ERRATA

Growth of voids in a ductile matrix: a review

P. GILORMINI, C. LICHT and P. SUQUET, *Arch. Mech.* **40**, pp. 43–80, 1988

Since all the corrections indicated in the proofs of the paper could not be included by the editor, the authors wish to point out the following errata to the reader. (Some additional minor misprints are not listed, for the sake of brevity.)

Page 45: lines 13 to 30 must be moved to page 44, above *1.2. Damage and micromechanics*, and n_j should be changed into n_i in Eq, (1.3).

Page 47: the last line above *2. Isolated Voids* should be read as follows: *ties, by J. Rice, B. Budiansky, J. W. Hutchinson, A. Needleman, and their coworkers.* A comma should replace the full stop at the end of 6[th] line from the bottom.

Page 48: a *1* is missing between *the* and $(\Sigma_{11}$], and a *2* has been omitted between *or* and $(\Sigma_{22} >$ in line 17.

Page 49: remove the *a* before generalized in line 11.

Page 53: change *is* into *was* in line 8, and $\in$ into $\dot{\varepsilon}$ in Eq. (2.5).

Page 57: change *1)*; *for* into *1), for* in line 6; the beginning of line 21 should be read as *to which a viod will tend has been* etc.; replace *ligh* by *high* in line 23.

Page 61, Eq. (3.4): read $s' = sl(s-l)$ instead of $s' = s(s-l)$.

Page 63: change *0.01,0.4* into *0.01 – 0.4* in line 7.

Page 66: change *in* into *into* (line 17), remove the comma after *i) approximate expressions* (line 16), and add one after *uniform strains* in the footnote.

Page 67: read $\dot{\tilde{\mathbf{u}}}^*$ instead of $\dot{\mathbf{u}}^*$ in the 2[nd] equation and instead of $*\dot{\tilde{\mathbf{u}}}*$ in the 5[th].

Page 68: read $s' = s/(s-l)$ instead of $s' = s(s-l)$ in the 4[th] equation, close the parenthesis in the 6[th], and change), into)), in the last line.

Page 69: $c_2 = -\frac{3}{2}^{\frac{s}{s-1}}$ should be replaced by $c_2 = -\frac{3}{2} G^{\frac{1}{s-1}}$.

Page 71: substitute $\dot{\tilde{E}}$ to $\tilde{E}_{\alpha\beta}$ two lines below Eq. (6.19).

Page 77: $\frac{2\mu}{\sigma^2}$ must be replaced by $\frac{2\mu}{\rho^2}$ in line 15, and E_{33} by $\dot{E}_{33}$ at the bottom of the page.

Page 79: substitute *dilatation* to *dilation* in ref. 3, *solids* to *solid* in ref. 5, and *composites* to *composities* in ref. [11].

Page 80: substitute *voids or inclusions* to *voids inclusions* in ref. [17].

Arch. Mech., 41, 1, pp. 37–59, Warszawa 1989

On the stress distribution in strongly anisotropic plates

A. BLINOWSKI (WARSZAWA)

A SYSTEM of INFINITE equations of a transversely isotropic plate of arbitrary thickness is proposed. Solution of the system expressed in terms of displacements satisfies the local equilibrium conditions, normal loading conditions in integral form, and modified boundary conditions across the thickness of the plate. Solutions in the form of infinite series are found for three practical cases and finite formulae for the model problems (loads having the form of eigenfunctions of the Laplace operator are qiven). In strongly anisotropic plates of large thickness-to-span ratio (of about 1/5) normal stress distributions considerably differ from the linear ones, stress maxima are higher then those predicted by the simplified theory, and the corresponding deflections are substantially different. The differences increase with increasing rigidity of the supports. Limits of applicablity of the simplified "engineering" theory are estimated.

Zaproponowano układ nieskończonych równań płyty transwersalnie izotropowej o dowolnej grubości. Rozwiązania układu w przemieszczeniach spełniają lokalne równania równowagi i całkowe warunki obciążeń normalnych oraz zmodyfikowane warunki brzegowe po grubości płyty. Znaleziono rozwiązania w postaci szeregów dla trzech zagadnień praktycznych oraz, zadane skończonymi wzorami, rozwiązania zadań modelowych (przy obciążeniach w postaci funkcji własnych operatora Laplace'a). Pokazano, że dla płyt o silnej anizotropii przy niewielkich rozpiętościach (rzędu 5 grubości) rozkłady naprężeń normalnych po grubości istotnie różnią się od liniowych, maksymalne wartości naprężeń zauważalnie przewyższają przewidywane przez teorię inżynierską, a ugięcia różnią się drastycznie od przewidywanych w modelu inżynierskim. Różnice są tym większe im sztywniejszy jest schemat zamocowania (przy zachowaniu rozpiętości i anizotropii). Oszacowano granice stosowalności inżynierskiego modelu.

Предложена система бесконечных уравнений трансверсально-изотропных пластин произвольной толщины. Решения в перемещениях удовлетворяют локальным уравнениям равновесия, интегральным условиям нормальной нагрузки и модифицированным краевым условиям по толщине пластины. Найдены решения в виде рядов для трех практических задач, а также решения, выражаемые конечными формулами для модельных задач (при нагрузках в виде собственных функций оператора Лапласа). Показывается, что для сильно анизотропных пластин при небольших пролетах — порядка 5-и толщин, распределения нормальных напряжений по толщине существенно отличаются от линейных. Максимальные значениях нормальных напряжений и прогибов существенно отличаются от величин предсказанных инженерной теорией, в случае прогибов эти результаты вообще несопоставимы. Наблюдаемые различия тем больше, чем жестче схема закрепления пластины. Производится оценка пределов применимости инженерной теории.

1. Introduction

IN PLATES and shells made of laminates and fibrous composites the shear moduli corresponding to the shear deformations which do not change the length of the reinforcement fibres, differ from the Young modulus corresponding to tension parallel to the fibres by at least one order of magnitude. Non-applicability of the Love–Kirchhoff hypothesis to structures of such kind may easily be demonstrated in the limiting case described by e.g. SPENCER [1], i.e. in the case of inextensible reinforcement fibres, when the entire normal stresses acting in cross-sections of a layer subject to bending are transmitted b infinitesimally thin surface layers. PIPKIN and EVERSTINE considered in paper [2] the problem of bending of a cantilever beam, reinforced by almost inextensible fibres and, using the boundary layer method, obtained quantitative results indicating a strong stress concentration in finite surface layers and a considerable contribution of shear deformations to the resulting beam deflection. The considerations remain practically unaltered in the case of cylindrical bending of a layer.

In the present paper a model will be proposed, enabling effective solution of the problems of bending of transversely isotropic layers and avoiding the necessity of utilization of the hypothesis of linear displacement distribution across the plate thickness. The model proposed, based on power series expansions of the displacements, is the result of unsuccessful attempts aimed at refining the simplified, "engineering" model by means of polynomial representations. Final results of the procedure make it clear that they probably could have been derived by applying to the anisotropic material the procedure used by A. I. LURIE [3] in the analysis of isotropic layers of finite thickness and thick plates.

This possibility was exploited by S. G. LEKHNITSKII [10] who, by applying an entirely different approach, derived the equations of infinite order for several functions which could be used to express the displacement field of a transversely isotropic infinite layer. For a particular case of a layer acted on by transversal forces, the set of functions reduces to a single function, and a series of operator functions — to a series of operators. It seems, however, that S. G. LEKHITSKII reducing the general problem of antisymmetrically loaded plate to the problem of a plate loaded by transversal forces, unnecessarily narrows the class of solutions to potential displacement fields $u_\alpha (\alpha = 1, 2)$ by assuming two of the three displacement functions sought for to vanish, while it would be sufficient to assume that they satisfy certain infinite differential equations. It is not clear, however, if it would then be possible to avoid the series of operator functions (that is, actually, double operator series). Unfortunately, S. G. LEKHITSKII does not suggest any practical methods of solution of the boundary value problems, or even any methods of determination of the displacement function for infinite layers. The approach to be proposed here resembles also that presented by L. H. DONNELL [4] but cannot be considered as its generalization to anisotropic materials.

2. Remarks on the simplified theory of transversely isotropic plates

In order to draw the attention of the reader to the "classical" assumptions and their influence on the form of the S. Germain equation in the case of transversal isotropy, let us follow the natural course of derivation of the simplified plate theory consisting in integration of the local equilibrium equations across the thickness of the plate. Other methods of derivation do not expose so clearly the role of the consecutive assumptions.

