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BRIEF NOTES

On Blinowski's second grade materials

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A THEORY of gradient sensitive fluids, first proposed by Blinowski, is extended to second grade elastic materials. Additional assumptions related to frame indifference are used in order to compute the dependence of the stress tensor on the internal energy, thus avoiding some indeterminations present in previous papers. Results are compared with those recently obtained by Dunn and Serrin.

1. Introduction

THE LIMITED aim of modelling a gradient sensitive fluid, in order to describe a thin interlay of such a material placed between two regions of ordinary liquid, led BLINOWSKI [1] to postulate, in an isothermal context, an energy balance containing an extra term, intended to describe non-standard interactions within the fluid.

However, his approach leads to the deduction of equations containing undefined tensor fields, not deducible from the stored energy function. This makes Blinowski's approach unsatisfactory for a wider understanding of second grade materials. DUNN and SERRIN [2], in a fundamental paper, formulate a thermodynamically admissible theory of such materials: introducing an extra flux into the energy balance, by use of the Clausius–Duhem inequality, they were able to deduce the form of the dependence of the stress tensor on the Helmholtz free energy.

In this paper we consider a second grade elastic body, rather than a fluid, and show how, simply adding a frame indifference hypothesis, it is possible to eliminate all indeterminations from Blinowski's model. Moreover, we show that in fact we are led to a set of field equations equivalent to those obtained by Dunn and Serrin, when restricted to isothermal processes. Our conclusion is that Blinowski's second grade materials constitute a special case of those studied by Dunn and Serrin.

2. Energy balance and field equations

We restrict our analysis to a second grade hyperelastic body $\mathscr{B}$, undergoing isothermal processes, and assume that an internal energy

$$\psi = \hat{\psi}(\mathbf{F}, \nabla\mathbf{F}) \tag{2.1}$$

is given, the function of the first and second deformation gradient. We postulate, following BLINOWSKI [1] and GREEN and RIVLIN [3], an energy balance valid for all motions χ_t and all subbodies $\mathscr{P}$, taking the form

$$(2.2)\qquad \frac{d}{dt}\left(\int_{\chi_t(\mathscr{P})} \varrho\left(\hat{\psi}+\frac{v^2}{2}\right)\right) = \int_{\partial\chi_t(\mathscr{P})} t(n)\cdot v + \int_{\partial\chi_t(\mathscr{P})} \Omega(n):\nabla v + \int_{\chi_t(\mathscr{P})} b\cdot v,$$

where $t(n)$ is the surface traction acting on $\partial\chi_t(\mathscr{P})$ and ϱ, b, v have obvious interpretations. The only non-standard term in Eq. (2.2) is

$$(2.3)\qquad \Omega(n):\nabla v, \qquad \Omega^{ip}(n)v_{i/p},$$

which represents an extra flux of energy, required to make the balance equation (2.2) compatible with Eq. (2.1). It was in fact proposed by GREEN and RIVLIN [3] who called Ω^{ip} ***multipolar stress***. This approach should be compared with the proposal made by TOUPIN [4, 5] along different lines.

Moreover, we assume frame indifference for $\hat{\psi}$, the traction t and the difference $b-a$, where a is the acceleration field.

As is well known, the assumption of frame-indifference on the energy balance is equivalent to appropriate invariance requirements on Eq. (2.2) upon superposition of rigid velocity fields.

Standard arguments [6] lead to the deduction of the following:

1) Cauchy's theorem

$$(2.3)\qquad t = Tn,$$

2) balance of linear momentum, taking the local form

$$\varrho a = \varrho b+\operatorname{div} T,$$

3) the existence of a third order tensor Σ such that

$$(2.4)\qquad \Omega(n) = n\Sigma, \qquad \Omega^{pq}(n_i) = n_i\Sigma^{ipq},$$

4) balance of angular momentum, implying that $A^{ik} := T^{ik}+\Sigma^{pik}/_p$ be symmetric:

$$(2.5)\qquad A^{ik} = A^{ki}.$$

We choose to satisfy the relation (2.5) by separately imposing that

$$(2.6)\qquad T^{ik} = T^{ki},$$

$$(2.7)\qquad \Sigma^{pik} = \Sigma^{pki}.$$

We notice that the assumptions (2.6) and (2.7) are absent from previous papers on the subject [1, 3]. We shall see that not only Eqs. (2.6) and (2.7) can be coherently satisfied but, more interestingly, they are exactly what is needed in order to compute explicitly the terms left undetermined in the paper by BLINOWSKI [1].

