

ERRATA

Growth of voids in a ductile matrix: a review

P. GILORMINI, C. LICHT and P. SUQUET, *Arch. Mech.* **40**, pp. 43–80, 1988

Since all the corrections indicated in the proofs of the paper could not be included by the editor, the authors wish to point out the following errata to the reader. (Some additional minor misprints are not listed, for the sake of brevity.)

Page 45: lines 13 to 30 must be moved to page 44, above 1.2. *Damage and micromechanics*, and n_j should be changed into n_i in Eq. (1.3).

Page 47: the last line above 2. *Isolated Voids* should be read as follows: *ties*, by J. Rice, B. Budiansky, J. W. Hutchinson, A. Needleman, and their coworkers. A comma should replace the full stop at the end of 6th line from the bottom.

Page 48: a 1 is missing between *the* and (Σ_{11}) , and a 2 has been omitted between *or* and $(\Sigma_{22} >$ in line 17.

Page 49: remove the *a* before *generalized* in line 11.

Page 53: change *is* into *was* in line 8, and ϵ into $\dot{\epsilon}$ in Eq. (2.5).

Page 57: change *l); for* into *l), for* in line 6; the beginning of line 21 should be read as *to which a void will tend has been* etc.; replace *ligh* by *high* in line 23.

Page 61, Eq. (3.4): read $s' = sl(s-l)$ instead of $s' = s(s-l)$.

Page 63: change *0.01,0.4* into *0.01–0.4* in line 7.

Page 66: change *in* into *into* (line 17), remove the comma after *i) approximate expressions* (line 16), and add one after *uniform strains* in the footnote.

Page 67: read $\dot{\mathbf{u}}^*$ instead of $\dot{\mathbf{u}}^*$ in the 2nd equation and instead of $\dot{\mathbf{u}}^*$ in the 5th.

Page 68: read $s' = s/(s-l)$ instead of $s' = s(s-l)$ in the 4th equation, close the parenthesis in the 6th, and change *)*, into $)$, in the last line.

Page 69: $c_2 = -\frac{3}{2} \frac{s}{s-1}$ should be replaced by $c_2 = -\frac{3}{2} G \frac{1}{s-1}$.

Page 71: substitute $\dot{\tilde{E}}$ to $\tilde{E}_{\alpha\beta}$ two lines below Eq. (6.19).

Page 77: $\frac{2\mu}{\sigma^2}$ must be replaced by $\frac{2\mu}{\rho^2}$ in line 15, and E_{33} by \dot{E}_{33} at the bottom of the page.

Page 79: substitute *dilatation* to *dilation* in ref. 3, *solids* to *solid* in ref. 5, and *composites* to *composities* in ref. [11].

Page 80: substitute *voids or inclusions* to *voids inclusions* in ref. [17].

On thermodynamics of elasto-plastic porous media (*)

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THE PAPER concerns a macroscopic description of fluid-saturated porous media via mixture theories extended by the volume fraction concept. Proceeding from second-grade materials, elasto-plasticity is taken into account by means of a multiplicative decomposition of the first and second solid deformation gradient. The constitutive theory is discussed on the basis of the example of an immiscible binary model consisting of an elasto-plastic solid matrix saturated by one viscous liquid. For this binary model the thermodynamical restrictions and several constitutive equations are offered assuming the system to be governed by a single temperature and constrained by an incompressibility condition for both constituents.

Praca dotyczy makroskopowego opisu ciał porowatych nasyconych cieczą przy użyciu teorii mieszanin uzupełnionej pojęciem udziału objętościowego. Wychodząc z teorii materiałów drugiego rzędu uwzględniono własności sprężysto-plastyczne drogą faktoryzacji pierwszego i drugiego gradientu odkształcenia. Teorię równań konstytutywnych omówiono na przykładzie binarnego modelu składającego się ze stałej, sprężysto-plastycznej matrycy nasyconej cieczą lepką. Dla takiego modelu binarnego przedstawiono ograniczenia termodynamiczne i szereg równań konstytutywnych zakładając, że układ jest określony przez jedną temperaturę, a oba jego składniki spełniają warunek nieściśliwości.

Работа касается макроскопического описания пористых тел насыщенных жидкостью при использовании теории смесей, пополненной понятием объемного участия. Исходя из теории материалов второго порядка, учтены упруго-пластические свойства путем факторизации первого и второго градиентов деформации. Теория определяющих уравнений обсуждена на примере бинарной модели, состоящей из твердой, упруго-пластической матрицы насыщенной вязкой жидкостью. Для такой бинарной модели представлены термодинамические ограничения и ряд определяющих уравнений, предполагая, что система определяется одной температурой, а оба ее компоненты удовлетворяют условию несжимаемости.

1. Introduction

IN THE FRAME of a macroscopic formulation [1], porous media models can be described via mixture theories [2–6] extended by the volume fraction concept [7–13]. Usually, such models are employed as an immiscible mixture of a porous elastic solid material saturated by an arbitrary number of fluids [10, 11]. The single constituents are assumed to be either compressible or incompressible. The constitutive equations for these models and the thermodynamical restrictions governing these constitutive equations are generally well known. However, there are only a few papers on elasto-plasticity for saturated porous media, cf. e.g., [14], most of them, e.g., [7, 14–16], using a rather simplifying thermodynamical approach which neglects the second-grade character [17] of the single constituents as introduced to mixture theories by MÜLLER [4]. In general, the definition of second-

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grade materials implies that the partial densities and density gradients of the fluid constituents and the first and second deformation gradients of the solid constituents are taken as independent constitutive variables. For immiscible mixtures such as porous media models, in addition, the volume fractions and volume fraction gradients are allowed to affect the mixture response [9–11].

The contribution of the present article is to show how an elasto-plastic solid material applies to the general theory stated above. Proceeding from second-grade materials, elasto-plasticity is taken into account by means of a multiplicative decomposition of the first and second solid deformation gradient. After a brief introduction to mixture theories including the volume fraction concept, the problem is discussed basing, for simplicity, on the example of an elasto-plastic solid matrix saturated by one viscous liquid. For this binary model the thermodynamical restrictions along with several constitutive equations are offered assuming the system to be governed by a single temperature and constrained by an incompressibility condition for both the solid and the liquid material. A simplified constitutive model related to the general one by means of a principle of constituent separation is included. Throughout this paper direct notation will be used, cf. e.g., [18].

2. Preliminaries

The present section offers a brief review of kinematics, balance laws and the entropy principle for mixtures together with the concept of volume fractions. In what follows, all introduced functions are assumed to be sufficiently smooth in space-time.

Consider \mathcal{B} as an arbitrary region of bulk volume V bounded by a surface $\partial\mathcal{B}$ of area A . Then, for a mixture of k immiscible constituents φ^i with particles X^i , each φ^i occupying its constituent volume V^i , a macroscopic formulation implies a model of superimposed continua where at any time t each spatial point \mathbf{x} of the current configuration is simultaneously occupied by k different particles X^i of constituents φ^i . These particles, of course, proceed from different reference positions \mathbf{X}_i . Thus each constituent is assigned its own motion

$$(2.1) \quad \mathbf{x} = \chi_i(\mathbf{X}_i, t).$$

The volume fractions

$$(2.2) \quad n^i = n^i(\mathbf{x}, t)$$

are defined as the local ratios of the constituent volumes V^i with respect to the bulk volume V :

$$(2.3) \quad V = \int_{\mathcal{B}} \sum_{i=1}^k dv^i = \int_{\mathcal{B}} \sum_{i=1}^k n^i dv.$$

For any model without voids the volume fractions are constrained by

$$(2.4) \quad \sum_{i=1}^k n^i = 1.$$

Given the relation (2.3), it must be noted that the definition of volume fractions as scalar quantities restricts the theory to models with an isotropic distribution of the different constituents. For the porous skeleton this restriction implies that only such solid materials can be considered that have an isotropic pore structure. Anisotropic pore structures might be taken into account using additional definitions like "structural permeability tensors" [19].

Associated with each φ^i is an effective density ϱ^{iR} which is defined as the mass of φ^i per unit of constituent volume and a partial or bulk density ϱ^i defined as the mass of φ^i per unit of bulk volume. The density functions are related by

$$(2.5) \quad \varrho^i = n^i \varrho^{iR}.$$

The density of the whole model or the mixture density, respectively, is

$$(2.6) \quad \varrho = \sum_{i=1}^k \varrho^i.$$

From the relation (2.1), each constituent is assigned its own velocity and acceleration field

$$(2.7) \quad \dot{\mathbf{x}}_i = \frac{\partial \mathbf{x}_i(\mathbf{X}_i, t)}{\partial t}, \quad \ddot{\mathbf{x}}_i = \frac{\partial^2 \mathbf{x}_i(\mathbf{X}_i, t)}{\partial t^2}.$$

Using the inverse motion function, an equivalent representation of Eqs. (2.7) is

$$(2.8) \quad \dot{\mathbf{x}}_i = \dot{\mathbf{x}}_i(\mathbf{x}, t), \quad \ddot{\mathbf{x}}_i = \ddot{\mathbf{x}}_i(\mathbf{x}, t).$$

The mean velocity

$$(2.9) \quad \dot{\mathbf{x}} = \frac{1}{\varrho} \sum_{i=1}^k \varrho^i \dot{\mathbf{x}}_i$$

represents the barycentric velocity of the mixture. The velocity of φ^i relative to the mean velocity field,

$$(2.10) \quad \mathbf{u}_i = \dot{\mathbf{x}}_i - \dot{\mathbf{x}},$$

is called diffusion velocity.

