Arch. Mech., 41, 1, pp. 3-23, Warszawa 1989

Creep rupture of metals under multi-axial state of stress (*)

A. LITEWKA (POZNAŃ)

THE AIM of the paper is to derive the constitutive equations of creep damage for solids subjected to a multi-axial state of stress. The set of equations proposed consists of the damage evolution equation, the yield criterion for the damaging solid and the equation specifying the additional strain due to damage growth. It was assumed that creep rupture occurs at critical values of the damage tensor components Ω_i lower than 1. The validity of the approach proposed was confirmed experimentally. To this end, the creep rupture test results obtained by various researchers for several metals were employed.

Celem pracy jest wyprowadzenie równań opisujących zniszczenie materiału uszkodzonego w warunkach pełzania w złożonym stanie naprężenia. W skład zaproponowanego zestawu równań wchodzi równanie ewolucji uszkodzenia, warunek plastyczności dla materiału uszkadzającego się oraz równanie określające dodatkowe odkształcenia spowodowane wzrostem uszkodzenia. Przyjęto, że zniszczenie przy pełzaniu następuje przy krytycznej kombinacji współrzędnych tensora uszkodzenia Ω_i, mniejszych niż 1. Poprawność zaproponowanego podejścia do problemu sprawdzono doświadczalnie. W tym celu wykorzystano wyniki badań zniszczenia przy pełzaniu uzyskane dla kilku metali przez różnych badaczy.

Целью работы является вывод уравнений описывающих разрушение поврежденного материала в условиях ползучести в сложном напряженном состоянии. В состав предложенной системы уравнений входят уравнение эволюции повреждения, условие пластичности для повреждающегося материала и уравнение, определяющее дополнительные деформации вызванные ростом повреждения. Принято, что разрушение при ползучести наступает при критической комбинации составляющих тензора повреждения Ω_i, меньших чем 1. Правильность предложенного подхода к проблеме проверена экспериментально. С этой целью использованы результаты исследований разрушения при ползучести полученные, для нескольких металлов, разными исследователями.

1. Introduction

CREEP PROCESSES, well-recognized at room temperature, are still scientific and technical problems when the structural elements are designed to operate at elevated temperature. It is known from the numerous experiments that the time during which the material can sustain stress is finite and this is the result of a growth of microdefects observed in the material structure. Metallographical inspection reveals that such a microstructural damage normally occurs in the form of fissures and voids which may coincide with grain boundaries. It is usually observed that the damage in the metal subjected to the constant load is accompanied by an increase of the strain rate and it is the so-called tertiary region of creep which can easily be detected when creep curves for the material tested are constructed. A similar process of material structure deterioration is observed not only in a creep at elevated temperature. As pointed out by LEMAITRE [1], three different phenomena

(*) This work was supported by the grant C.P.B.P. 02.01, t.1.10

should be distinguished: plastic damage associated with large plastic deformation, fatigue damage being the result of small irreversibilities in each cycle of loading and creep damage due to time exposure to constant load. The final result observed as a microstructural damage is similar in each of the processes mentioned above; however, the physical background of the damage evolution is different and therefore each type of damage is usually considered separately.

Since the pioneering papers by KACHANOV [2] and RABOTNOV [3] appeared, a great number of valuable theoretical and experimental results concerning, in particular, creep damage has been obtained. The detailed evaluation of those results can be found in survey papers like those of LEMAITRE [4], KRAJCINOVIC [5], BASISTA [6] or ŻUCHOWSKI [7]; however, the general conclusion which can be drawn from those papers is that the problem of a consistent description of the mechanical behaviour of damaging solids has not been exhausted yet. Particularly, great discrepancies are observed in an approach to the definition of the damage variable describing the internal state of the deteriorating material. The earliest definition based on a scalar representation of damage proposed by KACHANOV [2] is still used and recommended by LEMAITRE [4, 8], while KRAJCINOVIC [5, 9] is of the opinion that the vector damage variable is the most suitable one, as it is a logical generalization of the original scalar measure. However, it is undeniable that a tensor damage variable, proposed by VAKULENKO and KACHANOV [10] and developed by MURAKAMI and OHNO [11], BETTEN [12, 13] and LECKIE and ONAT [14], can store more information than the scalar or vector one and makes possible a rational description of the mechanical behaviour of the damaging material. The tensor damage variable and the theory of tensor function representations seem to be promising tools in obtaining a consistent description of the creep damage of a material subjected to a multi-axial state of stress. This line is followed by MURAKAMI and SANOMURA [15, 16], who formulated the constitutive equations of tertiary creep. Their theory was successfully used to describe the creep damage of copper and Nimonic 80A. However, the approach presented by Murakami and Sanomura is strongly affected by the classical concept where the power law of the damage evolution equation proposed by KACHANOV [2] and RABOTNOV [3] is used together with the notion of so-called net—or effective stresses. This results in a great number of material constants to be identified by fitting their numerical values to the experiment results obtained from several creep curves.

The aim of this paper is to derive the constitutive equations of creep damage for a solid subjected to a multi-axial state of stress. The set of the equations proposed consists of the damage evolution equation, the yield criterion for the damaging solid and the equation specifying the additional strain due to damage growth. All these equations are formulated by employing the theory of tensor function representations and contain the damage and stress tensors as independent variables. Instead of the concept of the net-stress tensor, being the modification of the Cauchy stress tensor accounting for the net area reduction, homogenization of the macroscopic material properties was performed. It was also assumed that creep rupture occurs at a critical value of the damage tensor components, not necessarily equal to unity as it is usually suggested. The approach presented in this paper is verified experimentally by employing the creep rupture test results available in the literature for various metals under uniaxial tension, pure shear and equal biaxial tension.

2. Mechanical behaviour of damage solid

The metallographical analysis of the microstructural damage that occurs in metals during creep performed by HAYHURST [17], DYSON, LOVEDAY and RODGERS [18], and DYSON and MCLEAN [19] revealed an oriented character of the microdefects of the material structure. This means that the overall mechanical response of the damaged material is anisotropic and the constitutitve equations of elasticity and plasticity should account for the specific damage-induced anisotropy. The formulation of such equations is discussed by the author in his previous papers [20, 21]. However, these equations are the starting point for the theory proposed in this paper, that is why the final form of the formulae derived in [20, 21] will be shown here.

It is assumed that the current state of the damaged material is described by the symmetric, second-order damage tensor **D**, similar to the one proposed by VAKULENKO and KACHANOV [10] and developed by MURAKAMI and OHNO [11]. However, the principal values D_1, D_2 and D_3 of the tensor proposed are different from those usually assumed and they are expressed by the relation

$$D_i = \frac{\Omega_i}{1-\Omega_i}, \quad i = 1, 2, 3. \quad D_i \in \langle 0, \infty), \tag{2.1}$$

where Ω_i are the principal values of the damage tensor $\mathbf{\Omega}$ defined by MURAKAMI and OHNO [11]. It is necessary to explain that two different forms of the damage tensor **D** and $\mathbf{\Omega}$ will be used in this paper. As pointed out by MURAKAMI and SANOMURA [15], the damage tensor $\mathbf{\Omega}$ is a suitable damage measure only when the damage evolution is considered. They also stated that when calculating strains in a damaged solid, the so-called damage effect tensor should be used. It should be noted that this problem is not finally solved as the form of the damage effect tensor described in [15] is the modification of that proposed by MURAKAMI and OHNO [11]. In his previous paper [20] the another also looked for an appropriate damage variable which could describe strength and stiffness reduction of anisotropically damaged solids. Finally it was found that the tensor **D** defined in [20, 21] is a suitable damage measure to be used for this purpose. This means that in this paper the damage tensor **D** will appear in the equations of elasticity and plasticity of damaged solids, whereas in the damaged evolution law the more convenient damage tensor $\mathbf{\Omega}$ will be used.

The final form of the equation of elasticity for the damaged material derived in [20] is

$$\boldsymbol{\epsilon} = -\frac{\nu}{E}\mathbf{I}\,\mathrm{tr}\boldsymbol{\sigma} + \frac{1+\nu}{E}\boldsymbol{\sigma} + \frac{D_1}{2(1+D_1)E}(\boldsymbol{\sigma}\mathbf{D}+\mathbf{D}\boldsymbol{\sigma}), \tag{2.2}$$

where $\boldsymbol{\epsilon}$, $\boldsymbol{\sigma}$ and **I** are the strain, stress and unit tensors, respectively, E is the Young modulus and ν the Poisson ratio of the matrix material.