By differentiation of Eq. (2.1) we have

$$(2.8)\qquad \dot{\psi} = \frac{\partial\hat{\psi}}{\partial F^h_{\ \alpha}}\dot{F}^h_{\ \alpha}+\frac{\partial\hat{\psi}}{\partial F^h_{\ \alpha\beta}}\dot{F}^h_{\ \alpha\beta},$$

where the Greek indices denote coordinates in the reference configuration of $\mathscr{B}$.

Since

$$(2.9)\qquad \dot{F}^h_{\ \alpha} = v^h|_p F^p_{\ \alpha}$$

and

$$\dot{F}^h{}_{\alpha\beta} = v^h|_{pl} F^p{}_\alpha F^l{}_\beta + v^h|_p F^p{}_{\alpha\beta}, \tag{2.10}$$

we have

$$\dot{\psi} = \frac{\partial\hat{\psi}}{\partial F^h{}_\alpha} v^h|_p F^p{}_\alpha + \frac{\partial\hat{\psi}}{\partial F^h{}_{\alpha\beta}} v^h|_{pl} F^p{}_\alpha F^l{}_\beta + \frac{\partial\hat{\psi}}{\partial F^h{}_{\alpha\beta}} v^h|_p F^p{}_{\alpha\beta}. \tag{2.11}$$

Next, applying twice the divergence theorem, we deduce the following identity:

$$\int\limits_{\partial\chi_t(\mathscr{P})} T_{ik}n^k v^i + n_k \Sigma^{kpq} v_{p/q} = \int\limits_{\chi_t(\mathscr{P})} T_{ik}|^k v^i + T_{ik} v^{i/k} + \Sigma^{kpq}|_k v_{p/q} + \Sigma^{kpq} v_{p/qk}. \tag{2.12}$$

Thus, in view of Eqs. (2.11) and (2.12) we may rewrite the energy balance in the form

$$\int\limits_{\chi_t(\mathscr{P})} \left(\varrho \frac{\partial\hat{\psi}}{\partial F^h{}_\alpha} F^p{}_\alpha + \varrho \frac{\partial\hat{\psi}}{\partial F^h{}_{\alpha\beta}} F^p{}_{\alpha\beta} - T^p{}_h - \Sigma^k{}_h{}^p|_k\right) v^h|_p + \left(\varrho \frac{\partial\hat{\psi}}{\partial F^h{}_{\alpha\beta}} F^p{}_\alpha F^l{}_\beta - \Sigma^l{}_h{}^p\right) v^h|_{pl} = 0. \tag{2.13}$$

We now require Eq. (2.13) to hold for every part $\mathscr{P}$ and all the velocity fields on $\chi_t(\mathscr{P})$, thus obtaining

$$T_h{}^p = A_h{}^p - \Sigma^l{}_h{}^p|_l, \tag{2.14}$$

where

$$A_h{}^p := \varrho \frac{\partial\hat{\psi}}{\partial F^h{}_\alpha} F^p{}_\alpha + \varrho \frac{\partial\hat{\psi}}{\partial F^h{}_{\alpha\beta}} F^p{}_{\alpha\beta} \tag{2.15}$$

and

$$\Sigma^l{}_h{}^p = \varrho \frac{\partial\hat{\psi}}{\partial F^h{}_{\alpha\beta}} F^p{}_\alpha F^l{}_\beta + B^l{}_h{}^p, \tag{2.16}$$

where $B^l{}_h{}^p$ is left undetermined, under the only condition that

$$B^l{}_h{}^p = -B^p{}_h{}^l. \tag{2.17}$$

It is well known that the frame indifference of the function $\hat{\psi}$ implies the symmetry of A, as defined by Eq. (2.15) [7, p. 399] [4]. On the other hand, denoting by $\Sigma^{(l}{}_h{}^{p)}$ and $\Sigma^{[l}{}_h{}^{p]}$ the symmetric and skew-symmetric parts of $\Sigma^l{}_h{}^p$, for each $h = 1, 2, 3$, we deduce from Eq. (2.16) that

$$\Sigma^{(l}{}_h{}^{p)} = \varrho \frac{\partial\hat{\psi}}{\partial F^h{}_{\alpha\beta}} F^p{}_\alpha F^l{}_\beta \tag{2.18}$$

$$\Sigma^{[l}{}_h{}^{p]} = B^l{}_h{}^p. \tag{2.19}$$

We now state as a separate Lemma and without proof a result of DUNN and SERRIN [2, Appendix A].

