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Existence in nonlocal elasticity

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An existence theorem on the displacement boundary value problems of homogeneous, isotropic, linear, nonlocal elasticity is given. After introducing the notation and some preliminaries, the displacement boundary value problems of homogeneous, isotropic, linear, nonlocal elasticity are defined. A Hilbert space in which the solutions of the considered boundary value problems are searched is defined and an inequality which plays an important role in the existence theory of elasticity is proven. Finally, it is shown that the bilinear form which appears in the weak formulation satisfies the requirements of the fundamental existence lemma (Lax-Milgram's Theorem).

Podano twierdzenie o istnieniu rozwiązania dla przemieszczeniowego zagadnienia brzegowego w liniowej lecz nielokalnej teorii sprężystości ciała jednorodnego i izotropowego. Zdefiniowano przestrzeń Hilberta, w której poszukuje się rozwiązania rozważanego zagadnienia brzegowego oraz wyprowadzono nierówność, która odgrywa istotną rolę w twierdzeniach o istnieniu rozwiązań teorii sprężystości. Wykazano na koniec, że forma bilinowa pojawiająca się w słabych sformułowaniach tej teorii spełnia wymagania podstawowego twierdzenia Laxa-Milgrama.

Приведена теорема о существовании решения для краевой задачи в перемещениях в линейной, но нелокальной теории упругости однородного и изотропного тела. Определено гильбертовое пространство, в котором ищется решения рассматриваемой краевой задачи, а также выведено наревенство, которое играет существенную роль в теоремах о существовании решений теории упругости. Наконец, показано, что билинейная форма, появляющаяся в слабых формулировках этой теории, удовлетворяет требованиям основной теоремы Лакса-Милгрема.

1. Introduction

One of the main streams of the advancement of science is to enhance the extent of the fundamental hypotheses of a theory when the theory proves to be insufficient in explaining the problems in its field. Although these enhancements bring a lot of complications, the efforts go on for the sake of explaining more phenomena. Similarly as in other branches of science, in continuum mechanics there is much research carried out in this direction. Here the nonlocal theory of continuum mechanics is perhaps the most recent study.

The nonlocal theory of continuum mechanics takes into account the nonlocal effects as its name suggests and differs from the local one in fundamental hypotheses. As is well known, in the local theory the fundamental conservation laws are assumed to be valid in any portion cut from the body. In the nonlocal theory, the assumption of the validity of conservation laws in any portion of the body has been abandoned and the conservation laws are assumed to be valid only over the whole body. With the assistance of the Clausius-Duhem inequality in global form and of some additional hypotheses, constitutive equations are obtained in such forms that the value of the dependent constitutive variables at a point are described by the value of the independent variables at all points in the neighborhood of the considered point. For a comprehensive account on nonlocal continuum mechanics, ERINGEN [1] and EDELEN [2] can be consulted.

The constitutive equations of various media obtained in the nonlocal approach can be found in the literature [3–11]. A number of problems solved in the frame of nonlocal continuum mechanics indicate the power of the theory. As is well known, the classical field theories fail to explain the phenomena at the atomic scale. For example, the propagation of waves with a very short wavelength is not dispersive in the classical theory. When the same problems are solved in the frame of nonlocal elasticity, a dispersion relation (a nonlinear relation between the wavelength and the phase velocity) is obtained, [12–16]. Another example is the crack problem. It is a well-known fact that there are some absurd stress singularities in the solutions of crack problems in the classical theory of elasticity [17, 18]. When these problems are handled in the frame of nonlocal elasticity, it has been shown that these singularities do not appear [19, 21]. The singularities in total strain energy and in stress appearing in the classical solutions of the dislocation problems [22] disappear in the solutions realized in nonlocal elasticity [23–25].

In this study, an existence theorem for the displacement boundary value problems of homogeneous, isotropic, linear, nonlocal elasticity will be given. A study of the conditions under which the solutions of a mathematical problem exist and are unique is of more than academical interest. This kind of studies describes the limitations of the mathematical models, the stability of the solutions, the consistence of the data, etc. Moreover, it is very important to know the space in which a problem has solutions, especially if one employs approximate methods, such as finite elements, finite differences, etc.

The existence and uniqueness of the boundary or initial value problems given by differential equations are studied extensively [26–29]. The boundary value problems of classical elasticity which are given by an elliptic partial differential equation are also studied from the existence and uniqueness points of view [30, 31]. The uniqueness theorems of the boundary value problem of linear elasticity can be found in [32]. In a previous work [37] we have given a uniqueness theorem for the initial-boundary value problems of nonlocal elastodynamics. In another work [38] a uniqueness theorem for the initial boundary value problem of nonlocal visco-elasticity can be found.

