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Group theoretic approach for solving time-independent free-convective boundary layer flow on a nonisothermal vertical flat plate

Y. Z. BOUTROS, M. B. ABD-EL-MALEK and N. A. BADRAN (ALEXANDRIA)

THE TRANSFORMATION group theoretic approach is applied to present an analysis of the problem of steady laminar free convection from a nonisothermal vertical flat plate, wherein a number of possible surface-temperature variations with position, T_w, are derived. The obtained set of nonlinear ordinary differential equations with the appropriate boundary conditions are solved numerically using a fourth-order Runge–Kutta scheme and the gradient method. Heat transfer results, for different values of Prandtl number Pr = 0.7, 1, 2, 6 and 10, are presented, as temperature and velocity distributions for two cases of surface-temperature variations with position. A plot of the Nusselt–Grashof relation against n, exponent of surface temperature variations with position, is illustrated for Pr = 0.7, 1 and 2. Comparison with other techniques is plotted and the variation of thermal boundary layer thickness, δ_T, with the Prandtl number, Pr, are plotted for the two cases of surface-temperature variations with position.

Podejście oparte na teorii grup zastosowano do analizy problemu ustalonego, swobodnego przepływu laminarnego wzdłuż nieizotermicznej płyty pionowej wprowadzając szereg możliwych skoków temperatury powierzchniowej T_w. Otrzymany w ten sposób układ nieliniowych równań różniczkowych zwyczajnych ze stosownymi warunkami brzegowymi rozwiązano numerycznie stosując schemat Rungego–Kutty czwartego rzędu i metodę gradientów. Wyniki dotyczące przepływu ciepła dla różnych wartości liczby Prandtla Pr = 0.7, 1, 2, 6 oraz 10 przedstawiono dla dwóch przypadków zmienności temperatury powierzchniowej. Podano wykresy zależności liczby Nusselta–Grasshoffa od parametru n, wykładnika zależności temperatury powierzchniowej od położenia, dla Pr = 0.7, 1, oraz 2. Rezultaty porównano z wynikami uzyskanymi innymi metodami, podając również zależność grubości termicznej warstwy powierzchniowej δ_T od liczby Prandtla Pr dla dwóch przypadków zmienności temperatury powierzchniowej z położeniem.

Подход, опирающийся на теорию групп, применен для анализа установившейся задачи свободного ламинарного течения вдоль неизотермической вертикальной плиты, вводя ряд возможных скачков поверхностной температуры T_w. Полученная таким образом система нелинейных обыкновенных дифференциальных уравнений с соответствующими граничными условиями решена численно, применяя схему Рунге–Кутта четвертого проядка и метод градиентов. Результаты, касающиеся течения тепла для разных значений числа Прандтля Pr = 0,7, 1, 2, 6 и 10, представлены для двух случаев переменности поверхностной температуры. Приведены диаграммы зависимости числа Нуссельта–Грасшофа от параметра n, показателя зависимости поверхностной температуры от положения, для Pr = 0,7, 1 и 2. Результаты сравнены с результатами, полученными другими методами, приводя тоже зависимость термической толщины поверхностного слоя δ_T от числа Прандтля Pr для двух случаев переменности поверхностной температуры с положением.

1. Introduction

SINCE SCHMIDT and BECKMANN [1] in 1930, a considerable amount of work has been done on steady free convective flow from a heated vertical plate. In 1953 OSTRACH [2] applied numerical solutions to solve the reduced equations in solving the problem of laminar free convection flow and heat transfer about a flat plate parallel to the direction of the generating body forces. Studies of surface temperature variations for the steady case have been pursued by numerous authors from various points of view. FINSTON [3] in 1956 and YANG [4] in 1960 carried out an original study of SCHMIDT and BECKMANN [1] through similarity solutions in two cases; (i) vertical plates and (ii) cylinders. In parallel, SPARROW and GREGG [5, 6] in 1956 and 1958 studied the same problem using numerical solutions. In 1963 BRINDLEY [7] extended the method widely used by MEKSYN [8], in 1961, for finding solutions in terms of asymptotic expansions to the problem of free convection in a boundary layer. One will find attractive discussions on the subject in LEVY [9], SCHUH [10], CHAPMAN and RUBESIN [11], BURMEISTER [12], and LIGHTHILL [13].