If Γ is a differentiable function of (\mathbf{x}, t) , then, owing to the relation (2.8)₁, k independent material time derivatives of Γ can be introduced:

$$(2.11) \quad \dot{\Gamma}_i = \frac{\partial \Gamma}{\partial t} + \text{grad} \Gamma \cdot \dot{\mathbf{x}}_i.$$

According to Eq. (2.9) the material time derivative of the mixture is

$$(2.12) \quad \dot{\Gamma} = \frac{\partial \Gamma}{\partial t} + \text{grad} \Gamma \cdot \dot{\mathbf{x}},$$

where

$$(2.13) \quad \dot{\Gamma}_i - \dot{\Gamma} = \text{grad} \Gamma \cdot \mathbf{u}_i.$$

In Eqs. (2.11)–(2.13) the symbol grad denotes partial differentiation with respect to the spatial position \mathbf{x} .

From the relation (2.1), the deformation gradient of φ^i and its inverse are

$$(2.14) \quad \begin{aligned} \mathbf{F}_i &= \text{Grad}_i \mathbf{x}, \\ \mathbf{F}_i^{-1} &= \text{grad } \mathbf{X}_i, \end{aligned}$$

Grad_i characterizing differentiation with respect to the reference position \mathbf{X}_i of φ^i .

Given Eq. (2.7)₁, the material velocity gradient of φ^i is defined by

$$(2.15) \quad (\mathbf{F}_i)'_i = \text{Grad}_i \dot{\mathbf{x}}_i.$$

From the relation (2.8)₁, the spatial velocity gradient of φ^i yields

$$(2.16) \quad \mathbf{L}_i = \text{grad } \dot{\mathbf{x}}_i,$$

and its symmetric and skew-symmetric parts are

$$(2.17) \quad \begin{aligned} \mathbf{D}_i &= \frac{1}{2} (\mathbf{L}_i + \mathbf{L}_i^T), \\ \mathbf{W}_i &= \frac{1}{2} (\mathbf{L}_i - \mathbf{L}_i^T), \end{aligned}$$

\mathbf{L}_i^T denoting the transpose of \mathbf{L}_i .

According to Truesdell's metaphysical principles, cf. e.g., [20, p. 221], the basic idea of the balance laws for mixtures is as follows:

On one hand the balance postulates are given for the single constituents separately, including convenient supply terms. Then, from the sum of the constituent balance equations, an equivalent balance law for the mixture can be derived which must have the same form as that for single continua. The mixture balance equation yields a constraint to the introduced supply terms. In what follows, the balance equations are given with respect to [13]:

Balance of mass:

$$(2.18) \quad \begin{aligned} \dot{\varrho}^i + \varrho^i \text{div } \dot{\mathbf{x}}_i &= \hat{\varrho}^i, \\ \sum_{i=1}^k \hat{\varrho}^i &= 0. \end{aligned}$$

Balance of momentum:

$$(2.19) \quad \begin{aligned} \text{div } \mathbf{T}^i + \varrho^i (\mathbf{b}^i - \ddot{\mathbf{x}}_i) + \hat{\mathbf{p}}^i &= \mathbf{0}, \\ \sum_{i=1}^k (\hat{\mathbf{p}}^i + \hat{\varrho}^i \dot{\mathbf{x}}_i) &= \mathbf{0}. \end{aligned}$$

Balance of moment of momentum:

$$(2.20) \quad \begin{aligned} \mathbf{T}^i &= \mathbf{T}^{Ti} - \hat{\mathbf{M}}^i, \\ \sum_{i=1}^k \hat{\mathbf{M}}^i &= \mathbf{0}. \end{aligned}$$

Balance of energy:

$$(2.21) \quad \rho^i \dot{\varepsilon}_i^i = -\hat{\mathbf{p}}^i \cdot \dot{\mathbf{x}}_i - \hat{\varrho}^i \left(\varepsilon^i + \frac{1}{2} \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i \right) + \mathbf{T}^i \cdot \mathbf{L}_i + \rho^i r^i - \operatorname{div} \mathbf{q}^i + \hat{e}^i, \\ \sum_{i=1}^k \hat{e}^i = 0.$$

In these equations $\hat{\varrho}^i$, $\hat{\mathbf{p}}^i$, $\hat{\mathbf{M}}^i$ and \hat{e}^i are the supply terms of mass, momentum, moment of momentum and energy representing the transfers to φ^i caused by the other constituents that occupy \mathbf{x} at time t . The quantities \mathbf{T}^i , \mathbf{b}^i , ε^i , r^i and \mathbf{q}^i are the partial Cauchy stresses, external body force densities, internal energy densities, external heat supplies and heat influx vectors of φ^i ; TRUESDELL [20] prefers the notion peculiar instead of partial.

After several discussions in literature on the correct form of the entropy inequality for mixtures, compare, for example, ERINGEN and INGRAM [21], GREEN and NAGHDI [5, 22, 23], BOWEN [3, 6], MÜLLER [4], TRUESDELL [24] and BOWEN and WIESE [25]; the nowadays commonly accepted generalization of the Clausius–Duhem inequality is the so-called BOWEN–TRUESDELL version [20]. One possible form of this inequality is [13]

$$(2.22) \quad \sum_{i=1}^k \frac{1}{\theta^i} \left[-\rho^i (\dot{\psi}_i^i + \dot{\theta}_i^i \eta^i) - \hat{\mathbf{p}}^i \cdot \dot{\mathbf{x}}_i + \hat{e}^i - \hat{\varrho}^i \left(\psi^i + \frac{1}{2} \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i \right) + \mathbf{T}^i \cdot \mathbf{L}_i - \frac{1}{\theta^i} \mathbf{q}^i \cdot \operatorname{grad} \theta^i \right] \geq 0.$$

For a detailed discussion of the balance laws and the entropy principle for mixtures the reader is referred to [6, 13, 20]. In the relation (2.22), θ^i are the absolute constituent temperatures and η^i , ψ^i the entropy and Helmholtz free energy densities of φ^i related by

$$(2.23) \quad \psi^i = \varepsilon^i - \theta^i \eta^i.$$

In case of thermodynamical processes governed by a single temperature $\theta = \theta^i$, it is convenient to use

$$(2.24) \quad \sum_{i=1}^k \left(-\dot{\Psi}_i^i - \dot{\theta} H^i - \hat{\mathbf{p}}^i \cdot \dot{\mathbf{x}}_i - \frac{1}{2} \hat{\varrho}^i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i - \rho^i \mathbf{K}^{Ti} \cdot \mathbf{L}_i - \frac{1}{\theta} \mathbf{h}^i \cdot \operatorname{grad} \theta \right) \geq 0$$

instead of the relation (2.22). Herein,

$$(2.25) \quad \Psi^i = \rho^i \psi^i, \\ H^i = \rho^i \eta^i$$

are the constituent free energy and entropy functions per unit of mixture volume,

$$(2.26) \quad \mathbf{K}^i = \frac{1}{\rho^i} (\Psi^i \mathbf{I} - \mathbf{T}^{Ti})$$

the chemical potential tensors as introduced by BOWEN and WIESE [25] and

$$(2.27) \quad \mathbf{h}^i = \mathbf{q}^i + \theta H^i \mathbf{u}_i$$

an influx vector which is the sum of the constituent heat influx and the diffusion entropy.

3. Decomposition of deformation gradients

In the frame of second-grade solids [17, 25], the response of the model is affected by the first and second solid deformation gradient (the subscript s denoting the solid):

$$(3.1) \quad \mathcal{R} = \mathcal{R}(\dots, \mathbf{F}_s, \text{Grad}_s \mathbf{F}_s, \dots).$$

Thus, if the solid deformation is not purely elastic but elasto-plastic, a multiplicative decomposition of \mathbf{F}_s and $\text{Grad}_s \mathbf{F}_s$ can be defined by the composition

$$(3.2) \quad (\mathbf{F}_s, \text{Grad}_s \mathbf{F}_s) = (\mathbf{F}_{se}, \mathbf{G}_{se}) \circ (\mathbf{F}_{sp}, \mathbf{G}_{sp}),$$

where

$$(3.3) \quad \mathbf{F}_s = \mathbf{F}_{se} \mathbf{F}_{sp}$$

and

$$(3.4) \quad \text{Grad}_s \mathbf{F}_s = (\mathbf{F}_{se} \mathbf{G}_{sp})^3 + [(\mathbf{G}_{se} \mathbf{F}_{sp})^{3T} \mathbf{F}_{sp}]^{23T}.$$

In Eq. (3.4), (\dots) is the symbol of a tensor of third order, $(\dots)^3$ indicates a contraction of the arguments in brackets towards a third-order tensor and $(\dots)^{23T}$ its transposition with respect to the indices 2 and 3.