2. Notation and some preliminaries

Throughout this paper Cartesian coordinates are used and conventional indicial notation is employed. An open, bounded, simply-connected region of three-dimensional Euclidean space occupied by the body will be denoted by B. For the boundary surface of the closure of B ($\bar{B}$) we use the symbol S and it is assumed that this surface is regular enough to permit to employ the Green–Gauss identity

$$\int_B \operatorname{div} \mathbf{V}\, dv = \int_S \mathbf{V} \cdot \mathbf{n}\, da, \tag{2.1}$$

where $\mathbf{V}$, $\mathbf{n}$, dv, da stand for a tensor-valued function, the unit outward normal vector of S, differential volume and area elements, respectively. Unless otherwise stated the subscripts have the range of the integers 1, 2, 3 and the repeated indices imply summation over the range. Indices following a comma indicate partial differentiation with respect to a space variable and for the position vector the notation $\mathbf{x}$ is used. u_i, ε_{ij}, t_{ij} stand for the components of the inifinitesimal displacement vector, the strain tensor and the stress tensor, respectively. f_i denotes the body forces and λ, μ are the elastic constants.

The set of functions which are continuous with their first derivatives, have a compact support in B and vanish near the boundary of $\bar{B}$ is denoted by $\mathring{C}_1(B)$. The notation $\mathring{C}_\infty(B)$ is used for the set of functions which are infinitely differentiable, have a compact support in B and vanish near the boundary of $\bar{B}$. The space $\mathring{H}_1(B)$ is defined as the closure of the set $\mathring{C}_1(B)$ with respect to the norm

$$(2.2) \qquad \|u\| = \sqrt{(u, u)}, \qquad u \in \mathring{C}_1(B),$$

where (.,.) is the inner product defined by

$$(2.3) \qquad (u, v) = \int_B u_{,i}(\mathbf{x}) v_{,i}(\mathbf{x})\, dv, \qquad u, v \in \mathring{C}_1(B).$$

Although the norm (2.2) gives the impression of a semi-norm, it can be easily shown that if

$$(2.4) \qquad \|u\| = 0,$$

then

$$(2.5) \qquad u = 0$$

almost everywhere in B. The notation $L_2(B)$ is used for the space which contains all functions square-integrable in B.

Now we wish to introduce the notion of weak solution of a boundary value problem in a restricted form appropriate to our purposes. Let us consider the equation

$$(2.6) \qquad Lu = f \quad \text{in } B$$

with the homogeneous boundary condition

$$(2.7) \qquad u = 0 \quad \text{on } S,$$

where L is a linear operator which transforms the elements of a Hilbert space H into $L_2(B)$. Let L^* be the adjoint of L and let $\phi \in \mathring{C}_\infty(B)$. A function $u \in H$ satisfying

$$(2.8) \qquad \int_B uL^*\phi\, dv = \int_B f\phi\, dv$$

for every $\phi \in \mathring{C}_\infty(B)$ is called a weak solution of the boundary value problem defined by Eqs. (2.6) and (2.7). For some purposes it is convenient to take $\phi \in H$ instead of $\phi \in \mathring{C}_\infty(B)$. For a precise definition and detailed information about weak solutions, the work [31] can be consulted.

As is well-known from the theory of integral equations, if a function u is given by the equation

$$(2.9) \qquad \int_B \alpha(\mathbf{x}, \mathbf{z}) u(\mathbf{z})\, dv(\mathbf{z}) = f(\mathbf{x}),$$

then, what we need to solve is a Fredholm integral equation of first kind. If the kernel $\alpha(\mathbf{x}, \mathbf{z})$ is subject to the condition below

$$(2.10) \qquad \alpha(\mathbf{x}, \mathbf{z}) = \alpha(\mathbf{z}, \mathbf{x}),$$

it is called a symmetric kernel. The following properties of symmetric kernels are employed in this study:

Mercer's Theorem: Let $\alpha(\mathbf{x}, \mathbf{z})$ be symmetric, continuous and such that the integral

$$\int_B |\alpha(\mathbf{x}, \mathbf{z})|^2 dv(\mathbf{x}) \leqslant D < \infty \tag{2.11$_1$}$$

is bounded in the set B. Moreover, if we have

$$\int_B \int_B \alpha(\mathbf{x}, \mathbf{z}) u(\mathbf{x}) u(\mathbf{z}) dv(\mathbf{x}) dv(\mathbf{z}) > 0 \tag{2.11$_2$}$$

for every $u \in L_2(B)$ not identically zero in B, then the kernel can he expressed as an infinite sum

$$\alpha(\mathbf{x}, \mathbf{z}) = \sum_{n=0}^{\infty} \frac{\phi_n(\mathbf{x})\phi_n(\mathbf{z})}{\alpha_n} \tag{2.12}$$

which is absolutely and uniformly convergent with respect to the pairs of variables $\mathbf{x}$ and $\mathbf{z}$.

For the proof of this theorem POGORZELSKI [33] can be consulted. It can also be shown that the set $\{\phi_n(x)\}_0^\infty$ appearing in Eq. (2.12) forms a complete, orthonormal base in $L_2(B)$ and α_n are positive numbers having no finite accumulation point.

As is well known, any $h \in L_2(B)$ can be expressed as an infinite sum

$$h(\mathbf{x}) = \sum_{n=0}^{\infty} h_n \phi_n(\mathbf{x}) \tag{2.13}$$

almost everywhere in B, where the set $\{\phi_n(\mathbf{x})\}_0^\infty$ is a complete, orthonormal base in $L_2(B)$ and

$$h_n = \int_B h(\mathbf{x})\phi_n(\mathbf{x}) dv. \tag{2.14}$$

The following theorem of functional analysis is very useful for our purposes.

