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Application of the unsteady lifting-lines method to arbitrary configurations of lifting surfaces

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THE EFFECTIVENESS of the unsteady lifting-lines method of solving the lifting surface integral equation is presented. This method, although assuming an unknown chordwise pressure distribution in a form of a series of preselected functions, leads to a system of linear algebraic equations, very similar to that obtained by the well-known doublet-lattice method discretizing directly the integral equation. It has been found that the present method is superior to the doublet-lattice method in terms of computational time for both planar and nonplanar lifting surfaces.

W pracy przedstawiono efektywność niestacjonarnej metody linii nośnych rozwiązywania równania całkowego powierzchni nośnej. W metodzie tej zakłada się nieznany rozkład ciśnienia wzdłuż cięciwy w postaci szeregu względem z góry dobranych funkcji, jednak wynikowy układ liniowych równań algebraicznych jest bardzo podobny do układu otrzymywanego w znanej metodzie siatki dipoli, która polega na bezpośredniej dyskretyzacji równania całkowego. Stwierdzono szybszą zbieżność prezentowanej metody w porównaniu z metodą siatki dipoli, zarówno dla płaskich, jak i niepłaskich powierzchni nośnych.

Представлена эффективность нестационарного метода несущих линий решения интегрального уравнения несущей поверхности. В этом методе предполагается неизвестное распределение давления вдоль хорды в виде ряда по отношению к заранее подобранным функциям, однако результирующая система линейных алгебраических уравнений очень подобная системе, получаемой в известном методе сетки диполей, которая заключается в непосредственной дискретизации интегрального уравнения. Констатирована быстрая сходимость предлагаемого метода по сравнению с методом сетки диполей, так для плоских, как и неплоских несущих поверхностей.

1. Introduction

SINCE THE EARLIEST days, aeroelastic stability analysis (flutter) has influenced the development of method used for the calculation of aerodynamic forces induced on bodies by unsteady flow. From the point of view of the capability of recently used numerical methods and equipment, an aircraft is still too complicated itself to be treated as a single body. Therefore, each component is considered separately, including aerodynamic interference effects only, if necessary. For flutter analysis, most important are unsteady aerodynamic forces acting on main lifting elements of an aircraft (wings and tail). These elements are usually thin enough to be treated as lifting surfaces (elements of infinitiesmal thickness). For low aspect ratio wings, and especially for swept wings, the most simplified two-dimensional model (airfoil) is unacceptable. The difference between two-dimensional and three-dimensional models is qualitative because of the stronger singularity occurring in the kernel of the integral equation, and also quantitative, because of the amount of computational time one order of magnitude greater. The numerical methods recently used for lifting surfaces are so much time-consuming that searching for more effective methods is still of present interest. The other reason issues from the iterative procedure of determining the stability condition (critical flutter velocity) during which the lifting surface integral equation has to be solved many times.

The aim of this work is to examine the effectiveness of the unsteady lifting-lines method [1] of solving the lifting surface integral equation and also to check whether this method can be applied to arbitrary configurations of lifting surfaces, as *T*-tail for example.

The unsteady lifting-lines method can also be used for predicting aerodynamic loadings caused by control surface motions [2]. The first preliminary results [15] for planar surfaces, although revealing sometimes strange and unexplained convergence behaviour, seem to be attractive but systematic analysis has not yet been made.

2. The unsteday lifting-lines method

Let $Oxyz$ be a rectangular coordinate system, oriented in such a way that the Ox axis and undisturbed flow velocity vector U are parallel and of the same direction. The lifting surface is placed in an inviscid flow at zero incidence. The surface can be nonplanar but is built up from lines parallel to the free-stream velocity. The lifting surface downwash $w(x, y, z)$ is given and the unknown function is the nondimensional pressure difference between the lower and upper side defined by

$$\Delta c_p(x, y, z) = \frac{2\Delta p(x, y, z)}{\varrho U^2},$$

where x, y, z are coordinates of the lifting surface point, p is the pressure, and ϱ stands for the flow density. The lifting surface integral equation [3] relates the pressure Δc_p to the normal velocity w:

$$\frac{w(x, y, z)}{U} = \frac{1}{8\pi}\iint_S \Delta c_p(\xi, \eta, \zeta) K(x_0, y_0, z_0) d\xi\, d\eta\, d\zeta, \tag{2.1}$$

where integration is carried out over the lifting surface S and the remaining symbols are defined as follows:

$$x_0 = x-\xi, \quad y_0 = y-\eta, \quad z_0 = z-\zeta,$$

$$K(x_0, y_0, z_0) = \frac{T_1 K_1(x_0, y_0, z_0)+T_2 K_2(x_0, y_0, z_0)}{r^2} e^{-i\frac{\omega x_0}{U}}, \tag{2.2}$$

$$r = \sqrt{y_0^2+z_0^2},$$

ω — circular frequency of surface oscillations.