It is known from several publications on continuum theories of single continua [26–34] that a multiplicative decomposition of \mathbf{F}_s into elastic parts \mathbf{F}_{se} and plastic parts \mathbf{F}_{sp} is connected with the suggestion of a stress-free intermediate configuration incompatible with the existence of partial derivatives such as

$$(3.5) \quad \begin{aligned} \mathbf{F}_{se} &= \text{grad}_{s_z} \mathbf{x}, \\ \mathbf{F}_{sp} &= \text{Grad}_s \mathbf{x}_{s_z}, \end{aligned}$$

\mathbf{x}_{s_z} denoting the position of particles X^s in the intermediate configuration and grad_{s_z} partial differentiation with respect to \mathbf{x}_{s_z} . In the case of homogeneous deformations, however, Eq. (3.5) generally holds. On the other hand, if one defines the intermediate configuration achieved from the actual one after a total removal of the external loads, the intermediate configuration is not stress-free in the frame of nonhomogeneous deformations but still contains residual stresses “defined as the permanent stresses remaining in the body after unloading” [30]. Then \mathbf{F}_{se} and \mathbf{F}_{sp} must be understood as the reversible and globally irreversible parts of \mathbf{F}_s . Following this, the decomposition (3.4) would imply

$$(3.6) \quad \text{Grad}_s \mathbf{F}_s = (\mathbf{F}_{se} \text{Grad}_s \mathbf{F}_{sp})^3 + \{[(\text{grad}_{s_z} \mathbf{F}_{se}) \mathbf{F}_{sp}]^{3T} \mathbf{F}_{sp}\}^{23T},$$

where

$$(3.7) \quad \begin{aligned} \mathbf{G}_{se} &= \text{grad}_{s_z} \mathbf{F}_{se}, \\ \mathbf{G}_{sp} &= \text{Grad}_s \mathbf{F}_{sp} \end{aligned}$$

has been used.

For convenience, in index notation, Eq. (3.6) yields

$$(3.8) \quad (F_s)_{\alpha B, c} = (F_{se})_{\alpha\alpha} (F_{sp})_{\alpha B, c} + (F_{se})_{\alpha\alpha, \beta} (F_{sp})_{\alpha B} (F_{sp})_{\beta c},$$

the small Latin subscripts belonging to the actual, the small Greek subscripts to the intermediate and the capital Latin subscripts to the reference configuration of φ^S . Note in passing that $\text{grad}_{S_z} \mathbf{F}_{S_e}$ and $\text{Grad}_S \mathbf{F}_{S_p}$ are proper second gradients with the symmetries

$$(3.9) \quad \begin{aligned} \text{grad}_{S_z} \mathbf{F}_{S_e} &= (\text{grad}_{S_z} \mathbf{F}_{S_e})^{23}_T, \\ \text{Grad}_S \mathbf{F}_{S_p} &= (\text{Grad}_S \mathbf{F}_{S_p})^{23}_T. \end{aligned}$$

Using the terminology of push-forward and pull-back transformations, cf. e.g., [32], the two terms of the right-hand side of Eq. (3.6) can be interpreted as follows: In the first term, the first basis of $\text{Grad}_S \mathbf{F}_{S_p}$ is transformed by push-forward with \mathbf{F}_{S_e} from the intermediate to the actual configuration; in the second term, the second and third bases of $\text{grad}_{S_z} \mathbf{F}_{S_e}$ are transformed by pull-back with \mathbf{F}_{S_p} from the intermediate to the reference configuration of φ^S .

However, proceeding from a general concept decoupling elasticity from plasticity, the stress-free intermediate configuration together with the definitions (3.2)–(3.4) must be applied as the conceptual basis for elasto-plastic porous media. Thus, in the frame of the constitutive theory for second-grade solids, the variables \mathbf{F}_S and $\text{Grad}_S \mathbf{F}_S$ can be replaced, in the sense of a mathematical model, by \mathbf{F}_{S_e} , \mathbf{F}_{S_p} and $\mathbf{G}_{S_e}^3$, $\mathbf{G}_{S_p}^3$ where $\mathbf{G}_{S_e}^3$ and $\mathbf{G}_{S_p}^3$ stand for the purely elastic and plastic parts of $\text{Grad}_S \mathbf{F}_S$. Therefore, (3.1) be rewritten in the form

$$(3.10) \quad \mathcal{R} = \mathcal{R}(\dots, \mathbf{F}_{S_e}, \mathbf{F}_{S_p}, \mathbf{G}_{S_e}^3, \mathbf{G}_{S_p}^3, \dots).$$

Given (3.10), \mathbf{F}_{S_e} , \mathbf{F}_{S_p} and $\mathbf{G}_{S_e}^3$, $\mathbf{G}_{S_p}^3$ are taken as independent constitutive variables. Without loss of generality, it is understood that $\mathbf{G}_{S_e}^3$ and $\mathbf{G}_{S_p}^3$ have the character of proper second gradients.

Thus

$$(3.11) \quad \begin{aligned} \mathbf{G}_{S_e}^3 &= (\mathbf{G}_{S_e}^3)^{23}_T, \\ \mathbf{G}_{S_p}^3 &= (\mathbf{G}_{S_p}^3)^{23}_T. \end{aligned}$$

The multiplicative concept (3.3) is compatible with the following additive decomposition of strain tensors, compare, for example, [30–34]: In a material description, the Langrangian strain of the solid

$$(3.12) \quad \mathbf{E}_S = \frac{1}{2} (\mathbf{F}_S^T \mathbf{F}_S - \mathbf{I})$$

can be decomposed into purely plastic parts \mathbf{E}_{S_p} and elastic parts \mathbf{E}_{S_e} defined as the difference between \mathbf{E}_S and \mathbf{E}_{S_p} :

$$(3.13) \quad \begin{aligned} \mathbf{E}_{S_p} &= \frac{1}{2} (\mathbf{F}_{S_p}^T \mathbf{F}_{S_p} - \mathbf{I}), \\ \mathbf{E}_{S_e} &= \frac{1}{2} (\mathbf{F}_S^T \mathbf{F}_S - \mathbf{F}_{S_p}^T \mathbf{F}_{S_p}), \\ \mathbf{E}_S &= \mathbf{E}_{S_e} + \mathbf{E}_{S_p}. \end{aligned}$$

Then, by push-forward of the Lagrangian strains with $\mathbf{F}_{Sp}^{T-1}(\dots)\mathbf{F}_{Sp}^{-1}$, an equivalent representation of Eqs (3.12) and (3.13) relative to the intermediate configuration is

$$\begin{aligned}
 \mathbf{\Gamma}_S &= \frac{1}{2}(\mathbf{F}_{Se}^T \mathbf{F}_{Se} - \mathbf{F}_{Sp}^{T-1} \mathbf{F}_{Sp}^{-1}), \\
 \mathbf{\Gamma}_{Sp} &= \frac{1}{2}(\mathbf{I} - \mathbf{F}_{Sp}^{T-1} \mathbf{F}_{Sp}^{-1}), \\
 \mathbf{\Gamma}_{Se} &= \frac{1}{2}(\mathbf{F}_{Se}^T \mathbf{F}_e - \mathbf{I}), \\
 \mathbf{\Gamma}_S &= \mathbf{\Gamma}_{Se} + \mathbf{\Gamma}_{Sp}.
 \end{aligned}
 \tag{3.14}$$

By push-forward of Eqs. (3.12) and (3.13) with $\mathbf{F}_S^{T-1}(\dots)\mathbf{F}_S^{-1}$ or, respectively, of Eqs. (3.14) with $\mathbf{F}_{Se}^{T-1}(\dots)\mathbf{F}_{Se}^{-1}$, we obtain the Almansian strain tensors of the spatial formulation

$$\begin{aligned}
 \mathbf{A}_S &= \frac{1}{2}(\mathbf{I} - \mathbf{F}_S^{T-1} \mathbf{F}_S^{-1}), \\
 \mathbf{A}_{Sp} &= \frac{1}{2}(\mathbf{F}_{Se}^{T-1} \mathbf{F}_{Se}^{-1} - \mathbf{F}_S^{T-1} \mathbf{F}_S^{-1}), \\
 \mathbf{A}_{Se} &= \frac{1}{2}(\mathbf{I} - \mathbf{F}_{Se}^{T-1} \mathbf{F}_{Se}^{-1}), \\
 \mathbf{A}_S &= \mathbf{A}_{Se} + \mathbf{A}_{Sp}.
 \end{aligned}
 \tag{3.15}$$

Furthermore, the multiplicative concept implies that we have only two proper spatial velocity gradients, namely \mathbf{L}_S and \mathbf{L}_{Spz} , the former relative to the actual, the latter relative to the intermediate configuration:

$$\begin{aligned}
 \mathbf{L}_S &= (\mathbf{F}_S)_S' \mathbf{F}_S^{-1}, \\
 \mathbf{L}_{Spz} &= (\mathbf{F}_{Sp})_S' \mathbf{F}_{Sp}^{-1}.
 \end{aligned}
 \tag{3.16}$$