The Lax–Milgram Theorem: Let H be a Hilbert space with the inner product (u, v). Further, let $B(u, v)$ be a bilinear form (that is, a form linear in both u and v) defined for $u \in H$, $v \in H$, and such that there exist constants $K > 0$, $\alpha > 0$, independent of u and v, so that for every $u \in H$, $v \in H$ we have

$$|B(u, v)| \leqslant K||u||\ ||v||, \tag{2.15}$$

$$B(u, u) \geqslant \alpha||u||^2. \tag{2.16}$$

Then every linear functional F bounded on H can be expressed in the form

$$F(u) = B(u, z), \quad u \in H, \tag{2.17}$$

where z is an element of the space H uniquely determined by the functional F. At the same time the inequality

$$||z|| \leqslant \frac{||F||}{\alpha} \tag{2.18}$$

holds, where $||F||$ is the norm of the functional F.

The proof of this theorem can be found in [30] or [31].

3. Displacement boundary value problems in nonlocal elasticity

In this section we wish to introduce the displacement boundary value problems of homogeneous, isotropic, linear, nonlocal elasticity and their weak solutions. The basic equations of linear, nonlocal elasticity are given by ERINGEN [12, 23, 24]. These equations consist of the displacement-strain relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}), \tag{3.1}$$

the stress-strain relation

$$t_{ij}(\mathbf{x}) = \int_B \alpha(|\mathbf{x}-\mathbf{x}'|)\{\lambda\varepsilon_{kk}(\mathbf{x}')\delta_{ij}+2\mu\varepsilon_{ij}(\mathbf{x}')\}dv' \tag{3.2}$$

the equilibrium equations

$$t_{ij,j}+f_i = 0, \quad t_{ij} = t_{ji}. \tag{3.3}$$

We assume that the interaction kernel $\alpha(|\mathbf{x}-\mathbf{x}'|)$ appearing in Eq. (3.2) satisfies the conditions of Mercer's theorem.

A displacement boundary value problem of homogeneous, isotropic, linear, nonlocal elasticity is to find a triplet $\{u_i, \varepsilon_{ij}, t_{ij}\}$ satisfying the equations (3.1)–(3.3) and the boundary condition

$$u_i(\mathbf{x}) = \hat{U}_i(\mathbf{x}), \quad \mathbf{x}\in S \tag{3.4}$$

for given f_i in B and $\hat{U}_i$ on S.

The weak solution of this problem is a vector-valued function $u_i(\mathbf{x})$ which satisfies the global energy balance equation

$$\int_B t_{ij}(\mathbf{x})e_{ij}(\mathbf{x})dv = \int_S t_{ij}(\mathbf{x})v_i(\mathbf{x})da_j+\int_B f_i(\mathbf{x})v_i(\mathbf{x})dv \tag{3.5}$$

for every v_i chosen from a Hilbert space which will be defined in the following section. If v_i s chosen as

$$v_i(\mathbf{x}) = 0, \quad \mathbf{x}\in S, \tag{3.6}$$

and if we employ the stress-strain relation (3.2), then, from Eq. (3.5)

$$\int_B\int_B \alpha(|\mathbf{x}-\mathbf{x}'|)\{\lambda\varepsilon_{kk}(\mathbf{x}')e_{mm}(x)+2\mu\varepsilon_{ij}(\mathbf{x}')e_{ij}(\mathbf{x})\}dv(\mathbf{x}')dv(\mathbf{x}) = \int_B f_i(\mathbf{x})v_i(\mathbf{x})dv \tag{3.7}$$

is obtained where

$$2e_{ij}(\mathbf{x}) = v_{i,j}(\mathbf{x})+v_{j,i}(\mathbf{x}), \quad \mathbf{x}\in B. \tag{3.8}$$

Equation (3.7) is fundamental in this study.

4. A function space

In this section we wish to introduce a Hilbert space of the weak solutions of the displacement boundary value problem of homogeneous, isotropic, linear, nonlocal elasticity. For $u, v\in \mathring{C}_1(B)$ we define an inner product as follows:

$$(u, v)_{\mathring{C}_1} = \int_B\int_B \alpha(\mathbf{x},\mathbf{z})u_{,i}(\mathbf{x})v_{,i}(\mathbf{z})dv(\mathbf{x})dv(\mathbf{z}). \tag{4.1}$$

We assume that the function $\alpha(\mathbf{x}, \mathbf{z})$ satisfies the conditions of Mercer's theorem. It can be easily shown that this inner product satisfies the following properties:

i) $$(u, v)_{\mathring{C}_1} = (v, u)_{\mathring{C}_1}.$$

ii) For any two real numbers a_1, a_2

$$(u, a_1 v_1 + a_2 v_2)_{\mathring{C}_1} = a_1 (u, v_1)_{\mathring{C}_1} + a_2 (u, v_2)_{\mathring{C}_1}.$$

iii) $(u, u)_{\mathring{C}_1} \geqslant 0$, $(u, u)_{\mathring{C}_1} = 0$, $u_{,i} = 0$ almost everywhere in B.