The mathematical technique used in the present analysis is the parameter-group transformation. The group methods, as a class of methods which lead to reduction of the number of the independent variables, were first introduced by BIRKHOFF [14, 15] in 1948 and 1960, respectively, where he made use of one-parametric transformation groups. Somewhat earlier, MORGAN [16] in 1952, presented a theory which has led to improvements over earlier similarity methods. In 1952 MICHAL [17] extended Morgan's theory. Later on, MORAN and GAGGIOLI [18, 19] in 1966 and 1968 presented a general technique for similarity analysis using group theory. Integral methods were first used in 1921 to solve boundary-layer problems by VON KÁRMÁN [20] and POHLHAUSEN [21]. GOODMAN [22–24] in 1957, 1961, and 1964 applied, extensively, the integral methods to solve one-dimensional transient heat conduction, whereas SFEIR [25] in 1976 considered the case of two-dimensional steady conduction. For additional discussion on integral methods, one may consult LONGFORD [26] and BURMEISTER [12], Chapter 8.

Although this review is not comprehensive, it is clear that all these investigations are limited to studies of similarity solutions since the similarity variables can give great physical insight with minimal efforts. In SHULMAN and BERKOVSKY [27] one finds vast summary tables of the variable and boundary conditions ensuring similarity problems.

In this work we present a general procedure for applying one-parametric group transformation to the set of governing partial differential equations and the boundary conditions. Under the transformation, the partial differential equations are reduced to simultaneous ordinary differential equations with the appropriate boundary conditions. The equations are then solved numerically using a fourth-order Runge–Kutta scheme and the gradient method given in ZETTL [28].

2. Formulation of the problem

Consider a natural convective, laminar boundary layer flow along an infinite vertical plate in an isothermal fluid of temperature $\bar{T}_\infty$, far from the plate. The plate has non-uniform surface temperature $\bar{T}_w > \bar{T}_\infty$ (i.e., heated plate case). Figure 1 illustrates this situation.

Under the assumption of constant fluid properties β (the volumetric coefficient of thermal expansion), ν (the kinematic viscosity), and α (the thermal diffusivity), along with the application of the Boussinesq and boundary layer approximation, the equations

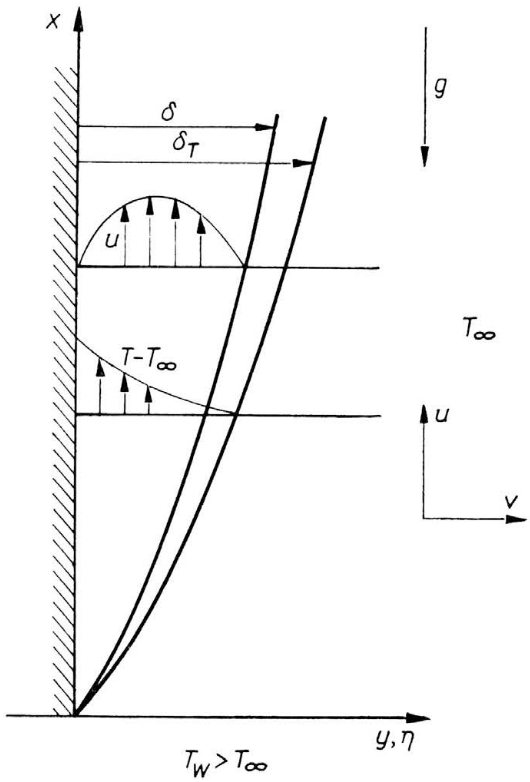

FIG. 1. Physical model of laminar boundary layer in free convection on a hot vertical flat plate.

expressing conservation of mass, momentum and energy for the physical model shown in Fig. 1, respectively, are as follows:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{2.1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \pm g\beta(\bar{T} - \bar{T}_\infty) + \nu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{2.2}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{2.3}$$

where the ($\pm$) denotes the heated plate case and cooled plate case, respectively, and g is the acceleration due to gravity.

