mathbf{e}(\mathbf{u})|^\alpha + |T|^\beta + |\nabla T|^\beta) d\mathbf{x}$$

for every $A \in \mathcal{A}_0$, $(\mathbf{u}, T) \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$. By taking the $\Gamma(L^\alpha(A)^N \times L^\beta(A))$-limit of these three terms we obtain

$$c_0 \int_A (|\mathbf{e}(\mathbf{u})|^\alpha + |T|^\beta + |\nabla T|^\beta) d\mathbf{x} \le F_\infty(\mathbf{u}, T, A)$$
$$\le c_1 \int_A (1 + |\mathbf{e}(\mathbf{u})|^\alpha + |T|^\beta + |\nabla T|^\beta) d\mathbf{x}$$

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for every $A \in \mathcal{A}_0$ and $(\mathbf{u},T) \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$.

Let now $B_\rho(\mathbf{u})$ be the ball in $\mathbb{R}^N$ with center at $\mathbf{x}$ and radius ρ, $|B_\rho(\mathbf{u})|$ denotes its Lebesgue measure. Furthermore, let us denote by $l_\mathbf{q}$ and l_ϵ the linear functions such that $l_\mathbf{q} : \mathbb{R}^N \to \mathbb{R}$, $l_\mathbf{q} = \mathbf{q}\cdot\mathbf{x}$, $l_\epsilon : \mathbb{E}^N_s \to \mathbb{R}$, $l_\epsilon = \boldsymbol{\epsilon}\,\mathbf{x}$. Then we get, cf. Remark 1.1 of [9]

$$(A) \qquad \lim_{\rho\to 0^+} \frac{1}{|B_\rho(\mathbf{u})|} \int\limits_{B_\rho(\mathbf{x})} f_\infty(\mathbf{x},\boldsymbol{\epsilon},\xi,\mathbf{q})d\mathbf{x} = f_\infty(\mathbf{x},\boldsymbol{\epsilon},\xi,\mathbf{q}),$$

a.e. $\mathbf{x} \in \mathbb{R}^N$, where $\xi \in \mathbb{R}$.

In virtue of the last relation we finally obtain

$$c_0(|\boldsymbol{\epsilon}|^\alpha + |\xi|^\beta + |\mathbf{q}|^\beta) \le f_\infty(\mathbf{x},\boldsymbol{\epsilon},\xi,\mathbf{q}) \le c_1(1 + |\boldsymbol{\epsilon}|^\alpha + |\xi|^\beta + |\mathbf{q}|^\beta).$$

It means that $F_\infty \in \mathcal{F}$ and the proof is complete. □

Now we are in a position to formulate a theorem which links d, Γ-convergence and ε-Yosida transform.

THEOREM 1. *Let $\{F_\delta\}_{\delta>0}$ be a sequence in $\mathcal{F}$ and $F_\infty \in \mathcal{F}$. Then the following conditions are equivalent:*

(1) $\lim_{\delta\to 0^+} d(F_\delta, F_\infty) = 0$;
(2) $\Gamma(L^\alpha \times L^\beta) \lim_{\delta\to 0^+} F_\delta = F_\infty$;
(3) $\lim_{\delta\to 0^+} (T_\varepsilon F_\delta)(\mathbf{u},T,A) = (T_\varepsilon F_\infty)(\mathbf{u},T,A)$
for each $\varepsilon > 0, (\mathbf{u},T) \in L^\alpha_{loc}(A)^N \times L^\beta_{loc}(A), A \in \mathcal{A}_0$.

Proof. The proof parallels that of Proposition 1 of DAL MASO and MODICA [9], with obvious extensions. Therefore it is omitted here. □

Random integral functionals

Let (Ω,Σ,P) be a fixed probability space, that is Ω is a set of elementary events, Σ is a σ-field of subsets of Ω and P is a probability measure on Σ.

A random integral functional is any measurable function $F : \Omega \to \mathcal{F}$ when Ω is equipped with the σ-field Σ and $\mathcal{F}$ with the Borel σ-field Σ_F generator by the distance d defined by Eq. (2.4), cf. [9].

If F is a random integral functional, the image measure $F_\#P$ on $\mathcal{F}$ defined by $(F_\#P)(S) = P(F^{-1}(S))$ for every $S \in \Sigma_F$, is called the *distribution law* of F. We shall write $F \sim G$ if F and G are random integral functionals having the same distribution law.

The additive group $\mathbb{Z}^N$ and the multiplicative group $\mathbb{R}^+$ act on $\mathcal{F}$ by the translation operator $\tau_\mathbf{z}$ ($\mathbf{z} \in \mathbb{Z}^N$) defined by

$$(2.8) \qquad (\tau_\mathbf{z} F)(\mathbf{u},T,A) = \int\limits_A f(\mathbf{x}+\mathbf{z},\mathbf{e}(\mathbf{u}),T,\nabla T)d\mathbf{x}$$

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and by the homothety operator ρ_ε ($\varepsilon > 0$) defined by

$$(\rho_\varepsilon F)(\mathbf{u}, T, A) = \int_A f\Big(\frac{\mathbf{x}}{\varepsilon}, \mathbf{e}(\mathbf{u}), T, \nabla T\Big) d\mathbf{x} \tag{2.9}$$

for every $(\mathbf{u}, T) \in W^{\alpha}_{loc}(\mathbb{R}^N)^N \times W^{1,\beta}_{loc}(\mathbb{R}^N)$, and $A \in \mathcal{A}_0$. We recall that $\mathbb{Z}$ stands for the set of integers. We observe that if the integrand f does not depend on T, but still depends on ∇T, then

$$(\tau_\mathbf{z} F)(\mathbf{u}, T, A) = F(\tau_\mathbf{z}\mathbf{u}, \tau_\mathbf{z} T, \tau_\mathbf{z} A),$$

where $(\tau_\mathbf{z}\mathbf{u})(\mathbf{x}) = \mathbf{u}(\mathbf{x} - \mathbf{z})$, $\tau_\mathbf{z} T(\mathbf{x}) = T(\mathbf{x} - \mathbf{z})$, $\tau_\mathbf{z} A = \{\mathbf{x} \in \mathbb{R}^N | \mathbf{x} - \mathbf{z} \in A\}$, and

$$(\rho_\varepsilon F)(\mathbf{u}, A) = \varepsilon^N(\rho_\varepsilon \mathbf{u}, \rho_\varepsilon T, \rho_\varepsilon A),$$

where $(\rho_\varepsilon \mathbf{u})(\mathbf{x}) = \dfrac{1}{\varepsilon}\mathbf{u}(\varepsilon\mathbf{x})$, $(\rho_\varepsilon T)(\mathbf{x}) = \dfrac{1}{\varepsilon} T(\varepsilon \mathbf{x})$, $\rho_\varepsilon A = \{\mathbf{x} \in \mathbb{R}^N | \varepsilon\mathbf{x} \in A\}$. In other words, during translation and homothety, T in (2.8) and (2.9) is treated as a parameter similarly to the case of periodic homogenization, cf. [4].

By virtue of Corollary 2.4 due to DAL MASO AND MODICA [9], we conclude that if F is a random integral functional and $\mathbf{z} \in \mathbb{R}^N$, $\varepsilon > 0$, then the functions $\tau_\mathbf{z} F$, $\rho_\varepsilon F : \Omega \to \mathcal{F}$ defined by

$$(\tau_\mathbf{z} F)(\omega) = \tau_\mathbf{z}(F(\omega)), \quad (\rho_\varepsilon F)(\omega) = \rho_\varepsilon(F(\omega)), \quad \forall \omega \in \Omega, \tag{2.10}$$

are random integral functionals. Furthemore, if G is another random integral functional such that $F \sim G$, then we have $\tau_\mathbf{z} F \sim \tau_\mathbf{z} G$ and $\rho_\varepsilon F \sim \rho_\varepsilon G$.

We say that $\{F_\varepsilon\}_{\varepsilon>0}$ is a *stochastic homogenization process* modelled on a fixed random integral functional F on Ω if $F_\varepsilon \sim \rho_\varepsilon F$ for every $\varepsilon > 0$, that is F_ε and $\rho_\varepsilon F$ have the same distribution law.

Let F be a random integral functional. We say that F is stochastically periodic if F and $\tau_\mathbf{z} F$ have the same law for every $\mathbf{z} \in \mathbb{Z}^N$.

Ergodicity is a well-established notion when applied to integrands. Here we need ergodicity in $\mathcal{F}$ with respect to $\mathbb{Z}^N$. After DAL MASO and MODICA [10] we say that a random integral functional $F \in \mathcal{F}$ is ergodic if $P[F \in S] = 0$ or 1 for every Σ_F-measurable subset S of $\mathcal{F}$ such that $\tau_\mathbf{z}(S) = S$ for every $\mathbf{z} \in \mathbb{Z}^N$.