Their symmetric parts are

$$\begin{aligned}
 \mathbf{D}_S &= \frac{1}{2}(\mathbf{L}_S + \mathbf{L}_S^T), \\
 \mathbf{D}_{Spz} &= \frac{1}{2}(\mathbf{L}_{Spz} + \mathbf{L}_{Spz}^T).
 \end{aligned}
 \tag{3.17}$$

\mathbf{D}_S thus defines the solid deformation rate of the spatial configuration and \mathbf{D}_{Spz} the purely plastic deformation rate of the intermediate configuration. By push-forward with $\mathbf{F}_{Se}^{T-1}(\dots)\mathbf{F}_{Se}^{-1}$, \mathbf{D}_{Spz} can be transformed to yield

$$\mathbf{D}_{Sp} = \frac{1}{2} \mathbf{F}_{Se}^{T-1} (\mathbf{L}_{Spz} + \mathbf{L}_{Spz}^T) \mathbf{F}_{Se}^{-1}.
 \tag{3.18}$$

Combining Eqs. (3.17)₁ and (3.18),

$$\mathbf{D}_{Se} = \mathbf{D}_S - \mathbf{D}_{Sp}
 \tag{3.19}$$

is a convenient measure for the elastic contribution to the spatial solid deformation rate. Following the concept of LIE or OLDROYD derivatives [30–34], respectively, Eqs. (3.17)₁, (3.18) and (3.19) can be expressed as follows:

$$\begin{aligned}
 \mathbf{D}_S &= \overset{\Delta}{\mathbf{A}}_S = (\mathbf{A}_S)'_S + \mathbf{L}_S^T \mathbf{A}_S + \mathbf{A}_S \mathbf{L}_S, \\
 \mathbf{D}_{Se} &= \overset{\Delta}{\mathbf{A}}_{Se} = (\mathbf{A}_{Se})'_S + \mathbf{L}_S^T \mathbf{A}_{Se} + \mathbf{A}_{Se} \mathbf{L}_S, \\
 \mathbf{D}_{Sp} &= \overset{\Delta}{\mathbf{A}}_{Sp} = (\mathbf{A}_{Sp})'_S + \mathbf{L}_S^T \mathbf{A}_{Sp} + \mathbf{A}_{Sp} \mathbf{L}_S, \\
 \mathbf{D}_S &= \mathbf{D}_{Se} + \mathbf{D}_{Sp} = \overset{\Delta}{\mathbf{A}}_{Se} + \overset{\Delta}{\mathbf{A}}_{Sp}.
 \end{aligned}
 \tag{3.20}$$

The symbol $\overset{\Delta}{(\dots)}$ defines the upper Lie derivatives relative to the actual configuration

$$\overset{\Delta}{(\dots)} = (\dots)'_S + \mathbf{L}_S^T (\dots) + (\dots) \mathbf{L}_S.
 \tag{3.21}$$

Of course, the concept of Lie derivatives, in a more classical terminology defined simply as convected derivatives, means nothing else than first pulling the respective tensorial object back to the reference configuration, then, taking the time derivative $(\dots)'_S$ and finally pushing it forward into the configuration where it comes from. Thus it is only natural that the upper plastic Lie derivatives

$$(\dots)_P^\Delta = (\dots)'_S + \mathbf{L}_{Spz}^T (\dots) + (\dots) \mathbf{L}_{Spz}
 \tag{3.22}$$

of the strain tensors \mathbf{F}_S , \mathbf{F}_{Sp} and \mathbf{F}_{Se} relative to the intermediate configuration, compare Eqs. (3.14), are equivalent to pulling Eqs. (3.20) back with $\mathbf{F}_{Se}^T(\dots) \mathbf{F}_{Se}$,

$$\begin{aligned}
 \left. \begin{aligned}
 (\mathbf{F}_S)_P^\Delta \\
 (\mathbf{F}_{Se})_P^\Delta \\
 (\mathbf{F}_{Sp})_P^\Delta
 \end{aligned} \right\} &= \mathbf{F}_{Se}^T \left\{ \begin{aligned}
 \overset{\Delta}{\mathbf{A}}_S \\
 \overset{\Delta}{\mathbf{A}}_{Se} \\
 \overset{\Delta}{\mathbf{A}}_{Sp}
 \end{aligned} \right\} \mathbf{F}_{Se}, \\
 (\mathbf{F}_S)_P^\Delta &= (\mathbf{F}_{Se})_P^\Delta + (\mathbf{F}_{Sp})_P^\Delta
 \end{aligned}
 \tag{3.23}$$

where from Eq. (3.17)₂

$$(\mathbf{F}_{Sp})_P^\Delta = \mathbf{D}_{Spz}.
 \tag{3.24}$$

Finally, by pull-back of the relation (3.23) with $\mathbf{F}_{Sp}^T(\dots) \mathbf{F}_{Sp}$, one obtains the strain rates of the solid reference configuration:

$$\begin{aligned}
 \left. \begin{aligned}
 (\mathbf{E}_S)'_S \\
 (\mathbf{E}_{Se})'_S \\
 (\mathbf{E}_{Sp})'_S
 \end{aligned} \right\} &= \mathbf{F}_{Sp}^T \left\{ \begin{aligned}
 (\mathbf{F}_S)_P^\Delta \\
 (\mathbf{F}_{Se})_P^\Delta \\
 (\mathbf{F}_{Sp})_P^\Delta
 \end{aligned} \right\} \mathbf{F}_{Sp}, \\
 (\mathbf{E}_S)'_S &= (\mathbf{E}_{Se})'_S + (\mathbf{E}_{Sp})'_S.
 \end{aligned}
 \tag{3.25}$$

Note that the left-hand side of the relation (3.25) can also be obtained by pulling Eqs. (3.20) back with $\mathbf{F}_S^T(\dots) \mathbf{F}_S$.

4. Constitutive assumptions

As concerns the constitutive theory, we restrict our attention to a binary model consisting of an elasto-plastic porous solid skeleton saturated by one viscous liquid (the subscript F denoting the liquid). The model under discussion be governed by the following properties:

In its reference positions $\mathbf{X}_i (i = s, f)$, the volume fractions n_{0i}^i and effective densities ϱ_{0i}^{iR} of both constituents are given and homogeneous:

$$(4.1) \quad \begin{aligned} \text{Grad}_i n_{0i}^i &= \mathbf{0}, \\ \text{Grad}_i \varrho_{0i}^{iR} &= \mathbf{0}. \end{aligned}$$

The constituents are assumed to be inert. Thus, mass exchanges are excluded:

$$(4.2) \quad \hat{\varrho}^i = 0.$$

The partial stresses are assumed to be symmetric so that

$$(4.3) \quad \hat{\mathbf{M}}^i = \mathbf{0}.$$

The model has but one single temperature

$$(4.4) \quad \theta = \theta^i.$$

The system be governed by an incompressibility condition for both the solid and the liquid material. Constituent incompressibility implies the effective densities to be constant during deformation:

$$(4.5) \quad \varrho^{iR} = \varrho_{0i}^{iR} = \text{const.}$$

Given Eq. (4.4), the temperature variation can be calculated from an energy balance of the whole system in which the sum of the energy supplies is naught. Using Eqs. (2.10)–(2.27) together with Eqs. (3.20)₄ and (4.2)–(4.4), this balance equation can be shown to yield

$$(4.6) \quad -\dot{\Psi}_S^S - \dot{\Psi}_F^F - \theta \dot{H} - H(\dot{\theta} + \theta \text{div} \dot{\mathbf{x}}) - \hat{\mathbf{p}}^F \cdot (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) - \varrho^S \mathbf{K}^S \cdot (\mathbf{D}_{Se} + \mathbf{D}_{Sp}) - \varrho^F \mathbf{K}^F \cdot \mathbf{D}_F + \varrho r^I - \text{div} \mathbf{h} = 0,$$

where

$$(4.7) \quad \begin{aligned} H &= H^S + H^F, \\ \mathbf{h} &= \mathbf{h}^S + \mathbf{h}^F, \\ r^I &= \frac{1}{\varrho} (\varrho^S r^S + \varrho^F r^F), \end{aligned}$$

r^I denoting the inner part of the external heat supplies as defined by TRUESDELL and TOUPIN [2, section 243]. From the incompressibility condition (4.5) together with Eq. (4.2), the balance of mass equations (2.18) prove to be balance equations for the volume fractions

$$(4.8) \quad \dot{n}_i^i + n^i \text{div} \dot{\mathbf{x}}_i = 0,$$

that can be integrated to yield

$$(4.9) \quad n^i = n_{0i}^i (\det \mathbf{F}_i)^{-1}.$$

It is important to note that constituent incompressibility does not imply macroscopic incompressibility of φ^i since from Eqs. (2.5) and (4.5), the bulk densities ϱ^i can still change through changes in volume fractions. On the other hand, as it is well known, incompressibility of all the constituents of the respective medium gives rise to a certain constraint to be incorporated into the entropy inequality of the model. Such a constraint has first been suggested by MILLS [35] in the frame of a mixture of incompressible Newtonian fluids

and, later, by CRAINE [36], ATKIN and CRAINE [37], BOWEN [10, 38] and DE BOER and EHLERS [13]. From Eq. (4.8) and the material time derivative of Eq. (2.4) following the motion of one of the both constituents, the constraint for the present model is

$$(4.10) \quad n^S \lambda \operatorname{div} \dot{\mathbf{x}}_S + n^F \lambda \operatorname{div} \dot{\mathbf{x}}_F + \lambda \operatorname{grad} n^F \cdot (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) = 0,$$

where λ is a Lagrangian multiplier with the dimension of a stress.