Consider a plate of thickness $2h$, its middle surface perpendicular to the x_3-axis which is parallel to the distinguished direction of transversal isotropy, Fig. 1. The constitutive equations written in the Cartesian coordinate system $\{x_1, x_2, x_3\}$ have the following form (cf. [5, 6]):

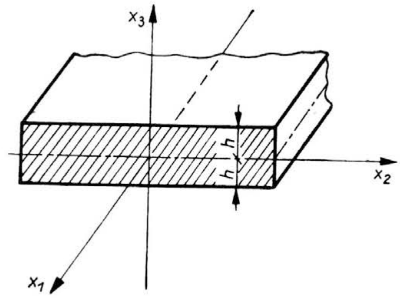

FIG. 1. Plate scheme, reference frame.

$$
\begin{aligned}
\sigma_{\alpha\beta} &= (\alpha\varepsilon_{\gamma\gamma}+\beta\varepsilon_{33})\,\delta_{\alpha\beta}+2\mu_2\,\varepsilon_{\alpha\beta},\\
\sigma_{\alpha 3} &= 2\mu_1\varepsilon_{\alpha 3},\\
\sigma_{33} &= \delta\varepsilon_{33}+\beta\varepsilon_{\gamma\gamma},
\end{aligned} \tag{2.1}
$$

where Greek indices take the values 1, 2, repeated indices denote summation. No assumptions are made concerning the values of elastic constants α, β, δ, μ_1, μ_2 as yet, though they should obviously satisfy the conditions of positive elastic energy and non-negative Poisson's ratio. The cases when the shear modulus μ_1 is smaller than the remaining elastic moduli by at least one order of magnitude will be of special interest in the analysis to follow.

Transversal deflection of the middle surface will be denoted by $w(x_1, x_2)$, and normal load applied to the plate — by $P(x_1, x_2)$.

In order to obtain the equations of the simplified plate theory it is necessary to make the following assumptions:

(a) $\sigma_{33} = 0$ plane stress condition,

(b) $\begin{cases} \sigma_{3\alpha}|_{x_3=\pm h} = 0, \\ f_\alpha = 0 \end{cases}$ no tangential load condition,

(c) $u_\alpha = -x_3 w_{,\alpha}$ straight normal condition.

Here f_α are components of body forces acting in the plane of the plate; comma denotes differentiation with respect to the corresponding Cartesian coordinates.

Condition (a) yields the following form of constitutive relation for $\sigma_{\alpha\beta}$:

$$\sigma_{\alpha\beta} = \left(\alpha-\frac{\beta^2}{\delta}\right)\varepsilon_{\gamma\gamma}\,\delta_{\alpha\beta}+2\mu_2\,\varepsilon_{\alpha\beta}. \tag{2.2}$$

Equations of equilibrium are written in the form

$$
\begin{aligned}
\sigma_{\alpha\beta,\beta}+\sigma_{\alpha 3,3} &= 0,\\
\sigma_{\alpha 3,\alpha}+\sigma_{33,3}+f_3 &= 0.
\end{aligned} \tag{2.3}
$$

On multiplying the first equation by x_3 and integrating by parts over the interval $(-h, h)$ (condition (b) being used), we obtain

$$M_{\alpha\beta,\beta}-S_\alpha = 0, \tag{2.4}$$

where

$$M_{\alpha\beta} \equiv \int\limits_{-h}^{h} \sigma_{\alpha\beta} x_3 \, dx_3, \qquad S_\alpha \equiv \int\limits_{-h}^{h} \sigma_{\alpha 3} \, dx_3.$$

Integration of the second equation (2.2) yields the relation

$$S_{\alpha,\alpha} + P = 0, \tag{2.5}$$

where

$$P \equiv \sigma_{33}\big|_{x_3=h} - \sigma_{33}\big|_{x_3=-h} + \int\limits_{-h}^{h} f_3 \, dx_3$$

is the normal load of the plate.

Substituting Eq. (2.4) into (2.5) and writing $\sigma_{\alpha\beta}$ (and then $M_{\alpha\beta}$) in terms of the derivatives of displacements u_α, and later, due to condition (a), in terms of the deflection derivatives, we obtain

$$\frac{2}{3} D_0 \Delta\Delta w = P, \tag{2.6}$$

where $D_0 \equiv (\alpha + 2\mu_2 - \beta^2/\delta)\, h^3$, and Δ is the two-dimensional Laplace operator.

The reason why the plate rigidity is denoted by $(2/3)D_0$ and not, as usual, by D, will be explained later. The inconsequent assumption of $\sigma_{\alpha 3}$ (and hence of S_α) being different from zero, despite the condition (c), will be not discussed here.

Let us stress the fundamental role of the assumption (a) which makes it possible to express the stresses $\sigma_{\alpha\beta}$ in terms of the derivatives of displacements u_α, and thus to reduce the problem to a two-dimensional one. Rigorous analysis would lead to the conclusion that assumption (a) implies the entire normal load to consist of suitably selected body forces. All a posteriori "implanted" stress distributions σ_{33} making it possible to satisfy the actual loading conditions at surfaces $x_3 = \pm h$ are of a rather "cosmetic" character, since (1) — they violate either the constitutive relations (e.g. it is assumed that $\beta = 0$), or the equilibrium conditions or, finally, the Beltrami–Michell equations; (2) — they are of no practical meaning from the point of view of the material strength conditions determined mainly by the normal stresses due to bending. Similar assumptions constitute the basis for the Reissner theory; however, replacement of assumption (c) by a weaker assumption of linear distribution of stresses throughout the thickness of the plate makes it possible to take into account the effect of shear deformations upon the value of plate deflection.

3. Differential equations of infinite order

In order to describe actual stress distributions in strongly anisotropic plates it is necessary to discard the hypothesis of linear displacement (stress) distributions across the plate thickness.

There are no reasons for replacing the linear distributions with e.g. cubic ones (or of any other particular power); to the contrary, it is probable that hyperbolic functions should be involved, cf. [2]. Thus let us keep all powers in the MacLaurin expansions of the displacement fields expressed as functions of x_3. Let

$$(3.1)\qquad \begin{aligned} u_\alpha &= -h\sum_{k=0}^{\infty}\Phi_\alpha^{(k)}(x_1, x_2)\left(\frac{x_3}{h}\right)^{2k+1},\\ u_3 &= \sum_{k=0}^{\infty}\Psi^{(k)}(x_1, x_2)\left(\frac{x_3}{h}\right)^{2k}. \end{aligned}$$

Making use of the fact that every three-dimensional vector field may uniquely be represented in the form of a sum of potential and solenoidal components, let us write (in two dimensions) the formula

$$\Phi_\alpha^{(k)} = \Phi'^{(k)}_{,\alpha}+\Phi''^{(k)}_{,\beta}\epsilon_{\alpha\beta},$$

where

$$\epsilon_{11} = \epsilon_{22} = 0, \qquad \epsilon_{12} = -\epsilon_{21} = 1,$$

and $\Phi'^{(k)}$, $\Phi''^{(k)}$ are certain scalar-valued functions. Let us assume the condition $\sigma_{33} = 0$ to be satisfied at each point of the body, i.e. the condition analogous with that used in the simplified theory.

It will be satisfied provided the relations hold true

$$(3.3)\qquad \Psi^{(k+1)} = \frac{h^2\beta}{\delta}\frac{\Delta\Phi'^{(k)}}{(2k+2)}.$$

Moreover, assume the first two equations of equilibrium (2.3) to be satisfied for $\alpha = 1, 2$. It should be noted that the equations are to be satisfied locally, at any point, not in the integral form. By evaluating the corresponding fields and performing the necessary differentiations we obtain

$$(3.4)\qquad \left(D_1\Delta\Phi'^{(k)}_{,\alpha}+\Phi'^{(k+1)}_{,\alpha}(2k+3)(2k+2)\right) +\left(\frac{h^2\mu_2}{\mu_1}\Delta\Phi''^{(k)}_{,\beta}+\Phi''^{(k+1)}_{,\beta}(2k+3)(2k+2)\right)\epsilon_{\alpha\beta} = 0,$$

$$k = 0, 1, 2, \ldots, \infty,$$

where

$$D_1 \equiv \frac{h^2}{\mu_1}\left[(\alpha+2\mu_2)-\frac{\beta}{\delta}(\beta+\mu_1)\right].$$

In view of the uniqueness of decomposition into the potential and solenoidal parts, both expresions in parentheses in (3.4) must simultaneously vanish.

Let us finally assume the surfaces $x_3 = \pm h$ to be free of shearing stresses, $\sigma_{\alpha 3}|_{x_3=\pm h} = 0$. The latter condition is satisfied provided

$$(3.5)\qquad -\sum_{k=0}^{\infty}(2k+1)\Phi'^{(k)}_{,\alpha}+\sum_{k=0}^{\infty}\Psi^{(k)}_{,\alpha}+\sum_{k=0}^{\infty}(2k+1)\Phi''^{(k)}_{,\alpha}\epsilon_{\alpha\beta} = 0.$$

Relations (3.3), (3.4), (3.5) enable us to express all functions $\Phi'^{(k)}$ and $\Psi^{(k)}$ in terms of derivatives of a single function $F(x_1, x_2)$, and all functions $\Phi''^{(k)}$ — in terms of derivatives of another function $G(x_1, x_2)$.