For $F \in \mathcal{F}$, $A \in \mathcal{A}_0$, $\xi \in \mathbb{R}$ and $(\mathbf{u}_0, T_0) \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$ we may consider the following Dirichlet problem:

$$m_\xi(F, \mathbf{u}_0, T_0, A) = \min\Big\{ \int_A f(\mathbf{x}, \mathbf{e}(\mathbf{u}(\mathbf{x})), \xi, \nabla T(\mathbf{x})) d\mathbf{x} \\ \Big| (\mathbf{u} - \mathbf{u}_0, T - T_0) \in W^{1,\alpha}_0(A)^N \times W^{1,\beta}_0(A) \Big\} \tag{2.11}$$

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We conclude that $m_\xi(F, \mathbf{u}_0, T_0)$ is continuous in F with respect to the metric d. We stress that in (2.11) $\xi \in \mathbb{R}$ plays the role of a parameter.

Let $Q_{1/\varepsilon}$ be the cube

$$Q_{1/\varepsilon} = \{\mathbf{x} \in \mathbb{R}^N : |x_i| < 1/\varepsilon,\ i = 1, ..., N\}$$

and $|Q_{1/\varepsilon}| = (2/\varepsilon)^N$ its Lebesgue measure. We recall that $l_\mathbf{q} = \mathbf{q}\cdot\mathbf{x}$ and $l_\epsilon = \boldsymbol{\epsilon}\,\mathbf{x}$, where $\mathbf{q} \in \mathbb{R}^N$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$.

After these lengthy, yet necessary preparations, we are in a position to state our main homogenization theorem.

THEOREM 2. *Let F be a random integral functional and define $F_\varepsilon = \rho_\varepsilon F$. If f is periodic in law, then F_ε converges P-almost everywhere as $\varepsilon \to 0^+$ to a random integral F_0. Moreover, there exist $\Omega' \subset \Omega$ of full measure such that the limit*

$$\lim_{\varepsilon \to 0^+} \frac{m_\xi(F(\omega), l_\epsilon, l_\mathbf{q}, Q_{1/\varepsilon})}{|Q_{1/\varepsilon}|} = f_0(\omega, \boldsymbol{\epsilon}, \xi, \mathbf{q}) \tag{2.12}$$

exists for every $\omega \in \Omega'$, $\xi \in \mathbb{R}$, $\mathbf{q} \in \mathbb{R}^N$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$ and

$$F_0(\omega)(\mathbf{u}, T, A) = \int_A f_0[\omega, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla T(\mathbf{x})] d\mathbf{x} \tag{2.13}$$

for every $\omega \in \Omega'$, $A \in \mathcal{A}_0$, $(\mathbf{u}, T) \in L^\alpha_{loc}(A)^N \times L^\beta_{loc}(A)$ with $\mathbf{u}|_A \in W^{1,\alpha}(A)^N$, $T_{|A} \in W^{1,\beta}(A)$. Additionally, if F is ergodic, then F_0 is or equivalently $f_0(\omega, \boldsymbol{\epsilon}, \xi, \mathbf{q})$ does not depend on ω and

$$f_0(\boldsymbol{\epsilon}, \xi, \mathbf{q}) = \lim_{\varepsilon \to 0} \int_\Omega \frac{m_\xi(F(\omega), l_\epsilon, l_\mathbf{q}, Q_{1/\varepsilon})}{|Q_{1/\varepsilon}|} \tag{2.14}$$

for every $\xi \in \mathbb{R}$, $\mathbf{q} \in \mathbb{R}^N$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$.

3. Proof the stochastic homogenization theorem and comments

Prior to passing to the proof of Theorem 2 we are going to provide useful comments and additional indispensable tools.

First we observe that similar theorem was formulated by DAL MASO and MODICA [10] for a much simpler case where

$$(\rho_\varepsilon F)(\omega)(T, A) = \int_A f\Big(\omega, \frac{\mathbf{x}}{\varepsilon}, \nabla T(\mathbf{x})\Big) d\mathbf{x}. \tag{3.1}$$

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The same authors stated a stronger result as Theorem 3 in their another paper [9]. More precisely, for $\{F_\varepsilon\}_{\varepsilon>0}$ a stochastic process modelled on a stochastically periodic random integral functional F, in [9], it was assumed that there exists $M > 0$ such that the two families of random functions

$$(F(\cdot)(T,A))_{T\in L^\beta_{loc}(\mathbb{R}^N)}, \qquad (F(\cdot)(T,B))_{T\in L^\beta_{loc}(\mathbb{R}^N)}$$

are independent wherever $A, B \in \mathcal{A}_0$ with $\mathrm{dist}(A,B) > M$. Then a counter-part of formula (2.14) holds. In fact, $\{F_\varepsilon\}$ converges in probability as $\varepsilon \to 0^+$ to the single functional $F_0 \in \mathcal{F}$ independent of ω. Now, the functional F_0 is easily deduced from (2.14) by deleting $\boldsymbol{\epsilon}$ and ξ.

Let us recall the motions of convergence in probability and convergence in law cf. [9] and the relevant references cited therein.

We say that a sequence of random integral functional $\{F_\varepsilon\}_{\varepsilon>0}$ converges in probability to a random integral functional F_∞ if

$$\lim_{\varepsilon\to 0^+} P\Big\{\omega\in\Omega | d(F_\varepsilon(\omega), F_\infty(\omega)) > \eta\Big\} = 0, \forall\eta > 0 \tag{3.2}$$

where d is the distance on $\mathcal{F}$. It is well-known that any sequence converging in probability contains a subsequence which converges pointwise almost everywhere.

We say that $\{F_\varepsilon\}_{\varepsilon>0}$ converges in law to F_∞ if the corresponding laws $\mu_\varepsilon = F_{\varepsilon\#}P$ converge weakly $- *$ as $\varepsilon \to 0^+$ to $\mu_\infty = F_{\infty\#}P$, i.e.,

$$\lim_{\varepsilon\to 0^+} \int\limits_{\mathcal{F}} \varphi(F) d\mu_\varepsilon(F) = \int\limits_{\mathcal{F}} \varphi(F) d\mu_\infty(F) \tag{3.3}$$

for every continuous function $\varphi: F \to \mathbb{R}$.

Equivalently we may write

$$\langle \mu_\varepsilon, \varphi\rangle \to \langle \mu_\infty, \varphi\rangle \quad \text{as} \quad \varepsilon \to 0$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing defined on $C^*(\mathcal{F}) \times C(\mathcal{F})$; $C(\mathcal{F})$ denotes the space of continuous functions on the compact space $\mathcal{F}$ with the supremum norm and $C^*(\mathcal{F})$ is its dual.

The interrelationship between these two types of covergence is well-known, cf. [9, Prop. 2. 9].

PROPOSITION 3. Let F_∞ be a constant random integral functional, that is there exists $F_0 \in \mathcal{F}$ such that $F_\infty(\omega) = F_0$ for P-almost all $\omega \in \Omega$. Then convergence in law and convergence in probability toward F_∞ are equivalent.

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Let us comment on the stronger version of the stochastic homogenization theorem due to DAL MASO and MODICA [9]. This theorem is unsatisfactory for two important reasons. First, the indenpedence at large distances is not always verified. Such is the case of chessboard structure with cells of random size sketched in Fig. 3 of [10]. Second, whilst convergence in probability is the best possible if we give as in [9] the hypotheses in terms of laws, the problem arises whether there is almost everywhere convergence in the case $F_\varepsilon = \rho_\varepsilon F$, a result well-known in the case of linear stochastic homogenization, cf. [9, 10] and the relevant references therein.

Both these difficulties are overcome by Theorem 1 of DAL MASO and MODICA [10] and our more general Theorem 2.

Nonlinear stochastic homogenization of random media was also performed by SAB [21]. In essence, this author considers integrands of the type $f(\omega, \mathbf{x}, \boldsymbol{\epsilon})$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$, convex with respect to $\boldsymbol{\epsilon}$. Essential novelty lies in admitting the linear growth in $\boldsymbol{\epsilon}$, thus allowing for the study of homogenization of perfectly plastic media with random distribution of microheterogeneities. Such an approach is confined to deformational theory of plasticity, sometimes called Hencky plasticity. SAB [21] observed a correspondence between periodic media and statistically homogeneous ergodic (S.H.E.) media. We observe that this class is larger than the class of media described by stochastically periodic random integral functionals. SAB'S [21] approach involves an N-dimensional dynamical system on Ω, sometimes called the measure preserving flow. This dynamical system is assumed to be ergodic. Having introduced the dynamical system, not necessarily ergodic, one can exploit the stochastic differential calculus.