Using Eq. (4.10) along with Eqs. (2.19)₂, (2.20)₁, (2.26), (3.20)₄, (4.2), (4.3) and (4.7), the entropy inequality (2.24) turns out to yield

$$(4.11) \quad -\dot{\Psi}_S^S - \dot{\Psi}_F^F - \dot{\theta} H - (\hat{\mathbf{p}}^F - \lambda \operatorname{grad} n^F) \cdot (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) \\ - (\varrho^S \mathbf{K}^S - n^S \lambda \mathbf{I}) \cdot (\mathbf{D}_{S_e} + \mathbf{D}_{S_p}) - (\varrho^F \mathbf{K}^F - n^F \lambda \mathbf{I}) \cdot \mathbf{D}_F - \frac{1}{\theta} \mathbf{h} \cdot \operatorname{grad} \theta \geq 0.$$

From the preceding considerations, the present model is defined by the balance equations for the volume fractions (4.8), the balance of momentum (2.19) when using Eqs. (2.26), (4.2) and (4.3), the energy balance (4.6) and the entropy inequality (4.11) together with the following set of independent constitutive equations which, from the principle of equipresence, must be functions of a common set of variables φ :

$$(4.12) \quad (\Psi^i, H^i, \varrho^i \mathbf{K}^i - n^i \lambda \mathbf{I}, \mathbf{h}, \hat{\mathbf{p}}^F - \lambda \operatorname{grad} n^F) = \mathcal{R}(\varphi).$$

Concerning the plastic range, a flow rule for \mathbf{D}_{S_p} must be added:

$$(4.13) \quad \mathbf{D}_{S_p} = \Lambda(\varphi, \dot{\boldsymbol{\sigma}}, \dot{\theta}_S, \kappa, \dots) \boldsymbol{\tau}.$$

In writing Eq. (4.13), it is understood that $\boldsymbol{\tau}$ is any stress measure, $\dot{\boldsymbol{\sigma}}$ any objective stress rate and κ a material or hardening parameter, respectively. The specification of $\boldsymbol{\tau}$ and $\dot{\boldsymbol{\sigma}}$ depends on the yield or failure condition in connection with both the loading criteria and the flow rule (associated or non-associated) used for the respective model. An extension of the set of parameters as used in Eqs. (4.13) including further parameters or internal variables is of course possible but not necessary for what follows since we are only interested in the thermodynamics of elasto-plastic porous media.

In the frame of second-grade materials, several suggestions have been made in literature for the possible choice of constitutive variables [6, 8, 10, 11, 37, 38]. For immiscible mixtures such as saturated porous media, however, the volume fractions and volume fraction gradients should be generally allowed to affect the response of the model [9–11]. Within the range of incompressible porous media, there seems to be some confusion since even BOWEN [10, 38] uses two principally different choices of constitutive variables which would apply to the present model in the frame of a purely elastic skeleton as follows:

$$(4.14) \quad \varphi_1 = \{\theta, \operatorname{grad} \theta, \mathbf{F}_S, \operatorname{Grad}_S \mathbf{F}_S, n^F, \operatorname{grad} n^F, \dot{\mathbf{x}}_S, \dot{\mathbf{x}}_F\}$$

is a modified version of BOWEN [10],

$$(4.15) \quad \varphi_2 = \{\theta, \operatorname{grad} \theta, \mathbf{F}_S, \operatorname{Grad}_S \mathbf{F}_S, \dot{\mathbf{x}}_S, \dot{\mathbf{x}}_F\}$$

corresponds to BOWEN [38]. Concerning Eq. (4.14), Bowen accepts a dependency between the arguments of φ_1 . Proceeding from the assumption that only independent variables should affect the response, we will deduce a modified version of Eq. (4.15) from a more general point of view.

For a compressible immiscible mixture consisting of an elastic skeleton saturated by a single viscous fluid

$$(4.16) \quad \varphi_3 = \{\theta, \text{grad } \theta, \mathbf{F}_S, \text{Grad}_S \mathbf{F}_S, n^F, \text{grad } n^F, \varrho^F, \text{grad } \varrho^F, \dot{\mathbf{x}}_S, \dot{\mathbf{x}}_F, \mathbf{L}_F\}$$

holds. Equation (4.16) corresponds to the compressible model of Bowen [11, 38] except that \mathbf{L}_F has been added due to fluid viscosity. Formally, Eq. (4.16) can be obtained by consulting CROSS [17] from an immiscible mixture of viscoelastic constituents using the symmetry group for second-grade fluids. Note in passing that n^S and $\text{grad } n^S$ are excluded since from Eq. (2.4):

$$(4.17) \quad \begin{aligned} n^S &= 1 - n^F, \\ \text{grad } n^S &= -\text{grad } n^F. \end{aligned}$$

However, the present model contains an incompressible liquid, i.e., ϱ^F and $\text{grad } \varrho^F$ are no longer independent but functions of n^F and $\text{grad } n^F$. Thus, Eq. (4.16) reduces to Eq. (4.14) except that \mathbf{L}_F has been included. From this point of view, Eq. (4.14) describes a compressible elastic skeleton saturated by one incompressible liquid. Since our solid material is incompressible,

$$(4.18) \quad \begin{aligned} n^F &= 1 - n_{0S}^S (\det \mathbf{F}_S)^{-1}, \\ \text{grad } n^F &= n_{0S}^S (\det \mathbf{F}_S)^{-1} \mathbf{F}_S^{T-1} (\mathbf{F}_S^{T-1} \text{Grad}_S \mathbf{F}_S)^{\perp} \end{aligned}$$

holds where

$$(4.19) \quad \text{Grad}_S (\det \mathbf{F}_S) = \det \mathbf{F}_S (\mathbf{F}_S^{T-1} \text{Grad}_S \mathbf{F}_S)^{\perp}$$

together with Eqs. (4.1), (4.9) and (4.17) has been used.

In Eq. (4.19), $(\dots)^{\perp}$ indicates a contraction of the arguments in brackets towards a vector. Again, for convenience, the index version of Eq. (4.19) is

$$(4.20) \quad [\det(F_S)_{ab}]_{,c} = [\det(F_S)_{ab}] (F_S^{-1})_{Ln} (F_S)_{nL,c}.$$

Given the relation (4.18), n^F and $\text{grad } n^F$ are no longer independent but functions of \mathbf{F}_S and $\text{Grad}_S \mathbf{F}_S$.

From these considerations, the relation (3.10) and (4.16) combine to yield

$$(4.21) \quad \varphi = \{\theta, \text{grad } \theta, \mathbf{F}_{Se}, \mathbf{F}_{Sp}, \overset{3}{\mathbf{G}}_{Se}, \overset{3}{\mathbf{G}}_{Sp}, \dot{\mathbf{x}}_S, \dot{\mathbf{x}}_F, \mathbf{L}_F\},$$

Eq. (4.21) thus representing the set of independent constitutive variables for an incompressible elasto-plastic porous skeleton saturated by an incompressible viscous liquid. In the frame of a purely elastic skeleton Eq. (4.21) reduces to Eq. (4.15) except that \mathbf{L}_F has been added.

Further modifications of Eq. (4.21) are due to the principle of material frame-indifference. Combining Eqs. (4.12) and (4.21), the principle requires that

$$(4.22) \quad \begin{aligned} &[\Psi^i, H^i, \mathbf{Q}(\varrho^i \mathbf{K}^i - n^i \lambda \mathbf{I}) \mathbf{Q}^T, \mathbf{Q} \mathbf{h}, \mathbf{Q}(\hat{\mathbf{p}}^F - \lambda \text{grad } n^F)] \\ &= \mathcal{A}[\theta, \mathbf{Q} \text{grad } \theta, \mathbf{Q} \mathbf{F}_{Se} \bar{\mathbf{Q}}^T, \bar{\mathbf{Q}} \mathbf{F}_{Sp}, \overset{3}{\mathbf{M}}_{Se}, \overset{3}{\mathbf{M}}_{Sp}, \dot{\mathbf{c}} + \dot{\mathbf{Q}} \mathbf{x} + \mathbf{Q} \dot{\mathbf{x}}_i, \mathbf{Q} \mathbf{L}_F \mathbf{Q}^T + \dot{\mathbf{Q}} \mathbf{Q}^T], \end{aligned}$$

where \mathbf{c} is an arbitrary time-dependent vector representing a translation, \mathbf{Q} a time-dependent proper orthogonal tensor representing a rigid rotation in terms of the actual configuration and $\bar{\mathbf{Q}}$ an independent proper orthogonal tensor specifying a rigid rotation in terms of the intermediate configuration.