Without going into extensive and tedious transformations, let us present the final formulae for displacements.

$$\tag{3.6} \begin{aligned} u_\alpha &= -h\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{x_3}{h}\right)^{2k+1}\left(D_1^k\overset{(k)}{\Delta}F_{,\alpha}+\epsilon_{\alpha\beta}\left(\frac{\mu_2}{\mu_1}h^2\right)^k\overset{(k)}{\Delta}G_{,\beta}\right),\\ u_3 &= \sum_{k=0}^{\infty}\frac{(-1)^kD_1^{k-1}}{(2k)!}\left[\frac{D_0}{\mu_1 h}-\frac{h^2\beta}{\delta}\left(\frac{x_3}{h}\right)^{2k}\right]\overset{(k)}{\Delta}F. \end{aligned}$$

Function G must satisfy the relation

$$\tag{3.7} \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\mu_2 h^2}{\mu_1}\right)^k\overset{(k)}{\Delta}G_{,\alpha} = 0.$$

In Eqs. (3.6), (3.7) symbol $\overset{(k)}{\Delta}$ denotes k-tuple application of the Laplace operator.

Careful readers may observe that the consideration could have been started from the Eqs. (3.6), (3.7) since, what can easily be seen by substitution, the displacement fields written in the form (3.6), with the additional condition (3.7), identically satisfy the original postulates

$$\tag{3.8} \begin{aligned} &(\mathrm{a}') \qquad \sigma_{33} = 0,\\ &(\mathrm{b}') \qquad \sigma_{\alpha\beta,\beta}+\sigma_{\alpha3,3} = 0,\\ &(\mathrm{c}') \qquad \sigma_{\alpha3}\big|_{x_3=\pm h} = 0. \end{aligned}$$

Equations (3.6) and (3.7) have been derived from the assumptions (3.1) and conditions (a′), (b′), (c′) and hence it seems that they should constitute the sufficient condition for simultaneous satisfaction of Eqs. (3.1) and (3.8); the problem of necessity of that condition requires additional analysis which, however, will not be dealt with in this paper. Observe moreover that Eq. (3.7) allows for a trivial solution $G \equiv 0$, though non-trivial solutions may also be found: let, for instance,

$$\tag{3.9} G(x_1, x_2) = A\exp\left(\frac{x_1}{a}+\frac{x_2}{b}\right).$$

Then

$$\tag{3.10} \sum_{k=0}^{\infty}(-1)^k\left(\frac{\mu_2}{\mu_1}h^2\right)^k\frac{\Delta G_{,\alpha}}{(2k)!} = \cos\left(\sqrt{\frac{\mu_2}{\mu_1}h^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}\right)G_{,\alpha},$$

so that Eq. (3.7) is satisfied provided

$$\tag{3.11} \frac{1}{a^2}+\frac{1}{b^2} = \frac{(2n+1)^2\pi^2}{4h^2}\frac{\mu_1}{\mu_2} \qquad (n = 1, 2, \ldots).$$

In accordance with the postulate (3.8 a′) the third equation of equilibrium is reduced to the form

$$\sigma_{\alpha 3,\alpha}+f_3 = 0. \tag{3.12}$$

Stress $\sigma_{\alpha 3}$ may be found if the displacements are known, hence the following relation must be satisfied

$$f_3 = \frac{D_0}{h}\sum_{k=0}^{\infty}\left[1-\left(\frac{x_3}{h}\right)^{2k+1}\right]\frac{(-1)^k D_1^k \overset{(k+2)}{\Delta} F}{(2k+2)!}. \tag{3.13}$$

It is seen that satisfaction of Eqs. (3.6), (3.7) is possible at the expense of impossibility of applying the normal loads in an arbitrary manner: they must be replaced by body forces suitably distributed across the thickness of the plate. We shall return to the problem later, new let us concentrate upon the determination of function $F(x_1, x_2)$.

Integrating Eq. (3.13) over the thickness and keeping in mind that $\sigma_{33} = 0$ we obtain the following differential equations of infinite order for the function $F(x_1, x_2)$:

$$2D_0\sum_{k=0}^{\infty}(-1)^k\frac{2k+2}{(2k+3)!}D_1^k \overset{(k+2)}{\Delta} F = P. \tag{3.14}$$

It is easily observed that by retaining the first term of the series ($k = 0$), Eq. (3.14) reduces to the Sophie Germain equation (2.6). Function F is determined by Eq. (3.14) to within the accuracy of at most one biharmonic function which makes the left-hand side of the equation identically zero; similarly to the Eq. (3.7), non-trivial solutions of the homogeneous equation (3.14) may also be found like, for instance, a function of the type of (3.9) under the condition that

$$\sqrt{D_1\left(\frac{1}{a^2}+\frac{1}{b^2}\right)} = \operatorname{tg}\sqrt{D_1\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}. \tag{3.15}$$

Relation (3.13) will be discussed in the following section.

4. Solution expressed in therms of eigenfunctions of the Laplace operator

Let the functions $f_n(x_1, x_2)$ $(n = 1, 2, \ldots)$ constitute a set of independent eigenfunctions of the Laplace operator corresponding to negative eigenvalues $-\lambda_n^2$,

$$\Delta f_n = -\lambda_n^2 f_n. \tag{4.1}$$

Let the load P be represented in the form of a series of eigenfunctions

$$P(x_1, x_2) = \sum_{n=1}^{\infty} P_n f_n(x_1, x_2). \tag{4.2}$$

Function $F(x_1, x_2)$ will be sought in the form of an analogous series

$$F(x_1, x_2) = \sum_{h=1}^{\infty} F_n f_n(x_1, x_2). \tag{4.3}$$

From the property of mutual independence of functions f_n it follows that the infinite order equation of the plate theory (3.14) will be satisfied if equations

$$(4.4)\qquad 2D_0 F_n \sum_{k=0}^{\infty} \frac{2k+2}{(2k+3)!} D_1^k \lambda_n^{2k+4} = P_n.$$

are fulfilled for any n, what implies

$$(4.5)\qquad F_n = \frac{P_n D_1^2}{2D_0} \frac{1}{\gamma_n(\gamma_n \operatorname{ch}\gamma_n - \operatorname{sh}\gamma_n)},$$

where

$$\gamma_n \equiv \sqrt{D_1}\,\lambda_n, \qquad \lambda_n \equiv \sqrt{\lambda_n^2} \geqslant 0.$$

Substitution of the representation (4.3) into the formulae for u_α and u_3 and summation of the series yields

$$(4.6)\qquad u_\alpha = -h \sum_{n=1}^{\infty} \frac{\operatorname{sh}\left(\gamma_n \frac{x_3}{h}\right)}{\gamma_n} F_n f_{n,\alpha} - h\left(\frac{x_3}{h}\right) F_{0,\alpha} + \frac{h}{6}\left(\frac{x_3}{h}\right)^3 D_1 \Delta F_{0,\alpha},$$

$$(4.7)\qquad u_3 = \frac{1}{D_1} \sum_{n=1}^{\infty} \left[\frac{D_0}{\mu_1 h} \operatorname{ch}\gamma_n - \frac{h^2\beta}{\delta} \operatorname{ch}\left(\gamma_n \frac{x_3}{h}\right)\right] F_n f_n$$
$$+ F_0 - \frac{1}{2}\left[\frac{D_0}{\mu_1 h} - \frac{h^2\beta}{\delta}\left(\frac{x_3}{h}\right)^2\right] \Delta F_0,$$

where $\Delta\Delta F_0 = 0$.

In Eqs. (4.6), (4.7) it was assumed that $G = 0$, and terms other than the biharmonic ones resulting from the homogeneous equations have been disregarded. This is due to the fact that the known solutions of (3.7) and of the homogeneous equation derived from (3.14) are eigenfunctions of the Laplace operator with positive eigenvalues. Taking them into account would lead to trigonometric functions of argument $\gamma_n x_3/h$ appearing in the formulae (4.6), (4.7). Such case will not be considered in this paper. Displacement fields (4.6), (4.7) correspond to the following stresses, moments, transverse shears and body forces:

$$(4.8)\qquad \sigma_{\alpha\beta} = \frac{h}{D_1} \sum_{n=1}^{\infty} F_n \gamma_n \operatorname{sh}\left(\gamma_n \frac{x_3}{h}\right) \left[\left(\alpha - \frac{\beta^2}{\delta}\right) f_n \delta_{\alpha\beta} - 2\mu_2 \frac{D_1}{\gamma_n^2} f_{n,\alpha\beta}\right]$$
$$- h\left(\frac{x_3}{h}\right)\left[\left(\alpha - \frac{\beta^2}{\delta}\right) \Delta F_0 \delta_{\alpha\beta} + 2\mu_2 F_{0,\alpha\beta}\right] + \frac{h\mu_2}{3}\left(\frac{x_3}{h}\right)^3 D_1 \Delta F_{0,\alpha\beta},$$

$$(4.9)\qquad \sigma_{\alpha 3} = \frac{D_0}{hD_1} \sum_{n=1}^{\infty} \left[\operatorname{ch}\gamma_n - \operatorname{ch}\left(\gamma_n \frac{x_3}{h}\right)\right] F_n f_{n,\alpha} - \frac{1}{2} \frac{D_0}{h}\left[1 - \left(\frac{x_3}{h}\right)^2\right] \Delta F_{0,\alpha},$$

$$M_{\alpha\beta} = \frac{2h^3}{D_1} \sum_{n=1}^{\infty} \frac{\gamma_n \operatorname{ch}\gamma_n - \operatorname{sh}\gamma_n}{\gamma_n} F_n \left[\left(\alpha - \frac{\beta^2}{\delta}\right) f_n \delta_{\alpha\beta} - \frac{2\mu_2 D_1}{\gamma_n^2} f_{n,\alpha\beta}\right] - \frac{2}{3} h^3 \left[\left(\alpha - \frac{\beta^2}{\delta}\right) \Delta F_0 \delta_{\alpha\beta} + 2\mu_2 F_{0,\alpha\beta}\right] + \frac{2h^3}{15} \mu_2 D_1 \Delta F_{0,\alpha\beta}, \tag{4.10}$$

$$S_\alpha = \frac{2D_0}{D_1} \sum_{n=1}^{\infty} \frac{\gamma_n \operatorname{ch}\gamma_n - \operatorname{sh}\gamma_n}{\gamma_n} F_n f_{n,\alpha} - \frac{2}{3} D_0 \Delta F_{0,\alpha}, \tag{4.11}$$

$$f_3 = \frac{D_0}{D_1^2 h} \sum_{n=1}^{\infty} \gamma_n^2 \left[\operatorname{ch}\gamma_n - \operatorname{ch}\left(\gamma_n \frac{x_3}{h}\right)\right] F_n f_n. \tag{4.12}$$