On the other hand, it can be verified that if

$u_{,i} = 0$ almost everywhere in B

then

$u = 0$ almost everywhere in B.

As is usual we define the norm of a $u \in \mathring{C}_1(B)$

$$\|u\|_{\mathring{C}_1} = \sqrt{(u, u)} \tag{4.2}$$

and the metric for every pair of $u, v \in \mathring{C}_1(B)$

$$\varrho(u, v) = \|u - v\|_{\mathring{C}_1}. \tag{4.3}$$

The closure of the set $\mathring{C}_1(B)$ with respect to the norm (4.2) will be denoted by $\mathring{V}_{C_1}(B)$. Since the accumulation points of all fundamental sequences with respect to the metric (4.3) are in $\mathring{V}_{C_1}(B)$, this space is complete. On the other hand, the inner product (4.1), the norm (4.2) and the metric (4.3) defined for the elements of $\mathring{C}_1(B)$ can be formally extended to the elements of the space $\mathring{V}_{C_1}(B)$. Therefore the space $\mathring{V}_{C_1}$ can be considered as a Hilbert space or Banach space.

The space obtained by the Cartesian product of three $\mathring{V}_{C_1}(B)$ will be denoted by $\mathring{\mathbf{V}}_{C_1}(B)$:

$$\mathring{\mathbf{V}}_{C_1}(B) = \mathring{V}_{C_1}(B) x \mathring{V}_{C_1}(B) x \mathring{V}_{C_1}(B). \tag{4.4}$$

5. Inequality of Korn's type

In this section we wish to prove a theorem analogous to Korn's inequality which plays an important role in the existence studies of the boundary value problems of classical elasticity [34, 35, 36]. This inequality can be generalized for nonlocal elasticity as follows:

THEOREM. *Let the kernel* $\alpha(\mathbf{x}, \mathbf{z})$ *appearing in the definition of the norm* (4.2) *have the property*

$$\frac{\partial \alpha(\mathbf{x}, \mathbf{z})}{\partial x_i} = -\frac{\partial \alpha(\mathbf{x}, \mathbf{z})}{\partial z_i}, \quad i = 1, 2, 3. \tag{5.1}$$

Then, for every $\mathbf{u} \in \mathring{\mathbf{V}}_{C_1}$, *the inequality*

$$\int_B \int_B \alpha(\mathbf{x}, \mathbf{z}) \{ \varepsilon_{ij}(\mathbf{x}) \varepsilon_{ij}(\mathbf{z}) \, dv(\mathbf{x}) \, dv(\mathbf{z}) \geqq \frac{1}{2} \|\mathbf{u}\|^2_{\mathring{V}_{C_1}} \tag{5.2}$$

holds, where

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad w_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}). \tag{5.3}$$

Proof. The following equalities are straightforward:

$$
\begin{aligned}
&\int_B \int_B \alpha(\mathbf{x}, \mathbf{z})\, \varepsilon_{ij}(\mathbf{x})\, \varepsilon_{ij}(\mathbf{z})\, dv(\mathbf{x})\, dv(\mathbf{z}) \\
&\qquad = \frac{1}{2} \int_B \int_B \alpha(\mathbf{x}, \mathbf{z}) \{u_{i,j}(\mathbf{x}) u_{i,j}(\mathbf{z}) + u_{i,j}(\mathbf{x}) u_{j,i}(\mathbf{z})\} dv(\mathbf{x})\, dv(\mathbf{z}),
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
&\int_B \int_B \alpha(\mathbf{x}, \mathbf{z})\, w_{ij}(\mathbf{x})\, w_{ij}(\mathbf{z})\, dv(\mathbf{x})\, dv(\mathbf{z}) \\
&\qquad = \frac{1}{2} \int_B \int_B \alpha(\mathbf{x}, \mathbf{z}) \{u_{i,j}(\mathbf{x}) u_{i,j}(\mathbf{z}) - u_{i,j}(\mathbf{x}) u_{j,i}(\mathbf{z})\} dv(\mathbf{x})\, dv(\mathbf{z}).
\end{aligned}
$$

By subtracting these equalities side by side, we obtain

$$
\begin{aligned}
&\int_B \int_B \alpha(\mathbf{x}, \mathbf{z}) \{\varepsilon_{ij}(\mathbf{x})\, \varepsilon_{ij}(\mathbf{z}) - w_{ij}(\mathbf{x})\, w_{ij}(\mathbf{z})\} dv(\mathbf{x})\, dv(\mathbf{z}) \\
&\qquad = \int_B \int_B \alpha(\mathbf{x}, \mathbf{z}) u_{i,j}(\mathbf{x}) u_{j,i}(\mathbf{z})\, dv(\mathbf{x})\, dv(\mathbf{z}).
\end{aligned}
\tag{5.5}
$$