If the surface is planar, then $T_1 = 1$, $T_2 = 0$. The planar part K_1 of the kernel is given by

$$K_1(x_0, y_0, z_0) = \left(1+\frac{x_0}{R}\right)e^{-ik_1u_1} - F(ik_1, u_1), \tag{2.3}$$

where:

$$R = \sqrt{x_0^2+\beta^2 r^2},$$

$$\beta^2 = 1-M^2 \quad (M \text{ — Mach number}),$$

$$k_1 = \frac{\omega r}{U},$$

$$u_1 = \frac{MR-x_0}{\beta^2 r},$$

$$F(v, u) = v\int_u^\infty \left(1-\frac{t}{\sqrt{1+t^2}}\right)e^{-vt}dt \tag{2.4}$$

(t denotes here the dummy integration variable). The nonplanar part K_2 of the kernel is defined by

$$K_2(x_0, y_0, z_0) = \left\{-2-\frac{x_0}{R}\left(2+\frac{\beta^2 r^2}{R^2}\right)-ik_1\left[u_1+\frac{x_0}{R}\left(\frac{Mr}{R}+u_1\right)\right]\right\}e^{-ik_1u_1} + F(ik_1, u_1)+G(ik_1, u_1), \tag{2.5}$$

where the function $G(v, u)$ is given by an integral

$$G(v, u) = v^2\int_u^\infty t\left(1-\frac{t}{\sqrt{1+t^2}}\right)e^{-vt}dt. \tag{2.6}$$

The functions T_1 and T_2 depend on the local deflection from a planar surface and are given by

$$T_1 = \cos(\mathbf{n}_S\mathbf{n}_R), \tag{2.7}$$

$$T_2 = \cos(\mathbf{r}\mathbf{n}_S)\cos(\mathbf{r}\mathbf{n}_R), \tag{2.8}$$

where $\mathbf{n}_S$ and $\mathbf{n}_R$ are unit normals to the surface at two points ξ, η, ζ and x, y, z, respectively. Vector $\mathbf{r}$ is built from components $(0, y_0, z_0)$. The kernel (2.2) of the integral equation (2.1) has a nonintegrable singularity along the line $y_0 = z_0 = 0$, $x_0 > 0$, and the finite part in Hadamard's sense has to be taken [4].

There are many works dealing with the lifting surface problem, a part of them is listed in a survey paper by LANDAHL and STARK [5], but essentially there are two groups of methods for solving the integral equation (2.1). The most popular representative of the first group is the doublet-lattice method [6] being a nonstationary extension of the well-known vortex-lattice method. The basic idea of these methods lies in a direct discretization of the integral equation in search of a finite number of Δc_p values in a set of arbitrary chosen points of the lifting surface. The methods of the second group assume the solution Δc_p as a linear combination of the known functions φ_j, with the unknown coefficients a_j:

$$\Delta c_p = \sum_{j=1}^{n} a_j\varphi_j(x, y, z),$$

and is called the lifting-surface method [7]. The set of n unknown coefficients is to be determined by any approximate method of solution of integral equations. Convergence of the method used strongly depends on a proper choice of preselected functions. It is easy to make such a choice for a lifting surface of a smooth boundary and of a continuous downwash function $w(x, y, z)$. Difficulties arise for wings with kinks in leading or trailing edges and also for control surface motion. The normal velocity distribution in such cases has lines or points of discontinuity resulting in solutions with singularities. This problem does not appear in the doublet-lattice method; however, in order to get sufficient accuracy, a large number of control points placed close to the discontinuity lines must be taken into account. The uniqueness of the solution (Kutta condition) in the lifting-surface method is achieved immediately if preselected functions all have zero trailing edge values. In the doublet-lattice method, uniqueness of the solution is achieved by the so-called 1/4–3/4 rule. According to this scheme, the control point is placed on the 3/4 chord of each panel and the lift distribution is concentrated on 1/4 chord line of the same panel (Fig. 1). Numerical calculations give satisfactory results but there is no rigorous justification of this rule.

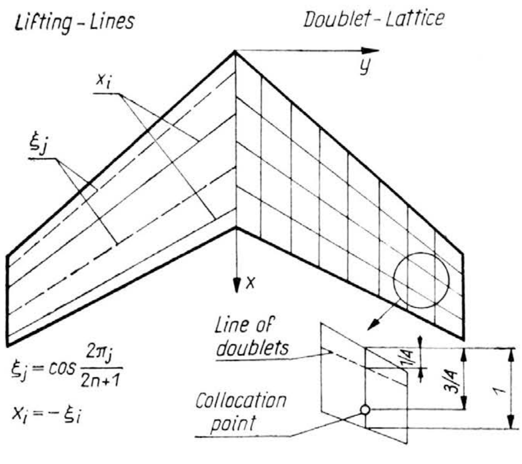


FIG. 1.

The unsteady lifting-lines method [1] shares some properties with both doublet-lattice and lifting-surface methods. The solution is assumed in a form of series

$$\Delta c_p(\xi, \sigma) = \frac{1}{b(\sigma)} \sqrt{\frac{1-\xi}{1+\xi}} \sum_{j=1}^{n} a_j(\sigma) P_{j-1}(\xi), \tag{2.9}$$

where ξ is the chordwise coordinate normalized to the interval $(-1, 1)$ in each section of the lifting surface, σ denotes the spanwise curvilinear coordinate, $b(\sigma)$ is a local semi-chord, and $P_j(\xi)$ are Jacobi polynomials orthogonal over the interval $(-1,1)$ with the weighing function $\sqrt{(1-\xi)/(1+\xi)}$. The unknown functions $a_j(\sigma)$ are to be found.

The normalization mentioned above of the streamwise coordinate ξ (or x) means that for each component of the lifting surface a new coordinate system ξ, σ (or x, σ) is created and each component forms in its own coordinate system a rectangle with a leading edge value $x = \xi = -1$ and a trailing edge value $x = \xi = 1$.