Substituting here the values of F_n from Eq. (4.5) we may write also the relations (4.6)–(4.12) in terms of P_n.

This is a suitable moment to return to the problem of estimation of the error resulting from the artificial scheme of application of the load enforced by the model. Let us first estimate the possible values of coefficients D_1 and γ_n. Elementary but time-consuming transformations lead to the following relation

$$D_0 = \frac{E_L h^3}{1 - \nu_{LL}^2}, \tag{4.13}$$

where E_L is Young's modulus corresponding to simple tension in the direction perpendicular to that of transversal isotropy, and Poisson's ratio ν_{LL} relates the strain measured in the direction perpendicular to the tensile forces (and the distinguished direction) with the strain measured in the direction of simple tension: for instance, in case of simple tension parallel to the axis x_1,

$$\begin{aligned} \sigma_{11} &= E_L \varepsilon_{11}, \\ \varepsilon_{22} &= -\nu_{LL} \varepsilon_{11}. \end{aligned} \tag{4.14}$$

If the plate considered is reinforced by fibres perpendicular to the x_3-axis, and Young's moduli of the reinforcement exceed by orders of magnitude the corresponding moduli of the matrix, the first rough estimate yields the results

$$\begin{aligned} E_L &\approx k E_Z, \\ \frac{1}{\mu_1} &\approx \frac{1-k}{\mu_M}, \end{aligned} \tag{4.15}$$

where k is the volumetric percentage of reinforcement, E_Z — Young's modulus of the reinforcement material, μ_M — shear modulus of the matrix material. If $\mu_M \ll E_Z$, we may write

$$D_1 \approx \frac{D_0}{\mu_1 h} \approx \frac{k(1-k) E_Z}{(1-\gamma_{LL}^2)\mu_M} h^2. \tag{4.16}$$

Assuming $k = 0.2$–0.3, $\nu_{LL} \approx 0.3$, $E_Z/\mu_M \approx 3 \cdot 10^2$, we obtain a reasonable estimate $D_1 \approx 50\ h^2$ (ratio $E_Z/\mu_M \approx 3 \cdot 10^2$ corresponds e.g. to carbon fibre reinforced epoxy resin [3]).

For a simply supported rectangular plate the following set of eigenfunctions of the Laplace operator may be used:

$$f_{(m,n)} = \sin\frac{m\pi x_1}{l_1}\sin\frac{n\pi x_2}{l_2},$$

where l_2, l_2 are dimensions of the plate. Then we obtain $\lambda_{(1,1)} = \pi\sqrt{1/l_1^2+1/l_2^2}$, and with $l_1 = l_2 = 10h$ (h denotes one half of the plate thickness) we finally obtain $\lambda_{(1,1)} = \pi\sqrt{2}/10h$, $\gamma_{(1,1)} = \pi$.

In case of circular symmetry (circular plate clamped at the circumference) $f_n = J_0(\lambda_n r/a)$ (cf. Sect. 5, Example III), J_0 denoting the Bessel function of order zero and λ_n — consecutive zeros of the Bessel function of order one $(J_1(\lambda_n) = 0)$, taking $a = 5h$ for the previously determined value of D_1 we obtain $\gamma_1 = 5.419$.

Assume now that in a certain case we obtain $\gamma_1 = 5$ and $D_1 = 50\,h^2$. On the basis of the previously discussed results, such values should be considered as extreme and correspond to thick, rigidly clamped plates. Let us estimate the order of magnitude of the ratio of mean normal stress at the surface to the applied load P. With reasonable accuracy this ratio may be replaced by the ratio of the first terms of the corresponding expansions

$$\frac{\sigma_{\alpha\alpha}}{2P} \approx \frac{D_1}{h^2}\frac{D_0-\mu_2}{2D_0}\frac{\operatorname{th}\gamma_n}{\gamma_n-\operatorname{th}\gamma_n} = \frac{D_1}{h^2}\frac{1+\nu_{LL}}{4}\frac{\operatorname{th}\gamma_n}{\gamma_n-\operatorname{th}\gamma_n} \approx 4.06. \tag{4.17}$$

This means that, in extreme cases, surface load does not exceed 25% of the mean stress at the outer surfaces $x_3 = \pm h$. Without committing any considerable errors in the evaluation of material strength, the stress field obtained may thus be modified by "implantation" of the stress field σ_{33} locally satisfying the third equilibrium equation and the surface conditions corresponding to the actual loading scheme of the plate. If, for instance, the plate was loaded at the surface $x_3 = -h$, the suitable assumption should be

$$\sigma_{33} = \frac{1}{2}\sum_{n=1}^{\infty} P_n\left(\frac{\frac{x_3}{h}\gamma_n\operatorname{ch}\gamma_n-\operatorname{sh}\left(\gamma\frac{x_3}{h}\right)}{\gamma_n\operatorname{ch}\gamma_n-\operatorname{sh}\gamma_n}-1\right)f_n. \tag{4.18}$$

The following equalities hold then true

$$\begin{aligned} \sigma_{33}|_{x_3=-h} &= P,\\ \sigma_{33}|_{x_3=h} &= 0,\\ \sigma_{\alpha 3,\alpha}+\sigma_{33,3} &= 0. \end{aligned} \tag{4.19}$$

To improve the accuracy, field $\sigma_{\alpha\beta}$ should be completed by the term $\sigma'_{\alpha\beta}$,

$$\sigma'_{\alpha\beta} = \beta/\delta\sigma_{33} = \sigma_{33}\nu_{TL}/(1-\nu_{LL}), \tag{4.20}$$

where ν_{TL} is the Poisson ratio governing the strain in the plane of isotropy under simple tension parallel to the distinguished direction. Obviously, such stress field usually does not satisfy the first and second equilibrium equations; however, the error committed is the smaller, the smaller are the derivatives of P with respect to variables x_α, and the smaller is the constant β as compared with δ.

Observe, moreover, that if the fictitious body force distribution simulating the load were of the form shown e.g. by Fig. 2, a risk could arise of generating fictitious normal stresses of the order of

$$(\beta/\delta)\int_0^{x_3} f_3(x_1, x_2, \xi)\,d\xi.$$

If, however, f_3 is expressed in terms of P_n (by inserting (4.5) into (4.12)), we obtain

$$f_3 = \frac{1}{2h}\sum_{n=1}^{\infty} \frac{\gamma_n\left[\operatorname{ch}\gamma_n - \operatorname{ch}\left(\gamma_n \frac{x_3}{n}\right)\right]}{\gamma_n \operatorname{ch}\gamma_n - \operatorname{sh}\gamma_n} P_n f_n. \tag{4.21}$$

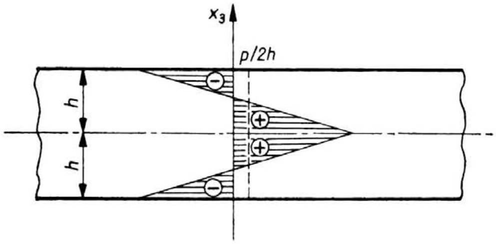

FIG. 2. "Pathological" loading of plate by fictitious body forces.

Coefficient of the term $P_n f_n$ at the middle surface of the plate lies within the interval (1, 1.5) for any value of γ_n, so that except for certain "pathological" cases (this author is unable to suggest any example), replacement of the actual load with fictitious body forces is equivalent to the transfer of the loads from the outer surface to the region of the middle surface of the plate.