Now we wish to show that the right hand side of Eq. (5.5) is positive. To this end, we consider the following identity:

$$
\begin{aligned}
u_{i,j}(\mathbf{x}) u_{j,i}(\mathbf{z}) &= \frac{\partial}{\partial z_i} \{u_{i,j}(\mathbf{x}) u_j(\mathbf{z}) - u_{k,k}(\mathbf{x}) u_i(\mathbf{z})\} \\
&\quad + u_{k,k}(\mathbf{x}) u_{l,l}(\mathbf{z}) + \frac{\partial}{\partial x_i} \{u_{i,j}(\mathbf{x}) u_j(\mathbf{z}) - u_{k,k}(\mathbf{x}) u_i(\mathbf{z})\}.
\end{aligned}
\tag{5.6}
$$

For a $u_i \in \mathring{C}_2(B)$ we have

$$
\begin{aligned}
I_1 &= \int_B \int_B \alpha(\mathbf{x}, \mathbf{z}) \frac{\partial}{\partial z_i} \{u_{i,j}(\mathbf{x}) u_j(\mathbf{z}) - u_{k,k}(\mathbf{x}) u_i(\mathbf{z})\} dv(\mathbf{x})\, dv(\mathbf{z}) \\
&= - \int_B \int_B \frac{\partial \alpha(\mathbf{x}, \mathbf{z})}{\partial z_i} \{u_{i,j}(\mathbf{x}) u_j(\mathbf{z}) - u_{k,k}(\mathbf{x}) u_i(\mathbf{z})\} dv(\mathbf{x})\, dv(\mathbf{z})
\end{aligned}
\tag{5.7}
$$

and

$$
\begin{aligned}
I_2 &= \int_B \int_B \alpha(\mathbf{x}, \mathbf{z}) \frac{\partial}{\partial x_i} \{u_{i,j}(\mathbf{x}) u_j(\mathbf{z}) - u_{k,k}(\mathbf{x}) u_i(\mathbf{z})\} dv(\mathbf{x})\, dv(\mathbf{z}) \\
&= - \int_B \int_B \frac{\partial \alpha(\mathbf{x}, \mathbf{z})}{\partial x_i} \{u_{i,j}(\mathbf{x}) u_j(\mathbf{z}) - u_{k,k}(\mathbf{x}) u_i(\mathbf{z})\} dv(\mathbf{x})\, dv(\mathbf{z}).
\end{aligned}
\tag{5.8}
$$

Under the assumption (5.1) it is clear that

$$
I_1 = -I_2.
$$

Consequently, from Eq. (5.6) we obtain

$$
\int_B \int_B \alpha(\mathbf{x}, \mathbf{z}) u_{i,j}(\mathbf{x}) u_{j,i}(\mathbf{z})\, dv(\mathbf{x})\, dv(\mathbf{z}) = \int_B \int_B \alpha(\mathbf{x}, \mathbf{z}) u_{k,k}(\mathbf{x}) u_{l,l}(\mathbf{z})\, dv(\mathbf{x})\, dv(\mathbf{z}).
\tag{5.9}
$$

On the other hand, since

$$\varepsilon_{ij}(\mathbf{x})\,\varepsilon_{ij}(\mathbf{z})-w_{ij}(\mathbf{x})\,w_{ij}(\mathbf{z}) = u_{i,j}(\mathbf{x})u_{j,i}(\mathbf{z}),$$

we arrive at

$$(5.10)\qquad \int\limits_B\int\limits_B \alpha(\mathbf{x},\mathbf{z})\{\varepsilon_{ij}(\mathbf{x})\,\varepsilon_{ij}(\mathbf{z})-w_{ij}(\mathbf{x})\,w_{ij}(\mathbf{z})\}\,dv(\mathbf{x})\,dv(\mathbf{z}) = \int\limits_B\int\limits_B \alpha(\mathbf{x},\mathbf{z})\,u_{k,k}(\mathbf{x})\,u_{l,l}(\mathbf{z})\,dv(\mathbf{x})\,dv(\mathbf{z}) \geqslant 0.$$

Similarly, since

$$\varepsilon_{ij}(\mathbf{x})\,\varepsilon_{ij}(\mathbf{z})+w_{ij}(\mathbf{x})\,w_{ij}(\mathbf{z}) = u_{i,j}(\mathbf{x})u_{i,j}(\mathbf{z}),$$

we have

$$(5.11)\qquad \int\limits_B\int\limits_B \alpha(\mathbf{x},\mathbf{z})\{\varepsilon_{ij}(\mathbf{x})\,\varepsilon_{ij}(\mathbf{z})+w_{ij}(\mathbf{x})\,w_{ij}(\mathbf{z})\}\,dv(\mathbf{x})\,dv(\mathbf{z}) = \int\limits_B\int\limits_B \alpha(\mathbf{x},\mathbf{z})\,u_{i,j}(\mathbf{x})\,u_{i,j}(\mathbf{z})\,dv(\mathbf{x})\,dv(\mathbf{z}) = ||\mathbf{u}||^2_{\mathring{V}_{C_1}}.$$

By summing the relations (5.10) and (5.11) side by side, we arrive at

$$(5.12)\qquad 2\int\limits_B\int\limits_B \alpha(\mathbf{x},\mathbf{z})\,\varepsilon_{ij}(\mathbf{x})\,\varepsilon_{ij}(\mathbf{z})\,dv(\mathbf{x})\,dv(\mathbf{z}) \geqslant ||\mathbf{u}||^2_{\mathring{V}_{C_1}}.$$