At first, the Galerkin scheme is applied, but only chordwise, with orthogonality condition to the set of Jacobi polynomials $Q_j(x)$ with the weighing function $\sqrt{(1+x)/(1-x)}$ over the interval $(-1,1)$. Both chordwise integrals involving the functions $P_j(\xi)$ and $Q_j(x)$ are calculated numerically according to the Gauss–Jacobi quadrature with the same number of abscissas, equal to the number of terms in the expansion (2.9). After that, the lifting surface integral equation leads to the set of one-dimensional integral equations:

$$\int_{L_S} \mathbf{K}(\omega)\mathbf{A}\mathbf{P}^T\mathbf{a}(\sigma)\,d\sigma = \mathbf{w}(\sigma), \tag{2.10}$$

where $\mathbf{K}$ — $n\times n$ matrix composed of kernel values $k_{ij} = K(x_i - \xi_j, y_0, z_0)$, $\mathbf{A}$ — diagonal matrix of Gauss–Jacobi quadrature weights, $\mathbf{P}$ — $n\times n$ matrix of Jacobi polynomial values corresponding to quadrature abscissas, $p_{ij} = P_{i-1}(\xi_j)$, $\mathbf{w}(\sigma)$ — vector of normal velocity values in quadrature abscissas, $w_i(\sigma) = w(x_i, y, z)$, L_s — length of the span of the lifting surface.

The abscissas of the quadrature are equal to (Fig. 1)

$$x_i = -\cos\left(\frac{2\pi i}{2n+1}\right), \quad \xi_j = -x_j, \quad i,j = 1,2,\ldots,n.$$

The next step is to discretize the spanwise integral in Eq. (2.10). The method employed is the same as that used in the doublet-lattice method. The lifting surface is divided into a set of narrow strips parallel to the free-stream velocity, assuming constant values of the functions $\mathbf{a}(\sigma)$ along each strip. The parabolic interpolation of the numerator of Eq. (2.2) is used and also an analytic evaluation of improper integrals is obtained. Since the error of interpolation grows very rapidly for points (x, y, z) and (ξ, η, ζ) closely spaced one to each other, the steady solution is substracted and obtained by using the horseshoe vortices.

After introducing a new set of unknowns

$$\mathbf{\Gamma}(\sigma) = \mathbf{A}\mathbf{P}^T\mathbf{a}(\sigma),$$

the set of linear algebraic equations is obtained, which is almost the same as that of the doublet-lattice method. The only difference comes from Gauss–Jacobi quadrature abscissas taken instead of a set of arbitrary chosen points along the chord lines. The meaning of the solution is also different. As a result, a set of coefficients of the expansion (2.9) is obtained instead of Δc_p values on the lifting surface. Consequently, the time of computation for the fixed number of unknowns is the same for both doublet-lattice and lifting-lines methods. However, a better accuracy of the lifting-lines method is observed in numerical examples.

Following the orthogonality condition of Jacobi polynomials

$$\mathbf{P}\mathbf{A}\mathbf{P}^T = \pi\mathbf{I},$$

the set of coefficients $\mathbf{a}(\sigma)$ can be obtained from the solution $\mathbf{\Gamma}(\sigma)$ without any knowledge about the quadrature weights

$$\mathbf{a}(\sigma) = \frac{1}{\pi}\mathbf{P}\mathbf{\Gamma}(\sigma). \tag{2.11}$$

Physical interpretation of the vector $\mathbf{\Gamma}(\sigma)$ can be found in [1].

3. Loads caused by control surfaces motion

The normal velocity induced on a lifting surface by control surfaces motion is no longer a continuous function of surface points. Therefore the Gauss-Jacobi quadrature cannot be used to evaluate the Galerkin method integrals involving $w(x, y, z)$. These integrals can be evaluated analytically assuming that

$$w(x, \sigma) = f(x, \sigma) H(x - e(\sigma)),$$

where $H(x-e)$ is the unit-step function (discontinuity appears at the point $x = e$ of the control surface leading edge) and $f(x, \sigma)$ is a continuous function of x. It can be easily verified, that in this case the right-hand side of Eqs. (2.10) equals

$$\mathbf{w}(\sigma) = \frac{1}{\pi} \mathbf{Q}^T \mathbf{g}(\sigma),$$

where

$$g_i(\sigma) = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} f(x, \sigma) Q_{i-1}(x) H(x-e) dx. \tag{3.1}$$

If the function $f(x, \sigma)$, for $\sigma = \text{const}$, can be approximated by a finite number of Jacobi polynomials, then the last integral (3.1) can be expressed as a linear combination of the following integrals:

$$\int_{e}^{1} \sqrt{\frac{1+x}{1-x}} Q_{k-1}(x) dx = \begin{cases} \sqrt{1-e^2} + \arccos(e), & \text{for} \quad k = 1, \\ \sqrt{1-e^2} \left[\dfrac{U_{k-1}(e)}{k} + \dfrac{U_{k-2}(e)}{k-1} \right], & \text{for} \quad k > 1, \end{cases}$$

where $U_k(e)$ denotes the value of the second kind Chebyshev polynomial of order k, in a point $x = e$.