5. Examples

I. Rectangular plate simply supported at the edges (Fig. 3)

For loads given in the form

$$P = P_{(m,n)} \sin\frac{m\pi x_1}{l_1} \sin\frac{n\pi x_2}{l_2} \tag{5.1}$$

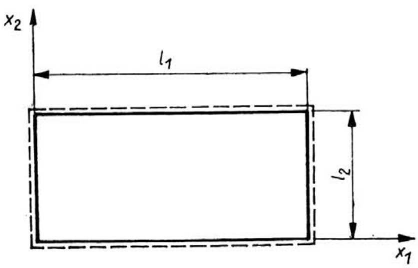

FIG. 3. Simply supported rectangular plate.

and with $F_0 = 0$, $G = 0$, we obtain

$$\sigma_{\alpha\beta} = h\frac{P_{(m,n)}D_1}{2D_0}\frac{\operatorname{sh}\left(\gamma_{(m,n)}\frac{x_3}{h}\right)}{\gamma_{(m,n)}\operatorname{ch}\gamma_{(m,n)}-\operatorname{sh}\gamma_{(m,n)}} \times\left[\left(\alpha-\frac{\beta^2}{\delta}\right)\sin\frac{m\pi x_1}{l_1}\sin\frac{n\pi x_2}{l_2}\delta_{\alpha\beta}-2\mu_2\frac{P_1}{\gamma^2_{(m,n)}}\left(\sin\frac{m\pi x_1}{2}\sin\frac{n\pi x_2}{l_?}\right)_{,\alpha\beta}\right], \tag{5.2}$$

$$w=\frac{P_{(m,n)}D_1^2}{2D_0}\frac{\sin\frac{m\pi x_1}{l_1}\sin\frac{n\pi x_2}{l_2}}{\gamma_{(m,n)}(\gamma_{(m,n)}-\operatorname{th}\gamma_{(m,n)})} \tag{5.3}$$

where

$$\gamma_{(m,n)} = \pi\sqrt{D_1\left(\frac{m^2}{l_1^2}+\frac{n^2}{l_2^2}\right)}.$$

Here w denotes the plate deflection at $x_3 = \pm h$; according to the simplified theory this is the deflection of the middle surface. In the present model this is not true, the differences being, however, of secondary importance.

Assuming similar load within the framework of the simplified plate theory (SPT), the following results are obtained:

$$\sigma_{\alpha\beta(\mathrm{SPT})} = \frac{3}{2}\frac{D_1}{D_0}\frac{x_3P_{(m,n)}}{\gamma^2_{(m,n)}} \times\left[\left(\alpha-\frac{\beta^2}{\delta}\right)\sin\frac{m\pi x_1}{l_1}\sin\frac{n\pi x_2}{l_2}\delta_{\alpha\beta}-2\mu_2\frac{D_1}{\gamma^2_{(m,n)}}\left(\sin\frac{m\pi x_1}{l_1}\sin\frac{n\pi x_2}{l_2}\right)_{,\alpha\beta}\right], \tag{5.4}$$

$$w_{(\mathrm{SPT})} = \frac{3}{2}\frac{D_1^2P_{(m,n)}}{D_0\gamma^4_{(m,n)}}\sin\frac{m\pi x_1}{l_1}\sin\frac{n\pi x_2}{l_2}. \tag{5.5}$$

Introduce the notations

$$C_{(m,n)} = \frac{w_{(\mathrm{SPT})}}{w} = \frac{3(\gamma_{(m,n)}-\operatorname{th}\gamma_{(m,n)})}{\gamma^3_{(m,n)}}, \tag{5.6}$$

$$d_{(m,n)} \equiv \frac{\sigma_{\alpha\beta(\mathrm{SPT})}}{\sigma_{\alpha\beta}} = \frac{3\gamma_{(m,n)}\left(\frac{x_3}{h}\right)}{\operatorname{sh}\left(\gamma_{(m,n)}\frac{x_3}{h}\right)}\frac{\gamma_{(m,n)}\operatorname{ch}\gamma_{(m,n)}-\operatorname{sh}\gamma_{(m,n)}}{\gamma^3_{(m,n)}}. \tag{5.7}$$

It is easily seen that for $\gamma_{(m,n)} \to 0$, i.e. for the plate of infinitesimal thickness, both magnitudes tend to unity. By disregarding the summation signs in Eqs. (4.7), (4.8), (4.10) it is found that the assumed form (5.1) of the eigenfunction yields the following identities:

$$\begin{aligned} u_3|_{\partial\Omega} &= 0,\\ \sigma_{\alpha\beta}n_\alpha n_\beta|_{\partial\Omega} &= 0,\\ M_{\alpha\beta}n_\alpha n_\beta|_{\partial\Omega} &= 0. \end{aligned} \tag{5.8}$$

Here $\mathbf{n}$ is the unit vector normal to $\partial\Omega$. It means that the boundary conditions of free support of the plate edges (rectangle $l_1 \times l_2$) are satisfied.

We are not going to discuss here the problem of non-vanishing twisting moments since it has been extensively discussed within the framework of the classical plate theory, and the present model does not contribute anything new to the problem. Obviously, result (5.8) remains valid if the load is represented by a finite sum of expressions of the type of (5.1), or even by an infinite series, provided the corresponding series for u_3 and $\sigma_{\alpha\beta}$ are uniformly convergent in the neighbourhood of the boundary of the region, thus securing the continuity of the corresponding functions at points approaching the boundary (for points lying at the boundary they are identically zero since all terms of the expansion vanish).

The case of a constant load may be represented in the from

$$P(x_1, x_2) = \frac{16P}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{(2m-1)\pi x_1}{l_1} \sin \frac{(2n-1)\pi x_2}{l_2}}{(2m-1)(2n-1)} \tag{5.9}$$

(cf. [8], p. 52, Eq. 1.442,1); it means that

$$P_{(m,n)} = \frac{16P}{\pi^2} \frac{1}{(2m-1)(2n-1)} \tag{5.10}$$

and

$$\gamma_{(m,n)} = \pi \sqrt{D_1 \left(\frac{(2m-1)^2}{l_1^2} + \frac{(2n-1)^2}{l_2^2} \right)}. \tag{5.11}$$

Finally,

$$\sigma_{\alpha\beta} = \frac{8hD_1 P}{\pi^2 D_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\operatorname{sh}\left(\gamma_{(m,n)} \frac{x_3}{h}\right)}{(2m-1)(2n-1)(\gamma_{(m,n)} \operatorname{ch} \gamma_{(m,n)} - \operatorname{sh} \gamma_{(m,n)})} \times \left[\left(\alpha - \frac{\beta^2}{\delta}\right) \sin \frac{(2m-1)\pi x_1}{l_1} \sin \frac{(2n-1)\pi x_2}{l_2} \delta_{\alpha\beta} - 2\mu_2 \frac{D_1}{\gamma_{(m,n)}^2} \left(\sin \frac{(2m-1)\pi x_1}{l_1} \sin \frac{(2n-1)\pi x_2}{l_2} \right)_{,\alpha\beta} \right], \tag{5.12}$$

$$w = \frac{8D_1^2 P}{\pi^2 D_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{(2m-1)\pi x_1}{l_1} \sin \frac{(2n-1)\pi x_2}{l_2}}{(2m-1)(2n-1)\gamma_{(m,n)}(\gamma_{(m,n)} - \operatorname{th} \gamma_{(m,n)})}. \tag{5.13}$$

Series (5.12), (5.13) and those corresponding to u_α and $M_{\alpha\beta}$ (not given here) are uniformly convergent in the neighbourhood of the boundary since the series of expansion coefficients are (at fixed values of m or n) absolutely convergent.

Distribution of stresses across the thickness of a square plate at point $x_1 = x_2 = l/2$ is shown in Fig. 4 a for the case of $D_1 = 50\ h^2$, $l_1 = l_2 = 10h$. For comparison, the distributions corresponding to the first harmonic term ($m = n = 1$) is also given, together with the values corresponding to both cases and resulting from the simplified theory. In Fig. 4 b is shown the variation of horizontal displacements across the thickness. The simplified theory is seen to yield the stresses less by about 30% from those following from the model

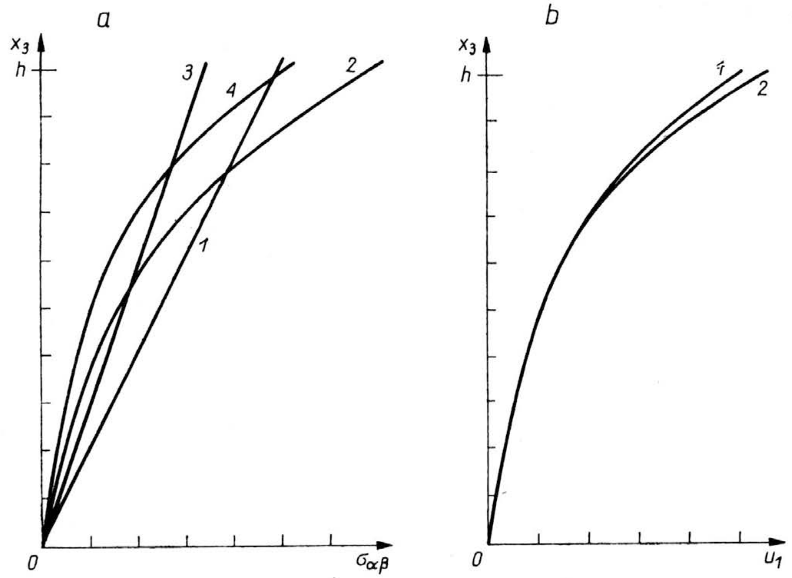

FIG. 4. a. Normal stress distribution across plate thickness at the center of a simply supported square plate (upper part; lower part antisymmetrical); conventional units. 1 — constant load, simplified model, 2 — constant load, proposed model, 3 — first harmonic term (load $P\sin(\pi x_1/l)\sin(\pi x_2/l)$) simplified model, 4 — first harmonic term — proposed model.
b. Horizontal displacements u for $x_1 = l/4$, $x_2 = l/4$ Conventional units (lower part — antisymmetric): 1 — constant load, 2 — first harmonic term.

proposed here, the latter being supposed to be more accurate. The difference in displacements is substantial: for uniformly distributed load the displacements are 4.417 times greater than those following from the simplified, classical theory, and in the case of loading by the first harmonic term this ratio is even greater, 4.818. These numbers should not be considered as surprising if we realize the fact that the simplified theory does not account for the effect of shearing stresses on the plate displacement; this effect should be of primary importance in the considered case of small value of μ_1 and under the assumed loading and support conditions.