It should be noted that Eq. (2.2) is linear with respect to **D** but, as pointed out in [20], such a simple form of the constitutive equation described very accurately the elastic behaviour of the cracked solids. The effective elastic constants of orthotropically-damaged material can be calculated by employing the well-known representation for the fourth-order tensor function. The appropriate linear, with respect to **D**, relation derived in [20] has the form

$$A_{ijkl} = -\frac{\nu}{E}\delta_{ij}\delta_{kl} + \frac{1+\nu}{2E}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{D_1}{4(1+D_1)E}(\delta_{ik}D_{jl} + \delta_{jl}D_{ik} + \delta_{il}D_{jk} + \delta_{jk}D_{il}), \tag{2.3}$$

where δ_{ij} is the Kronecker delta and A_{ijkl} is a well-known fourth-order elastic anisotropy tensor. The validity of the equations (2.2) and (2.3) was verified experimentally by using the models simulating the damaged material [20].

Taking into account Eq. (2.2), the elastic strain energy calculated for a homogenized equivalent material, possessing the same elastic properties as the damaged solid, has the form

$$\Phi_e = \frac{1-2\nu}{6E}\operatorname{tr}^2\boldsymbol{\sigma} + \frac{1+\nu}{2E}\operatorname{tr}\mathbf{S}^2 + \frac{D_1}{2(1+D_1)E}\operatorname{tr}\boldsymbol{\sigma}^2\mathbf{D}, \tag{2.4}$$

where $\mathbf{S}$ is the stress deviator.

It was proposed in [21, 22] that strength reduction of the cracked solid can be described by means of the yield criterion formulated as an isotropic scalar function of the stress tensor $\boldsymbol{\sigma}$ and the damage tensor $\mathbf{D}$. The assumed yield criterion has the form

$$C_1\operatorname{tr}^2\boldsymbol{\sigma} + C_2\operatorname{tr}\mathbf{S}^2 + C_3\operatorname{tr}\mathbf{D}\boldsymbol{\sigma}^2 - \sigma_0^2 = 0, \tag{2.5}$$

where C_1, C_2 and C_3 are the material constants and σ_0 is the uniaxial yield stress for the matrix material. The simplest way to determine C_1, C_2 and C_3 is to specify Eq. (2.5) for the prescribed states of stress such as uniaxial tension in two mutually perpendicular principal directions of the material structure symmetry and for a biaxial tension. As a result, the following set of linear equations is obtained:

$$\begin{aligned} C_1 + \frac{2}{3}C_2 + D_1C_3 &= (\sigma_0/\sigma_{10})^2, \\ C_1 + \frac{2}{3}C_2 + D_2C_3 &= (\sigma_0/\sigma_{20})^2, \\ 4C_1 + \frac{2}{3}C_2 + (D_1+D_2)C_3 &= (\sigma_0/T_0)^2, \end{aligned} \tag{2.6}$$

where σ_{10} and σ_{20} are the respective yield stresses for damaged material loaded uniaxially in the principal directions corresponding to the values D_1 and D_2, and T_0 stands for the yield stress for the biaxial uniform tension. Those yield stresses required to calculate C_1, C_2 and C_3 were determined from two theoretical models described in [21, 22]. In the first model, named the mechanical one, the failure modes associated with the plastic zones developing between the adjacent cracks were analysed and, as a result, the respective yield stresses were obtained. The second model, referred to as a theoretical one, consists in comparing the strain energy (2.4) calculated for the equivalent material with that obtained for the matrix material. The curves of σ_{10} versus Ω_1 and also the relevant experimental results determined for square and quincuncial crack arrangements are shown in Figs. 1 and 2. The numerical value of Ω_1 according to Murakami's definition [11], is calculated as a ratio of crack area and the original cross section taken in the plane perpendicular to the principal direction 1. The character of these curves, and also the fact that the crack

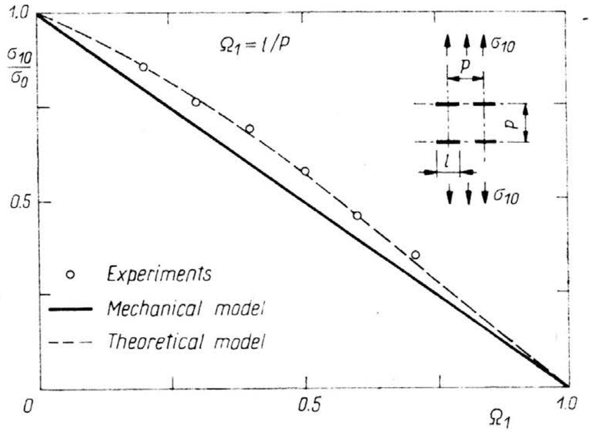

FIG. 1. Uniaxial yield stress versus damage tensor component Ω_1 for square crack arrangement.

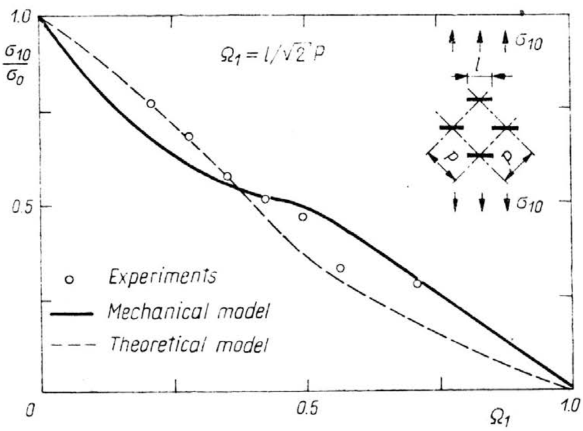

FIG. 2. Uniaxial yield stress versus damage tensor component Ω_1 for quincuncial crack arrangement.

arrangement in the damaged material is usually the combination of those two patterns assumed in the models analysed, justifies the assumption that the yield stresses σ_{10}, σ_{20} and T_0 can be calculated from the simplified relations

$$\sigma_{10} = T_0 = (1-\Omega_1)\sigma_0, \qquad (2.7)$$
$$\sigma_{20} = (1-\Omega_2)\sigma_0.$$

The above result concerning T_0 is not taken directly from Figs. 1 and 2 but from the additional considerations presented in [21, 22] and also those shown in [23] for perforated materials with regular arrays of circular cavities. It should be noted that for a given crack arrangement much more accurate results than those expressed by Eqs. (2.7) are available in [21, 22, 23]. However, as it has already been mentioned, the crack array in the damaged material is generally an unknown combination of various simple patterns and in such a situation the values given by the relations (2.7) can be considered as sufficiently accurate.

3. Damage evolution equation

The formulation of the appropriate damage evolution equation as a function

$$\dot{\mathbf{\Omega}} = \mathbf{F}(\boldsymbol{\sigma}, \mathbf{\Omega}, T, \alpha) \tag{3.1}$$

is a crucial problem of the damage mechanics. In Eq. (3.1), $\dot{\mathbf{\Omega}}$ stands for the time derivative of the damage tensor $\mathbf{\Omega}$, T is the temperature and α is a strain hardening parameter. There are many attempts to formulate the explicit form of the function (3.1) but most of them like those of SDOBYREV [24], HAYHURST [17] or LECKIE and HAYHURST [25] are the generalization of the classical theory proposed by KACHANOV [2] and RABOTNOV [3] to describe creep rupture under uniaxial tension. The only promising theory, based on the tensorial nature of damage and enabling the analysis of tertiary creep of solid subjected to a comlex loading as proposed by MURAKAMI and SANOMURA [15], is handicapped by a great number of various material constants.

The damage evolution equation proposed in this paper is formulated for a given constant temperature as a tensor function:

$$\dot{\mathbf{\Omega}} = \mathbf{F}(\boldsymbol{\sigma}, \mathbf{\Omega}), \tag{3.2}$$

where, for simplicity, the strain hardening parameter is omitted. The most general mathematical form of the function (3.2) obtained on the basis of the tensor function representations [26] can be written as follows

$$\dot{\mathbf{\Omega}} = \sum_i \alpha_i \mathbf{G}_i, \qquad i = 1, 2, \ldots, 9, \tag{3.3}$$

where $\mathbf{G}_i$ is a set of nine tensor generators

$$\mathbf{I}, \quad \boldsymbol{\sigma}, \quad \boldsymbol{\sigma}^2, \quad \mathbf{\Omega}, \quad \mathbf{\Omega}^2, \quad \boldsymbol{\sigma}\mathbf{\Omega}+\mathbf{\Omega}\boldsymbol{\sigma}, \quad \boldsymbol{\sigma}^2\mathbf{\Omega}+\mathbf{\Omega}\boldsymbol{\sigma}^2,$$
$$\mathbf{\Omega}^2\boldsymbol{\sigma}+\boldsymbol{\sigma}\mathbf{\Omega}^2, \quad \boldsymbol{\sigma}^2\mathbf{\Omega}^2+\mathbf{\Omega}^2\boldsymbol{\sigma}^2$$

and α_i are the polynomial functions of the scalar invariants

$$\mathrm{tr}\boldsymbol{\sigma}, \quad \mathrm{tr}\boldsymbol{\sigma}^2, \quad \mathrm{tr}\boldsymbol{\sigma}^3, \quad \mathrm{tr}\mathbf{\Omega}, \quad \mathrm{tr}\mathbf{\Omega}^2, \quad \mathrm{tr}\mathbf{\Omega}^3, \quad \mathrm{tr}\boldsymbol{\sigma}\mathbf{\Omega}, \quad \mathrm{tr}\boldsymbol{\sigma}\mathbf{\Omega}^2 \quad \mathrm{tr}\boldsymbol{\sigma}^2\mathbf{\Omega}, \quad \mathrm{tr}\boldsymbol{\sigma}^2\mathbf{\Omega}^2.$$

There is no need to look for the damage evolution equation in such a general form. Taking into account some experimental observations, Eq. (3.3) can be written in a shorter and more convenient form. The basic experimentally-stated facts which made it possible will be summarized below.