Besides the right-hand side modification of the integral equation, proper interpretation of the solution is necessary in the case of discontinuous downwash. There exists a logarithmic singularity $\ln|x-e|$ of the solution at the control surface leading edge and without any additional information a great number of terms of the expansion (2.9) would be needed to achieve the acceptable accuracy. However, in the very first approach use can be made of the known result for the airfoil with control surface in two-dimensional flow [8]. Such a known, singular term is taken as a singular part Δc_{pS} of the solution to the lifting surface equation (in chord direction):

$$\Delta c_{pS}(x, \sigma) = [b_0(\sigma) + b_1(\sigma)(x-e) + b_2(\sigma)(x-e)^2 + \ldots] \Lambda(x, e), \tag{3.2}$$

where $b_i(\sigma)$ are known coefficients, and the function $\Lambda(x, e)$ is given by

$$\Lambda(x, e) = \frac{1}{2} \ln \left| \frac{1 - xe + \sqrt{1-e^2}\sqrt{1-x^2}}{1 - xe - \sqrt{1-e^2}\sqrt{1-x^2}} \right|.$$

This function has a logarithmic singularity $\ln|x-e|$ and equals zero at the leading and trailing edges. The total difference of pressure (solution to be found) is a sum of a regular part:

(3.3) $$\Delta c_{pR}(x, \sigma) = \frac{1}{b(\sigma)} \sqrt{\frac{1-x}{1+x}} \sum_{i=1}^{n} c_i(\sigma) P_{i-1}(x)$$

and the singular part Δc_{pS} (3.2). The coefficients obtained from Eqs. (2.10) are equal to

$$a_i(\sigma) = c_i(\sigma) + d_i(\sigma), \quad i = 1, 2, \ldots, n,$$

where $d_i(\sigma)$ are coefficients of an expansion of the singular part of the solution in a set of Jacobi polynomials:

$$\Delta c_{pS}(x, \sigma) = \frac{1}{b(\sigma)} \sqrt{\frac{1-x}{1+x}} \sum_{i=1}^{\infty} d_i(\sigma) P_{i-1}(x),$$

$$d_i(\sigma) = \frac{b(\sigma)}{\pi} \int_{-1}^{1} \Delta c_{pS}(x, \sigma) P_{i-1}(x) dx.$$

Knowing the coefficients $d_i(\sigma)$, the solution Δc_p can be written in the form

$$\Delta c_p(x, \sigma) = \frac{1}{b(\sigma)} \sqrt{\frac{1-x}{1+x}} \sum_{i=1}^{n} [a_i(\sigma) - d_i(\sigma)] P_{i-1}(x) + \Delta c_{pS}(x, \sigma).$$

For any fixed value of σ it is possible to evaluate coefficients $d_i(\sigma)$ analytically:

$$d_i(\sigma) = \frac{b(\sigma)}{\pi} \sum_{j=0}^{L} b_j(\sigma) C^j_{i-1}(e),$$

where $L+1$ denotes the number of terms in the expansion (3.2). The integrals

$$C^j_i(e) = \int_{-1}^{1} (x-e)^j P_i(x) A(x, e) dx$$

satisfy the following recursive formula:

$$C^j_i(e) = 2C^{j+1}_{i-1}(e) + 2eC^j_{i-1}(e) - C^j_{i-2}(e),$$

where

$$C^0_i(e) = \begin{cases} \pi\sqrt{1-e^2}, & \text{for} \quad i = 0, \\ \dfrac{\pi}{2}\sqrt{\dfrac{1+e}{1-e}}\left[\dfrac{1}{i+1}\big(Q_i(e) - Q_{i+1}(e)\big) + \dfrac{1}{i}\big(Q_{i-1}(e) - Q_i(e)\big)\right], & \text{for} \quad i > 0, \end{cases}$$

$$C^0_{-1}(e) = -C^0_0(e), \qquad C^1_{-1}(e) = -C^1_0(e),$$

Essentially, the point of the method of handling the discontinuous boundary condition $w(x, y, z)$ is that the partial sum (from $n+1$ to infinity) of the slow converging series (2.9) has been replaced by the partial sum corresponding to the two-dimensional solution. Only the singularity at the control surface leading edge can be treated in this way. The side edges singularities require more closely spaced spanwise strips in order to achieve satisfactory accuracy (the same problem appears in the double-lattice method).

4. Analytical continuation of the kernel

The formulae (2.3) ... (2.8) describing the kernel (2.2) of the lifting surface integral equation are valid also for complex values of the frequency ω. However, because of the convergence condition of the integrals (2.4) and (2.6), $\mathrm{Re}(v) \geqslant 0$, only the motion with growing amplitude is allowed. For exponentially decaying motion, an analytical continuation of functions $F(v, u)$ and $G(v, u)$ onto the left-half of the complex v-plane has to be evaluated. DSEMARAIS [9] proposed an exponential approximation of the integrand of Eq. (2.4) of the form

$$1-\frac{1}{\sqrt{1+t^2}} \approx \sum_{i=1}^{12} \alpha_i e^{-2\gamma^i t}. \tag{4.1}$$

The values of the coefficients $\alpha_i (i = 1, 2, \ldots, 12)$, γ are given in [9].

This approximation results in the rational approximation for the function $F(v, u)$:

$$(4.2)\qquad \begin{aligned} F(v, u) &= ve^{-vu} \sum_{i=1}^{12} \frac{\alpha_i e^{-2^i \gamma u}}{2^i\gamma+v}, && \text{for} \quad u \geqslant 0, \\ F(v, u) &= -2-2v^2 \sum_{i=1}^{12} \frac{\alpha_i}{(2^i\gamma)^2-v^2}+e^{-vu}\left(2+v\sum_{i=1}^{12} \frac{\alpha_i e^{2^i\gamma u}}{2^i\gamma-v}\right), && \text{for} \quad u < 0. \end{aligned}$$

The rational approximation (4.2) gives satisfactory results within the range $\pi/4 < |\mathrm{Arg}(v)| < 3\pi/4 (-\infty < u < \infty)$ and, therefore, can be very useful for determining the pre- and post-flutter behaviour of aeroelastic systems.