In order to prove our Theorem 2 we need a few additional results, cf. [1, 10].

A set function $\mu : \mathcal{A}_0 \to \mathbb{R}$ is said to be *subadditive* if

$$\mu(A) \le \sum_{k \in K} \mu(A_k) \tag{3.4}$$

for every $A \in \mathcal{A}_0$ and for every finite family $\{A_k\}_{k\in K}$ in $\mathcal{A}_0$ such that

$$A_k \subset A \ \ \forall k \in K, \qquad A_j \cap A_k = \emptyset \ \ \forall j, k \in K,\ j \ne k, \qquad |A - \bigcup_{k\in K} A_k| = 0.$$

Let $\mathcal{M} = \mathcal{M}(c)$ be the family of subadditive functions $\mu : \mathcal{A}_0 \to \mathbb{R}$ such that

$$0 \le \mu(A) \le c|A| \qquad \forall A \in \mathcal{A}_0$$

where $c > 0$ is a fixed constant. We denote by Σ_M the trace on $\mathcal{M}$ of the product Σ-algebra of $\mathbb{R}^{\mathcal{A}_0}$.

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Let (Ω, Σ, P) be a given probability space. $A\,(\Sigma, \Sigma_M)$-measurable map $\mu: \Omega \to \mathcal{M}$ is called a *subadditive process.*

The group $\mathbb{Z}^N$ acts on $\mathcal{M}$ by the formula

$$(\tau_{\mathbf{z}}\mu)(A) = \mu(\tau_{\mathbf{z}}A). \tag{3.5}$$

If $(-\mu)$ is subadditive then μ is called superadditive.

We say that a subadditive process is ergodic if $P[\mu \in S] = 0$ or 1 for Σ_M-measurable subset S of $\mathcal{M}$ such that $\tau_{\mathbf{z}}S = S$ for every $\mathbf{z} \in \mathbb{Z}^N$.

Essential role in the proof of Theorem 2. will play the following proposition, which is substantially the subadditive ergodic theorem due to AKCOGLU and KRENGEL [1].

PROPOSITION 4. Let $\mu: \Omega \to \mathcal{M}$ be a subadditive process. If μ is periodic in law, that is μ and $\tau_{\mathbf{z}}\mu$ have the same law for every $\mathbf{z} \in \mathbb{Z}^N$, then there exists a Σ-measurable function $\Phi: \Omega \to \mathbb{R}$ and a subset $\Omega' \subset \Omega$ of full measure such that

$$\lim_{\varepsilon \to 0^+} \frac{\mu(\omega)\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} = \lim_{t \to +\infty} \frac{\mu(\omega)(tQ)}{|tQ|} = \Phi(\omega) \tag{3.6}$$

for every $\omega \in \Omega'$ and for every cube $Q \in \mathbb{R}^N$. Moreover, if μ is ergodic then Φ is constant. □

For the proof the reader is refered to DAL MASO and MODICA [10].

Proof of Theorem 2. We divide it into two steps.

STEP 1. The random integrand $f(\omega, \mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q})$ does not depend on ξ. Then $m_\xi(F(\omega), l_\epsilon, l_{\mathbf{q}}, Q_{1/\varepsilon})$ appearing in Eq. (2.12) does not involve ξ and simply write $m(F(\omega), l_\epsilon, l_{\mathbf{q}}, Q_{1/\varepsilon})$. Now our proof is an extension of the proof of Theorem I due to DAL MASO and MODICA [10]. We recall that $l_{\mathbf{q}} = \mathbf{q} \cdot \mathbf{x}$, $l_\epsilon = \boldsymbol{\epsilon}\,\mathbf{x}$.

Let us fix $\mathbf{q} \in \mathbb{R}^N$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$ and define

$$\mu_{\mathbf{p}}(\omega)(A) = m(F(\omega), l_\epsilon, l_{\mathbf{q}}, A), \quad \forall \omega \in \Omega,\ \forall A \in \mathcal{A}_0$$

where $\mathbf{p} = (\boldsymbol{\epsilon}, \mathbf{q})$. Then $\mu_{\mathbf{p}}(\omega) \in \mathcal{M}(c)$ with $c = c_1(1 + |\mathbf{q}|^\alpha + |\boldsymbol{\epsilon}|^\beta)$ for every $\omega \in \Omega$, and $\mu_{\mathbf{p}}: \Omega \to \mathcal{M}$ is (Σ, Σ_M)-measurable since $m(\cdot, l_\epsilon, l_{\mathbf{q}}, A)$ is continuous

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on $\mathcal{F}$ equipped with the distance d. For every $\mathbf{z} \in \mathbb{Z}^N$, $\omega \in \Omega$, $A \in \mathcal{A}_0$ we have

$$(\tau_{\mathbf{z}}\mu_{\mathbf{p}})(\omega)(A) = \mu_{\mathbf{p}}(\omega)(\tau_{\mathbf{z}}A) = \min_{(\mathbf{u},T)}\Big\{(\tau_{\mathbf{z}}F)(\omega)(\tau_{-\mathbf{z}}\mathbf{u}, \tau_{-\mathbf{z}}T, A)$$

$$\mid \tau_{-\mathbf{z}}\mathbf{u} - \tau_{-\mathbf{z}}l_{\epsilon} \in W_0^{1,\alpha}(A)^N,\ \tau_{-\mathbf{z}}T - \tau_{-\mathbf{z}}l_{\mathbf{q}} \in W_0^{1,\beta}(A)\Big\}$$

$$= \min_{(\mathbf{v},R)}\Big\{(\tau_{\mathbf{z}}F)(\omega)(\mathbf{v} + l_{\epsilon}(\mathbf{z}), R + l_{\mathbf{q}}(\mathbf{z}), A)$$

$$\mid \mathbf{v} - l_{\epsilon} \in W_0^{1,\alpha}(A)^N,\ R - l_{\mathbf{q}} \in W_0^{1,\beta}(A)\Big\}.$$

Since the integrand of F depends only on $\mathbf{x}$, $\mathbf{e}(\mathbf{u})$ and ∇T, therefore

$$(\tau_{\mathbf{z}}F)(\mathbf{v} + l_{\epsilon}(\mathbf{z}), R + l_{\mathbf{q}}(\mathbf{z}), A) = (\tau_{\mathbf{z}}F)(\omega)(\mathbf{v}, R, A).$$

Hence

$$(\tau_{\mathbf{z}}\mu_{\mathbf{p}})(\omega)(A) = m((\tau_{\mathbf{z}}F)(\omega), l_{\epsilon}, l_{\mathbf{q}}, A),$$

for every $\mathbf{z} \in \mathbb{Z}^N$, $\omega \in \Omega$, $A \in \mathcal{A}_0$. Thus $\mu_{\mathbf{p}}$ is periodic in law because $\tau_{\mathbf{z}}F$ and F have the same law and $m(\cdot, l_{\epsilon}, l_{\mathbf{q}}, A)$ is continuous on $\mathcal{F}$.

In virtue of Proposition 4 we conclude that there exist a subset $\Omega'_{\mathbf{p}} \subset \Omega$ of full measure and a Σ-measurable function $\Phi_{\mathbf{p}} :\ \Omega \to \mathbb{R}$ such that

$$\lim_{t\to+\infty} \frac{\mu_{\mathbf{p}}(\omega)(tQ)}{|tQ|} = \Phi_{\mathbf{p}}(\omega)$$

for every $\omega \in \Omega'_{\mathbf{p}}$ and for every cube $Q \in \mathbb{R}^N$. Let now $Q_{1/\varepsilon}$ be the cube defined in Sec. 2 and let $f_0 :\ \Omega \times \mathbb{E}^N_s \times \mathbb{R}^N \to \mathbb{R}$ be the function defined by

$$f_0(\omega, \epsilon, \mathbf{q}) = \limsup_{\varepsilon\to 0^+} \frac{\mu_{\mathbf{p}}(\omega)(Q_{1/\varepsilon})}{|Q_{1/\varepsilon}|} \quad \forall(\omega, \epsilon, \mathbf{q}) \in \Omega \times \mathbb{E}^N_s \times \mathbb{R}^N.$$