HAYHURST [17] performed creep rupture experiments for an Al-Mg-Si alloy subjected to multiaxial tensile states of stress at a temperature of 483°K. He stated that uniformly distributed grain-boundary cracks could be observed on planes which were inclined at approximately 90° to the principal stress directions, and the plane of most intense cracking was found to be perpendicular to the largest principal stress. The examination of equally biaxially-loaded specimens showed a uniform distribution of grain boundary cracks and it was difficult to assign a preferential direction of growth.

DYSON and MCLEAN [19] studied metallographically the specimens of Nimonic 80A subjected to tension and torsion at 1023°K. They revealed that all the cracked grain boundaries were inclined at an angle greater than 30° to the maximum principal axis in both tension and torsion specimens. It justifies the assumption that only tensile stresses could develop microstructural damage in metals. This conclusion is well supported by the results obtained by JOHNSON, HENDERSON and MATHUR [27] for copper under uniaxial compressive stress. Careful examination of the microstructure of the specimens subjected to compressive stress for the times in excess of the tensile uniaxial rupture time did not reveal any evidence of grain boundary voids or material deterioration.

HAYHURST [17] additionally studied the micrographs taken of the planes perpendicular to the surface of the plate specimens subjected to plane states of stress. He showed that in all tests the grain boundary cracks had grown on planes which were at an angle 90° to the direction of the maximum tensile stress.

All those results prove that the principal values of the damage tensor and their directions are closely connected with the positive principal values of the stress tensor. On the other hand, as stated by KRAJCINOVIC [5], the effect of the stress applied, observed as a growth of microdefect density, could be changed only at the expense of externally supplied energy. It seems reasonable to identify this energy with the strain energy accumulated in the solid subjected to a given state of stress with account for the current state of the material, described by the damage tensor.

The above experimental results, together with the purely mathematical basis supplied by the theory of the tensor function representations, made it possible to derive an appropriate form of the damage evolution equation. It will be proved in this paper that the damage evolution equation

$$\dot{\mathbf{\Omega}} = B_1 \Phi_e^m \mathbf{I} + B_2 \Phi_e^n \boldsymbol{\sigma}^* \tag{3.4}$$

accounting for both isotropic and anisotropic damage is sufficiently general to describe the creep rupture behaviour of the metals. Equation (3.4) contains the modified stress tensor $\boldsymbol{\sigma}^*$ expressed in terms of its positive principal values. This means that in the case of negative principal values of the stress tensor $\boldsymbol{\sigma}$, the relevant principal values of the modified tensor $\boldsymbol{\sigma}^*$ are equal to zero. The exponents m, n and multipliers B_1 and B_2 are the material constants, whereas Φ_e is the strain energy expressed by Eq. (2.4).

The assumed form (3.4) of the general expression (3.3) contains only two tensor generators $\mathbf{I}$ and $\boldsymbol{\sigma}^*$ and the scalars α_i are the simple polynomial functions of three invariants $\operatorname{tr}\boldsymbol{\sigma}$, $\operatorname{tr}\mathbf{S}^2$ and $\operatorname{tr}\boldsymbol{\sigma}^2\mathbf{D}$. According to HAYAHURST [17], SDOBYREV [24] and LECKIE and HAYHURST [25], the damage evolution is affeted not only by the positive principal stresses but also by the second invariant of the stress deviator and the first invariant of the stress tensor. This means that a simple form of the damage evolution equation (3.4) accounts for the combined effect of all the factors considered in the damage mechanics as most important.

Further considerations presented in this paper are based on the following form of the damage evolution equation:

$$\dot{\mathbf{\Omega}} = \left[\frac{1-2\nu}{6E}\operatorname{tr}^2\boldsymbol{\sigma} + \frac{1+\nu}{2E}\operatorname{tr}\mathbf{S}^2 + \frac{D_1}{2(1+D_1)E}\operatorname{tr}\boldsymbol{\sigma}^2\mathbf{D}\right]^n B_2\boldsymbol{\sigma}^*, \tag{3.5}$$

where the isotropic damage expressed by the term $B_1\mathbf{I}$ is omitted. This is justified as the metallographical inspection of damaging solids [17, 19] detected mainly the oriented damage associated with the positive values of the principal stresses.

Three different values of the exponent n were assumed, and as a result the following forms of Eq. (3.5) were obtained:

$$\dot{\mathbf{\Omega}} = \left[\frac{1-2\nu}{3(1+\nu)}\,\mathrm{tr}^2\boldsymbol{\sigma}+\mathrm{tr}\mathbf{S}^2+\frac{D_1}{(1+\nu)(1+D_1)}\,\mathrm{tr}\boldsymbol{\sigma}^2\mathbf{D}\right]K_2\boldsymbol{\sigma}^* \tag{3.6}$$

for $n = 1$,

$$\dot{\mathbf{\Omega}} = \left\{\frac{(1-2\nu)^2}{9(1+\nu)^2}\,\mathrm{tr}^4\boldsymbol{\sigma}+\mathrm{tr}^2\mathbf{S}^2+\frac{2(1-2\nu)}{3(1+\nu)}\,\mathrm{tr}^2\boldsymbol{\sigma}\,\mathrm{tr}\mathbf{S}^2 + \left[\frac{2(1-2\nu)}{3(1+\nu)^2}\,\mathrm{tr}^2\boldsymbol{\sigma}+\frac{2}{1+\nu}\,\mathrm{tr}\mathbf{S}^2\right]\frac{D_1}{1+D_1}\,\mathrm{tr}\boldsymbol{\sigma}^2\mathbf{D}\right\}K_2\boldsymbol{\sigma}^* \tag{3.7}$$

for $n = 2$ and

$$\dot{\mathbf{\Omega}} = \left\{\frac{(1-2\nu)^3}{27(1+\nu)^3}\,\mathrm{tr}^6\boldsymbol{\sigma}+\mathrm{tr}^3\mathbf{S}^2+\frac{(1-2\nu)^2}{3(1+\nu)^2}\,\mathrm{tr}^4\boldsymbol{\sigma}\,\mathrm{tr}\mathbf{S}^2 + \frac{1-2\nu}{1+\nu}\,\mathrm{tr}^2\boldsymbol{\sigma}\,\mathrm{tr}^2\mathbf{S}^2+\left[\frac{(1-2\nu)^2}{3(1+\nu)^3}\,\mathrm{tr}^4\boldsymbol{\sigma} + \frac{3}{1+\nu}\,\mathrm{tr}^2\mathbf{S}^2+\frac{2(1-2\nu)}{(1+\nu)^2}\,\mathrm{tr}^2\boldsymbol{\sigma}\,\mathrm{tr}\mathbf{S}^2\right]\frac{D_1}{1+D_1}\,\mathrm{tr}\boldsymbol{\sigma}^2\mathbf{D}\right\}K_2\boldsymbol{\sigma}^* \tag{3.8}$$

for $n = 3$, where

$$K_2 = \left(\frac{1+\nu}{2E}\right)^n B_2.$$

It should be noted that Eqs. (3.7) and (3.8) represent the simplified forms of Eq. (3.5) derived for $n = 2$ and 3 where the terms containing $\mathbf{D}^2$ and $\mathbf{D}^3$ are neglected. A detailed analysis of numerical values of those terms together with relevant multipliers showed that they are relatively small in comparison with these retained in Eqs. (3.7) and (3.8).

At this moment it is necessary to explain that two different damage tensors are left in Eqs. (3.6), (3.7) and (3.8) as the principal values of both tensors $\mathbf{\Omega}$ and $\mathbf{D}$ are related by the very simple equation (2.1). Therefore there is no need to rearrange these equations in order to obtain the uniform expressions in terms of either $\mathbf{\Omega}$ or $\mathbf{D}$.