The approximation (4.1) can also be used for obtaining the rational approximation of the function $G(v, u)$:

$$\begin{aligned} G(v, u) &= v^2 e^{-vu} \sum_{i=1}^{12} \frac{\alpha_i e^{-2^i\gamma u}}{2^i\gamma+v}\left(u+\frac{1}{2^i\gamma+v}\right), \qquad \text{for} \quad u \geqslant 0, \\ G(v, u) &= -2+2v^2 \sum_{i=1}^{12} \alpha_i \frac{(2^i\gamma)^2+v^2}{[(2^i\gamma)^2-v^2]^2} \\ &\quad + e^{-vu}\left[2(1+vu)+v^2 \sum_{i=1}^{12} \alpha_i \frac{e^{2^i\gamma u}}{2^i\gamma-v}\left(u-\frac{1}{2^i\gamma-v}\right)\right], \qquad \text{for} \quad u < 0. \end{aligned}$$

The calculations based on the rational approximation (4.2) confirm the existence of the pure aerodynamic singularity of the solution to the linear algebraic equations discretizing Eq. (2.10), in some isolated points of the complex ω-plane, corresponding to the decaying motion. Such aerodynamic singularity was for the first time observed by UEDA [16] for the Mach number of 0.8 and a rectangular wing with an aspect ratio of three. Present calculations reveal more points of aerodynamic singularity also for lower Mach numbers.

5. Aerodynamic coefficients

The generalized aerodynamic forces are given by

$$q_{ij} = \iint_S h_i(x, y, z) \Delta c_{pj}(x, y, z) dS, \tag{5.1}$$

where $h_i(x, y, z)$ is a deflection of the lifting surface (corresponding to the i-th natural frequency) and $\Delta c_{pj}(x, y, z)$ is the unsteady loading caused by the motion associated with j-th deflection mode. If the function h_i is approximated by polynomials, then the integrals (5.1) can be evaluated analytically. The generalized force corresponding to $h_i = 1$ has a simple interpretation of the aerodynamic force, and for the linear function h_i the aerodynamic moment is obtained. These two cases are interesting because for them most experimental data are available.

For a given σ, i.e., for a given chordwise section, the normal force and moment with respect to the chord's point of coordinate x_M are given by

$$N(\sigma) = \pi a_1(\sigma) \varrho U^2/2,$$

$$M(\sigma) = \frac{\pi}{2}[a_2(\sigma) - b(\sigma)(1 + 2x_M) a_1(\sigma)] \varrho U^2/2.$$

The force acting on a control surface is the sum

$$N_{CS}(\sigma) = N_S(\sigma) + N_R(\sigma),$$

where $N_S(\sigma)$ is the normal force induced by the singular part of the loading, Δc_{pS} (3.2) and $N_R(\sigma)$ is caused by the regular part Δc_{pR} (3.3). These forces are given by

$$N(\sigma) = \sqrt{1-e^2} \sum_{i=0}^{L} b_j(\sigma) S_j(e, e)/(j+1),$$

where the integrals

$$S_j(c, e) = \int_c^1 \frac{(x-e)^j}{\sqrt{1-x^2}} dx$$

are to be evaluated according to the recursive formula

$$S_j(c, e) = \frac{1}{j}\left[\sqrt{1-c^2}(c-e)^{j-1} + (1-2j) e S_{j-1} + (j-1)(1-e^2) S_{j-2}\right],$$

$$S_0(c, e) = \arccos(c),$$

$$S_1(c, e) = \sqrt{1-c^2} - e \arccos(c).$$

The regular part of the normal force is the following:

$$N_R(\sigma) = \sum_{i=1}^{n} [a_i(\sigma) - d_i(\sigma)] D_{i-1}(e),$$

where

$$D_i(e) = \begin{cases} \arccos(e) - \sqrt{1-e^2}, & \text{for} \quad i = 0, \\ \sqrt{1-e^2}\left[\dfrac{U_{i-1}(e)}{i} - \dfrac{U_i(e)}{i+1}\right], & \text{for} \quad i > 0, \end{cases}$$

$$D_{-1}(e) = -D_0(e).$$

The hinge moment $M_{CS}(\sigma)$ can also be calculated analytically:

$$M_{CS}(\sigma) = \left\{ \sqrt{1-e^2} \sum_{j=0}^{L} \frac{b_j(\sigma)}{j+2} S_{j+1}(e, e) + \frac{1}{2} \sum_{i=1}^{n} [a_i(\sigma) - d_i(\sigma)][D_{i-2}(e) + D_i(e)] \right.$$
$$\left. + [(e-x_0)N_S(\sigma) - x_0 N_R(\sigma)]b(\sigma) \right\} \varrho U^2/2,$$

where x_0 denotes the nondimensional hinge line abscissa in the σ-section.

The nondimensional coefficients corresponding to the unit deflection are defined as follows:

$$C_N = 2N/(\varrho U^2 S_R) \qquad \text{lift coefficient},$$
$$C_M = 2M/(\varrho U^2 S_R l_R) \quad \text{moment coefficient},$$

where S_R and l_R denote the reference area and reference length, respectively.