We observe that the functions

$$\mathbf{p} = (\epsilon, \mathbf{q}) \to \frac{\mu_{\mathbf{p}}(\omega)(A)}{|A|} \quad (\omega \in \Omega,\ A \in \mathcal{A}_0)$$

are convex and equibounded between 0 and $c_1(1 + |\mathbf{q}|^{\beta} + |\epsilon|^{\alpha})$, hence locally equicontinuous. The convexity follows from the convexity in $(\mathbf{u}, T)$ of $F(\omega)(\mathbf{u}, T, A)$. Consequently $f_0(\omega, \epsilon, \mathbf{q})$ is convex in $(\epsilon, \mathbf{q})$. Let us set

$$\Omega' = \bigcap_{\mathbf{p}\in\mathbb{Q}^N} \Omega'_{\mathbf{p}}$$

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where $\mathbb{Q}$ is the set of rational numbers. We have $P(\Omega') = 1$ and

$$\lim_{\varepsilon\to 0^+} \frac{\mu_{\mathbf{p}}(\omega)\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} = f_0(\omega, \boldsymbol{\epsilon}, \mathbf{q})$$

for every $\omega \in \Omega'$, $\mathbf{p} = (\boldsymbol{\epsilon}, \mathbf{q}) \in \mathbb{E}_s^N \times \mathbb{R}^N$ and for every cube Q in $\mathbb{R}^N$. Furthermore, we get

$$\mu_{\mathbf{p}}(\omega)\left(\frac{1}{\varepsilon}Q\right) = \left(\frac{1}{\varepsilon}\right)^N m((\rho_\varepsilon F)(\omega), l_\epsilon, l_{\mathbf{q}}, Q).$$

Hence, since $\rho_\varepsilon F = F_\varepsilon$, we obtain

$$\lim_{\varepsilon\to 0^+} \frac{m(F_\varepsilon(\omega), l_\epsilon, l_{\mathbf{q}}, Q)}{|Q|} = f_0(\omega, \boldsymbol{\epsilon}, \mathbf{q})$$

for every $\omega \in \Omega'$, $\boldsymbol{\epsilon} \in \mathbb{E}_s^N$, $\mathbf{q} \in \mathbb{R}^N$ and for every cube in $\mathbb{R}^N$. In virtue of Proposition 2, for every $\omega \in \Omega'$ there exists an integral functional $F_0(\omega) \in \mathcal{F}$ such that $F_\varepsilon(\omega)$ $\Gamma(L^\alpha \times L^\beta)$ converges to $F_0(\omega)$ as $\varepsilon \to 0^+$. More precisely, there exists such a subsequence still denoted by F_ε.

Let us calculate the integrand $g_0(\omega, \mathbf{x}, \boldsymbol{\epsilon}, \mathbf{q})$ of $F_0(\omega)$. Fix $\omega \in \Omega'$ and set

$$Q_\rho(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^N : |y_i - x_i| < \rho,\ i = 1, ..., N\}.$$

Taking into account formula (A) of Sec. 2 and the continuity of $m(\cdot, l_\epsilon, l_{\mathbf{q}}, A)$, we conclude that there exist a subset $\mathcal{N}$ of $\mathbb{R}^N$ with $|\mathcal{N}| = 0$ such that

$$\begin{aligned} g_0(\omega, \mathbf{x}, \boldsymbol{\epsilon}, \mathbf{q}) &= \lim_{\rho\to 0^+} \frac{m(F_0(\omega), l_\epsilon, l_{\mathbf{q}}, Q_\rho(\mathbf{x}))}{|Q_\rho(\mathbf{x})|} \\ &= \lim_{\rho\to 0^+} \lim_{\varepsilon\to 0^+} \frac{m(F_\varepsilon(\omega), l_\epsilon, l_{\mathbf{q}}, Q_\rho(\mathbf{x}))}{|Q_\rho(\mathbf{x})|} \\ &= \lim_{\rho\to 0^+} \lim_{\varepsilon\to 0^+} \frac{\mu_{\mathbf{p}}(\omega)\left(\frac{1}{\varepsilon}Q_\rho(\mathbf{x})\right)}{\left|\frac{1}{\varepsilon}Q_\rho(\mathbf{x})\right|} = f_0(\omega, \boldsymbol{\epsilon}, \mathbf{q}) \end{aligned}$$

for every $\mathbf{x} \in \mathbb{R}^N \setminus \mathcal{N}$, $\boldsymbol{\epsilon} \in \mathbb{E}_s^N$, $\mathbf{q} \in \mathbb{R}^N$. Thus we get

$$F_0(\omega)(\mathbf{u}, T, A) = \int_A f_0[\omega, \mathbf{e}(\mathbf{u}(\mathbf{x})), \nabla T(\mathbf{x})] d\mathbf{x}$$

for every $\omega \in \Omega'$, $A \in \mathcal{A}_0$, $(\mathbf{u}, T) \in L^\alpha_{loc}(\mathbb{R}^N)^N \times L^\beta_{loc}(\mathbb{R}^N)$ such that $(\mathbf{u}, T)|_A \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$.

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If F is ergodic, then $\mu_{\mathbf{p}}$ is ergodic and on account of Proposition 4, $\Phi_{\mathbf{p}}$, and thus also f_0 do not depend on ω.

STEP 2. Let now $f = f(\omega, \mathbf{x}, \boldsymbol{\epsilon}, T, \mathbf{q})$. On account of Proposition 2, for a fixed $\omega \in \Omega$, there exists a subsequence of $\{F_\varepsilon\}_{\varepsilon>0}$, still denoted by $\{F_\varepsilon\}$ such that $F_\varepsilon(\omega)\ \Gamma(L^\alpha \times L^\beta)$ converges to

$$F_\infty(\omega)(\mathbf{u}, T, A) = \int\limits_{\Omega} f_0[\omega, \mathbf{x}, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla T(\mathbf{x})] d\mathbf{x}$$

for each $A \in \mathcal{A}_0$, $(\mathbf{u}, T) \in L^\alpha_{loc}(\mathbb{R}^N)^N \times L^\beta_{loc}(\mathbb{R}^N)$ with $(\mathbf{u}, T)|_A \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$.

Let now $T \in C^1(\mathbb{R}^N)$ and consider the function, cf. BRAIDES [4] in the case of periodic homogenization

$$f_T(\omega, \mathbf{x}, \boldsymbol{\epsilon}, \mathbf{q}) = f(\omega, \mathbf{x}, \boldsymbol{\epsilon}, T(\mathbf{x}), \mathbf{q}).$$

To the function f_T we may apply Step 1 and write

$$\lim_{\varepsilon \to 0^+} \frac{m(F_T(\omega), l_\epsilon, l_\mathbf{q}, Q_{1/\varepsilon})}{|Q_{1/\varepsilon}|} = f_{0T}(\omega, \boldsymbol{\epsilon}, \mathbf{q}) \tag{3.7}$$

for every $\omega \in \Omega'$, $(\boldsymbol{\epsilon}, \mathbf{q}) \in \mathbb{E}^N_s \times \mathbb{R}^N$, and

$$F_{0T}(\mathbf{u}, R, A) = \int\limits_{A} f_0[\omega, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla R(\mathbf{x})] d\mathbf{x}$$

Still by Step 1 and (3.7) we get

$$f_\infty(\omega, \mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q}) = f_{0\xi}(\omega, \boldsymbol{\epsilon}, \mathbf{q}) = f_0(\omega, \boldsymbol{\epsilon}, \xi, \mathbf{q}).$$

Ergodicity implies that f_0 does not depend on ω. Thus the proof of Theorem 2 is complete. □

REMARK 1.

(i) The easiest way of proving ergodicity of F is to verify a mixing condition (or independence at large distances), cf. [9].

(ii) A random integrand f is ergodic if it satisfies the following mixing condition [9]:

$$\lim_{\substack{|\mathbf{z}| \to +\infty \\ \mathbf{z} \in \mathbb{Z}^N}} P\Big(\{\omega \in \Omega | f(\omega, \mathbf{x}_i, \boldsymbol{\epsilon}_i, \xi, \mathbf{q}_i) > s_i\ \forall i \in I,$$

$$f(\omega, \mathbf{y}_j + \mathbf{z}, \boldsymbol{\Delta}_j, \xi, \mathbf{r}_j) > t_j\ \forall j \in J\}\Big)$$

$$= P\Big(\{\omega \in \Omega | f(\omega, \mathbf{x}_i, \boldsymbol{\epsilon}_i, \xi, \mathbf{q}_i) > s_i\ \forall i \in I\}\Big)$$

$$\times P\Big(\{\omega \in \Omega | f(\omega, \mathbf{y}_j, \boldsymbol{\Delta}_j, \xi, \mathbf{r}_j) > t_j\ \forall j \in J\}\Big)$$

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for every pair of finite families $\{(\mathbf{x}_i, \boldsymbol{\epsilon}_i, \mathbf{q}_i\}_{i\in I}$ and $\{(\mathbf{y}_j, \boldsymbol{\Delta}_j, \mathbf{r}_j\}_{j\in J}$ in $\mathbb{R}^N \times \mathbb{E}_s^N \times \mathbb{R}^N$. Here $\xi \in \mathbb{R}$ is treated as a parameter. Then one can extend Theorem III due to DAL MASO and MODICA [10] and prove that, for instance, if f is ergodic, then F is ergodic.
Ergodicity of F also follows if a measure preserving ergodic flow on Ω is introduced. Then the integrand is ergodic and the ergodicity of F is satisfied.