4. Rupture criterion

According to KACHANOV's theory [2], it is usually assumed that the final rupture of the material in creep under constant load occurs when at least one principal value of the damage tensor $\mathbf{\Omega}$ is equal to unity. However, there is experimental evidence which contradicts such an assumption. For example, PIATTI, BERNASCONI and COZZARELLI [28] are of the opinion that there is a so-called critical value of the scalar damage variable corresponding to the rupture of the element. LEMAITRE [4] suggests that for metals this value is less than 1 and varies from 0.2 to 0.8. A more precise conclusion is drawn by DYSON and MCLEAN [19] from metallographical inspection of broken Nimonic 80A samples subjected to tension and torsion. They found that the total volume of cavities had been too small to reduce the net cross-sectional area to zero as Kachanov supposed. Furthermore, detailed quantitative assessment of the damage growth during creep up to rupture at a temperature of 1237°K for a refractory alloy IN100 can be found in the papers by CHABOCHE [29, 30]. From the diagrams of the damage scalar variable versus the time shown in those papers, it is seen that for the time very close to the rupture time the experimentally-determined value of the damage scalar parameter is less than 0.3. Besides, for the whole life-time of the elements the damage was smaller for higher stress applied than that detected in the samples subjected to the lower stress.

HAYHURST [17] in his experiments with aluminium alloy observed that close to the final rupture, the microcracks had deformed in shear bands which had been inclined approximately at 45° to the direction of the applied stress. He stated that the boundary micro-fissures had grown in a stable manner during creep to the size and distribution at which the collapse condition for the current structure of the specimen had been satisfied. It is worthwhile to mention that the shear bands developing at rupture and detected in a microscale by HAYHURST [17] are similar to those analysed in a macroscale in the papers [31, 32] where macroscopic models simulating the damage materials were tested.

To explain all those experimental observations, it was assumed in this paper that the onset of rupture is observed when the material yield stress, continuously decreasing due to the damage growth, becomes equal to the stress actually applied. In a general case of the multiaxial state of stress, the rupture of the element begins at a critical combination of the damage tensor components which can be determined from the yield criterion (2.5). This means that the yield criterion (2.5) is assumed as a collapse condition for the deteriorated material structure as supposed by HAYHURST [17].

This new approach to the problem of creep rupture is explained for the plane state of stress in Fig. 3. Let us assume at the beginning that the material at a given temperature

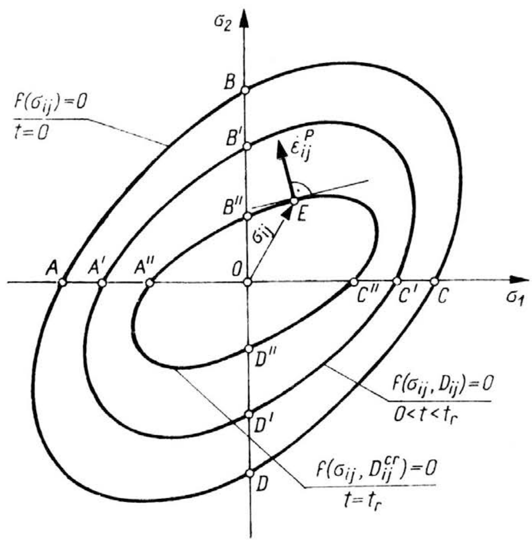

FIG. 3. Yield surfaces for damaging solid.

is perfectly elastic-plastic and obeys the Huber–Mises yield criterion which is represented in Fig. 3 as an ellipse $ABCD$. The actual loading is described by a stress tensor σ_{ij} and corresponds to the point E. This loading, not exceeding the matrix material yield stress, results in the instantaneous elastic strain and then primary and secondary creep is observed. However, from the beginning of the creep process microstructural damage and the stable reduction of the overall material strength occurs. This process of damage growth is described by the damage evolution equation (3.4) and the relevant strength reduction is expressed by the yield criterion (2.5) represented in Fig. 3 for the time $0 < t < t_r$ by the curve $A'B'C'D'$. In a limit case when the yield surface is reduced to such a size that it touches the point E (curve $A''B''C''D''$ in Fig. 3), the critical value of the damage tensor D_{ij}^{cr} is obtained and plastic flow represented by the plastic strain rate tensor $\dot{\varepsilon}_{ij}^{p}$ begins. Some ductile metals when cracked, particularly for a great amount of damage, exhibit pronounced brittleness and then the onset of plastic flow is hardly detected because the rupture is accompanied by very small plastic deformation. This is observed at lower stress where accumulation of damage sufficient to reduce the strength of the material to the level of the stress applied required a long period of time. At high stress, in comparison with the material yield stress, failure occurs relatively quickly because the total amount of damage necessary to reduce the overall strength of the material is smaller and this results in greater macroscopic ductility of the material at rupture.

It is clear from the above considerations that the rupture condition of the material consists of two equations: the damage evolution equation (3.4) or (3.5) and the yield criterion for damaging material (2.5). The rupture time for the element subjected to the multi-axial state of stress according to Eq. (3.5) can be calculated from the equation

$$(4.1) \qquad t_r = \frac{1}{B_2 \sigma_1} \left\{ \int \left[\frac{1-2\nu}{6E} \operatorname{tr}^2 \boldsymbol{\sigma} + \frac{1+\nu}{2E} \operatorname{tr} \mathbf{S}^2 + \frac{D_1^{cr}}{2(1+D_1^{cr})E} \operatorname{tr} \boldsymbol{\sigma}^2 \mathbf{D}^{cr} \right]^{-n} d\Omega_1^{cr} - C \right\},$$

where C is the constant to be calculated from the initial condition $\Omega_1 = 0$ for $t = 0$. The critical values of the damage tensor components D_{ij}^{cr} can be calculated from the yield criterion

$$(4.2) \qquad C_1 \operatorname{tr}^2 \boldsymbol{\sigma} + C_2 \operatorname{tr} \mathbf{S}^2 + C_3 \operatorname{tr} \mathbf{D}^{cr} \boldsymbol{\sigma}^2 - \sigma_0^2 = 0$$

corresponding to the curve $A''B''C''D''$ in Fig. 3. As the ratios of the principal values of $\mathbf{\Omega}$ and $\boldsymbol{\sigma}^*$, according to the damage evolution equation (3.5), are the same, the set of equations (4.1) and (4.2) is sufficient to calculate the rupture time for the multi-axial state of stress. To this end, the elastic and plastic characteristics of the matrix material are required together with only one additional constant B_2 contained in the damage evolution equation (3.5).

A similar approach to the problem of creep rupture was proposed by BUI-QUOC and BIRON [33] but their promising results were limited only to the uniaxial state of stress.

5. Strains in tertiary creep

The strains in creep are usually considered when theoretical tertiary creep curves are constructed so as to fit the experimental results (MURAKAMI and SANOMURA [15], CHABOCHE [30] and HAYHURST [34]). However, there are at least two reasons to formulate appropriate equations describing strains in a tertiary creep region. The necessity to account for strains arises when analysing the rupture of damaging solids subjected to a nonhomogeneous complex state of the stress. This problem was considered theoretically and experimentally by HAYHURST [34], LECKIE and HAYHURST [35, 36] and LECKIE and WOJEWÓDZKI [37]. They found that after initial elastic response, interaction between the elastic and creep strains results in stress redistributions. Moreover, a further stress redistribution can be expected as a result of the softening of the deteriorated material. This stress redistribution is the first reason why the analysis of strains becomes one of the most important problems of the damage mechanics. The second reason is connected with engineering practice and strong limitations placed by some design regulations on strains in elements and structures.

The instantaneous strain and also the creep strain in primary and secondary regions are described by the well known equations and they are beyond the scope of this paper. For this reason, only so-called accelerated creep in a tertiary region will be considered further. It is seen from Eq. (2.2) that strain in the damaging material consists of the instantaneous elastic strain represented by the first two terms and additional strain which is associated with the damage growth. This additional strain

$$\boldsymbol{\epsilon}^D = \frac{D_1}{2(1+D_1)E} (\boldsymbol{\sigma}\mathbf{D}+\mathbf{D}\boldsymbol{\sigma}) \tag{5.1}$$

is a nonlinear function of time and increases with the increasing damage according to the damage evolution equation (3.4). Strain in tertiary creep is explained for the uniaxial tension in Fig. 4. It is seen that the additional strain $\boldsymbol{\epsilon}^D$ appears at the very beginning of