6. Existence of the solutions

For existence of the solutions of the displacement boundary value problems of homogeneous, isotropic, linear nonlocal elasticity, it is sufficient to show that the bilinear form

$$(6.1)\qquad \begin{gathered} B(\mathbf{u},\mathbf{v}) = \int\limits_B\int\limits_B \alpha(|\mathbf{x},\mathbf{x}'|)\{\lambda\varepsilon_{kk}(\mathbf{x}')e_{ll}(\mathbf{x})+2\mu\varepsilon_{ij}(\mathbf{x}')e_{ij}(\mathbf{x})\}\,dv(\mathbf{x}')\,dv(\mathbf{x}), \\ \mathbf{u},\mathbf{v}\in\mathring{\mathbf{V}}_{C_1}(B) \end{gathered}$$

which appears in the weak formulation of these problems satisfies the conditions of the Lax–Milgram Theorem given by the inequalities (2.11) and (2.16). To this end we wish to rearrange Eq. (6.1) employing Mercer's Theorem. According to the Mercer's Theorem, the kernel $\alpha(|\mathbf{x}-\mathbf{x}'|)$ can be expressed as follows:

$$(6.2)\qquad \alpha(|\mathbf{x}-\mathbf{x}'|) = \sum_{n=0}^{\infty}\frac{\phi_n(\mathbf{x})\phi_n(\mathbf{x}')}{\alpha_n}.$$

Let us define

$$(6.3)\qquad \eta^n_{ij} = \int\limits_B \varepsilon_{ij}(\mathbf{x})\phi_n(\mathbf{x})\,dv,\qquad \nu^n_{ij} = \int\limits_B e_{ij}(\mathbf{x})\phi_n(\mathbf{x})\,dv.$$

If we introduce Eq. (6.2) into Eq. (6.1) and consider the relations (6.3), we obtain

$$(6.4)\qquad B(\mathbf{u},\mathbf{v}) = \sum_{n=0}^{\infty}\alpha_n^{-1}\{\lambda\eta^n_{kk}\nu^n_{ll}+2\mu\eta^n_{ij}\nu^n_{ij}\}.$$

With the following definitions

$$(6.5)\quad \mathbf{M} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix}, \quad \boldsymbol{\gamma}_n = \begin{bmatrix} \eta^n_{11} \\ \eta^n_{22} \\ \eta^n_{33} \\ \sqrt{2}\eta^n_{12} \\ \sqrt{2}\eta^n_{13} \\ \sqrt{2}\eta^n_{23} \end{bmatrix}, \quad \boldsymbol{\beta}_n = \begin{bmatrix} \nu^n_{11} \\ \nu^n_{22} \\ \nu^n_{33} \\ \sqrt{2}\nu^n_{12} \\ \sqrt{2}\nu^n_{13} \\ \sqrt{2}\nu^n_{23} \end{bmatrix},$$

we obtain

$$(6.6)\qquad B(\mathbf{u}, \mathbf{v}) = \sum_{n=0}^{\infty} \alpha_n^{-1} \boldsymbol{\beta}_n^T \mathbf{M} \boldsymbol{\gamma}_n .$$

It can be easily shown that the eigenvalues of the matrix $\mathbf{M}$ are

$$(6.7)\qquad 3\lambda+2\mu,\ 2\mu .$$

Let $\varkappa_1$ denote the absolutely largest eigenvalue of the matrix $\mathbf{M}$, then

$$(6.8)\qquad |\boldsymbol{\beta}_n \mathbf{M} \boldsymbol{\gamma}_n| \leqslant \varkappa_1 |\boldsymbol{\beta}_n^T \cdot \boldsymbol{\gamma}_n| \leqslant \varkappa_1 |\boldsymbol{\beta}_n|\, |\boldsymbol{\gamma}_n|$$

can be written. Introducing this inequality into Eq. (6.6) and employing the Schwarz inequality we obtain

$$(6.9)\qquad |B(\mathbf{u}, \mathbf{v})| \leqslant \varkappa_1 \sum_{n=0}^{\infty} \alpha_n^{-1} |\boldsymbol{\beta}_n|\, |\boldsymbol{\gamma}_n| \leqslant \varkappa_1 \left\{ \sum_{n=0}^{\infty} \frac{\beta_{ni}\beta_{ni}}{\alpha_n} \right\}^{1/2} \left\{ \sum_{n=0}^{\infty} \frac{\gamma_{ni}\gamma_{ni}}{\alpha_n} \right\}^{1/2} .$$