II. Cylindrical bending of a plate strip (infinite cantilever plate)

Consider the plane problem shown in Fig. 5. In absence of the normal loads relation (4.5) is useless. However, the expression for the moment M_{11} distrubution may readily be written in the form

(5.14) $$M_{11} = S(x_l - l).$$

Assume $G = 0$, function F will be sought in the form

(5.15) $$F = \sum_{n=0}^{\infty} F_n \cos\frac{(2n+1)\pi x_1}{2l} + C,$$

whence

$$\gamma_n = \pi\sqrt{D_1}\,\frac{2n+1}{2l}.$$

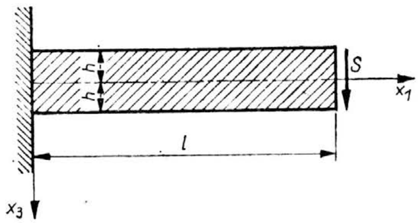

FIG. 5. Plate strip (infinite cantilever plate)

Displacements are written as follows:

$$u_1 = \frac{h}{\sqrt{D_1}} \sum_{n=0}^{\infty} F_n \operatorname{sh}\left(\gamma_n \frac{x_3}{h}\right) \sin\frac{(2n+1)\pi x_1}{2l}, \tag{5.16}$$

$$u_3 = \frac{1}{D_1} \sum_{n=0}^{\infty} F_n \left[\frac{D_0}{\mu_1 h} \operatorname{ch}\gamma_n - \frac{h^2\beta}{\delta} \operatorname{ch}\left(\gamma_n \frac{x_3}{h}\right)\right] \cos\frac{(2n+1)\pi x_1}{2l} + C, \tag{5.17}$$

so that, in view of the fact that $u_2 = 0$, we obtain

$$\sigma_{11} = \frac{D_0}{h^2 D_1} \sum_{n=0}^{\infty} F_n \gamma_n \operatorname{sh}\left(\gamma_n \frac{x_3}{h}\right) \cos\frac{(2n+1)\pi x_1}{2l}. \tag{5.18}$$

Integration yields the result

$$M_{11} = \frac{2D_0}{D_1} \sum_{n=0}^{\infty} F_n \left(\operatorname{ch}\gamma_n - \frac{\operatorname{sh}\gamma_n}{\gamma_n}\right) \cos\frac{(2n+1)\pi x_1}{2l}. \tag{5.19}$$

Using the formula (1.444,6), p. 53 in [8](1)

$$\sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2} - |x|\right) \quad \text{for} \quad -\pi \leqslant x < \pi. \tag{5.20}$$

formula (5.14) is rewritten in the form

$$M_{11} = -\frac{8lS}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos\frac{(2k+1)\pi x_1}{2l}}{(2k+1)^2}. \tag{5.21}$$

Comparison of the relations (5.19) and (5.21) leads to the formula

$$F_n = -\frac{SD_1^2}{D_0 l} \frac{1}{\gamma_n(\gamma_n \operatorname{ch}\gamma_n - \operatorname{sh}\gamma_n)}. \tag{5.22}$$

(1) This formula given in [8] is erroneous, summation should start from $k = 0$ and not $k = 1$, cf. the formula (0.234,2), p. 21 in that book, for $x = 0$ both the formulae should be identical.

On substituting this results into the Eqs. (5.16), (5.19) it is readily found that the corresponding series converge and satisfy the following boundary conditions:

(5.23)
$$M_{11}|_{x_1=l} = 0,$$
$$u_\alpha|_{x_1=0} = 0.$$

The problem of series (5.18) determining the stress σ_{11} is slightly more complicated. Beyond any doubt, it diverges at $x_1 = 0$ thus reflecting, as it could be expected, the singular character of stress at the clamped corner of the plate. Let us replace a non-trivial discussion of the behaviour of that series in vicinity of the point $x_1 = 1$ with argumentation based on "experimental mathematics" and refer to Fig. 6, where the numerically determined maximum stress value is shown for $D_1 = 50\ h^2$, $l = 10\ h$. The curve was plotted by summing up 100 teims of the series (5.18) at 100 points. Thus the following boundary condition is satisfied

(5.24)
$$\sigma_{11}|_{x_1=l} = 0,$$
$$M_{11}|_{x_1=l} = 0.$$

Let us finally consider the condition for vertical displacements u_3 at the cross-section $x_1 = 0$. This condition cannot be satisfied for the entire cross-section for any value of C, let us demand it to be satisfied for $w \equiv u_3|_{x_3=\pm h}$.

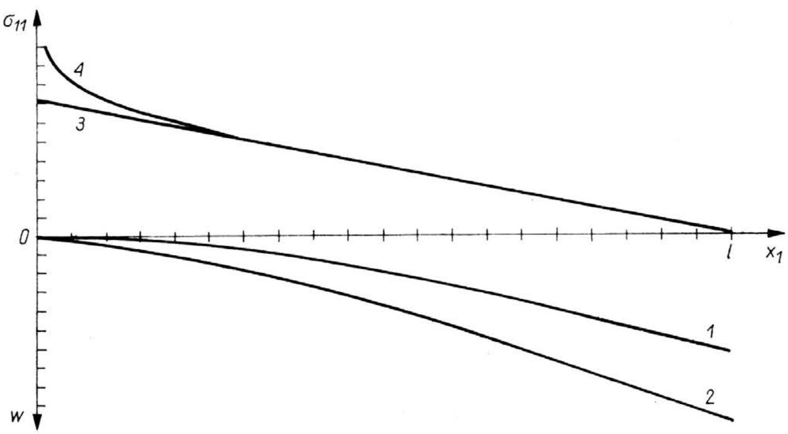

FIG. 6. Deflection and stresses in a plate strip. 1 — deflection — simplified model, 2 — deflection — proposed model, 3 — normal stress at the surface — simplified model, 4 — normal stress at the surface — proposed model.

Suitable choice of C leads to

(5.25)
$$w = \frac{SD_1^2}{D_0 l} \sum_{n=0}^{\infty} \frac{1-\cos\dfrac{(2n+1)\pi x_1}{2l}}{\gamma_n(\gamma_n - \operatorname{th}\gamma_n)}.$$

Diagram of the function w is shown in Fig. 6 together with the graphs illustrating the behaviour of the deflection and stress $\sigma_{11\,\max}$ as determined by the simplified theory. By replacing the condition $u_3 = 0$ for $x_1 = 0$ with the condition $w = 0$ for $x_1 = 0$, the original problem is replaced with the right-hand side of the problem shown in Fig. 8. It should be noted that the stress variation across the thickness at $x_1 = 0.1$ shown in Fig. 7 does not substantially differ from that derived from the simplyfied theory. The corresponding diagram for $x = 0.5\, l$ is not given since both curves for σ_{11} and σ_{11SPT} are practically identical. Slightly larger differences appear as far as displacements are concerned but, in spite of similar geometric dimensions of the present problem, they are much smaller than the differences established in the previous example. It may easily be observed that if

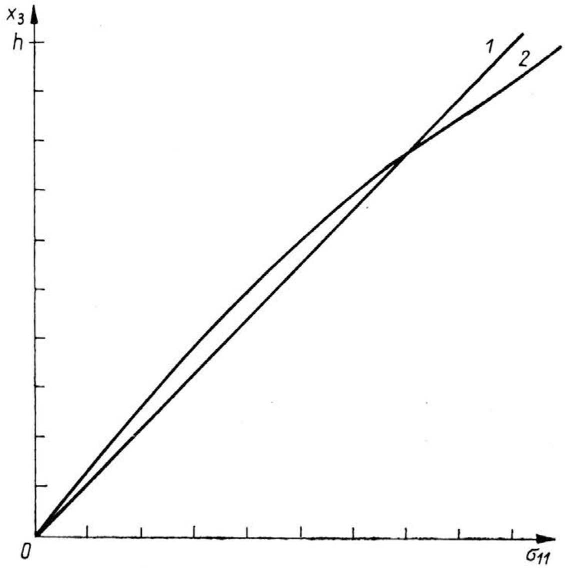

FIG. 7. Plate strip. Stress distribution within the thickness for $x_1 = 0.1\, l$. Upper part of the plate; lower part antisymmetrical. Conventional units. 1 — simplified model, 2 — proposed model.

the cantilever plate considered here was loaded at its end by a moment instead of the vertical force, the simplified solution would exactly satisfy the boundary conditions of the model and both solutions would coincide. This observation as well as the formerly analyzed example lead to the conclusion that the differences between the "accurate" solutions derived here and the simplified solutions increase with increasing rigidity of the edge supported since, at the same geometric dimensions and identical values of D_1, γ_n increases with increasing λ_n. This conclusion will be illustrated in the following section by an example of a plate with clamped edges.