(iii) It seems that Theorem 2 can be weakened by assuming the convexity of integrands only with respect to $\boldsymbol{\epsilon} \in \mathbb{E}_s^N$ and $\mathbf{q} \in \mathbb{R}^N$. Then appropriate conditions on f are specified by BRAIDES [4]. This author performed the so-called periodic nonuniform homogenization. It means that after homogenization the integrand f_0 depends additionally on the macroscopic variable $\mathbf{x} \in V$, where V denotes a domain in $\mathbb{R}^N$ occupied by the considered body. The authors of the present paper are not aware whether any results of this type are available for random media; we mean here non-uniform stochastic homogenization of functionals. The only available approach is due to BOURGEAT *et al.* [3]. These authors introduced the motion of stochastic two-scale convergence in the mean which allows for treating the media remaining macroscopically inhomogeneous.

4. Thermoelastic stochastically periodic composite

Theorem 1 is general and covers a broad class of stochastic microstructures. In the remaining part of the paper we will focus on specific microstructures. More precisely, we consider a two-phase thermoelastic composite, occupying a domain $A \subset \mathbb{R}^N$. The phases are located randomly in periodic cubic cells with a given distribution. The dimensionless parameter ε is equal to the ratio of the length of the cell l and the characteristic dimension of the body L, $\varepsilon = l/L$.

The classical Duhamel-Neumann relations are satisfied at an arbitrary point $\mathbf{x} \in A$:

$$\boldsymbol{\sigma} = \boldsymbol{\lambda}^{\varepsilon}(\omega, \mathbf{x})\mathbf{e} - \beta^{\varepsilon}(\omega, \mathbf{x})T, \quad \boldsymbol{\beta}^{\varepsilon}(\omega, \mathbf{x}) \equiv \boldsymbol{\lambda}^{\varepsilon}(\omega, \mathbf{x})\boldsymbol{\alpha}^{\varepsilon}(\omega, \mathbf{x}), \tag{4.1}$$

where the strain-displacement is linear

$$e_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{4.2}$$

The Fourier relation takes now the form

$$\mathbf{q}^{\varepsilon}(\omega, \mathbf{x}) = -\varkappa^{\varepsilon}(\omega, \mathbf{x})\nabla T, \tag{4.3}$$

where $\omega \in \Omega$. Here, $\boldsymbol{\sigma}$ is the stress tensor; $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ are the tensors of the elastic moduli and thermal expansion; T is the temperature increment; $\mathbf{q}$ is the heat flux whiles $\varkappa$ denotes the conductivity tensor.

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We recall that

$$\boldsymbol{\lambda}^{\varepsilon}(\omega,\mathbf{x}) = \boldsymbol{\lambda}\Big(\omega,\frac{\mathbf{x}}{\varepsilon}\Big), \quad \varkappa^{\varepsilon}(\omega,\mathbf{x}) = \varkappa\Big(\omega,\frac{\mathbf{x}}{\varepsilon}\Big), \quad etc.$$

The random functional F^{ε}, given by:

$$F^{\varepsilon}(\omega)(\mathbf{u},T,A) \equiv \begin{cases} \displaystyle\int_A f^{\varepsilon}(\omega,\mathbf{x},\mathbf{e}(\mathbf{u});T,\nabla T)d\mathbf{x} & \text{if } (\mathbf{u},T) \in H^1(A)^N \times H^1(A) \\ +\infty & \text{otherwise} \end{cases} \tag{4.4}$$

where

$$f^{\varepsilon}(\omega,\mathbf{x},\mathbf{e}(\mathbf{u});T,\nabla T) = \frac{1}{2}\{[\mathbf{e}(\mathbf{u}) - T\boldsymbol{\alpha}^{\varepsilon}(\omega,\mathbf{x})]^{\top}\boldsymbol{\lambda}^{\varepsilon}(\omega,\mathbf{x}) \cdot [\mathbf{e}(\mathbf{u}) - T\boldsymbol{\alpha}^{\varepsilon}(\omega,\mathbf{x})] + (\nabla T)^{\top}\varkappa^{\varepsilon}(\omega,\mathbf{x})\nabla T\}. \tag{4.5}$$

Under usual symmetry and coercivity assumptions pertaining to matrices λ and κ the integrand $f(\omega,\mathbf{x},\boldsymbol{\epsilon},\xi,\mathbf{q})$ is convex in $(\boldsymbol{\epsilon},\xi,\mathbf{q}) \in \mathbb{E}^N_s \times \mathbb{R} \times \mathbb{R}^N$ and $f \in \mathcal{F}$ where $\alpha = \beta = 2$. The convexivity in ξ results from linearity of the transformation $(\boldsymbol{\epsilon},\xi) \longrightarrow (\boldsymbol{\epsilon} - \xi\boldsymbol{\alpha})$. Here the superscript $\top$ stands for transposition.

The moduli $\boldsymbol{\lambda}$, $\varkappa$, $\boldsymbol{\alpha}$ possess the usual properties, cf. [11-13].

We have

$$\boldsymbol{\sigma} = \frac{\partial f^{\varepsilon}}{\partial \mathbf{e}}, \qquad -\mathbf{q}^{\varepsilon} = \frac{\partial f^{\varepsilon}}{\partial \nabla T}. \tag{4.6}$$

Now we introduce the stochastically periodic structure as follows: let $(X^{\varepsilon}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^N}$ be a family of independent random variables defined on a probability space (Ω,Σ,P)

$$\begin{aligned} &P\{\omega\in\Omega: \ \ X^{\varepsilon}_{\mathbf{k}}(\omega) = 1\} = c_1, \\ &P\{\omega\in\Omega: \ \ X^{\varepsilon}_{\mathbf{k}}(\omega) = 0\} = 1 - c_1 = c_2 \end{aligned} \tag{4.7}$$

for every $\varepsilon > 0$, $\mathbf{k} \in \mathbb{Z}^N$ and for $c_1 \in]0,1[$ fixed.

For every $\varepsilon > 0$ and $\mathbf{k} \in \mathbb{Z}^N$, let $Q^{\varepsilon}_{\mathbf{k}}$ be the cube in $\mathbb{R}^N$ defined by

$$Q^{\varepsilon}_{\mathbf{k}} = \{\mathbf{x}\in\mathbb{R}^N: \quad \varepsilon k_i \le x_i < \varepsilon(k_i+1), \quad i = 1,...,N\} \tag{4.8}$$

and denote by $I^{\varepsilon}_{\mathbf{k}}$ its characteristic function.

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Furthermore, let us define the stochastically periodic characteristic function

$$\chi^{\varepsilon}(\omega, \mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} X^{\varepsilon}_{\mathbf{k}}(\omega) I^{\varepsilon}_{\mathbf{k}}(\mathbf{x}), \qquad \omega \in \Omega,\ \mathbf{x} \in \mathbb{R}^N \tag{4.9}$$

where $I^{\varepsilon}_{\mathbf{k}}(\mathbf{x})$ is periodic in $\mathbf{x}$. More precisely, $I^{\varepsilon}_{\mathbf{k}}$ is the characteristic function of the cube $Q^{\varepsilon}_{\mathbf{k}}$:

$$I^{\varepsilon}_{\mathbf{k}}(\mathbf{u}) = \begin{cases} 1, & \text{if } \mathbf{x} \in Q^{\varepsilon}_{\mathbf{k}}, \\ 0, & \text{otherwise.} \end{cases}$$

We also set

$$\chi^{\varepsilon}(\omega, \mathbf{x}) = \chi\Big(\omega, \frac{\mathbf{x}}{\varepsilon}\Big), \tag{4.10}$$

where $\chi(\omega, \cdot)$ is a 1-periodic function, since

$$I^{\varepsilon}_{\mathbf{k}}(\mathbf{x}) = I^{1}_{\mathbf{k}}\Big(\frac{\mathbf{x}}{\varepsilon}\Big).$$

REMARK 2. If $\mathbf{y} = \mathbf{x}/\varepsilon$, one recovers the microvariable well-known in periodic homogenization [22]. □