It can be easily verified that this last inequality can be written equivalently as follows:

$$(6.10)\qquad |B(\mathbf{u}, \mathbf{v})| \leqslant \varkappa_1 \|\mathbf{u}\|_{\mathring{V}_{C_1}} \|\mathbf{v}\|_{\mathring{V}_{C_1}}$$

which means that the condition (2.15) of the Lax–Milgram theorem is satisfied for

$$(6.11)\qquad K = \varkappa_1 = \sup\{|3\lambda+2\mu|, |2\mu|\}.$$

To show that the condition (2.16) is satisfied, we consider that the expression

$$(6.12)\qquad B(\mathbf{u}, \mathbf{u}) = \int_B \int_B \alpha(|\mathbf{x}-\mathbf{x}'|)\{\lambda\varepsilon_{kk}(\mathbf{x}')\,\varepsilon_{ll}(\mathbf{x}) + 2\mu\varepsilon_{ij}(\mathbf{x}')\,\varepsilon_{ij}(\mathbf{x})\}\,dv(\mathbf{x}')\,dv(\mathbf{x})$$

can be written equivalently as

$$(6.13)\qquad B(\mathbf{u}, \mathbf{u}) = \sum_{n=0}^{\infty} \alpha_n^{-1} \{\boldsymbol{\gamma}_n^T \mathbf{M} \boldsymbol{\gamma}_n\}.$$

Since $\alpha_n > 0$ for all n, both eigenvalues of $\mathbf{M}$ must be positive, i.e.,

$$(6.14)\qquad 3\lambda+2\mu > 0, \qquad \mu > 0$$

to ensure that

$$(6.15)\qquad B(\mathbf{u}, \mathbf{u}) > 0$$

for a $\mathbf{u} \in \mathring{\mathbf{V}}_{C_1}$ not identically zero in B. Let $\varkappa_2$ denote the smallest eigenvalue of the matrix $\mathbf{M}$. The following inequality is obvious:

$$(6.16)\qquad \boldsymbol{\gamma}_n^T \mathbf{M} \boldsymbol{\gamma}_n \geqslant \varkappa_2 |\boldsymbol{\gamma}_n|^2 = \varkappa_2 \eta^n_{ij} \eta^n_{ij} .$$

By introducing Eq. (6.16) into Eq. (6.13),

$$(6.17)\qquad B(\mathbf{u},\mathbf{u}) \geqslant \varkappa_2 \sum_{n=0}^{\infty} \alpha_n^{-1} \eta_{ij}^n \eta_{ij}^n = \varkappa_2 \int_B \int_B \alpha(|\mathbf{x}-\mathbf{x}'|)\, \varepsilon_{ij}(\mathbf{x}')\, \varepsilon_{ij}(\mathbf{x})\, dv(\mathbf{x}')\, dv(\mathbf{x})$$

is obtained. On the other hand, by considering the generalized Korn inequality (5.12), we arrive at

$$(6.18)\qquad B(\mathbf{u},\mathbf{u}) \geqslant \frac{\varkappa_2}{2} \int_B \int_B \alpha(|\mathbf{x}-\mathbf{x}'|)\, u_{i,j}(\mathbf{x}')\, u_{ij}(\mathbf{x}')\, dv(\mathbf{x}) = \frac{\varkappa_2}{2} \|\mathbf{u}\|^2_{\mathring{V}C_1}$$

which indicates that the condition (2.16) of the Lax–Milgram theorem is satisfied for

$$(6.19)\qquad \alpha = \frac{\varkappa_2}{2}, \qquad \varkappa_2 = \inf\{3\lambda + 2\mu, 2\mu\}.$$

Thus we have proved:

Boundary value problems of nonlocal elasticity with homogeneous boundary conditions posses a unique solution if the interaction kernel is a positive definite kernel and the elastic constants obey the inequalities (6.14).

7. Conclusions and remarks

In this study we have shown that the displacement boundary value problems of homogeneous, isotropic, linear, nonlocal elasticity possess a unique weak solution under some conditions. For this purpose, a new function space is defined and in this space an important inequality, which is a generalization of the inequality known as Korn's Inequality in classical elasticity, is proven.

The function space denoted by $\mathring{\mathbf{V}}_{C_1}$ is a corner-stone of this study. The analysis of the properties of this space is left to a further paper. Accordingly, we also have not mentioned about the dual space of $\mathring{\mathbf{V}}_{C_1}(B)$. However, it is quite clear that some elements of this space exist in the distibution sense, what means that the solution of some displacement boundary value problems cannot be represented by means of continuous functions. If the continuum hypothesis is considered, this seems to be a lack of nonlocal theory. But, according to the opinion of the author, this situation is an advantage of the nonlocal theory which enables us to deal with problems without absurd singularities in stress for which the continuity fails (e.g., dislocation and crack problems). On the other hand, as is clear from Eq. (3.7), if $\mathring{\mathbf{V}}_{C_1}(B)$ is restricted to $\mathring{H}_1(B)$, then the dual space of $\mathring{\mathbf{V}}_{C_1}(B)$ becomes $L_2(B)$.

Finally we wish to point out that the uniqueness and existence conditions of the field equations of nonlocal elasticity are the same. As is well known, the same situation arises in elliptic partial differential equations.