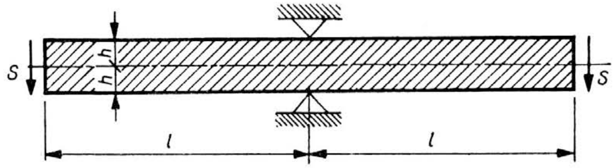

FIG. 8. Accurate loading scheme corresponding to the solution.

III. Circular plate clamped at the edge

From the example given in the preceding section it is seen that, in the case of a clamped edge, the principal problem consists in satisfying the boundary condition $u_\alpha = 0$. In case of circular symmetry the problem is reduced to

$$u_r|_{r=a} = 0. \tag{5.26}$$

The second condition, similarly to the previous example, may be taken in the weaker form

$$u_3\Big|_{\substack{r=a\\ x_3=\pm h}} \equiv w|_{r=a} = 0. \tag{5.27}$$

For a circular plate of radius a condition (5.26) is satisfied automatically provided the radial displacements are expressible by a uniformly convergent Fourier–Bessel series

$$u_r = \sum_{k=1}^{\infty} u_{r(k)} J_1\left(\lambda_k \frac{r}{a}\right), \tag{5.28}$$

with J_1 denoting the Bessel function of the first kind and first order, and λ_k — consecutive roots of equation $J_1(\lambda_k) = 0$. Assume now that the axi-symmetric load of the circular plate may be represented in the form

$$P = \sum_{k=1}^{\infty} P_k J_0\left(\lambda_k \frac{r}{a}\right), \tag{5.29}$$

i.e. in the form of the Dini series (with constant $H = 0$). Using Eq. (4.6) and selecting properly the function F_0 = const we obtain

$$w = \frac{D_1^2}{2D_0} \sum_{k=1}^{\infty} \frac{P_k}{\gamma_k(\gamma_k - \operatorname{th}\gamma_k)} \left(J_0\left(\lambda_k \frac{r}{a}\right) - J_0(\lambda_k)\right), \tag{5.30}$$

with $\gamma_k = \sqrt{D_1}\,\lambda_k/a$, satisfying the condition (5.27). Condition (5.26) is fulfilled identically provided the corresponding series is uniformly convergent. Expression for stresses has the form

$$\sigma_{\alpha\beta} = \frac{hD_1}{2D_0} \sum_{k=1}^{\infty} \frac{P_k \operatorname{sh}\left(\gamma_k \frac{x_3}{h}\right)}{\gamma_k \operatorname{ch}\gamma_k - \operatorname{sh}\gamma_k} \left\{\left[\left(\alpha - \frac{\beta^2}{\delta}\right)\delta_{\alpha\beta} + 2\mu_2 \frac{x_\alpha x_p}{r^2}\right] J_0\left(\lambda_k \frac{r}{a}\right) + 2\mu_2 \frac{\sqrt{D_1}}{\gamma_k} \frac{1}{r} J_1\left(\lambda_k \frac{r}{a}\right)\left(\delta_{\alpha\beta} - \frac{x_\alpha x_\beta}{r^2}\right)\right\}. \tag{5.31}$$

For the Dini series we have at our disposal effective formulae for evaluating the expansion coefficients [9]; in our case

$$P_k = \frac{2}{J_0^2(\lambda_k)} \int_0^r \frac{r}{a^2} P\left(\frac{r}{a}\right) J_0\left(\lambda_k \frac{r}{a}\right) dr. \tag{5.32}$$

Confining our considerations to the first term of (5.29), i.e. assuming P to have the form $P\,J_0(\lambda_1 r/a)$, we obtain the following values of w and of the stress $\sigma_{rr\,\max}$ at the center of the plate ($r = 0$):

$$w = \frac{D^2}{2D_0}\frac{P}{\gamma_1(\gamma_1 - \operatorname{th}\gamma_1)}(1 - J_0(\lambda)),$$

(5.33)

$$\sigma_{rr(\max)} = \frac{PD_1}{2h^2}\frac{\operatorname{th}\gamma_1}{\gamma_1 - \operatorname{th}\gamma_1},$$

where $\lambda_1 = 3.8317060$, $J_0(\lambda_1) = -0.40275$, and γ_1 is found from the relation following the formula (5.30).

Values of c_n and d_n are expressed by the same formulae as in the first example, thus

$$c_1 \equiv \frac{w_{(\mathrm{SPT})}}{w} = \frac{3(\gamma_1 - \operatorname{th}\gamma_1)}{\gamma_1^3},$$

(5.34)

$$d_1 \equiv \frac{\sigma_{rr(\max,\,\mathrm{SPT})}}{\sigma_{rr(\max)}} = \frac{3(\gamma_1 \operatorname{th}\gamma_1 - 1)}{\gamma_1^2}.$$

Assuming, as before, $D_1 = 50\ h^2$, $2a = 10\ h$, we obtain $\gamma_1 = 5.4188503$ and, finally, $c_1 = 0.083320$, $d_1 = 0.451235$. The difference in deflections is striking, and in stresses — at least considerable. This is not due to any particular form of loading or to the circular symmetry of the problem, but mainly follows from clamping of the edges. It is easily verified that for a square plate with the same characteristics as in the first example, but with rigidly clamped edges, the first eigenfunction should be assumed in the form

$$f_{(11)} = \cos\frac{2\pi x_1}{l}\cos\frac{2\pi x_2}{l}$$

(for a plate occupying the region $-l/2 \leqslant x_1 \leqslant l/2$: $-l/2 \leqslant x_2 \leqslant l/2$; $l = 10\ h$), and the corresponding values would be $\gamma_{(1,1)} = 2\pi$, $c_{(1,1)} = 0.0639$, $d_{(1,1)} = 0.410$.

Observe that the solution procedure proposed cannot be applied if the load $P = \text{const}$. Then Eq. (5.32) yields the result $P_m = 0$ for $\lambda_m \neq 0$, while for $\lambda_m = 0$ formulae (5.30) and (5.31) for w and $\sigma_{\alpha\beta}$ become invalid. Let us approach this case in a different manner and start from expanding F into the Dini series

(5.35) $$F(r) = \sum_{m=1}^{\infty} F_m J_0\left(\lambda_m \frac{r}{a}\right).$$

Assuming $F_0 = \text{const}$, $G = 0$ and calculating S_r from Eq. (4.11) we obtain

(5.36) $$S_r = -\frac{2D_0}{D_1^{3/2}}\sum_{n=1}^{\infty} F_n(\gamma_n \operatorname{ch}\gamma_n - \operatorname{sh}\gamma_n)J_1\left(\lambda\frac{r}{a}\right).$$

On the other hand, simple equilibrium considerations yield, with $P = \text{const}$ for any $r \leqslant a$, the relation

(5.37) $$\pi r^2 P + S_r 2\pi r = 0$$

that is

(5.38) $$S_r = -\frac{rP}{2} = -\frac{Pa}{2}\left(\frac{r}{a}\right).$$

Expansion of the function $f(x) = x$ into the Fourier–Bessel series is known (cf. [9], p. 581), thus we obtain

$$S_r = aP \sum_{n=1}^{\infty} \frac{J_1\left(\lambda_n \frac{r}{a}\right)}{\lambda_n J_0(\lambda_n)}. \tag{5.39}$$

Comparison of (5.36) and (5.39) yields

$$F_n = -\frac{D_1^2 P}{2D_0} \frac{1}{\gamma_n J_0(\lambda_n)(\gamma_n \operatorname{ch}\gamma_n - \operatorname{sh}\gamma_n)}. \tag{5.40}$$

