Arch. Mech., 42,3, pp. 397-399, Warszawa 1990

BRIEF NOTES

A note on the linear stabilities of the solitary and cnoidal wave solutions to the two-dimensional KdV equation

DAXIN WU and SHIH-LIANG WEN (OHIO)

IT IS SHOWN that the solitary and cnoidal traveling wave solutions of the two-dimensional kdV equation are linearly stable with respect to a class of traveling wave perturbations, the results of Jeffrey and Kakutani, and of Drazin are used.

1. Introduction

JEFFREY and KAKUTANI [1] have shown that the solitary traveling wave solutions of the one-dimensional KdV equation are linearly stable. DRAZIN [2] demonstrated that the cnoidal traveling wave solutions to the one-dimensional KdV equation are also linearly stable. We are not aware of the corresponding two-dimensional results in the literature. In this note, we shall show that the perturbation equations involved in the two-dimensional case can be converted into a form which is identical to that of Jeffrey and Kakutani, and of Drazin, by considering infinitesimal perturbations of the wave in a class of traveling waves described by Eqs. (2.5) and (3.2). Therefore, we claim that both the two-dimensional solitary traveling wave solutions and cnoidal traveling wave solutions are linearly stable with respect to this class of perturbations.

2. Solitary traveling wave solution

Consider the two-dimensional KdV equation

$$(u_t+6uu_x+u_{xxx})_x+3u_{yy} = 0. \tag{2.1}$$

It is well known that (see, for example CHEN and WEN [1]).

$$u_0(Z) = (a^2/2)\,\mathrm{sech}^2 Z$$

is a solitary traveling wave solution to Eq. (2.1), where $Z = 1/2\ (ax+by-\omega t)$, a and b are wave numbers, ω is the frequency and satisfies the equation

$$\omega = a^3+3b^2/a. \tag{2.2}$$

We note that $u_0(Z) \to 0$ as $|Z| \to \infty$.

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Now, we superimpose a small disturbance $v(x, y, t)$ upon this solution

$$u = u_0(Z)+v(x, y, t). \tag{2.3}$$

We make an assumption on v (call it assumption C) that

$$|v| \ll |u_0|, \quad |v_x| \to 0 \quad \text{as} \quad |Z| \to \infty,$$

and $|v_{xx}|$ is bounded for all x, y, $t \geqslant 0$. Substituting Eq. (2.3) into Eq. (2.1), and using assumption C and the fact that u_0 is a solution of Eq. (2.1), we obtain

$$v_{xt}+12u_{0x}v_x+6u_0v_{xx}+6u_{0xx}v+v_{xxxx}+3v_{yy} = 0. \tag{2.4}$$

Let

$$v(x, y, t) = v(Z, t) = f(Z)e^{\sigma t}, \tag{2.5}$$

where σ is a constant and $f \in C^4$. Then we get

$$\{-a\omega/4f''(Z)+\sigma a/2f'(Z)+a^4/16f^{(4)}(Z) \\ +3b^2/4f''(Z)+a^2/4[3a^2\text{sech}^2Zf(Z)]_{ZZ}\}e^{\sigma t} = 0, \tag{2.6}$$

where the assumption C is applied. Integrating Eq. (2.6) with respect to Z, we have the following equation

$$f'''(Z)+(3/4a^4\text{sech}^2Z-a\omega/4+3b^2/4)16/a^4f'(Z) \\ +(\sigma a/2-3/2a^4\text{sech}^2Z\tanh Z)16/4f(Z) = 0, \tag{2.7}$$

where we have assumed that $f, f', f''' \to 0$ as $|Z| \to \infty$.

By Eq. (2.2) we note

$$\frac{-a\omega+3b^2}{4} = -\frac{a^4}{4},$$

and let $\alpha = -8\sigma/a^3$; Eq. (2.7) then can be transformed into

$$f'''-4(1-3\text{sech}^2Z)f'-(24\text{sech}^2Z\tanh Z+\alpha)f = 0,$$

which is identical with Eq. (3.2.13) on p. 624 of JEFFREY and KAKUTANI paper [1].

3. Cnoidal traveling wave solution

The cnoidal traveling wave solution of Eq. (2.1) is of the form [3]:

$$u_0(Z) = u_3+(u_1-u_2)dn^2[1/a\sqrt{12(u_1-u_3)}\,(Z_1-Z_2), K],$$

where

$$u_2 \leqslant u_0 \leqslant u_1, \quad u_0(Z_1) = u_1, \quad K^2 = \frac{u_1-u_2}{u_1-u_3}, \quad u_1 > u_2 > u_3,$$

and

$$Z = ax+by-\omega t.$$

Let

$$u = u_0(Z)+v(x, y, t).$$

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Then the corresponding linearized equation for v is

$$v_{xt}+12u_{0x}v_x+6u_0v_{xx}+6u_{0xx}v+v_{xxxx}+3v_{yy} = 0.$$

Integrating the above equation with respect to x, we get

$$v_t+6(u_0v)_x+v_{xxx}+3\int v_{yy}dx = A, \tag{3.1}$$

where A is an integration constant. Assuming that

$$v(x, y, t) = e^{-i\sigma t}\psi(Z), \tag{3.2}$$

choosing A to be zero and substituting Eq. (3.2) into Eq. (3.1) and using Eq. (2.2), we obtain

$$\frac{d^3\psi}{dZ^3}+\left(\frac{6}{a^2}u_0-1\right)\frac{d\psi}{dZ}+\left(\frac{6}{a^2}\frac{du_0}{dZ}-\frac{i\sigma}{a^3}\right)\psi = 0. \tag{3.3}$$

Notice

$$6/a^2u_0-1 = 1-U+(6/a^2u_0-2+U),$$

where U is the constant in DRAZIN paper [2]. If we let

$$S = \frac{6}{a^3}u_0-2+U,$$

then Eq. (3.3) becomes

$$\frac{d^3\psi}{dZ^3}+(1-U+S)\frac{d\psi}{dZ}+\left(\frac{dS}{dZ}-\frac{i\sigma}{a^3}\right)\psi = 0. \tag{3.4}$$

Since $cn\psi = \sqrt{1-sn^2\psi}$, $dn\psi = \sqrt{1-k^2sn^2\psi}$ for some constant k, the functions $cn^2\psi$ and $dn^2\psi$ have the same properties, namely, they are both even and periodic. Hence Eq. (3.4) is identical with Eq. (12) on page 94 of DRAZIN paper [2]. Eq. (3.4) is a linear homogeneous ordinary differential equation for ψ. It is a Floquet system whose general properties have been intensitvely studied for a long time. Eq. (3.4) has been thoroughly analyzed by DRAZIN [2]. Therefore we conclude, by applying Drazin's result, that the cnoidal traveling wave solutions to the two-dimensional kdV equation is linearly stable with respect to infinitesimal perturbations of the form described in Eq. (3.2).

References

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MATHEMATICS DEPARTMENT
OHIO UNIVERSITY, OHIO, USA.

Received April 14, 1990.

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