The third-order tensors \mathbf{M}_{Se}^3 and \mathbf{M}_{Sp}^3 are defined by

$$(4.23) \quad \begin{aligned} \mathbf{M}_{Se}^3 &= \{[(\mathbf{Q}\mathbf{G}_{Se})^3 \overline{\mathbf{Q}}^T]^{23} \overline{\mathbf{Q}}^T\}^{23}, \\ \mathbf{M}_{Sp}^3 &= (\overline{\mathbf{Q}}\mathbf{G}_{Sp})^3. \end{aligned}$$

Using standard arguments, cf. e.g., [6], Eqs. (4.12), (4.13), (4.21) and (4.22) finally combine to

$$(4.24) \quad \begin{aligned} (\Psi^i, H, \rho^i \mathbf{K}^i - n^i \lambda \mathbf{I}, \mathbf{h}, \hat{\mathbf{p}}^F - \lambda \text{grad} n^F) &= \mathcal{R}(\phi), \\ \mathbf{D}_{Sp} &= \Lambda(\phi, \dot{\boldsymbol{\sigma}}, \dot{\theta}_S, \kappa, \dots) \boldsymbol{\tau}, \\ \phi &= \{\theta, \text{grad} \theta, \mathbf{F}_{Se}, \mathbf{F}_{Sp}, \mathbf{G}_{Se}^3, \mathbf{G}_{Sp}^3, \dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S, \mathbf{D}_F\}. \end{aligned}$$

Equations (4.24) represent the constitutive assumptions as used for the subsequent considerations.

5. Thermodynamical restrictions

For the present model, thermodynamical restrictions result from the dissipation principle (4.11) together with the constitutive assumptions (4.24). In the frame of mixture theories, however, the general procedure proves to be rather laborious, producing plenty of lengthy formulae that are beyond the scope of this paper. Thus, only the main results are listed below. Using standard arguments as introduced to the thermodynamics of single continua by COLEMAN and NOLL [39], extended by arguments as used by BOWEN [6] and combined with several symmetry and skew-symmetry conditions [13], it can be shown that

$$(5.1) \quad \begin{aligned} \Psi^S &= \Psi^S(\theta, \text{grad} \theta, \mathbf{F}_{Se}, \mathbf{F}_{Sp}, \dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S), \\ \Psi^F &= \Psi^F(\theta, \text{grad} \theta, \mathbf{F}_{Se}, \mathbf{F}_{Sp}, \dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S), \\ \Psi^I, H &= f(\theta, \mathbf{F}_{Se}, \mathbf{F}_{Sp}) \end{aligned}$$

and

$$(5.2) \quad H = - \frac{\partial \Psi^I}{\partial \theta},$$

where

$$(5.3) \quad \Psi^I = \Psi^S + \Psi^F$$

is the inner part of the free energy of the mixture. In addition,

$$(5.4) \quad \rho^S \mathbf{K}^S - n^S \lambda \mathbf{I} = - \frac{\partial \Psi^I}{\partial \mathbf{F}_{Se}} \mathbf{F}_{Se}^T - \frac{\partial \Psi^S}{\partial (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S)} \otimes (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S)$$

and thus from (2.20), (2.26) and (4.3),

$$(5.5) \quad \mathbf{T}^S = (\Psi^S - n^S \lambda) \mathbf{I} + \frac{\partial \Psi^I}{\partial \mathbf{F}_{Se}} \mathbf{F}_{Se}^T + \frac{\partial \Psi^S}{\partial (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S)} \otimes (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S).$$

The remaining or dissipative part of the entropy inequality yields

$$(5.6) \quad -\frac{1}{\theta} \operatorname{grad} \theta \cdot \mathbf{m} - (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) \cdot \hat{\mathbf{f}}^F - \mathbf{D}_{S_p} \cdot \left[\varrho^S \mathbf{K}^S - n^S \lambda \mathbf{I} + \mathbf{F}_{S_e} \frac{\partial \Psi^I}{\partial \mathbf{F}_{S_p}} \mathbf{F}_{S_p}^T \mathbf{F}_{S_e}^T \right. \\ \left. - (\mathbf{F}_{S_e} \mathbf{F}_{S_e}^T - \mathbf{I}) \frac{\partial \Psi^I}{\partial \mathbf{F}_{S_e}} \mathbf{F}_{S_e}^T + \frac{\partial \Psi^S}{\partial (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S)} \otimes (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) \right] \\ - \mathbf{D}_F \cdot \left[\varrho^F \mathbf{K}^F - n^F \lambda \mathbf{I} + \frac{\partial \Psi^F}{\partial (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S)} \otimes (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) \right] \geq 0,$$

where

$$(5.7) \quad \mathbf{m} = \mathbf{h} + \theta \frac{\partial \Psi^S}{\partial \theta} \mathbf{u}_S + \theta \frac{\partial \Psi^F}{\partial \theta} \mathbf{u}_F, \\ \hat{\mathbf{f}}^F = \hat{\mathbf{p}}^F - \lambda \operatorname{grad} n^F + \mathbf{F}_{S_e}^{T-1} \left(\frac{\partial \Psi^F}{\partial \mathbf{F}_{S_e}} \mathbf{G}_{S_e} \right)^{\frac{1}{3}} + \mathbf{F}_{S_e}^{T-1} \mathbf{F}_{S_p}^{T-1} \left(\frac{\partial \Psi^F}{\partial \mathbf{F}_{S_p}} \mathbf{G}_{S_p} \right)^{\frac{1}{3}}.$$

The inequality (5.6) is now evaluated near the thermodynamical equilibrium for the mixture [6] or the so-called mixture equilibrium, respectively. This state defined as the minimum state for the inequality (5.6) implies that ϕ from Eq. (4.24)₃ reduces to ϕ_0 ,

$$(5.8) \quad \phi_0 = \{ \theta, \operatorname{grad} \theta = \mathbf{0}, \mathbf{F}_{S_e}, \mathbf{F}_{S_p}, \mathbf{G}_{S_e}^3, \mathbf{G}_{S_p}^3, \dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S = \mathbf{0}, \mathbf{D}_F = \mathbf{0} \},$$

and

$$(5.9) \quad \mathbf{D}_{S_p} = \mathbf{0}.$$

In the elastic range, Eq. (5.9) generally holds. In the plastic range, Eq. (5.9) holds in case of neutral loading of hardening materials. In a strong equilibrium state, i. e., if the left-hand side of the inequality (5.6) vanishes, it is easily seen that

$$(5.10) \quad \mathbf{m}(\phi_0) = \mathbf{0}, \\ \hat{\mathbf{f}}^F(\phi_0) = \mathbf{0}, \\ (\varrho^F \mathbf{K}^F - n^F \lambda \mathbf{I})(\phi_0) = \mathbf{0}.$$

Concerning the general procedure, compare, for example, [6, 13].

Near the equilibrium state, a linear expansion of Ψ^S , Ψ^F , \mathbf{m} , $\hat{\mathbf{f}}^F$ and $(\varrho^F \mathbf{K}^F - n^F \lambda \mathbf{I})$ about $\phi = \phi_0$ yields

$$(5.11) \quad \Psi^S = \Psi^S(\theta, \mathbf{F}_{S_e}, \mathbf{F}_{S_p}), \\ \Psi^F = \Psi^F(\theta, \mathbf{F}_{S_e}, \mathbf{F}_{S_p}), \\ \mathbf{m}(\phi) = -\beta_\theta(\phi_0) \operatorname{grad} \theta + \theta \alpha_\theta(\phi_0) (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S), \\ \hat{\mathbf{f}}^F(\phi) = -\alpha_\theta(\phi_0) \operatorname{grad} \theta - \alpha_v(\phi_0) (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S), \\ (\varrho^F \mathbf{K}^F - n^F \lambda \mathbf{I})(\phi) = -2\mu^F(\phi_0) \mathbf{D}_F^D - \nu^F(\phi_0) (\mathbf{D}_F \cdot \mathbf{I}),$$

where the principle of material frame-indifference (4.22) combined with Eq. (4.24)₃ along with several arguments as explained in [6, 13] has been used. In Eq. (5.11)₅,

$$(5.12) \quad \mathbf{D}_F^D = \mathbf{D}_F - \frac{1}{3} (\mathbf{D}_F \cdot \mathbf{I}) \mathbf{I}.$$