For large λ_n the value of $J_0(\lambda_n)$ is of the order of $\lambda_n^{-1/2}$ (cf. [9]), hence the series

$$u_\alpha = -\frac{hD_2^{3/2}P}{2D_0} \sum_{n=1}^{\infty} \frac{\operatorname{sh}\left(\gamma_n \frac{x_2}{h}\right) J_1\left(\lambda_n \frac{r}{a}\right) \frac{x_\alpha}{r}}{\gamma_n J_0(\lambda_n)(\gamma_n \operatorname{ch}\gamma_n - \operatorname{sh}\gamma_n)} \tag{5.41}$$

may be found by means of a comparison test with an absolutely convergent series to be uniformly convergent in the neighbourhood of $r = a$, i.e. condition (5.26) is satisfied. By suitable selection of constant C condition (5.27) may also be satisfied, thus the following deflection of the plate is found:

$$w = -\frac{D_1^2 P}{2D_0} \sum_{n=1}^{\infty} \frac{1}{\gamma_n(\gamma_n - \operatorname{th}\gamma_n)} \left(\frac{J_0\left(\lambda_n \frac{r}{a}\right)}{J_0(\lambda_n)} - 1\right), \tag{5.42}$$

while the stresses at the surface are given by the formula

$$\sigma_{rr}\big|_{x_3=h} = -\frac{D_1^2 P}{2D_0} \sum_{n=1}^{\infty} \frac{\operatorname{th}\gamma_n}{\gamma_n - \operatorname{th}\gamma_n} \frac{1}{J_0(\lambda_n)} \times \left[\frac{D_0}{h^2 D_1} J_0\left(\lambda_n \frac{r}{a}\right) - \frac{2\mu_2}{\sqrt{D_1}\left(\frac{r}{a}\right)\gamma_n} \frac{h}{a} J_1\left(\lambda_n \frac{r}{a}\right)\right]. \tag{5.43}$$

It is easily found that this series diverges (as $1/\lambda_n \approx 1/(n+1/4)\pi$) at $r = a$ so that, in compliance with our intuition, singularities in the rigidly clamped corners of the cros-section appear; the singularity at $r = 0$ is apparent. Analogous results of the simplified theory have the form

$$w_{\mathrm{SPT}} = \frac{3Pa^4}{128D_0}\left(1 - \frac{r^2}{a^2}\right) \tag{5.44}$$

and

$$\sigma_{rr\,\mathrm{SPT}}\big|_{x_3=h} = x_3 \frac{3Pa^2}{16D_0}\left[\frac{D_0}{h^3}\left(1 - \frac{2r^2}{a^2}\right) - \mu_2\left(1 - \frac{r^2}{a^2}\right)\right]. \tag{5.45}$$

In Figs. 9 a and 10 are shown the graphs of $\sigma_{rr}|_{x_3=h}$ and w as functions of r following from the application of our model and from the simplified, classical theory, with the same parameters as before, $D_1 = 50\,h^2$, $a = 5\,h$, and the value of $h^3\,\mu_2/D = (1-\nu_{LL})/2$ equal to 0.35.

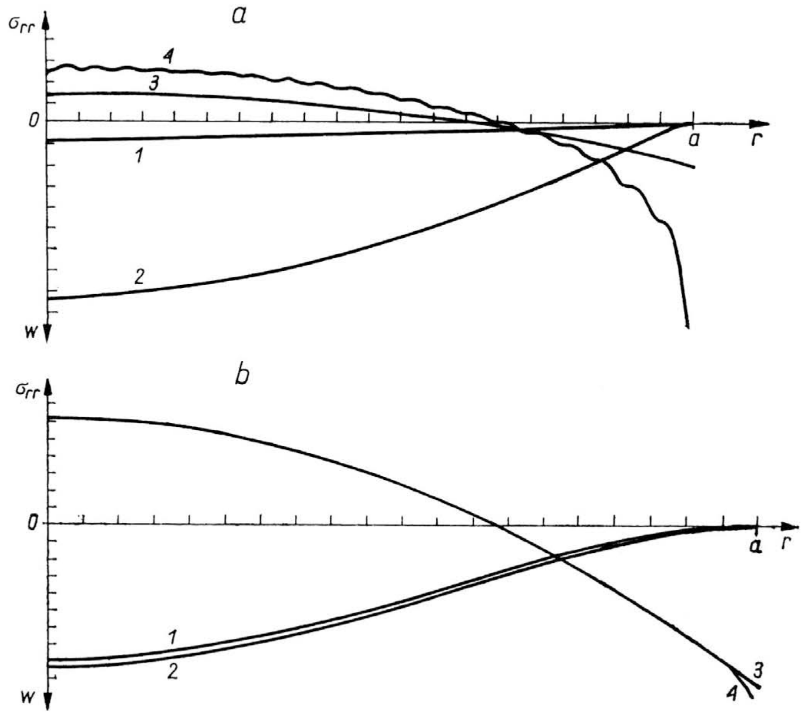

FIG. 9. Deflection and stresses in a circular clamped plate, constant load. 1 — deflection — simplified theory, 2 — deflection — proposed model, 3 — σ_{rr}-stresses at the surface — simplified model, 4 — σ_{rr}-stresses at the surface — proposed model. Units arbitrary chosen.
(a) $D_1 = 50\ h^2$, $h^3\mu_2/D_0 = 0.35$, $a = 5\ h$; (b) $D_1 = 3\ h^2$, $h^3\ \mu_2/D_0 = 0.35$, $a = 20\ h$.

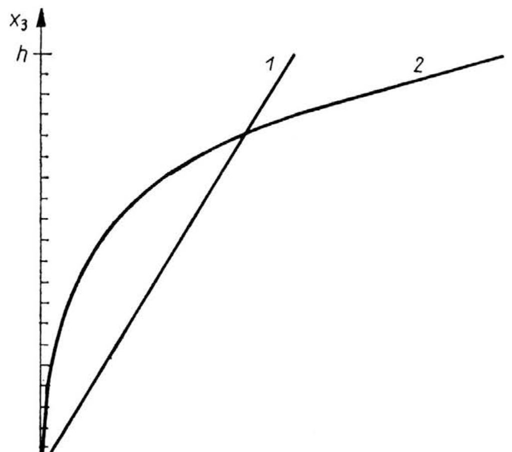

FIG. 10. σ_{rr}-stress distribution across thickness of a clamped circular plate under constant load, $r = 0.1$ a; conventional units (corresponding to case (a), Fig. 9). 1 — simplified model, 2 — proposed model.

Periodicity of the stress diagram is of a numerical nature; the series converges so slowly that 40 terms were not sufficient; general character of the curve has been established but determination of the accurate form of the curve was too time-consuming to be continued.

6. Discussion

Let us consider the limits of applicability of the simplified theory. Consider the results derived in the preceding section as accurate and treat them as reference data for estimating the errors of the simplified theory. Determine the values of c_n and d_n for the least favourable conditions of clamped edge of the plate. Results corresponding to the first harmonic term (first eigenfunction) may be considered to be representative for practically encountered load distributions. Assume then $\lambda_1 = 2\sqrt{2}\pi/l$ (this was the value for a clamped square plate, analogous value for a circular plate would be $2\cdot 3.8317/d$, d — plate diameter).

Retaining in expressions (5.6) and (5.7) for c_1 and d_1 only the first terms of expansions of hyperbolic functions into power series, the following estimates of the results of the simplified, classical theory are obtained:

$$\frac{2h}{l} \leqslant 0.5\sqrt{\frac{\delta\mu_1 e_w}{(\alpha+2\mu_2)\delta-\beta(\beta+u_1)}}, \tag{6.1}$$

$$\frac{2h}{l} \leqslant 1.23\sqrt{\frac{\delta\mu_1 e_\sigma}{(\alpha+2\mu_2)\delta-\beta(\beta+\mu)}}. \tag{6.2}$$

Here e_w and e_σ denote the maximal permissible relative errors of the simplified theory,

$$e_w = \max\left|\frac{w-w_{\mathrm{SPT}}}{w}\right|, \tag{6.3}$$

$$e_\sigma = \max\left|\frac{\sigma-\sigma_{\mathrm{SPT}}}{\sigma}\right|. \tag{6.4}$$

In particular, for an isotropic material for which $D_1 = (2-\nu)h^2/(1-\nu)$ and $\nu = 0.3$, the permissible errors of 10% in w and σ implies, according to relations (6.1) and (6.2), the conditions

$$\begin{aligned} 2h/l &\leqslant 1/10 \quad \text{for} \quad e_w \leqslant 0.1,\\ 2h/l &\leqslant 1/4 \quad \text{for} \quad e_\sigma \leqslant 0.1. \end{aligned} \tag{6.5}$$

The graphs of σ_{rr} (for $x_3 = h$) and deflection w of a thin, almost isotropic, circular clamped plate under constant load are shown in Fig. 9b. The calculated stresses are seen to coincide with the results of the simplified theory (except for the neighbourhood of the singularity), deviations of the displacements from the more accurate values comply fairly well with the estimates (6.1) and (6.5). In cases of less constrained systems (free or simply supported boundaries) the more "liberal" conditions (6.4) are obtained. To conclude, let us mention that the results of the simplified model concerning the deflections may be improved by means of the Reissner model (in case of cylindrical bending — by means of the Timoshenko beam model with plate bending rigidity replacing the usual beam rigidity); however, the errors resulting from the assumptions concerning linear displacement and stress variations leading to false stress estimation in the subsurface regions cannot be eliminated this way.

In this paper it has been demonstrated that the model considered enables us to detect singularities what is not feasible within the framework of any plate theory based on finite polynomial expansions; such result is also inaccessible by means of the boundary layer method as proposed by PIPKIN and EVERSTINE (cf. [6], p. 24].

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