The boundary conditions appropriate to the problem are

$$\begin{aligned} &\bar{v} = 0, \quad \bar{u} = 0, \quad \bar{T}_w = \bar{T}_w(x) \quad \text{at} \quad \bar{y} = 0, \\ &\bar{u} = 0, \quad \bar{T} = \bar{T}_\infty \quad \text{as} \quad \bar{y} \to \infty. \end{aligned} \tag{2.4}$$

Dimensionalize the variables according to

$$\begin{aligned} &x = \bar{x}/L, \quad y = (\mathrm{Gr})^{\frac{1}{4}}\bar{y}/L, \quad T = (\bar{T} - \bar{T}_\infty)/\Delta T, \quad \theta = T/T_w, \\ &u = \bar{u}/U, \quad v = (\mathrm{Gr})^{\frac{1}{4}}\bar{v}/U, \end{aligned}$$

where L is some arbitrary reference length, $\Delta T = T_{\text{ref}} - T_{\infty}$, T_{ref} is some arbitrary reference temperature, U is the characteristic velocity given by $U = (g\beta L\Delta T)^{\frac{1}{2}}$, Gr is the Grashof number defined by

$$\text{Gr} = g\beta L^3 \Delta T/\nu^2. \tag{2.5}$$

In dimensionless form, Eqs. (2.1) to (2.3) become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.6}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = T + \frac{\partial^2 u}{\partial y^2}, \tag{2.7}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\text{Pr}}\frac{\partial^2 T}{\partial y^2}, \tag{2.8}$$

where Pr is the Prandtl number defined by

$$\text{Pr} = \nu/\alpha. \tag{2.9}$$

The boundary conditions become

$$\begin{aligned} &v = 0, \quad u = 0, \quad T = T_w(x) \quad &&\text{at} \quad y = 0, \\ &u = 0, \quad T = 0 &&\text{as} \quad y \to \infty. \end{aligned} \tag{2.10}$$

From Eq. (2.6) it is seen that there exists a nondimensional stream function $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{2.11}$$

Eqs. (2.7) and (2.8) become

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \theta T_w + \frac{\partial^3 \psi}{\partial y^3}, \tag{2.12}$$

$$T_w\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} + \theta\frac{\partial \psi}{\partial y}\frac{\partial T_w}{\partial x} - T_w\frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{1}{\text{Pr}}T_w\frac{\partial^2 \theta}{\partial y^2}, \tag{2.13}$$

and the boundary conditions (2.10) become

$$\begin{aligned} &\frac{\partial \psi}{\partial x}(x, 0) = \frac{\partial \psi}{\partial y}(x, 0) = 0, \quad \theta(x, 0) = 1, \\ &\lim_{y\to\infty}\frac{\partial \psi}{\partial y}(x, y) = 0, \quad \lim_{y\to\infty}\theta(x, y) = 0. \end{aligned} \tag{2.14}$$

3. Solution of the problem

The method of solution depends on the application of one-parametric group transformation to the system of partial differential equations (2.12) and (2.13). Under this transformation the two independent variables will be reduced by one and the system of equations transforms into a system of ordinary differential equations in only one independent variable which is the similarity variable. For more details about the method we recommend HANSEN [29].

3.1. The group systematic formulation

First, the procedure is initiated with the group G_1, a class of one-parametric group a with the form

$$G_1 : \bar{S} = C^s(a) S + K^s(a), \tag{3.1}$$

where S stands for x, y, ψ, θ, T_w and the C and the K are real-valued and differentiable with respect to the parameter of the group a.