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References

1. A. C. ERINGEN, *Continuum physics*, vol. IV. Part III. [Ed. A. C. ERINGEN], Academic Press Inc., London 1976.
2. D. G. B. EDELEN, *Continuum physics*, vol. IV, Part II, [Ed. A. C. ERINGEN], Academic Press, London 1976.
3. A. C. ERINGEN, D. G. B. EDELEN, *On nonlocal elasticity*, Int. J. Engn. Sci. **10**, 233, 1972.
4. A. C. ERINGEN, *On nonlocal fluid mechanics*, Int. J. Engn. Sci., **10**, 561, 1972.
5. A. Ü. ERDEM, M. DIKMEN, *Constitutive equations of nonlocal micromorphic elastic dielectrics. II*, METU J. Pure and Appl. Sci., **11**, 313, 1978.
6. H. DEMIRAY, *On nonlocal theory of quasi-static elastic dielectrics*, Int. J. Engn. Sci., **10**, 258, 1972.
7. A. C. ERINGEN, *Memory dependent nonlocal elastic solids*, Int. J. Engn. Sci., **15**, 579, 1977.
8. A. C. ERINGEN, *On nonlocal thermoelasticity*. Int. J. Engn. Sci., **12**, 1063, 1974.
9. F. A. BALTA, E. S. SUHUBI, *Theory of nonlocal generalized thermoelasticity*, Int. J. Engn. Sci., **15**, 579, 1977.
10. A. C. ERINGEN, *Nonlocal polar elastic continua*, Int. J. Engn. Sci., **10**, 1, 1972.
11. A. C. ERINGEN, *On nonlocal plasticity*, Int. J. Engn. Sci., **19**, 1461, 1981.
12. A. C. ERINGEN, *Linear theory of nonlocal elasticity and dispersion of plane waves*, Int. J. Engn. Sci., **10**, 425, 1972.
13. A. C. ERINGEN, *Linear theory of nonlocal microelasticity and dispersion of plane waves*, Lett. Appl. Engn. Sci., **1**, 129, 1973.
14. J. L. NOWINSKI, *On nonlocal theory of longitudinal waves in an elastic circular bar*, Acta Mechanica, **52**, 189, 1984.
15. J. L. NOWINSKI, *On the nonlocal theory of wave propagation in elastic plates*, ASME J.Applied Mech,. **51**, 606, 1984.
16. J. L. NOWINSKI, *On the nonlocal aspects of the propagation of Love waves*, Int. Journ. Engn. Sci., **22**, 383, 1984.
17. G. C. SIH, *Mechanics of fracture*, vol. 1., Nordhoff Int. Pub., Netherland 1973.
18. I. N. SNEDDON, M. LOWENGRUB, *Crack problems in the classical theory of elasticity*, John Wiley and Sons Inc., New York 1969.
19. A. C. ERINGEN, C. G. SPEZIALE, B. S. KIM, *Crack tip problems in nonlocal elasticity*, J. Mech. Phys. Solids, **25**, 339, 1977.
20. C. N. ARI, *High field gradient phenomena in nonlocal media*, Dissertation, Princeton University, New Jersey 1981.
21. A. C. ERINGEN, *Line crack subject to shear*, Int. J. Fracture, **14**, 367, 1978.
22. R. W. LARDNER, *Mathematical theory of dislocations and fracture*, University of Toronto Press, Toronto 1974.
23. A. C. ERINGEN, *Edge dislocation in nonlocal elasticity*, Technical Report, nr 42, Princeton University, New Jersey 1976.
24. A. C. ERINGEN, *Screw dislocation in nonlocal elasticity*, Technical Report, nr 41, Princeton Universit, New Jersey 1976.
25. A. C. ERINGEN, F. BALTA, *Screw dislocation in nonlocal hexagonal crystals*, Crystal Lattice Defects, 183, 1978.
26. C. MIRANDA, *Partial differential equations*, Springer Verlag, Berlin 1970.
27. D. GILBARG, N. S. TRUDINGER, *Elliptic partial differential equations of second order*, Springer-Verlagy, Berlin 1977.
28. L. HÖRMANDER, *Linear partial differential operators*, Springer-Verlag, Berlin 1969.
29. D. D. KELLOGG, *Foundations of potential theory*, Springer-Verlag, Berlin 1967.
30. G. FICHERA, *Existence theorems in elasticity*, Encyclopedia of Physics, vol. VI a/2, Springer-Verlag, Berlin 1972.
31. K. REKTORYS, *Variational methods in mathematics, science and engineering*, D. Reidel Publishing Comp., Dordrecht 1975.
32. R. J. KNOPS, L. E. PAYNE, *Uniqueness theorems in linear elasticity*, Springer-Verlag, Berlin 1971.

33. W. POGORZELSKI, *Integral equations and their applications*, vol. 1, Pergamon Press, Oxford-London 1966.
34. M. E. GURTIN, *Linear theory of elasticity*, Encyclopedia of Physics, VI a/2 [ed. C. TRUESDELL], Springer-Verlag, Berlin 1972.
35. I. HLAVÀCEK, J. NECAS, *On inequalities of Korn's type*, Arch. Rat. Mech. Anal., **36**, 305, 1970.
36. L. E. PAYNE, H. W. WEINBERGER, *On Korn's inequality*, Arch. Rational. Mech. Anal., **8**, 89, 1961.
37. S. B. ALTAN, *Uniqueness in the linear theory of nonlocal elasticity*, Bull. Tech. Univ. Istanbul. **37**, 375, 1984.
38. S. B. ALTAN, *A uniqueness theorem in the linear theory of nonlocal visco-elasticity*, Bull. Tech. Univ. Istanbul, **38**, 233, 1985.

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