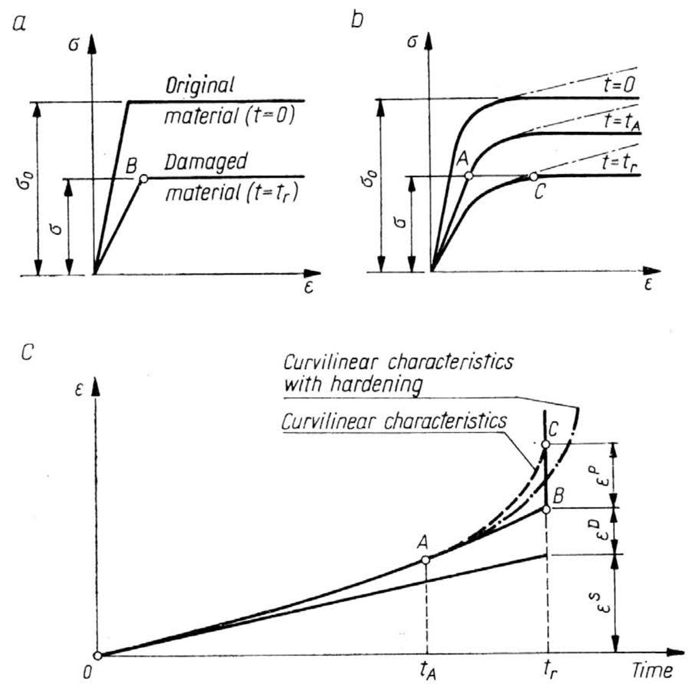

FIG. 4. Tertiary creep curves for various material models.

the creep process and increases steadily, resulting in the curvilinear diagram OAB of the total strain versus time. In the case of the perfectly elastic-plastic material as shown in Fig. 4a, in the point B where the yield stress of the deteriorated material is equal to the stress applied σ, plastic flow occurs. The appropriate plastic strain rate can be calculated from the associated flow law

$$\dot{\varepsilon}_{ij}^{p} = \lambda \frac{\partial f(\sigma_{ij}, D_{ij})}{\partial \sigma_{ij}}, \tag{5.2}$$

where $f(\sigma_{ij}, D_{ij})$ is the yield criterion (2.5) and λ is a scalar multiplier. The plastic strain rate $\dot{\varepsilon}_{ij}^{p}$ obtained from Eq. (5.2) is much greater than $\dot{\varepsilon}_{ij}^{D}$ calculated from Eq. (5.1) together with the stabilized creep strain rate ε_{ij}^{s}. This means that from the point B, corresponding to the rupture time, the curve ε versus time is vertical as shown in Fig. 4c.

However, the stress-strain diagrams for metals in their original state, and particularly after micro-cracking, are curvilinear as shown in Fig. 4b. Thus small plastic deformation appears starting from point A at time t_A shorter than the rupture time t_r. In such a case, the corner obtained in point B is smoothed, and for curvilinear material characterictics without hardening the expected creep curve is represented in Fig. 4c by the dashed line. The rupture time t_r predicted from the perfectly elastic-plastic model should then be very close to that determined for the real material. For a material which exhibits hardening (Fig. 4b), the rupture time could be longer as shown by the chain line in Fig. 4c.

It is seen from the above considerations that the tertiary creep is a rather complicated combination of stabilized creep, additional strains due to material structure deterioration and plastic flow. The approach to creep rupture presented in this paper requires experimental verification not only for uniaxial tension but also for the multi-axial state of stress. Such a verification requires appropriate experimental results collected from specially designed multi-axial creep tests. Analysis of the existing literature of the subject shows that complete experimental verification of the proposed theory is not possible at the moment due to the lack of complete experimental data concerning the elastic, plastic and creep characteristics of metals at elevated temperature. However, it is worthwhile to check the validity of this approach to the extent the available experimental data allow. To this end, a comparison of theoretical results with the experimental data collected for several metals by MURAKAMI and SANOMURA [15], LEMAITRE [4], CHABOCHE [29, 30], DYSON and MCLEAN [19], HAYHURST [17] and SALIM [38] is presented in the next sections.

6. Unixial tension

The three forms (3.6), (3.7) and (3.8) of the proposed damage evolution equation specified for the uniaxial tension lead to the same differential equation:

$$\frac{1-\Omega_1}{a-a\Omega_1+b\Omega_1^2}\, d\Omega_1 = K\sigma^{2n+1}dt, \tag{6.1}$$

where $K = \dfrac{K_2}{(1+\nu)^n}$, $a = 1$, $b = n = 1$, 2 or 3 and σ is the applied stress. As a solution of this equation the following function is obtained:

$$-\frac{1}{2b}\ln\left|\Omega_1^2-\frac{a}{b}\Omega_1+\frac{a}{b}\right|-\frac{1-2\frac{b}{a}}{b\sqrt{4\frac{b}{a}-1}}\operatorname{arctg}\frac{2\frac{b}{a}\Omega_1-1}{\sqrt{4\frac{b}{a}-1}}=K\sigma^{2n+1}t+C, \tag{6.2}$$

where C is a constant to be calculated from the initial condition $\Omega_1 = 0$ for the time $t = 0$.

Equation (6.2) together with the uniaxial yield stress for the damaged material $(2.7)_1$ represents the rupture criterion in the case of uniaxial tension. This means that the rupture time can be calculated from the equation

$$t_r=\left[-C-\frac{1}{2b}\ln\left|(\Omega_1^{cr})^2-\frac{a}{b}\Omega_1^{cr}+\frac{a}{b}\right|-\frac{1-2\frac{b}{a}}{b\sqrt{4\frac{b}{a}-1}}\operatorname{arctg}\frac{2\frac{b}{a}\Omega_1^{cr}-1}{\sqrt{4\frac{b}{a}-1}}\right]\frac{1}{K\sigma^{2n+1}}, \tag{6.3}$$

where Ω_1^{cr} is a critical value of the appropriate damage tensor component calculated according to Eq. $(2.7)_1$ from the relation

$$\Omega_1^{cr}=1-\frac{\sigma}{\sigma_0}. \tag{6.4}$$

It is seen from these equations that to calculate the rupture time t_r the material yield stress σ_0 and the constant K are required. The yield stress σ_0 can easily be determined experimentally for the material at a given temperature, and the constant K can be calculated from only one creep test up to rupture.

It was found that Eqs. (6.2) and (6.3) written for $n = 1$ in the form

$$-\frac{1}{2}\ln|\Omega_1^2-\Omega_1+1|+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\Omega_1-1}{\sqrt{3}}+\frac{\pi}{6\sqrt{3}}=K\sigma^3t \tag{6.5}$$

can be applied to describe the creep rupture behaviour of a commercially pure copper at a temperature of 523°K. The theoretical predictions concerning the rupture time calculated from Eqs. (6.5) and (6.4) were compared with the experimental results collected by MURAKAMI and SANOMURA [15]. The theoretical creep curves calculated from the equation

$$\varepsilon=\varepsilon_0+\dot{\varepsilon}^s t+\frac{D_1^2\sigma}{(1+D_1)E} \tag{6.6}$$

for $E = 80000$ MPa and for the instantaneous and primary creep strain ε_0 and also for $\dot{\varepsilon}^s$ taken from the experimental curves are compared with the experimental results in Fig. 5. The uniaxial yield stress σ_0 cannot be found among the experimental data presented by MURKAMI and SANOMURA [15], therefore two creep curves were used to determine from Eqs. (6.4) and (6.5) the constant $K = 7.6\cdot 10^{-9}$ 1/MPa^{3}h and the yield stress of undamaged material $\sigma_0 = 120$ MPa.

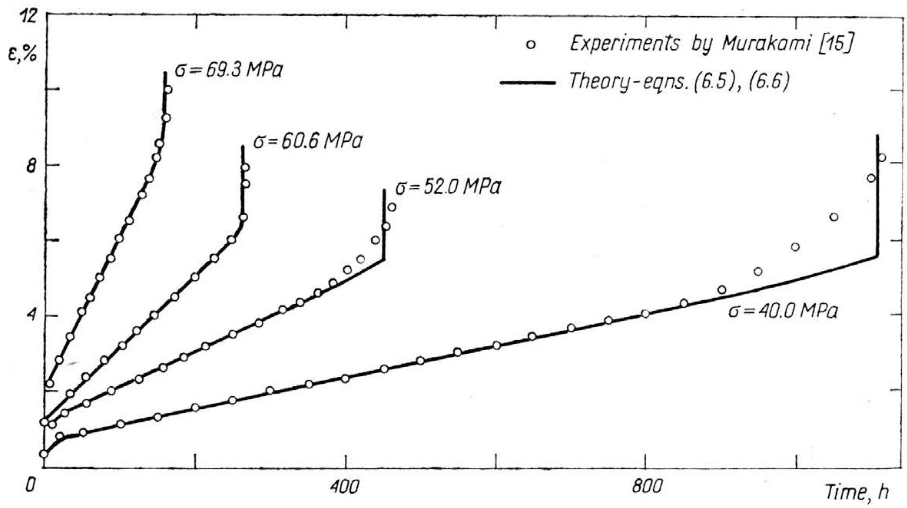

FIG. 5. Creep curves for copper at 523°K.