3.2. The invariance analysis

The transformations in G_1, (3.1), are for the dependent and independent variables only. To transform the differential equation, transformations for the derivatives can be obtained directly from G_1 via chain rule operations:

$$\begin{aligned} \bar{S}_i &= (C^s/C^i) S_i, && i = x, y, \\ \bar{S}_{ij} &= (C^s/C^i C^j) S_{ij}, && i = x, y \quad \text{and} \quad j = x, y, \\ \bar{S}_{ijk} &= (C^s/C^i C^j C^k) S_{ijk}, && i = x, y, \quad j = x, y \quad \text{and} \quad k = x, y, \end{aligned} \tag{3.2}$$

where S stands for ψ, θ and T_w.

Equation (2.12) is said to be transformed invariantly under Eqs. (3.1) and (3.2) whenever

$$\bar{\psi}_y \bar{\psi}_{yx} - \bar{\psi}_x \bar{\psi}_{yy} - \bar{\theta} \bar{T}_w - \bar{\psi}_{yyy} = H_1(a) [\psi_y \psi_{yx} - \psi_x \psi_{yy} - \theta T_w - \psi_{yyy}], \tag{3.3}$$

for some function H_1 (a) which may be constant. Substitution from Eqs. (3.1) and (3.2) into Eq. (3.3) yields

$$\begin{aligned} &[(C^\psi)^2/C^x(C^y)^2] \psi_y \psi_{yx} - [(C^\psi)^2/C^x(C^y)^2] \psi_x \psi_{yy} - (C^\theta C^T] \theta T_w \\ &\qquad - [C^\psi/(C^y)^3] \psi_{yyy} - R_1 = H_1(a) [\psi_y \psi_{yx} - \psi_x \psi_{yy} - \theta T_w - \psi_{yyy}], \end{aligned} \tag{3.4}$$

where

$$R_1 = [C^\theta K^T] \theta + [C^T K^\theta] T_w.$$

Invariance of Eq. (3.4) implies

$$R_1 \equiv 0,$$

which is satisfied by taking

$$K^T = K^\theta = 0, \tag{3.5}$$

and

$$[(C^\psi)^2/C^x(C^y)^2] = [C\psi/(C^y)^3] = [C^\theta C^T] \equiv H_1(a). \tag{3.6}$$

In a similar manner, the invariant transformation of Eq. (2.13) gives

$$\begin{aligned} &(C^T C^\psi C^\theta / C^y C^x) [T_w \psi_y \theta_x + \theta (T_w)_x \psi_y - T_w \psi_x \theta_y] - \frac{1}{\mathrm{Pr}} [C^T C^\theta_{\blacksquare} / (C^y_{\blacksquare})^2] T_w \theta_{yy} - R_2 \\ &\qquad = H_2(a) \left[T_w \psi_y \theta_x + \theta (T_w)_x \psi_y - T_w \psi_x \theta_y - \frac{1}{\mathrm{Pr}} T_w \theta_{yy} \right], \end{aligned} \tag{3.7}$$

where

$$(3.8)\qquad R_2 = [K^0 C^\psi C^T/C^y C^x]\psi_y (T_w)_x - [K^T C^\psi C^0/C^x C^y]\psi_x \theta_y + [K^T C^\psi C^0/C^x C^y]\psi_y \theta_x - \frac{1}{\Pr}[K^T C^0/(C^y)^2]\theta_{yy}.$$

For invariability, we should have

$$(3.9)\qquad H_2(a) \equiv [C^T C^\psi C^0/C^y C^x] = [C^T C^\theta/(C^y)^2],$$

and

$$R_2 \equiv 0, \quad \text{which yields} \quad K^T = K^\theta = 0.$$

Moreover, the boundary conditions (2.14) are also invariant in form whenever the condition

$$K^y = 0,$$

is appended to Eqs. (3.5), (3.6) and (3.9).