LEMMA. Let C^{lph} be a third order tensor such that

$$C^{iph} = C^{lhp}.$$

Then

$$C^{[lp]h} = C^{(lh)p} - C^{(ph)l}$$

and

$$C^{lph} = C^{(lp)h} + C^{(lh)p} - C^{(ph)l}.$$

We define

$$C^{lph} := \Sigma^l{}_k{}^p g^{kh} = \Sigma^{lhp}, \tag{2.20}$$

where g^{kh} is the metric tensor. In view of the relation (2.7) the Lemma stated above yields

$$\Sigma^{[l}{}_h{}^{p]} = C^{[lp]k} g_{kh} = [C^{(lk)p} - C^{(pk)l}] g_{kh}. \tag{2.21}$$

Since

$$C^{(lp)h} = \Sigma^{(l}{}_s{}^{p)} g^{sh} = \varrho \frac{\partial \hat{\psi}}{\partial F^s{}_{\alpha\beta}} F^l{}_\alpha F^p{}_\beta g^{sh},$$

from Eqs. (2.19) and (2.21) we deduce that

$$B^l{}_h{}^p = \varrho \frac{\partial \hat{\psi}}{\partial F^s{}_{\alpha\beta}} [F^l{}_\alpha F^k{}_\beta g^{sp} - F^p{}_\alpha F^k{}_\beta g^{sl}] g_{kh} \tag{2.22}$$

and

$$\Sigma^l{}_h{}^p = \varrho \frac{\partial \hat{\psi}}{\partial F^s{}_{\alpha\beta}} [F^p{}_\alpha F^l{}_\beta g^{sk} + F^l{}_\alpha F^k{}_\beta g^{sp} - F^p{}_\alpha F^k{}_\beta g^{sl}] g_{kh}. \tag{2.23}$$

The last equality, together with Eq. (2.15). can be inserted back into (2.14), thus giving the full expression of the symmetric Cauchy stress

$$T_h{}^p = \varrho \frac{\partial \hat{\psi}}{\partial F^h{}_\alpha} F^p{}_\alpha + \varrho \frac{\partial \hat{\psi}}{\partial F^h{}_{\alpha\beta}} F^p{}_{\alpha\beta} - \left\{ \varrho \frac{\partial \hat{\psi}}{\partial F^s{}_{\alpha\beta}} [F^p{}_\alpha F^l{}_\beta g^{sk} + F^l{}_\alpha F^k{}_\beta g^{sp} - F^p{}_\alpha F^k{}_\beta g^{sl}] g_{kh} \right\}\Big|_l .$$

We notice that the additional assumptions (2.6) anp (2.7) have been used to deduce the form of the dependence of T on the energy function $\hat{\psi}$ which was left undetermined in previous papers.

An interesting aspect of our result is that the expression of T we have obtained coincides exactly with that given by DUNN and SERRIN [2, p. 114], which was derived in a wider thermodynamical context, using the Clausius–Duhem inequality. In order to see this, we should identify our ψ with the Helmholtz free energy, which is certainly possible for isothermal processes. This shows how, under appropriate additional assumptions, Blinowski's theory may be better clarified and reconciled with othes approaches.

References

1. A. BLINOWSKI, *On the surface behaviour of gradient-sensitive liquids*, Arch. Mech., **25**, 259–268, 1973.
2. J. E. DUNN and J. SERRIN, *On the thermomechanics of interstitial working*, Arch. Rational Mech. Anal., **88**, 95–133, 1985.
3. A. M. GREEN and R. S. RIVLIN, *Multipolar continuum mechanics*, Arch. Rational. Mech. Anal., **17**, 113–147, 1964.
4. R. A. TOUPIN, *Elastic materials with couple-stresses*, Arch. Rational Mech. Anal., **11**, 385–414, 1962.
5. R. A. TOUPIN, *Theories of elasticity with couple-stresses*, Arch. Rational Mech. Anal., **17**, 85–112, 1964.
6. A. M. GREEN and R. S. RIVLIN, *On Cauchy's equations of motions*, J. Appl. Math. and Physics (ZAMP), **15**, 290–292, 1964.
7. C. TRUESDELL and W. NOLL, *The non-linear field theories of mechanics*, Handbuch der Physik III/3.

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