Much more complete experimental verification was obtained for Eq. (6.2) specified for $n = 2$

$$(6.7)\qquad -\frac{1}{4}\ln\left|\Omega_1^2-\frac{1}{2}\Omega_1+\frac{1}{2}\right|+\frac{3}{2\sqrt{7}}\operatorname{arctg}\frac{4\Omega_1-1}{\sqrt{7}}+0.03159 = K\sigma^5 t.$$

This equation was used to describe the creep behaviour of the refractory alloy IN10 and Nimonic 80A.

The experimental data obtained by LEMAITRE [4] and CHABOCHE [29, 30] include the diagrams of the rupture time versus the uniaxial stress at various temperatures. Unfortunately no hint concerning the material yield stress at the test temperature can be found in their papers. That is why the constants K and σ_0 were calculated from Eqs. (6.7) and (6.4) using two experimental results for each temperature. Eventually the following results were obtained for the refractory alloy IN100:

$$\left.\begin{aligned} K &= 1.6\cdot 10^{-15}\,1/\mathrm{MPa}^5\mathrm{h} \\ \sigma_0 &= 700\ \mathrm{MPa} \end{aligned}\right\} \quad \text{for} \quad 1173^\circ\mathrm{K},$$

$$\left.\begin{aligned} K &= 1.05\cdot 10^{-13}\,1/\mathrm{MPa}^5\mathrm{h} \\ \sigma_0 &= 400\ \mathrm{MPa} \end{aligned}\right\} \quad \text{for} \quad 1273^\circ\mathrm{K},$$

$$\left.\begin{aligned} K &= 0.7\cdot 10^{-11}\,1/\mathrm{MPa}^5\mathrm{h} \\ \sigma_0 &= 250\ \mathrm{MPa} \end{aligned}\right\} \quad \text{for} \quad 1373^\circ\mathrm{K}.$$

The above numerical values were used to calculate from Eqs. (6.7) and (6.4) the theoretical curves of the rupture time versus the uniaxial stress. The comparison of these curves with the experimental results by LEMAITRE [4] and CHABOCHE [29] collected at various temperatures is shown in Fig. 6.

The same equation (6.7) was also used to describe the creep rupture of Nimonic 80A at a temperature of 1023°K. The experimental data gathered for the uniaxial tension by

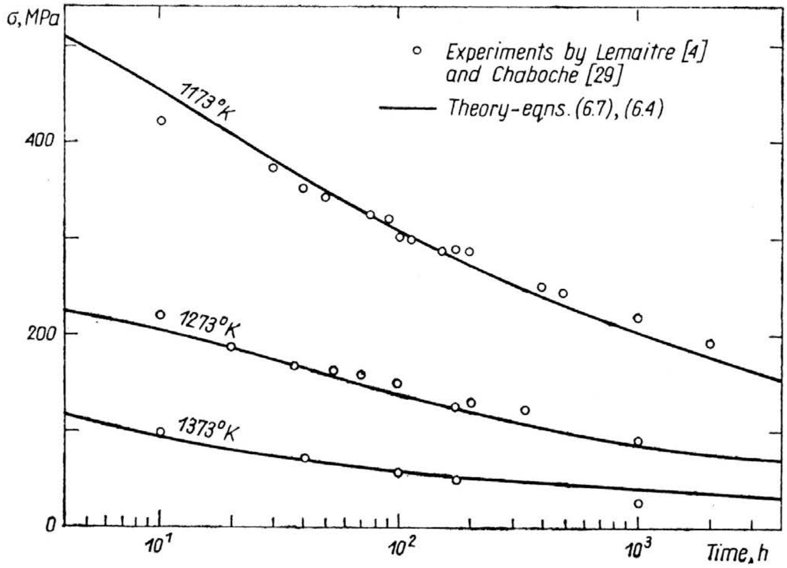

FIG. 6. Rupture time versus uniaxial stress for refractory alloy IN100 at various temperatures.

DYSON and MCLEAN [19] made it possible to calculate the constant $K = 3.8 \cdot 10^{-15}$ 1/MPa⁵h and the uniaxial yield stress $\sigma_0 = 800$ MPa. To this end two experimental results concerning the rupture time were used. These two constants together with Eqs. (6.7) and (6.4) enabled to determine the theoretical curve of the rupture time versus the effective stress $\bar{\sigma}$ calculated according to the Huber–Mises criterion. The comparison of this curve with the experimental results by DYSON and MCLEAN [19] is shown in Fig. 9.

The third form of the damage evolution equation written for $n = 3$

$$(6.8) \qquad -\frac{1}{6}\ln\left|\Omega_1^2 - \frac{1}{3}\Omega_1 + \frac{1}{3}\right| + \frac{5}{3\sqrt{11}}\operatorname{arctg}\frac{6\Omega_1 - 1}{\sqrt{11}} - 0.03594 = K\sigma^7 t$$

was used to describe the creep rupture behaviour of Al-Mg-Si alloy at 483°K. To this end, the experimental results obtained by HAYHURST [17, 34] were used. As the material yield stress $\sigma_0 = 149.4$ MPa for the aluminium alloy tested was given in the paper [34], only one experimental result was used to calculate the constant $K = 3.08 \cdot 10^{-16}$ 1/MPa⁷h. The creep curves calculated from Eqs. (6.4), (6.6) and (6.8) for $E = 60060$ MPa, as determined by HAYHURST [34], are shown in Fig. 7. A comparison of the theoretical curve of the rupture time versus the applied stress with the appropriate experimental results is shown in Fig. 10.

To make the verification of the proposed theory more complete, in the case of uniaxial tension the experimental results collected by SALIM [38] for 1% Cr.Mo.V steel at a temperature of 838°K were used. The creep rupture behaviour of this material is described by Eqs. (6.8). The theoretical creep curves determined from Eqs. (6.4), (6.6) and (6.8) for various uniaxial stresses are compared with the experimental results in Fig. 8. The numerical calculations were performed for $\sigma_0 = 600$ MPa, $K = 1.55 \cdot 10^{-20}$ 1/MPa⁷h and $E = 10^5$ MPa. No experimental points are shown in Fig. 8 because, as stated by

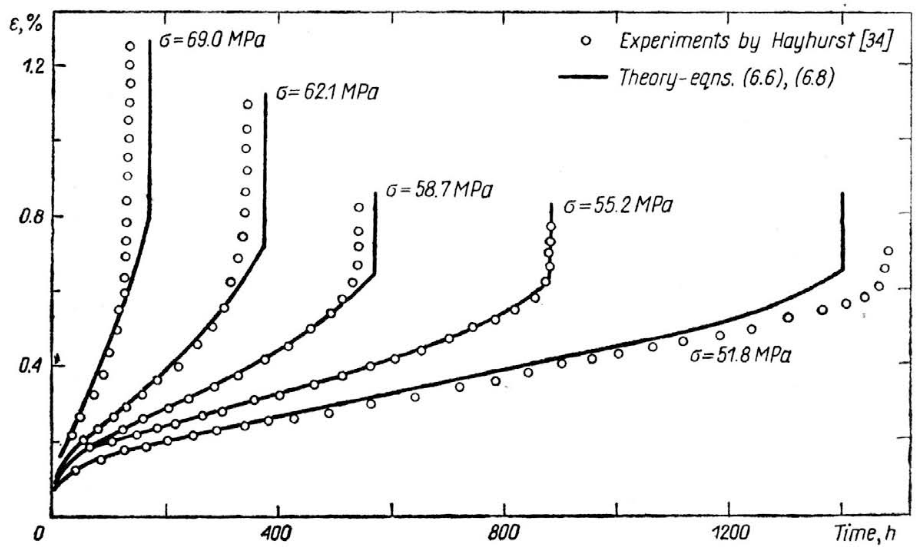

FIG. 7. Creep curves for Al-Mg-Si alloy at 483°K.

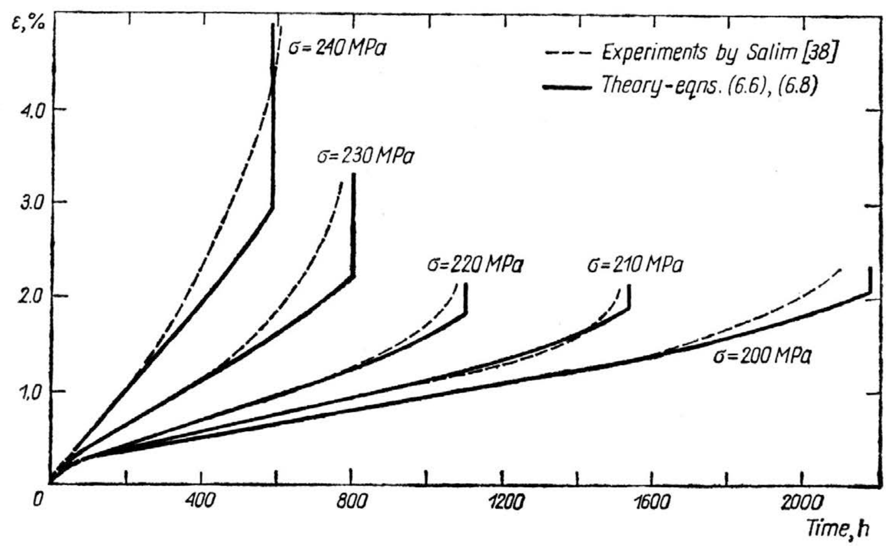

FIG. 8. Creep curves for 1% Cr.Mo.V. steel at 838°K.