From Eqs. (2.20), (2.26), (4.3), (5.5) and (5.11), we obtain the following results:

$$\begin{aligned}
 \mathbf{T}^S &= (\Psi^S - n^S \lambda) \mathbf{I} + \frac{\partial \Psi^I}{\partial \mathbf{F}_{S_e}} \mathbf{F}_{S_e}^T, \\
 \mathbf{T}^F &= (\Psi^F - n^F \lambda) \mathbf{I} + 2\mu^F \mathbf{D}_F^D + \nu^F (\mathbf{D}_F \cdot \mathbf{I}) \mathbf{I}, \\
 \mathbf{h} &= -\beta_\theta \text{grad } \theta + \theta \alpha_\theta (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) - \theta \frac{\partial \Psi^S}{\partial \theta} \mathbf{u}_S - \theta \frac{\partial \Psi^F}{\partial \theta} \mathbf{u}_F, \\
 \hat{\mathbf{p}}^F &= -\alpha_\theta \text{grad } \theta - \alpha_v (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) + \lambda \text{grad } n^F \\
 &\quad - \mathbf{F}_{S_e}^{T-1} \left(\frac{\partial \Psi^F}{\partial \mathbf{F}_{S_e}} \mathbf{G}_{S_e} \right)^{\frac{1}{2}} - \mathbf{F}_{S_e}^{T-1} \mathbf{F}_{S_p}^{T-1} \left(\frac{\partial \Psi^F}{\partial \mathbf{F}_{S_p}} \mathbf{G}_{S_p} \right)^{\frac{1}{2}}.
 \end{aligned}
 \tag{5.13}$$

Given Eqs. (2.27), (4.7)₂ and (5.7)₁, we have two different formulae for the influx vector \mathbf{h} , namely:

$$\begin{aligned}
 \mathbf{h} &= \mathbf{m} - \theta \frac{\partial \Psi^S}{\partial \theta} \mathbf{u}_S - \theta \frac{\partial \Psi^F}{\partial \theta} \mathbf{u}_F, \\
 \mathbf{h} &= \mathbf{q}^S + \mathbf{q}^F + \theta H^S \mathbf{u}_S + \theta H^F \mathbf{u}_F.
 \end{aligned}
 \tag{5.14}$$

In assuming

$$\mathbf{q}^S + \mathbf{q}^F = -\beta_\theta \text{grad } \theta,
 \tag{5.15}$$

it follows from Eqs. (2.10), (4.7)₁, (5.2) and (5.3) together with Eqs. (5.7)₁ and (5.14) that

$$\begin{aligned}
 \Pi^S &= -\left(\frac{\partial \Psi^S}{\partial \theta} + \alpha_\theta \right), \\
 \Pi^F &= -\left(\frac{\partial \Psi^F}{\partial \theta} - \alpha_\theta \right).
 \end{aligned}
 \tag{5.16}$$

Thus, it is felt that α_θ is an entropy coupling parameter for the present single temperature model; β_θ is the coefficient of thermal conductivity for the whole system. However, since $\mathbf{m} = \mathbf{h} = \mathbf{q}^S + \mathbf{q}^F$ is the heat influx whenever $\dot{\mathbf{x}}_F = \dot{\mathbf{x}}_S = \dot{\mathbf{x}}$ and, therefore, $\mathbf{u}_F = \mathbf{u}_S = \mathbf{0}$, Eq. (5.15) also represents the constitutive equation for the heat influx for the model without diffusion.

Given the relations (5.6) and (5.9), the parameters α_v , β_θ , μ^F and ν^F are constrained by

$$\alpha_v \geq 0, \quad \beta_\theta \geq 0, \quad \mu^F \geq 0, \quad \nu_{\frac{3}{2}}^F \geq 0.
 \tag{5.17}$$

As shown in [13], α_v can be determined by the coefficient of permeability k which, of course, is not a constant but depends on ϕ_0 :

$$\alpha_v = \frac{\rho^F}{k} |\mathbf{b}|.
 \tag{5.18}$$

In Eq. (5.18),

$$\mathbf{b} = \mathbf{b}^S = \mathbf{b}^F
 \tag{5.19}$$

has been assumed.

The quantities μ^F and ν^F are the macroscopic shear and bulk viscosity parameters of the liquid. Note in passing that even if the liquid is incompressible, usually

$$(5.20) \quad \mathbf{D}_F \cdot \mathbf{I} = \operatorname{div} \dot{\mathbf{x}}_F \neq 0.$$

Therefore, ν^F must be generally incorporated into the theory. Finally, in case of plastic loading we have

$$(5.21) \quad \mathbf{D}_{Sp} \neq \mathbf{0}$$

instead of Eq. (5.9).

Given $\phi = \phi_0$, it follows from (5.6) along with (2.20), (2.26), (4.3) and (5.10) that

$$(5.22) \quad \mathbf{D}_{Sp} \cdot \left[\mathbf{T}^S - (\Psi^S - n^S \lambda) \mathbf{I} - \mathbf{F}_{Se} \frac{\partial \Psi^I}{\partial \mathbf{F}_{Sp}} \mathbf{F}_{Sp}^T \mathbf{F}_{Se}^T + (\mathbf{F}_{Se} \mathbf{F}_{Se}^T - \mathbf{I}) \frac{\partial \Psi^I}{\partial \mathbf{F}_{Se}} \mathbf{F}_{Se}^T \right] \geq 0,$$

where the last two terms in square brackets must yield a symmetric tensor. However, if $\phi \neq \phi_0$, the inequality (5.22) can be understood as a sufficient condition, a restriction for the rate of plastic work.

6. Simplified constitutive theory

Considering applications of the preceding theory, it would appear that some simplifications are necessary. Since the constitutive equations (5.13) and the restriction (5.22) depend on the choice of the free energy functions Ψ^S and Ψ^F , it seems to be reasonable to introduce simplifications with respect to these functions. Up to now the principle of equipresence has been used in the sense of a "rule to guide us when we come to set up constitutive equations in the first place" [40, p. 135]. However, there are other theories like the theory of "multiphase mixtures" of PASSMAN, NUNZIATO and WALSH [41] where the principle of equipresence is generally substituted by another principle called the "principle of phase separation". Porous media models as discussed in the preceding sections have been introduced in the same sense as in theories of multiphase mixtures, namely, as mixtures with immiscible constituents. Thus, to simplify the constitutive equations for the present model, one may apply a modified version of the principle of phase separation cited above.

We now introduce a "principle of constituent separation" by the fact that the free energy densities ψ^S and ψ^F depend on its own constituent variables only. Thus, from Eqs. (2.25)₁, (5.11)₁ and (5.11)₂

$$(6.1) \quad \begin{aligned} \Psi^S(\theta, \mathbf{F}_{Se}, \mathbf{F}_{Sp}) &= \varrho^{SR} n^S(\mathbf{F}_{Se}, \mathbf{F}_{Sp}) \psi^S(\theta, \mathbf{F}_{Se}, \mathbf{F}_{Sp}), \\ \Psi^F(\theta, \mathbf{F}_{Se}, \mathbf{F}_{Sp}) &= \varrho^{FR} n^F(\mathbf{F}_{Se}, \mathbf{F}_{Sp}) \psi^F(\theta). \end{aligned}$$

Note in passing that the volume fractions as kinematic variables must depend on both the elastic and the plastic parts of the solid deformation gradient, compare Eqs. (3.3), (4.9) and (4.18)₁. Note further that $\psi^S(\theta, \mathbf{F}_{Sp}, \mathbf{F}_{Se})$ corresponds to a model first proposed by GREEN and NAGHDI [42].

Using Eq. (6.1) along with the relations (2.5), (2.25)₁, (3.3), (4.5), (4.9), (4.17)₁ and (5.3),

$$(6.2) \quad \frac{\partial \Psi^I}{\partial \mathbf{F}_{Se}} = - \left(\Psi^S - n^S \frac{\Psi^F}{n^F} \right) \mathbf{F}_{Se}^{T-1} + \varrho^S \frac{\partial \psi^S}{\partial \mathbf{F}_{Se}}$$

holds where

$$(6.3) \quad \frac{\partial n^S}{\partial \mathbf{F}_{Fe}} = -n^S \mathbf{F}_{Se}^{T-1}$$

has been used.

Thus, Eqs. (5.13)₁ and (5.13)₂ reduce to

$$(6.4) \quad \begin{aligned} \mathbf{T}^S &= -n^S p \mathbf{I} + \varrho^S \frac{\partial \psi^S}{\partial \mathbf{F}_{Se}} \mathbf{F}_{Se}^T, \\ \mathbf{T}^F &= -n^F p \mathbf{I} + 2\mu^F \mathbf{D}_F^D + \nu^F (\mathbf{D}_F \cdot \mathbf{I}) \mathbf{I}, \end{aligned}$$

where

$$(6.5) \quad p = \lambda - \frac{\Psi^F}{n^F}$$

can be interpreted as the unspecified hydrostatic pressure acting on the whole model.