6. Results and discussion

The convergence of the lifting-lines method is compared with the doublet-lattice method for incompressible flow. At first, the calculations were performed for a 25° swept, untapered wing with an aspect ratio of 2.94, without control surfaces. The wing oscillates in pitch about the root-midchord with reduced frequency $k = 0.372$, referred to the semichord. The number of elements along the wing semispan is fixed at five. Figure 2 shows

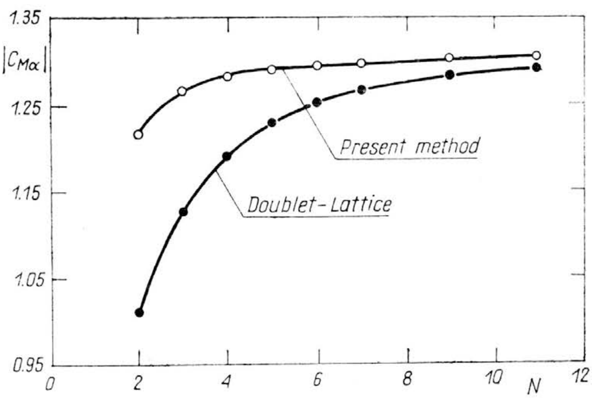


FIG. 2.

the dependency of the moment coefficient (with respect to the pitching axis) on the number N of lifting lines (chordwise elements). Figure 3 shows the dependency of the hinge moment coefficient on the number of lifting lines. The wing is the same as in the previous case but has a full span, 30% chord, trailing edge control surface. Only the control surface oscillates about the hinge line. Figure 4 illustrates the convergence of the lifting-lines method for a nonplanar configuration of lifting surfaces. The yawing moment coefficient of the T-tail is presented. The T-tail is that investigated by STARK [11] and oscillates about an axis

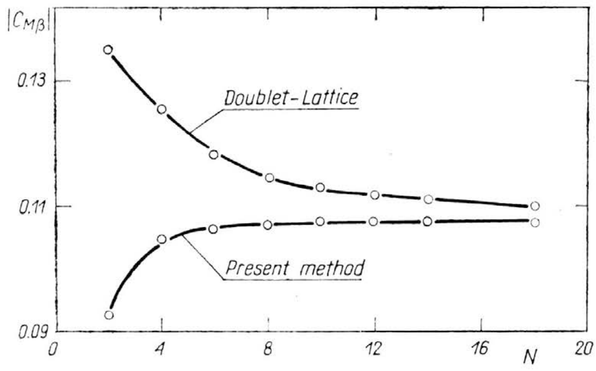


FIG. 3.

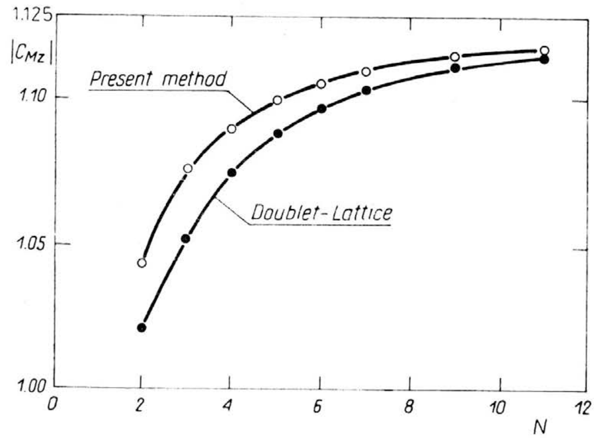
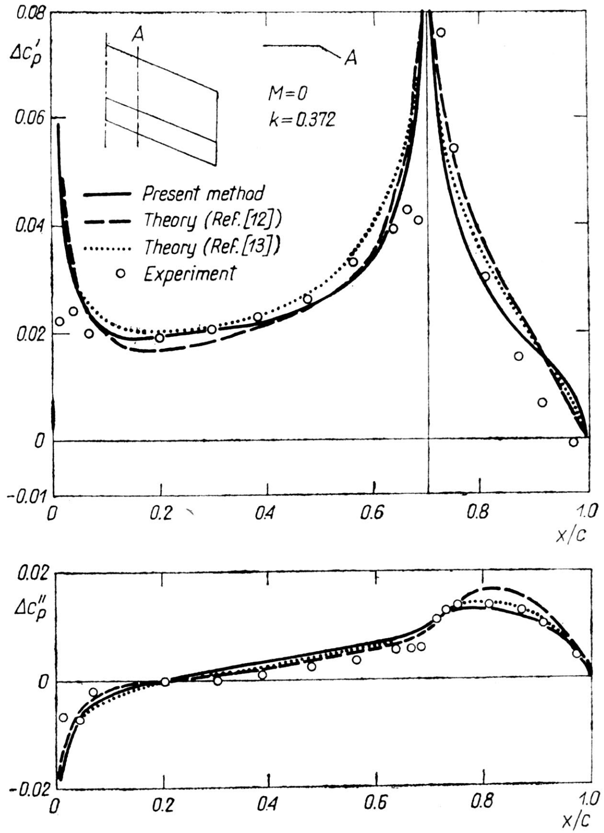


FIG. 4.

parallel to the z-axis of the coordinate system, passing through the center of the tip chord of the fin. Both numbers of elements along the fin span and the horizontal stabilizer semi-span are equal to five. The reduced frequency $k = 0.3$ is based on one third of the span of the fin.