SALIM [38], all creep data lie accurately on the experimental curves, hence the individual readings were not shown in the figures presented in his paper.

The good agreement of the theoretical predictions with the experimental results obtained for uniaxial tension confirms the validity of the proposed theory of creep rupture. However, a more complete verification of the proposed approach can be obtained by comparing the theoretical predictions with the creep rupture results under multi-axial loading.

7. Pure shear

High standard experimental results concerning the creep rupture of Nimonic 80A in torsion at a temperature of 1023°K were obtained by DYSON and MCLEAN [19], and these results were used to check the validity of the proposed theory in the case of pure shear. The damage evolution equation (3.7) written for $n = 2$ and specified for the pure shear, where $\sigma_1 = -\sigma_2 = \tau$, $\mathrm{tr}\boldsymbol{\sigma} = 0$, $\mathrm{tr}\mathbf{S}^2 = 2\tau^2$ and $\mathrm{tr}\boldsymbol{\sigma}^2\mathbf{D} = \tau^2 D_1$ gives the same differential equation as obtained for the uniaxial tension (6.1) where $\sigma = \tau$, and with the folowing values of the coefficients:

$$a = 4(1+\nu)^2, \quad b = 4(1+\nu).$$

This means that the analysis of creep rupture behaviour in pure shear requires the Poisson ratio of the undamaged material. Because of the lack of the relevant experimental results for Nimonic 80A at 1023°K, it was assumed that $\nu = 0.3$. The final form of the damage evolution equation for pure shear was as follows:

$$(7.1) \quad -\ln|\Omega_1^2 - 1.3\Omega_1 + 1.3| + 0.7473\,\mathrm{arctg}(1.068\Omega_1 - 0.6938) + 0.7157 = 10.4\,K\tau^5 t,$$

where $K = 3.8 \cdot 10^{-15} 1/\mathrm{MPa}^5\mathrm{h}$ is the same material constant as determined for uniaxial tension of Nimonic 80A in the previous section of this paper.

The rupture time was calculated from Eq. (7.1) inserting the critical value Ω_1^{cr} obtained from the yield criterion (2.5) and taking into account the appropriate yield stresses (2.7) required to specify the constants C_1, C_2, C_3 from the set of equations (2.6). However,

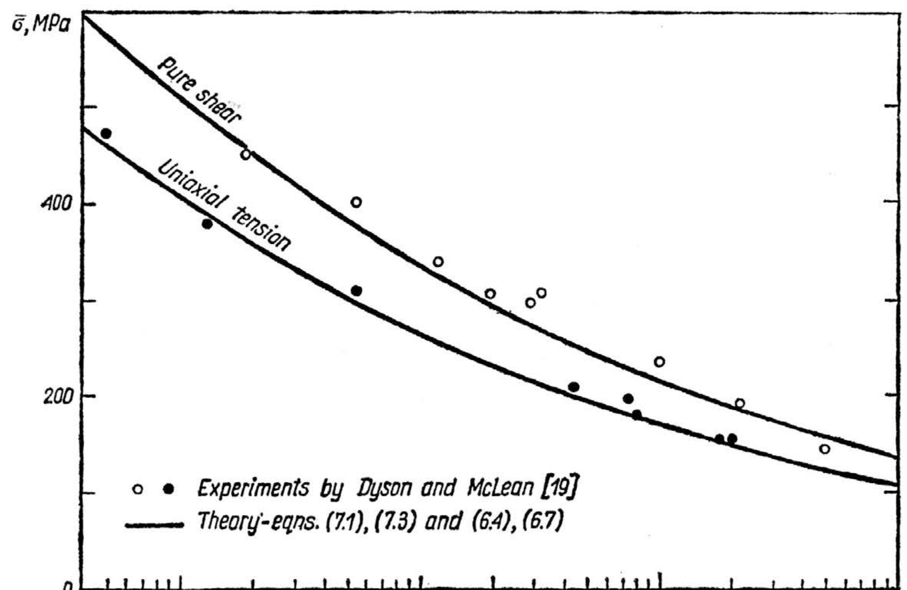

FIG. 9. Rupture time versus effective stress for Nimonic 80A at 1023°K under uniaxial tension and pure shear.

it was found that a very good approximation of the critical value Ω_1^{cr} can be obtained from a semi-empirical formula

$$\Omega_1^{cr} = \frac{2\tau_0 - 2\tau}{2\tau_0 - \tau}. \tag{7.2}$$

where $\tau_0 = \sigma_0/\sqrt{3}$ is a yield stress in pure shear of undamaged material.

The theoretical curve of the rupture time versus the effective stress $\bar{\sigma}$ calculated according to the Huber–Mises criterion, together with the experimental results by DYSON and MCLEAN [19] and also the relevant results obtained for the uniaxial tension, are shown in Fig. 9.

8. Equal biaxial loading

To verify the applicability of the proposed theory in the case of uniform biaxial tension, the experimental results of HAYHURST [17] for Al-Mg-Si alloy at 483°K were employed. To this end, the damage evolution equation (3.8) written for $n = 3$ was specified for the equal biaxial tension, where $\sigma_1 = \sigma_2 = T$, $\mathrm{tr}\boldsymbol{\sigma} = 2T$, $\mathrm{tr}\mathbf{S}^2 = 2T^2/3$ and $\mathrm{tr}\boldsymbol{\sigma}^2\mathbf{D} = 2T^2D_1$. As a result, the differential equation (6.1) was obtained, where $\sigma = T$ and

$$a = 8 - 24\nu + 24\nu^2 - 8\nu^3,$$

$$b = 24 - 48\nu + 24\nu^2.$$

Although HAYHURST supplies in his papers [17, 34] almost complete material characteristics of the undamaged material at the test temperature, there is no information about

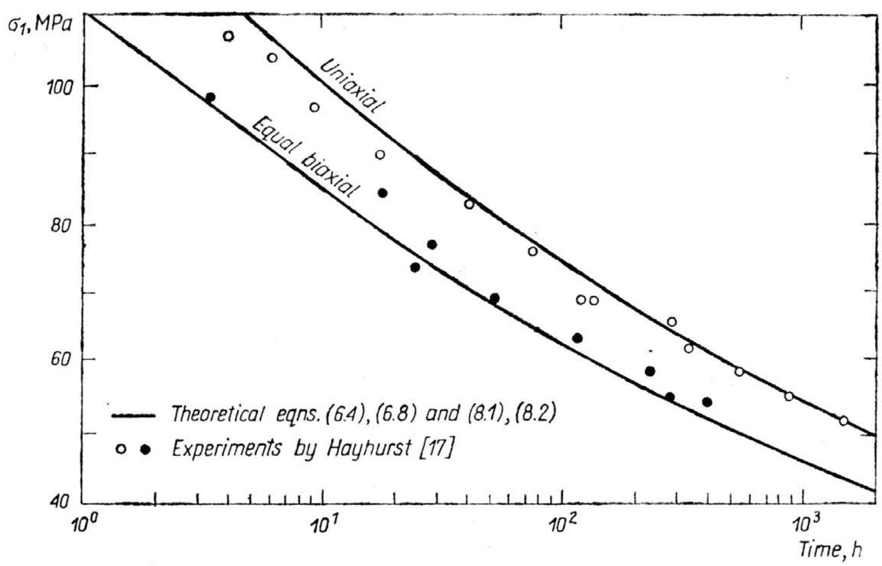

FIG. 10. Rupture time versus maximum principal stress for Al-Mg-Si alloy at 483°K under uniaxial and equal biaxial tension.

the Poisson ratio of the aluminium alloy tested. Therefore, it was assumed that $\nu = 0.3$, and the final form of the function (6.2) was as follows:

$$(8.1) \qquad -0.04252\ln|\Omega_1^2-0.2333\Omega_1+0.2333| +0.1602\,\mathrm{arctg}(2.135\Omega_1-0.2484)-0.02279 = KT^7t,$$

where $K = 3.08 \cdot 10^{-16}$ 1/MPa^{7}h is the same material constant as determined for the uniaxial tension for Al-Mg-Si alloy at the temperature 483°K.