For $\varepsilon > 0$ the values of the coefficients $\boldsymbol{\lambda}^{\varepsilon}$, $\boldsymbol{\alpha}^{\varepsilon}$ and $\boldsymbol{\varkappa}^{\varepsilon}$ may be determined by the function $\chi^{\varepsilon}(\omega, \mathbf{x})$ and positive definite moduli $\boldsymbol{\lambda}^{(i)}$, $\boldsymbol{\beta}^{(i)}$, $\boldsymbol{\alpha}^{(i)}$, $\boldsymbol{\varkappa}^{(i)}$, $i = 1, 2$, characterizing each of the two phases we write

$$\begin{aligned}
\boldsymbol{\lambda}^{\varepsilon}(\omega, \mathbf{x}) &= \boldsymbol{\lambda}^{(1)}\chi^{\varepsilon}(\omega, \mathbf{x}) + \boldsymbol{\lambda}^{(2)}(1 - \chi^{\varepsilon}(\omega, \mathbf{x})),\\
\boldsymbol{\beta}^{\varepsilon}(\omega, \mathbf{x}) &= \boldsymbol{\beta}^{(1)}\chi^{\varepsilon}(\omega, \mathbf{x}) + \boldsymbol{\beta}^{(2)}(1 - \chi^{\varepsilon}(\omega, \mathbf{x})),\\
\boldsymbol{\alpha}^{\varepsilon}(\omega, \mathbf{x}) &= \boldsymbol{\alpha}^{(1)}\chi^{\varepsilon}(\omega, \mathbf{x}) + \boldsymbol{\alpha}^{(2)}(1 - \chi^{\varepsilon}(\omega, \mathbf{x})),\\
\boldsymbol{\varkappa}^{\varepsilon}(\omega, \mathbf{x}) &= \boldsymbol{\varkappa}^{(1)}\chi^{\varepsilon}(\omega, \mathbf{x}) + \boldsymbol{\varkappa}^{(2)}(1 - \chi^{\varepsilon}(\omega, \mathbf{x})).
\end{aligned}$$

The moduli $\lambda^{(i)}$, $\beta^{(i)}$, $\alpha^{(i)}$, $\varkappa^{(i)}$ are constant.

Now we are in a position to apply Theorem 1. Particularly, we have

$$(\rho_{\varepsilon}F)(\omega)(\mathbf{u}, T, A) = \int_A f\Big(\omega, \frac{\mathbf{x}}{\varepsilon}, \mathbf{e}(\mathbf{u}); T, \nabla T) d\mathbf{x}, \tag{4.11}$$

where

$$f^{\varepsilon}(\omega, \mathbf{x}, \mathbf{e}(\mathbf{u}); T, \nabla T) = f\Big(\omega, \frac{\mathbf{x}}{\varepsilon}, \mathbf{e}(\mathbf{u}); T, \nabla T\Big),$$

$$\begin{aligned}
f\Big(\omega, \frac{\mathbf{x}}{\varepsilon}, \mathbf{e}(\mathbf{u}); T, \nabla T\Big) &= \frac{1}{2}\Big[\mathbf{e}(\mathbf{u}) - T\boldsymbol{\alpha}\Big(\omega, \frac{\mathbf{x}}{\varepsilon}\Big)\Big]^{\top} \cdot \boldsymbol{\lambda}\Big(\omega, \frac{\mathbf{x}}{\varepsilon}\Big)\\
&\cdot \Big[\mathbf{e}(\mathbf{u}) - T\boldsymbol{\alpha}\Big(\omega, \frac{\mathbf{x}}{\varepsilon}\Big)\Big] + + \frac{1}{2}(\nabla T)^{\top} \boldsymbol{\varkappa}\Big(\omega, \frac{\mathbf{x}}{\varepsilon}\Big)\nabla T.
\end{aligned} \tag{4.12}$$

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The Γ-limit of $F^{\varepsilon}_{\varepsilon>0}$ is now given by Theorem 1, where now f^{ε} is defined by (3.12).

The limit functional F_0 is given by a non-random integral functional:

$$(4.13)\quad F_0(\omega)(\mathbf{u},T,A) \equiv \begin{cases} \int\limits_A f_0(\mathbf{e}(\mathbf{u});T,\nabla T)d\mathbf{x} & \text{if } (\mathbf{u},T)\in H^1(A)^3\times H^1(A), \\ +\infty & \text{otherwise.} \end{cases}$$

The integrand $f_0(\cdot;\cdot,\cdot)$ is given by

$$(4.14)\quad f_0(\mathbf{E};\theta,\mathbf{S}) = \lim_{\varepsilon\to 0}\int\limits_{\Omega}\min_{\mathbf{u},T}\Big\{\frac{1}{|Q_{1/\varepsilon}|}\int\limits_{Q_{1/\varepsilon}} f(\omega,\mathbf{x},\mathbf{e}(\mathbf{u});\theta,,\nabla T)d\mathbf{x}dP(\omega)$$

$$|\mathbf{u}=\mathbf{E}\cdot\mathbf{x},\quad T=\mathbf{S}\cdot\mathbf{x}\quad\text{on}\quad \partial Q_{1/\varepsilon},(\mathbf{u},T)\in\mathcal{U}\Big\}$$

where $\mathbf{E}\in\mathbb{E}^N_s$, $\theta\in\mathbb{R}$, $\mathbf{S}\in\mathbb{R}^N$.

REMARK 3. The necessary conditions for the existence of minimum in (4.14) are the following Euler equations

$$(4.15)\quad \left.\begin{aligned} &\operatorname{div}[\boldsymbol{\lambda}(\mathbf{x},\omega)\cdot(\mathbf{e}(\widetilde{\mathbf{u}})+\mathbf{E}-\boldsymbol{\alpha}(\mathbf{x},\omega)\theta)]=0\\ &\operatorname{div}[\varkappa(\mathbf{x},\omega)\cdot(\nabla\widetilde{T}+\mathbf{S})]=0\end{aligned}\right\}\quad\text{in } Q_{1/\varepsilon}$$

with

$$\widetilde{\mathbf{u}}=\mathbf{0}\qquad \widetilde{T}=0\quad\text{on}\quad\partial Q_{1/\varepsilon.}$$

Since the problem is linear, we look for the fields $\widetilde{\mathbf{u}}$ and $\widetilde{T}$ in the following abberative form

$$(4.16)\qquad \widetilde{u}_i=-\varphi^{(mn)}_i E_{mn}+\phi_i\theta,\qquad \widetilde{T}=\psi^m S_m.$$

Then

$$(4.17)\qquad \begin{aligned} &e_{ij}(\widetilde{\mathbf{u}})=-e_{ij}(\varphi^{(mn)})E_{mn}+e_{ij}(\phi)\theta,\\ &\nabla_i\widetilde{T}=-\partial_i\psi^m Q_m.\end{aligned}$$

Substituting (4.17) into (4.15) we get:

$$(4.18)\qquad \begin{cases} -[\partial_i\lambda_{ijkl}e_{kl}(\varphi^{(mn)})]E_{mn}+(\partial_i\lambda_{ijmn})E_{mn}=0 & \forall\mathbf{E}\in\mathbb{E}^N_s\\ -(\partial_i\lambda_{ijkl}\alpha_{kl})\theta+[\partial_i\lambda_{ijmn}e_{mn}(\phi)]\theta=0 & \forall\theta\in\mathbb{R}\\ \partial_i[\varkappa_{ij}(-\partial_j\psi^m)]S_m+\partial_i(\varkappa_{ij})S_i=0 & \forall\mathbf{S}\in\mathbb{R}^N.\end{cases}$$

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Hence we get the system of equations posed on $Q_{1/\varepsilon}$

$$\begin{aligned} &\partial_i \lambda_{ijkl} \partial_{(k} \varphi_{l)}^{(mn)} = \partial_i \lambda_{ijmn} \\ (4.19) \qquad &\partial_i \lambda_{ijmn} \partial_{(k} \phi_{l)} = \partial_i \lambda_{ijkl} \alpha_{kl} \\ &\partial_i \varkappa_{ij} \partial_j \psi^m = \partial_i \varkappa_{im} \end{aligned}$$

with the homogeneous boundary conditions on $\partial Q_{1/ve}$ for the unknown fields $\varphi_i^{(mn)}$, ϕ_i, ψ. □