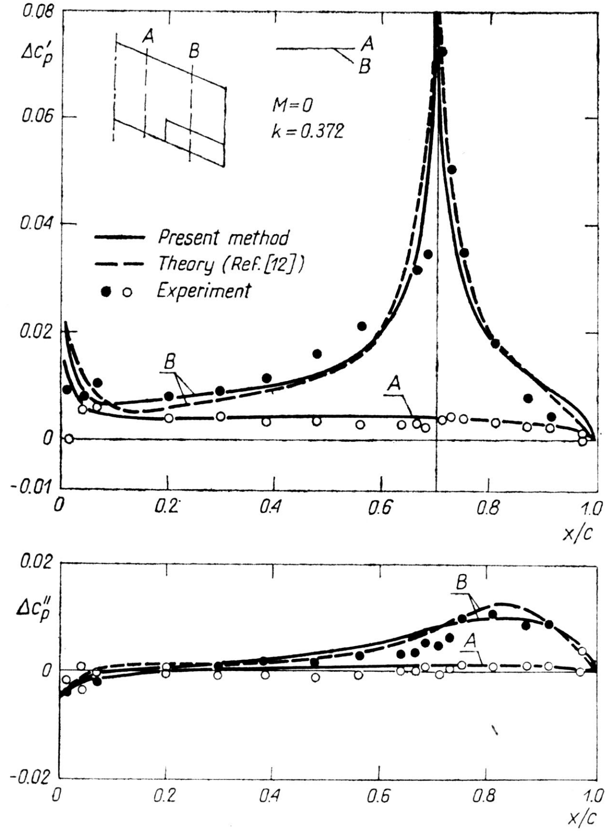
The convergence of all aerodynamic coefficients calculated by using the lifting-lines method is faster than that obtained from the doublet-lattice method, for both planar and nonplanar configurations of lifting surfaces.

For the 25° swept, untapered wing with an aspect ratio of 2.94, the unsteady pressure distribution has been calculated and compared with the experimental results by FÖRSCHING *et al.* [10]. For the same wing, two other theoretical pressure distributions are also shown. The first one is obtained by using the lifting-surface method [13] and the second one is based on the potential function panel method [12]. The complex unsteady pressure distribution is written as $\Delta c_p = \Delta c'_p + i\,\Delta c''_p$ and both real and imaginary parts are shown for four different control surfaces motions. The wing has two control surfaces both of the same chord length (30% of the wing chord) and covering the entire span of the wing. The



0.08
$\Delta c_p'$
0.06
0.04
0.02
0
-0.01
A
M=0
k=0.372
Present method
Theory (Ref. [12])
Theory (Ref. [13])
Experiment
0
0.2
0.4
0.6
0.8
1.0
x/c
0.02
$\Delta c_p''$
0
-0.02

FIG. 5.



0.08
$\Delta c_p'$
0.06
0.04
0.02
0
-0.01
A
B
M=0
k=0.372
Present method
Theory (Ref. [12])
Experiment
0
0.2
0.4
0.6
0.8
1.0
x/c
0.02
$\Delta c_p''$
0
-0.02

FIG. 6.

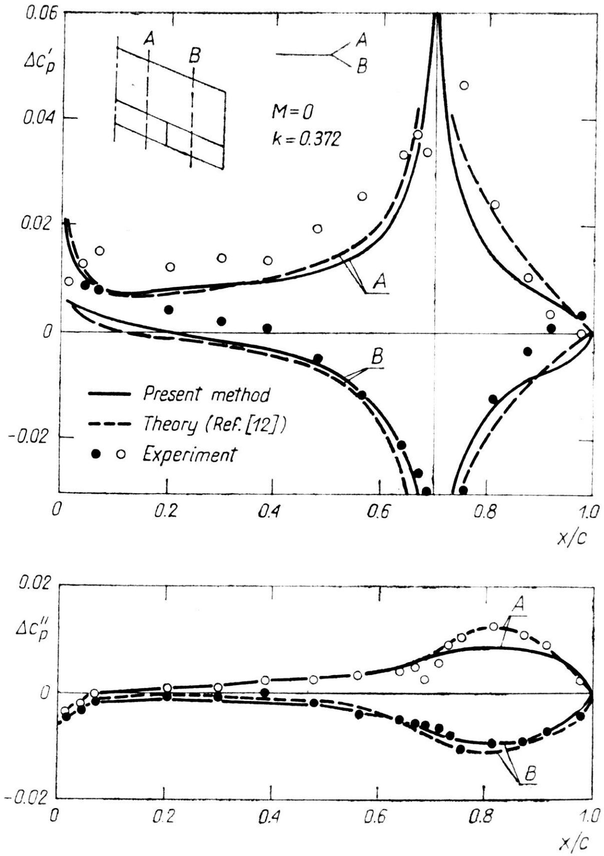

0.06
$\Delta c_p'$
0.04
0.02
0
-0.02
A
B
A
B
M=0
k=0.372
A
B
Present method
Theory (Ref. [12])
Experiment
0
0.2
0.4
0.6
0.8
1.0
x/c
0.02
$\Delta c_p''$
0
-0.02
A
B
0
0.2
0.4
0.6
0.8
1.0
x/c

FIG. 7.

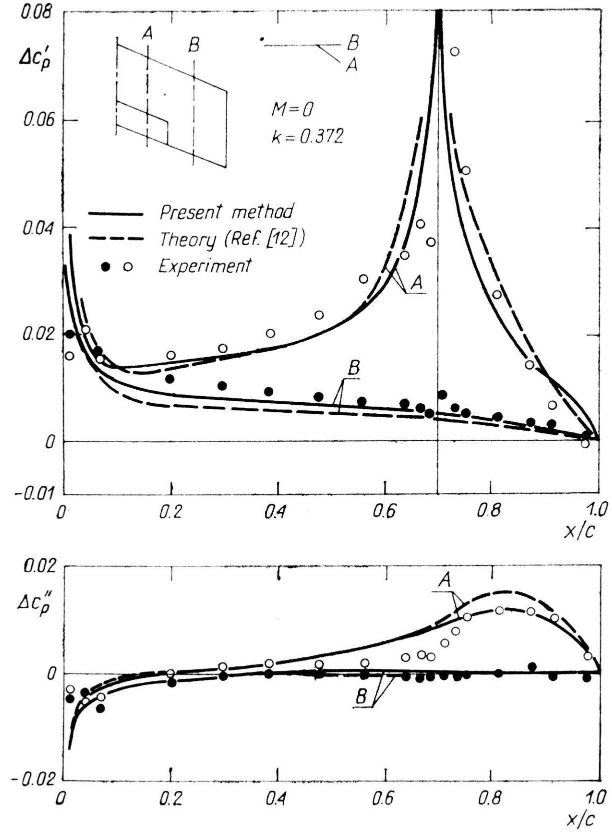

0.08
$\Delta c_p'$
0.06
0.04
0.02
0
-0.01
A
B
B
A
M=0
k=0.372
Present method
Theory (Ref. [12])
Experiment
A
B
0
0.2
0.4
0.6
0.8
1.0
x/c
0.02
$\Delta c_p''$
0
-0.02
A
B
0
0.2
0.4
0.6
0.8
1.0
x/c

FIG. 8.