The rupture time for the biaxially-loaded specimens were calculated from Eq. (8.1) for the critical values of Ω_1 determined according to Eq. $(2.7)_1$

$$(8.2) \qquad \Omega_1^{cr} = 1-\frac{T}{\sigma_0}.$$

In Fig. 10, the relevant theoretical curve of the rupture time versus the maximum principal stress is compared with the experimental results obtained by HAYHURST [17].

9. Conclusions

The damage evolution equation, formulated as a tensor function of the stress and damage tensor, together with the yield criterion for the damaging material constitute the set of equations which make possible a full description of creep rupture behaviour of metals at elevated temperature. The application of the proposed theory requires conventional data concerning the elastic and plastic characteristics of the original material at a test temperature. The only additional material constant K required to calculate the rupture time and to construct the creep curves for the multi-axial state of stress, can be easily determined by employing the single creep curve obtained from the uniaxial creep rupture test. A comparison of the theoretical prediction with the test results gathered by various researchers for several metals like steel, aluminium alloy, copper and nickel alloy, confirmed the validity of the theory proposed. The general conclusion is that the derived equations can be used successfully in the case of creep rupture of metals under proportional multi-axial loading. The experimental verification presented in this paper and based on the perfectly elastic-plastic model of metals showed good agreement of the theoretical and experimental results. More accurate results concerning, in particular, creep curves could be obtained when the curvi-linear material characteristics are taken into account.

References

1. J. LEMAITRE, *Damage modelling for prediction of plastic or creep fatigue failure in structures*, Trans. 5th Int. Conf. SMiRT, Berlin, 1979, North-Holland, Amsterdam 1979, vol. L, L5/1*b, 1–8.
2. L. M. KACHANOV, *On time to rupture in creep conditions*, Izv. AN SSSR, O.T.N., 8, 26–31, 1958 [in Russian].
3. Y. N. RABOTNOV, *Creep rupture*, Proc. 12th Int. Congress Appl. Mech., ed. M. HETENYI, W.G. Vincenti, Stanford 1968, Springer, Berlin 1968, 342–349.
4. J. LEMAITRE, *How to use damage mechanics*, Nucl. Eng. Design, **80**, 233–245, 1984.
5. D. KRAJCINOVIC, *Continuum damage mechanics*, Appl. Mech. Rev., **37**, 1, 1–6, 1984.

6. M. BASISTA, *On continuum models of material damage*, IFTR Reports, 40/1984, Warszawa 1984 [in Polish].
7. R. ŻUCHOWSKI, *Analysis of failure processes of metals under conditions of thermal fatigue*, Sci. Papers Inst. Mat. Sci. Appl. Mech., Technical University of Wrocław, Monographs, 18, 1986 [in Polish].
8. J. LEMAITRE, *Coupled elasto-plasticity and damage constitutive equations*, Comp. Meth, Appl. Mech. Eng., **51**, 31–49, 1985.
9. D. KRAJCINOVIC, *Constitutive equations for damaging materials*, J. Appl. Mech., **50**, 355–360, 1983.
10. A. A. VAKULENKO, M. L. KACHANOV, *Continuum theory of medium with cracks*, Izv. An SSSR, M.T.T., 4, 159–166, 1971 [in Russian].
11. S. MURAKAMI, N. OHNO, *A continuum theory of creep and creep damage*, in: Creep in Structures, eds. A. R. S. PONTER, D. R. HAYHURST, Springer, Berlin 1981, 422–444.
12. J. BETTEN, *Net-stress analysis in creep mechanics*, Ing.-Archiv, **52**, 405–419, 1982.
13. J. BETTEN, *Damage tensors in continuum mechanics*, J. Méc. Théor. Appl., **2**, 1, 13–32, 1983.
14. F. A. LECKIE, E. T. ONAT, *Tensorial nature of damage measuring internal variables*, in.: Physical Non--Linearities in Structural Analysis, eds. J. HULT, J. LEMAITRE, Springer, Berlin 1981, 140–155.
15. S. MURAKAMI, Y. SANOMURA, *Creep and creep damage of copper under multiaxial state of stress*, in: Plasticity Today, eds. A. SAWCZUK, G. BIANCHI., Elsevier Appl. Sci., London 1985, 535–551.
16. S. MURAKAMI, Y. SANOMURA, *Analysis of coupled effect of plastic damage and creep damage in Nimonic 80A at finite deformation*, Eng. Fract. Mech., **25**, 5/6, 693–704, 1986.
17. D. R. HAYHURST, *Creep rupture under multi-axial states of stress*, J. Mech. Phys. Solids, **20**, 381–390, 1972.
18. B. F. DYSON, M. S. LOVEDAY, M. J. RODGERS, *Grain boundary cavitation under various states of applied stress*, Proc. Roy. Soc. Lond. **A349**, 245–259, 1976.
19. B. F. DYSON, D. MCLEAN, *Creep of Nimonic 80A in torsion and tension*, Met. Sci., **11** 2, 37–45, 1977.
20. A. LITEWKA, *Effective material constants for orthotropically damaged elastic solid*, Arch. Mech., **37**, 6, 631–642, 1985.
21. A. LITEWKA, *On stiffness and strength reduction of solids due to crack devolopment*, Eng. Fract. Mech., **25**, 5/6, 637–643, 1986.
22. A. LITEWKA, J. MORZYŃSKA, E. ROGALSKA, *Theoretical and experimental study of mechanical properties of cracked solids*, Eng. Trans., **35**, 4, 705–719, 1987 [in Polish].
23. A. LITEWKA, A. SAWCZUK, *A yield criterion for perforated sheets*, Ing.-Archiv., **50**, 393–400, 1981.
24. V. P. SDOBYREV, *The creep rupture of EI-437B alloy under complex stress*, Izv. AN SSSR, O.T.N., 4, 92–97, 1958.
25. F. A. LECKIE, D. A. HAYHURST, *Constitutive equations for creep rupture*, Acta Metall., **25**, 1059–1070, 1977.
26. A. J. M. SPENCER, *Theory of invariants*, in: Continuum Physics, vol. 1, ed. C. ERINGEN, Academic Press, New York 1971, 239–353.
27. A. E. JOHNSON, J. HENDERSON, V. D. MATHUR, *Combined stress creep fracture of a commercial copper at 250 deg Centigrade*, The Eng., **202**, (5248), 261–265, 1956.
28. G. PIATTI, G. BERNASCONI, F. A. COZARELLI, *Damage equations for creep rupture in steels*, Trans. 5th Int. Conf. SMiRT, Berlin 1979, North-Holland, Amsterdam 1979, vol. L, L11/4, 1–9.
29. J. L. CHABOCHE, *Le concept de contrainte effective appliqué à l'élasticité et à la viscoplasticité en présence d'un endommagement anisotrope*, in: Mechanical Behaviour of Anisotropic Solids, ed. J. P. BOEHLER, Edit. CNRS, Paris 1982, 737–760.
30. J. L. CHABOCHE, *Continuous damage mechanics—a tool to describe phenomena before crack initiation*, Nucl. Eng. Design, **64**, 233–247, 1981.
31. A. LITEWKA, J. MORZYŃSKA, *Failure modelling of material with oriented damage*, Eng. Trans., **33**, 1/2, 81–99, 1985 [in Polish].
32. A. LITEWKA, J. MORZYŃSKA, *On yielding and fracture of damaged materials*, Trans. 8th Int. Conf. SMiRT, Brussels 1985, North-Holland, Amsterdam 1985, vol. L, L6/2, 281–286.
33. T. BUI-QUOC, A. BIRON, *Predictions of creep behavior of some stainless steels on the basis of short-term tensile properties*, Trans. 5th Int. Conf. SMiRT, Berlin 1979, North-Holland, Amsterdam 1979, vol. L, L5/1, 1–8.

34. D. R. HAYHURST, *Stress redistribution and rupture due to creep in a uniformly stretched thin plate containing a circular hole*, Trans. ASME, J. Appl. Mech., **40**, 244–250, 1973.
35. F. A. LECKIE, D. R. HAYHURST, *Creep rupture of structures*, Proc. R. Soc. Lond. **A340**, 323–347, 1974.
36. D. R. HAYHURST, F. A. LECKIE, *The effect of creep constitutive and damage relationships upon the rupture time of solid circular torsion bar*, J. Mech. Phys. Solids, **21**, 431–446, 1973.
37. F. LECKIE, W. WOJEWÓDZKI, *Estimates of rupture life-constant load*, Int. J. Solids Struct., **11**, 12, 1357–1365, 1975.
38. A. SALIM, *Creep design of thick walled* 1% Cr.Mo.V. *steel cylinders under internal pressure at* 565°C, Trans. 8th Int. Conf. SMiRT, Brussels 1985, North-Holland, Amsterdam 1985, vol. L, L3/4, 131–139.

TECHNICAL UNIVERSITY OF POZNAŃ,
INSTITUTE OF BUILDING STRUCTURES.

Received April 30, 1987.