Furthermore, it can be shown with the aid of the relations (2.5), (2.25)₁, (3.3), (3.4), (4.18) and (6.1)₂ that

$$(6.6) \quad \mathbf{F}_{Se}^{T-1} \left(\frac{\partial \Psi^F}{\partial \mathbf{F}_{Se}} \mathbf{G}_{Se}^3 \right)^{\frac{1}{2}} + \mathbf{F}_{Se}^{T-1} \mathbf{F}_{Sp}^{T-1} \left(\frac{\partial \Psi^F}{\partial \mathbf{F}_{Sp}} \mathbf{G}_{Sp}^3 \right)^{\frac{1}{2}} = \frac{\Psi^F}{n^F} \text{grad } n^F,$$

where

$$(6.7) \quad \begin{aligned} \frac{\partial \Psi^F}{\partial \mathbf{F}_{Se}} &= n^S \frac{\Psi^F}{n^F} \mathbf{F}_{Se}^{T-1}, \\ \frac{\partial \Psi^F}{\partial \mathbf{F}_{Sp}} &= n^S \frac{\Psi^F}{n^F} \mathbf{F}_{Sp}^{T-1}, \end{aligned}$$

together with

$$(6.8) \quad \frac{\partial n^S}{\partial \mathbf{F}_{Sp}} = -n^S \mathbf{F}_{Sp}^{T-1}$$

have been used.

Given Eqs. (5.13)₄, (6.5) and (6.6), the momentum supply or the interacting force per unit of mixture volume between the solid and the liquid material, respectively, yields

$$(6.9) \quad \hat{\mathbf{p}}^F = p \text{grad } n^F - \alpha_0 \text{grad } \theta - \alpha_v (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S).$$

Finally, using the same procedure as to obtain Eq. (6.2),

$$(6.10) \quad \frac{\partial \Psi^I}{\partial \mathbf{F}_{Sp}} = - \left(\Psi^S - n^S \frac{\Psi^F}{n^F} \right) \mathbf{F}_{Sp}^{T-1} + \varrho^S \frac{\partial \psi^S}{\partial \mathbf{F}_{Sp}}$$

holds and thus, from the relations (5.22) and (6.5),

$$(6.11) \quad \mathbf{D}_{Sp} \cdot \left[\mathbf{T}^S + n^S p \mathbf{I} - \varrho^S \mathbf{F}_{Se} \frac{\partial \psi^S}{\partial \mathbf{F}_{Sp}} \mathbf{F}_{Sp}^T \mathbf{F}_{Se}^T + (\mathbf{F}_{Se} \mathbf{F}_{Se}^T - \mathbf{I}) \varrho^S \frac{\partial \psi^S}{\partial \mathbf{F}_{Se}} \mathbf{F}_{Se}^T \right] \geq 0.$$

In the inequality (6.11), the last two terms in square brackets must form again a symmetric tensor. It should be noted that the results (6.4), (6.9) and (6.11) of the simplified model differ from the results that would have been achieved from the usual simplifying assumptions, namely when the second solid deformation gradient or $\overset{3}{\mathbf{G}}_{Se}$ and $\overset{3}{\mathbf{G}}_{Sp}$, respectively, is omitted from the independent variables (4.24)₃. In that case Ψ^F proves to be independent of \mathbf{F}_{Sp} , \mathbf{F}_{Se} what, of course, is an unrealistic result as far as n^F is not a constant, cf. Eq. (6.1)₂.

In the isotropic range, as it is well known, the functional dependence of ψ^S on \mathbf{F}_{Se} , \mathbf{F}_{Sp} can be substituted by

$$(6.12) \quad \psi^S(\theta, \mathbf{F}_{Se}, \mathbf{F}_{Sp}) \rightarrow \psi^S(\theta, \mathbf{E}_{Se}, \mathbf{E}_{Sp}).$$

Then, since $\mathbf{E}_{Se} = \mathbf{E}_{Se}(\mathbf{F}_{Se}, \mathbf{F}_{Sp})$ and $\mathbf{E}_{Sp} = \mathbf{E}_{Sp}(\mathbf{F}_{Sp})$, cf. Eqs. (3.13), the restriction for the rate of plastic work yields

$$(6.13) \quad \mathbf{D}_{Sp} \cdot \left[\mathbf{T}^S + n^S p \mathbf{I} - \varrho^S \mathbf{F}_{Se} \frac{\partial \psi^S}{\partial \mathbf{T}_{Sp}} \mathbf{F}_{Se}^T \right] \geq 0$$

where, from the chain rule,

$$(6.14) \quad \frac{\partial \psi^S}{\partial \mathbf{F}_{Se}} = \left(\frac{\partial \mathbf{E}_{Se}}{\partial \mathbf{F}_{Se}} \right)^T \frac{\partial \psi^S}{\partial \mathbf{E}_{Se}} = \mathbf{F}_{Se} \mathbf{F}_{Sp} \frac{\partial \psi^S}{\partial \mathbf{E}_{Se}} \mathbf{F}_{Sp}^T$$

and

$$(6.15) \quad \frac{\partial \psi^S}{\partial \mathbf{F}_{Sp}} = \left(\frac{\partial \mathbf{E}_{Se}}{\partial \mathbf{F}_{Sp}} \right)^T \frac{\partial \psi^S}{\partial \mathbf{E}_{Se}} + \left(\frac{\partial \mathbf{E}_{Sp}}{\partial \mathbf{F}_{Sp}} \right)^T \frac{\partial \psi^S}{\partial \mathbf{E}_{Sp}} = (\mathbf{F}_{Se}^T \mathbf{F}_{Se} - \mathbf{I}) \mathbf{F}_{Sp} \frac{\partial \psi^S}{\partial \mathbf{E}_{Se}} + \mathbf{F}_{Sp} \frac{\partial \psi^S}{\partial \mathbf{E}_{Sp}}$$

together with

$$(6.16) \quad \frac{\partial \psi^S}{\partial \mathbf{E}_{Sp}} = \left(\frac{\partial \mathbf{T}_{Sp}}{\partial \mathbf{E}_{Sp}} \right)^T \frac{\partial \psi^S}{\partial \mathbf{T}_{Sp}} = \mathbf{F}_{Sp}^{-1} \frac{\partial \psi^S}{\partial \mathbf{T}_{Sp}} \mathbf{F}_{Sp}^{T-1}$$

has been used. As concerns the inequality (6.13), a possible interpretation of

$$(6.17) \quad \mathbf{Y}^S = \varrho^S \frac{\partial \psi^S}{\partial \mathbf{T}_{Sp}}$$

is that this quantity represents the back-stress tensor of kinematically hardening solids.

However, assuming circumstances where ψ^S is independent of \mathbf{E}_{Sp} such that

$$(6.18) \quad \psi^S = \psi^S(\theta, \mathbf{E}_{Se}),$$

the restriction for the rate of plastic work reduces to

$$(6.19) \quad \mathbf{D}_{Sp} \cdot (\mathbf{T}^S + n^S p \mathbf{I}) \geq 0.$$

The relations (6.18) and (6.19) correspond to a liquid-saturated elastic ideal-plastic skeleton as proposed by DE BOER and KOWALSKI [14] or DE BOER and EHLERS [16].

Note in passing that, as usual in theories of constrained materials, the rate of plastic work depends on the so-called extra stresses of the solid,

$$(6.20) \quad \mathbf{T}_E^S = \mathbf{T}^S + n^S p \mathbf{I},$$

which are functions of the deformation process. Thus, from Eqs. (6.4)₁ and (6.14)

$$(6.21) \quad \mathbf{T}_E^S = \varrho^S \frac{\partial \psi^S}{\partial \mathbf{A}_{Se}},$$

where

$$(6.22) \quad \frac{\partial \psi^S}{\partial \mathbf{E}_{Se}} = \mathbf{F}_{Sp}^{-1} \mathbf{F}_{Se}^{-1} \frac{\partial \psi^S}{\partial \mathbf{A}_{Se}} \mathbf{F}_{Se}^{T-1} \mathbf{F}_{Sp}^{T-1}$$

has been used. Equation (6.20), however, which also represents the partial or peculiar solid stresses in the frame of empty porous media with constituent incompressibility or the effective stresses in the frame of soil-mechanical problems, generally contains deviatoric as well as hydrostatic parts, the latter due to possible variations of porosity.

7. Concluding remarks

A unified macroscopic approach to thermodynamics of saturated elasto-plastic porous media has been presented. Especially the simplified constitutive model must be suggested as a convenient tool for applications.

However, since in finite theories of porous media even the elastic strains are not necessarily small, an elasticity law of the Hookean type does not cover the possible range of elastic deformations. Imagine, for example, an empty sponge of incompressible rubber material, then, even the hydrostatic deformations can be of such magnitude that within a macroscopic formulation any elasticity law of the Hookean type must fail. Thus, it seems to be more convenient to prefer elasticity laws of the MURNAGHAN type [43] or SIMO-PISTER type [44]. A modified version of the latter extended towards finite elasticity of porous media will be presented by the author in [45].

To cover the plastic range, a yield or failure condition for ductile materials has been proposed by DE BOER and KOWALSKI [14]; brittle or granular materials such as soil or concrete have been discussed by DE BOER and EHLERS [16]. An improved yield condition together with a non-associated flow rule has been given by DE BOER [46]; it reflects the results of three-dimensional shear testing including the effects of extension and compression as obtained in the frame of soil mechanics, for example, by experiments on clay or sand, cf. e.g., [47–49]. A complete description of incompressible, liquid-saturated elasto-plastic porous media together with a detailed analysis of the elastic as well as the plastic response will be the subject of a forthcoming report [50]. Within the preceding theory, the problem of fluid flow through rigid porous media has been discussed in [13].

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