Knowing the solution of the "cell problems" i.e. the functions $\varphi_i^{(mn)}$, ϕ_i, ψ on $\partial Q_{1/\varepsilon}$, which obviously are the functions of $\mathbf{u}$, ω and ε, we obtain the unique fields

$$\begin{aligned} \mathbf{e}(\mathbf{u}) &= \mathbf{e}(\widetilde{\mathbf{u}}) + \mathbf{E}, \\ \nabla T &= \nabla \widetilde{T}_{\mathbf{S}} \end{aligned}$$

and

$$\begin{aligned} (4.20) \quad f(\omega, \mathbf{x}, \mathbf{e}(\mathbf{u}); \theta, \nabla T) &= \frac{1}{2}[(e_{ij} - \alpha_{ij}\theta)\lambda_{ijkl}(e_{kl} - \alpha_{kl}\theta) + \partial_k T \varkappa_{kl} \partial_l T] \\ &= \frac{1}{2}[\lambda_{ijkl}(-\partial_{(k}\varphi_{l)}^{(mn)} E_{mn} + E_{kl} + \partial_{(k}\phi_{l)}\theta - \alpha_{kl}\theta) \\ &\cdot (-\partial_{(i}\varphi_{j)}^{(mn)} E_{mn} + E_{ij} + \partial_{(i}\phi_{j)}\theta - \alpha_{ij}\theta) \\ &+ (-\partial_i \psi^j S_j + S_i)\varkappa_{ik}(-\partial_k \psi^l S_l + S_k)] \\ &= \frac{1}{2}\Big\{\lambda_{ijkl}[(I_{klmn} - \partial_{(k}\varphi_{l)}^{(mn)})E_{mn} + (\partial_{(k}\phi_{l)} - \alpha_{kl})\theta] \\ &\cdot [(I_{ijmn} - \partial_{(i}\varphi_{j)}^{(mn)})E_{mn} + (\partial_{(i}\phi_{j)} - \alpha_{ij})\theta] \\ &\qquad + \varkappa_{ik}[(I_{kl} - \partial_k \psi^l)S_l][(I_{mi} - \partial_m \psi^i)S_m]\Big\}. \end{aligned}$$

The tensors I_{kl} and I_{klmn} are unit tensors in the proper spaces, namely:

$$I_{ij} = \delta_{ij} \qquad I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Substituting (4.20) into (4.14) we arrive at:

$$(4.21) \qquad f_0(\mathbf{E}; \theta, \mathbf{S}) = \frac{1}{2}[\boldsymbol{\lambda}^* \cdot (\mathbf{E} - \boldsymbol{\alpha}^*\theta) \cdot (\mathbf{E} - \boldsymbol{\alpha}^*\theta) + \varkappa^* \cdot \mathbf{S} \cdot \mathbf{S}],$$

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where the *macroscopic moduli* are determined by

(4.22)
$$\begin{aligned}
\boldsymbol{\lambda}^* &= \lim_{\varepsilon\to 0}\int\limits_{\Omega}\frac{1}{|Q_{1/\varepsilon}|}\int\limits_{Q_{1/\varepsilon}}\boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\varphi)d\mathbf{x}dP(\omega),\\
\boldsymbol{\alpha}^* &= \lim_{\varepsilon\to 0}\int\limits_{\Omega}\frac{1}{|Q_{1/\varepsilon}|}\int\limits_{Q_{1/\varepsilon}}\boldsymbol{\lambda}\cdot(\boldsymbol{\alpha}-\nabla\phi)d\mathbf{x}dP(\omega),\\
\boldsymbol{\beta}^* &= \lim_{\varepsilon\to 0}\int\limits_{\Omega}\frac{1}{|Q_{1/\varepsilon}|}\int\limits_{Q_{1/\varepsilon}}\boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\varphi)\cdot\boldsymbol{\alpha}d\mathbf{x}dP(\omega),\\
\varkappa^* &= \lim_{\varepsilon\to 0}\int\limits_{\Omega}\frac{1}{|Q_{1/\varepsilon}|}\int\limits_{Q_{1/\varepsilon}}\varkappa\cdot(\mathbf{I}-\nabla\psi)d\mathbf{x}dP(\omega).
\end{aligned}$$

The above abbreviated notation should be understood as, e.g.,

$$\lambda^*_{ijkl} = \lim_{\varepsilon\to 0}\int\limits_{\Omega}\frac{1}{|Q_{1/\varepsilon}|}\int\limits_{Q_{1/\varepsilon}}\lambda_{ijmn}(\omega,\mathbf{x})[I_{mnkl}-\partial_{(m}\varphi^{(kl)}_{n)}(\varepsilon,\omega,\mathbf{x})]d\mathbf{x}dP(\omega).$$

Here we have exploited the following properties of the solutions of the cell problems

$$\int\limits_{Q_{1/\varepsilon}}\boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\varphi)\cdot\nabla\varphi d\mathbf{x}=0,\qquad \int\limits_{Q_{1/\varepsilon}}\boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\varphi)\cdot\nabla\phi d\mathbf{x}=0,$$

$$\int\limits_{Q_{1/\varepsilon}}\boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\phi)\cdot\nabla\phi d\mathbf{x}=0,\qquad \int\limits_{Q_{1/\varepsilon}}\boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\phi)\cdot\nabla\varphi d\mathbf{x}=0,$$

$$\int\limits_{Q_{1/\varepsilon}}\boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\psi)\cdot\nabla\psi d\mathbf{x}=0.$$

One can set: $\mathbf{E}=\mathbf{e}(\mathbf{u})$, $\theta=T$, $\mathbf{S}=\nabla T$ and then the macroscopic potential given by (4.21) yields the macroscopic stresses and heat flux fields in the form:

(4.23)
$$\boldsymbol{\sigma}^0=\frac{\partial f_0}{\partial\mathbf{e}},\qquad \mathbf{q}^0=-\frac{\partial f_0}{\partial\nabla T},$$

where

(4.24)
$$\sigma^0=\boldsymbol{\lambda}^*\cdot[\mathbf{e}(\mathbf{u})-\boldsymbol{\alpha}^*T],\qquad q^0=\varkappa^*\nabla T.$$

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5. One-dimensional case

Let assume that all material properties as well as displacement and temperature fields are scalar functions of one variable, denoted further by x. Then for every $\varepsilon > 0$, let $(X_k^\varepsilon)_{k\in Z}$ be a family of independent random variables defined on the probabilistic space (Ω, Σ, P)

(5.1)
$$P\{\omega \in \Omega: \ X_k^\varepsilon(\omega) = 1\} = c_1, \qquad P\{\omega \in \Omega: \ X_k^\varepsilon(\omega) = 0\} = 1 - c_1 = c_2$$

for every $k \in Z$ and for $c_1 \in]0, 1[$ fixed.

For every $\varepsilon > 0$ and $k \in Z$, let Q_k^ε be the interval in $\mathbb{R}$ defined by

$$Q_{1/\varepsilon} = \{-j \le x < j\}, \qquad |Q_{1/\varepsilon}| = \frac{1}{2j}. \tag{5.2}$$

Denote by I_k^ε its characteristic function. The stochastically periodic characteristic function is given by

$$\chi^\varepsilon(\omega, x) = \sum_{k\in Z} X_k^\varepsilon(\omega) I_k^\varepsilon(x) \qquad \omega \in \Omega, \ \ x \in \mathbb{R}, \tag{5.3}$$

where $I_k^\varepsilon(x)$ is periodic in x. Moreover, we set

$$\chi^\varepsilon(\omega, x) = \chi\Big(\omega, \frac{x}{\varepsilon}\Big), \tag{5.4}$$

where $\chi(\omega, \cdot)$ is 1-periodic function, because

$$I_k^\varepsilon(x) = I_k^1\Big(\frac{x}{\varepsilon}\Big).$$

Then the coefficients λ^ε, α^ε and $\varkappa^\varepsilon$ are determined by the function $\chi^\varepsilon(\omega, x)$ and positive constants $\lambda^{(i)}$, $\beta^{(i)}$, $\alpha^{(i)}$, $\varkappa^{(i)}$, $i = 1, 2$, as follows

$$\begin{aligned}
\lambda^\varepsilon(\omega, x) &= \lambda^{(1)}\chi^\varepsilon(\omega, x) + \lambda^{(2)}(1 - \chi^\varepsilon(\omega, x)),\\
\beta^\varepsilon(\omega, x) &= \beta^{(1)}\chi^\varepsilon(\omega, x) + \beta^{(2)}(1 - \chi^\varepsilon(\omega, x)),\\
\alpha^\varepsilon(\omega, x) &= \alpha^{(1)}\chi^\varepsilon(\omega, x) + \alpha^{(2)}(1 - \chi^\varepsilon(\omega, x)),\\
\varkappa^\varepsilon(\omega, x) &= \varkappa^{(1)}\chi^\varepsilon(\omega, x) + \varkappa^{(2)}(1 - \chi^\varepsilon(\omega, x)).
\end{aligned} \tag{5.5}$$

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In this case the macroscopic potential is expressed by

$$(5.6)\quad f_0(E;\theta,Q) = \lim_{j\to\infty}\int\limits_{\Omega} \min_{u,T} \frac{1}{2}\frac{1}{2j}\int\limits_{-j}^{j} [(u' - \alpha(\omega,x)\theta)\lambda(\omega,x) \cdot (u' - \alpha(\omega,x)\theta) + T'\varkappa(\omega,x)T']dx dP(\omega),$$

subject to

$$(5.7)\qquad u(-j) = -jE, \qquad u(j) = jE, \qquad T(-j) = -jS, \qquad T(j) = jS.$$