It is obvious that when $K^y = 0$, the transformation of the boundary condition $\theta(x, 0) = 1$ implies, that $\theta(\bar{x}, 0) = 1$, which is only satisfied if

$$(3.10)\qquad C^\theta = 1.$$

Combining Eq. (3.6), and (3.9) and invoking the result (3.10), we get

$$(3.11)\qquad C^x = C^y C^\psi, \quad C^T = C^\psi/(C^y)^3.$$

Therefore, Eqs. (2.12), (2.13) and the boundary conditions (2.14) are invariant in form under the group

$$(3.12)\qquad G_1: \begin{cases} \bar{x} = [C^y C^\psi]x + K^x, \\ \bar{y} = [C^y]y, \\ \bar{\psi} = [C^\psi]\psi + K^\psi, \\ \bar{T}_w = [C^\psi/(C^y)^3]T_w, \\ \bar{\theta} = \theta. \end{cases}$$

3.3. The absolute invariants

First, consider the absolute invariant of the independent variables, which is called "the similarity variable". According to a basic theorem from group theory, see [19], the new independent variable, $\eta(x, y)$, is an absolute invariant of a one-parametric group if, and only if, $\eta(x, y)$ satisfies the following first-order differential equation:

$$(3.13)\qquad (\alpha_1 x + \alpha_2)\frac{\partial \eta}{\partial x} + \alpha_3 y \frac{\partial \eta}{\partial y} = 0,$$

where

$$\alpha_1 = \frac{\partial C^x}{\partial a}(a^0).$$

$$\alpha_2 = \frac{\partial K^x}{\partial a}(a^0),$$

$$\alpha_3 = \frac{\partial C^y}{\partial a}(a^0),$$

and a^0 is the identity element of the group.

The standard techniques for linear partial differential equation indicate that two possible, general classes of solutions may be obtained for Eq. (3.13). Accordingly, we have two forms of the similarity variable η leading to two cases of similarity representation.

CASE 1. Corresponds to $\alpha_1 \neq 0$

The solution of Eq. (3.13) in this case gives

$$\eta = f\big(y(Ax+B)^m\big), \tag{3.14}$$

or, simply taking f to be the identity function, we have

$$\eta = y\pi_1(x), \tag{3.15}$$

where

$$\pi_1(x) = (Ax+B)^m, \tag{3.16}$$

and the constants A, B, and m are given by

$$A = \alpha_1, \quad B = \alpha_2, \quad m = -\alpha_3/a_1. \tag{3.17}$$

The constants A and B may be chosen arbitrarily.

CASE 2. Corresponds to $\alpha_1 = 0$

The solution of Eq. (3.13) in this case gives

$$\eta = f(Kye^{rx}). \tag{3.18}$$

Again, taking f to be the identity function, we get

$$\eta = y\pi_2(x), \tag{3.19}$$

where

$$\pi_2(x) = Ke^{rx}, \tag{3.20}$$

and $r = -\alpha_3/\alpha_2$, K is a positive constant.

3.4. The complete set of absolute invariants

The importance of the absolute invariants lies in the fact that they become the similarity variables, i.e., the variables of the similarity representations. Besides, the absolute invariant η of the independent variables, the complete set of absolute invariants of the group includes also three independent $g: g_1, g_2$ and g_3 corresponding to the three dependent variables $\theta(x, y)$, $\psi(x, y)$ and $T_w(x)$.

The procedure to be followed in deriving g is similar to that used in obtaining η.