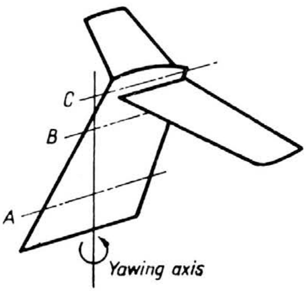


FIG. 9.

span of the inner control surface is equal to 47% of the semispan of the wing. Figure 5 corresponds to both control surfaces oscillating in phase about their leading edges, with the same amplitude of 0.82°. The *A*-section for the experimental data corresponds to 28% of the semispan and for the present method is at 29% of the semi span of the wing. Figure 6 shows the pressure distribution caused by the inner control surface oscillations with an amplitude of 0.82°. The loading corresponding to the outer control surface oscillations with an amplitude of 0.66° is shown in Fig. 7. Figure 8 shows the pressure distribution due to both control surfaces oscillating in anti-phase with an amplitude of 0.66°. The *B*-section is placed at 70% of the semispan of the wing for the experimental data, and at 67% of the semispan for the present method. Essentially, none of the methods compared is superior to the others in terms of their accuracy.

For a nonplanar configuration, a comparison is made of the pressure distribution for the *T*-tail oscillating in yaw (Fig. 9) with the results of the doublet-lattice method and with the experimental data. All data for comparison have been taken from Ref. [14]. The pressure distributions are shown in Figs. 10, 11 and 12, for three stations shown in Fig. 9, and corresponding to 25%, 65% and 95% of the fin span.

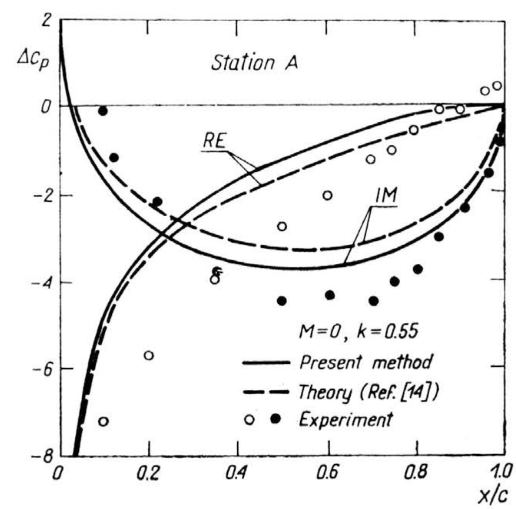


FIG. 10.

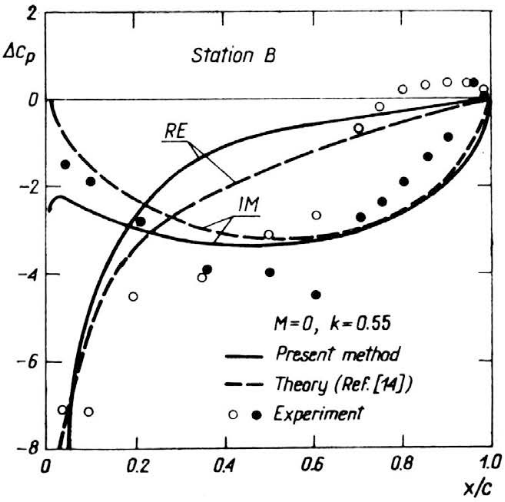


FIG. 11.

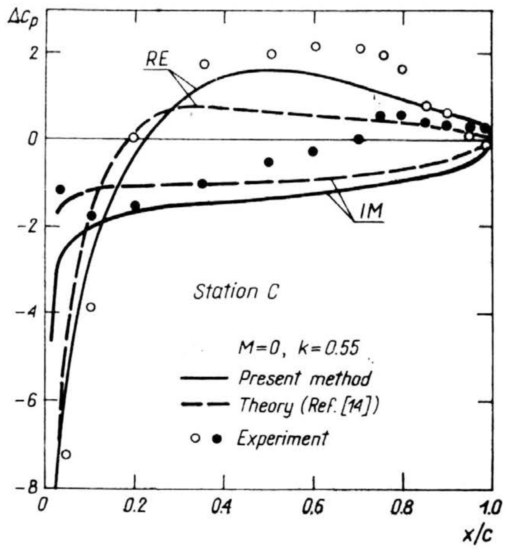


FIG. 12.

7. Conclusions

The agreement between the lifting-lines method and the experimental data for the simple motions of the wing and the control surfaces is generally good and, therefore, one can expect a satisfactory accuracy of generalized aerodynamic forces calculated by this method. The complicated shape of deflection modes of the lifting surface, especially for higher natural frequencies, requires more terms in the expansion (2.9) and also more spanwise elements. Consequently, the lifting-lines method can be considered superior to the doublet-lattice method in application to flutter calculations because of the higher efficiency in terms of computational time.

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