Here $u' = du/dx$, etc. To find the minimum in (5.6) we solve the following Euler equations

$$(5.8)\qquad [\lambda(\omega,x)u'(x) - \beta(\omega,x)\theta]' = 0 \text{ and } [\varkappa(\omega,x)T'(x)]' = 0,\ \forall x \in]-j;j[$$

with the following boundary conditions

$$(5.9)\qquad \begin{array}{ll} u(-j) = -jE, & u(j) = jE, \\ T(-j) = -jS, & T(j) = jS. \end{array}$$

After straightforward calculations we get

$$(5.10)\qquad u' = \frac{d_1}{\lambda(\omega,x)} + \alpha(\omega,x)\theta,$$

where

$$(5.11)\qquad d_1 = \Big[2Ej - \theta\int\limits_{-j}^{j}\alpha(\omega,x)dx\Big]\Big(\int\limits_{-j}^{j}\frac{dx}{\lambda(\omega,x)}\Big)^{-1},$$

and

$$(5.12)\qquad T' = \frac{d_2}{\varkappa(\omega,x)},$$

$$(5.13)\qquad d_2 = 2Qj\Big(\int\limits_{-j}^{j}\frac{dx}{\varkappa(\omega,x)}\Big)^{-1}.$$

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Hence

$$\int_\Omega \min_{u,T}\Big\{\frac{1}{2}\frac{1}{2j}\int_{-j}^{j}[(u'-\alpha(\omega,x)\theta)\lambda(\omega,x)(u'-\alpha(\omega,x)\theta)+T'\varkappa(\omega,x)T']dxdP(\omega)$$

$$|u(-j)=-jE,\ \ u(j)=jE,\ \ T(-j)=-jS,\ \ T(j)=jS\Big\}$$

$$=\frac{1}{2}\frac{1}{2j}\Big[\Big(\int_{-j}^{j}[\lambda(\omega,x)]^{-1}dx\Big)^{-1}\Big(2Ej-\theta\int_{-j}^{j}\alpha(\omega,x)dx\Big)^2$$

$$+\Big(\int_{-j}^{j}[\varkappa(\omega,x)]^{-1}dx\Big)^{-1}S^2 j\Big]$$

$$=\frac{1}{2}\Big\{\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\lambda^{(1)}X_k^\varepsilon(\omega)I_k^1(x)+\lambda^{(2)}(1-X_k^\varepsilon(\omega)I_k^1(x))\Big)^{-1}\Big]^{-1}E^2$$

$$+2\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\lambda^{(1)}X_k^\varepsilon(\omega)I_k^1(x)+\lambda^{(2)}(1-X_k^\varepsilon(\omega)I_k^1(x))\Big)^{-1}\Big]^{-1}E$$

$$\cdot\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\alpha^{(1)}X_k^\varepsilon(\omega)I_k^1(x)+\alpha^{(2)}(1-X_k^\varepsilon(\omega)I_k^1(x))\Big)\Big]\theta$$

$$+\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\lambda^{(1)}X_k^\varepsilon(\omega)+\lambda^{(2)}(1-X_k^\varepsilon(\omega))\Big)^{-1}\Big]^{-1}$$

$$\cdot\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\alpha^{(1)}X_k^\varepsilon(\omega)I_k^1(x)+\alpha^{(2)}(1-X_k^\varepsilon(\omega)I_k^1(x))\Big)\Big]^2\theta^2$$

$$+\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\varkappa^{(1)}X_k^\varepsilon(\omega)I_k^1(x)+\varkappa^{(2)}(1-X_k^\varepsilon(\omega)I_k^1(x))\Big)^{-1}\Big]^{-1}S^2\Big\}$$

for any $\varepsilon\in\Omega$.

Taking into account that $(X_k^\varepsilon)_{k\in Z}$ is the family of independent random variables for every ε, $I_k^1(x)$ is one-periodic characteristic function and applying the strong law of large numbers [9] to the limit $j\to\infty$, we obtain:

(5.14) $$f_0(E;\theta,S)$$

$$=(c_1\lambda_1^{-1}+c_2\lambda_2^{-1})^{-1}E^2-2(c_1\lambda_1^{-1}+c_2\lambda_2^{-1})^{-1}(c_1\alpha_1+c_2\alpha_2)E\theta$$

$$+(c_1\lambda_1^{-1}+c_2\lambda_2^{-1})^{-1}(c_1\alpha_1+c_2\alpha_2)^2\theta^2+(c_1\varkappa_1^{-1}+c_2\varkappa_2^{-1})^{-1}S^2.$$

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The final form of the integrand f_0 is expressed by

$$f_0(E;\theta,S) = \lambda^* E^2 - 2\lambda^*\alpha^* E\theta + \lambda^*\alpha^{*2}\theta^2 + \varkappa^* S^2, \tag{5.15}$$

where

$$\lambda^* = (c_1\lambda_1^{-1} + c_2\lambda_2^{-1})^{-1}, \qquad \beta^* = \lambda^*\alpha^*$$
$$\alpha^* = c_1\alpha_1 + c_2\alpha_2, \qquad \varkappa^* = (c_1\varkappa_1^{-1} + c_2\varkappa_2^{-1})^{-1}.$$

6. Final remarks

Except cases such as one-dimensional, one has to resort to approximation of effective moduli. Elaboration of upper and lower bounds for effective thermoelastic moduli remains open. Though media with random microstructure were studied by many authors, see for instance TORQUATO [23], yet application of mathematically rigorous homogenization methods seem to be at its very beginning. The lack of lucid bounding methods is thus not surprising. From Theorem 1 and its specific form applied to two-phase composites with random microstructure we conclude that the method applied is not applicable to nonstationary thermoelasticity.

To solve the problem of stochastic homogenization of equations of coupled, nonstationary thermoelasticity, one has to use either the method of G-convergence or the stochastic two-scale convergence in the mean, cf. [2,3,15,18].

The general stochastic homogenization Theorem 1 was applied to classical, linear, stationary thermoelasticity. This theorem can also be used to physically nonlinear thermoelestic materials with random microstructures provided the the deformations are small.

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The journal *ARCHIVES OF MECHANICS* (ARCHIWUM MECHANIKI STOSOWANEJ) deals with the printing of original papers which should not appear in other periodicals.

As a rule, the volume of a paper should not exceed 40 000 typographic signs, that is about 20 type-written pages, format: 210×297 mm, leaded. The papers should be submitted in two copies.

They must be set in accordance with the norms established by the Editorial Office. Special importance is attached to the following directions:

1. The title of the paper should be as short as possible.
2. The text should be preceded by a brief introduction; it is also desirable that a list of notations used in the paper should be given.
3. The formula number consists of two figures: the first represents the section number and the other the formula number in that section. Thus the division into subsections does not influence the numbering of formulae. Only such formulae should be numbered to which the author refers throughout the paper, and also the resulting formulae. The formula number should be written on the left-hand side of the formula; round brackets are necessary to avoid any misunderstanding. For instance, if the author refers to the third formula of the set (2.1), a subscript should be added to denote the formula, viz. $(2.1)_3$.
4. All the notations should be written very distinctly. Special care must be taken to write small and capital letters as precisely as possible. Semi-bold type should be underlined in black pencil. Explanations should be given on the margin of the manuscript in case of special type face.
5. It has been established to denote vectors by semi-bold type. Trigonometric functions are denoted by sin, cos, tg and ctg, inverse functions – by arc sin, arc cos, arc tg and arc ctg; hyperbolic functions are denoted by sh, ch, th and cth, inverse functions – by Arsh, Arch, Arth and Arcth.
6. Figures in square brackets denote reference titles. Items appearing in the reference list should include the initials of the first name of the author and his surname, also the full title of the paper (in the language of the original paper); moreover;
   a) In the case of books, the publisher's name, the place and year of publication should be given, e.g., 5. S. Ziemba, *Vibration analysis*, PWN, Warszawa 1970;
   b) In the case of a periodical, the full title of the periodical, consecutive volume number, current issue number, pp. from ... to ..., year of publication should be mentioned; the annual volume number must be marked in black pencil so as to distinguish it from the current issue number, e.g., 6. M. Sokołowski, *A thermoelastic problem for a strip with discontinuous boundary conditions*, Arch. Mech., **13**, 3, 337–354, 1961.
7. The authors should enclose a summary of the paper. The volume of the summary is to be about 100 words.
8. The authors are kindly requested to enclose the figures prepared on diskettes (format WMF, EMF, GIF, PCX, BitMaP, EPS or PostScript).

Upon receipt of the paper, the Editorial Office forwards it to the reviewer. His opinion is the basis for the Editorial Committee to determine whether the paper can be accepted for publication or not.

The printing of the paper completed, the author receives 25 copies of reprints free of charge. The authors wishing to get more copies should advise the Editorial Office accordingly, not later than the date of obtaining the galley proofs.

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