Since, from the group (3.12), θ is itself an absolute invariant, then we have

$$g_1(x, y; \theta) = \theta(\eta). \tag{3.22}$$

The following form for the invariants g_2 and g_3 in terms of the x, ψ, and T_w variables can be established:

(3.23) $$g_2(x, \psi) = \phi_1(\psi/\Gamma(x)) = F(\eta),$$

and

(3.24) $$g_3(x, T_w) = \phi_2(T_w/\omega(x)) = T(\eta).$$

Without loss of generality, the ϕ in Eqs. (3.23) and (3.24) can be selected to be the identity functions.

Then Eqs. (3.23) and (3.24) reduce to

(3.25) $$\psi(x, y) = \Gamma(x)F(\eta),$$

(3.26) $$T_w(x) = \omega(x)T(\eta).$$

Since $\omega(x)$ and $T_w(x)$ are independent of y whereas η depends on it, then T must be equal to a constant, and

(3.27) $$T_w(x) = T_0\omega(x).$$

The functions $\Gamma(x)$ in Eq. (3.25) and $\omega(x)$ in Eq. (3.27) are to be determined to get a similarity representation.

Finally, to obtain the similarity representation, let us reduce of the system of equations (2.12) and (2.13) to a system of ordinary differential equations. This can be achieved as follows:

Substitution for θ, ψ and T_w and their partial derivatives from Eqs. (3.25) and (3.27) into Eq. (2.12) yields, after dividing by $\pi^3\Gamma$, where π and Γ are π_1 and Γ_1, respectively, for case 1, π_2 and Γ_2 for case 2, and rearranging the terms, the following equation

(3.28) $$F''' + \left(\frac{T_0\omega}{\pi_3\Gamma}\right)\theta + \left(\frac{\Gamma'}{\pi}\right)FF'' - \left(\frac{\Gamma'}{\pi} + \frac{\Gamma\pi'}{\pi^2}\right)F'^2 = 0,$$

where the primes mean differentiation of each function with respect to its own variable.

In Eq. (3.28), the first term has the constant coefficient unity. Therefore, for this equation to be reduced to an expression in the single independent variable η, it is necessary that the remaining coefficients be constant. This results from the fact that Γ, π and ω are independent of y. Thus we have

(3.29) $$\frac{\Gamma'}{\pi} = C_1,$$

(3.30) $$\frac{\Gamma\pi'}{\pi^2} = C_2,$$

(3.31) $$\frac{T_0\omega}{\Gamma\pi^3} = C_3,$$

where C_1, C_2 and C_3 are constants to be determined. Substituting Eqs. (3.29)–(3.31) into Eq. (3.28), we get

(3.32) $$F''' + C_3\theta + C_1FF'' - (C_1 + C_2)F'^2 = 0.$$

CASE 1. $\pi_1 = (Ax+B)^m$

From Eq. (3.29)

$$\Gamma(x) = \frac{C_1}{(m+1)A}(Ax+B)^{m+1}.$$

Substitution into Eq. (3.30) yields

$$C_2 = \frac{m}{m+1}C_1,$$

and from Eq. (3.31) we deduce that the function $\omega(x)$ has the form

$$\omega(x) = \frac{C_1 C_3}{(m+1)AT_0}(Ax+B)^{4m+1}. \tag{3.33}$$

Though the constants T_0 and C_3 are arbitrary, they may be equal to unity. Then $\omega(x)$ will take the form

$$\omega(x) = \frac{C_1}{(m+1)A}(Ax+B)^{4m+1}. \tag{3.34}$$

Without loss of generality, we can take

$$C_1 = 4(m+1) \quad \text{and} \quad C_2 = 4m, \tag{3.35}$$

which, when substituted into Eq. (3.32), yield

$$F'''+\theta+4(m+1)FF''-4(2m+1)F'^2 = 0. \tag{3.36}$$

Similarly, for Eq. (2.13) we get the following ordinary differential equation:

$$\frac{1}{\mathrm{Pr}}\theta''-C_4F'\theta+C_1F\theta' = 0, \tag{3.37}$$

where the constant C_4 is given by

$$C_4 = \frac{\Gamma\omega'}{\pi\omega} = 4(4m+1). \tag{3.38}$$

Then Eq. (3.37) reduces to

$$\frac{1}{\mathrm{Pr}}\theta''-4(4m+1)F'\theta+4(m+1)F\theta' = 0. \tag{3.39}$$

Now, if we put $4m+1 = n$, then the equations for $T_w(x)$, $\eta(x, y)$ and $\Gamma(x)$ take the form

$$T_w(x) = \frac{4}{A}(Ax+B)^n, \tag{3.40}$$

$$\eta(x, y) = y(Ax+B)^{(n-1)/4}, \tag{3.41}$$

$$\Gamma_1(x) = \frac{4}{A}(Ax+B)^{(n+3)/4}. \tag{3.42}$$

Now the problem of Case 1 reduces to solving the equations

$$F''' + (n+3)FF'' - 2(n+1)F'^2 + \theta = 0,$$

(3.43)
$$\frac{1}{\Pr}\theta'' - 4nF'\theta + (n+3)F\theta' = 0$$

with the boundary conditions

(3.44)
$$F(0) = F'(0) = 0, \quad \theta(0) = 1,$$
$$F'(\infty) = 0, \quad \theta(\infty) = 0.$$

The boundary layer characteristics are

(a) the vertical velocity

(3.45)
$$u = \frac{4}{A}(Ax+B)^{(n+1)/2}F'(\eta).$$

(b) The horizontal velocity

(3.46)
$$v = -(Ax+B)^{(n-1)/4}[(n+3)F + (n-1)\eta F'].$$

(c) The coefficient of heat transfer

(3.47)
$$g = -(Ax+B)^{(5n-1)/4}\theta'(0).$$

Equations (3.43) are the same as those obtained by YANG [4] and SPARROW and GREGG [6] using different methods.

CASE 2. $\pi_2 = Ke^{rx}$

Following the same procedure as that used in Case 1, Eqs. (2.12) and (2.13) become

(3.48)
$$F''' + C_3\theta + C_1FF'' - (C_1+C_2)F'^2 = 0,$$

(3.49)
$$\frac{1}{\Pr}\theta'' - C_4\theta F' + C_1F\theta' = 0,$$

which are similar to those obtained by [4] and SPARROW and GREGG [6] using different techniques. Here the quantities C_1, C_2, C_3 and C_4 are defined by Eqs. (3.29)–(3.31) and (3.38).

From Eqs. (3.29) and (3.30) we find

$$C_1 = C_2, \quad C_4 = 3C_2 + C_1, \quad \Gamma_2 = K\frac{C_1}{r}e^{rx}$$

and from Eq. (3.31) we get

(3.50)
$$\omega(x) = K^4\frac{C_2C_3}{T_0 r}e^{4rx}.$$

The possible form for T_w is

(3.51)
$$T_w = \frac{K^4}{r}e^{4rx}.$$

Without loss of generality, values of C_2, C_3 and α_3 may be assigned. With $C_3 = 1$, $\alpha_3 = -1$ and $C_2 = 1$, Eqs. (3.48) and (3.49) become

$$F''' + \theta + FF'' - 2F'^2 = 0, \qquad \frac{1}{\text{Pr}}\theta'' - 4\theta F' + F\theta' = 0 \tag{3.52}$$

with the same boundary conditions as those in Eqs. (3.44).

The boundary layer characteristics are

(a) The vertical velocity

$$u = \frac{1}{r} e^{2rx} F'(\eta). \tag{3.53}$$

(b) The horizontal velocity

$$v = -e^{4rx}(F + \eta F'). \tag{3.54}$$

(c) The coefficient of heat transfer

$$q = -\frac{1}{r} e^{5rx}\theta'(0). \tag{3.55}$$

4. Numerical results

Equations (3.43) with the boundary conditions (3.44), for Case 1, and Eqs. (3.52) with the same boundary conditions (3.44), for Case 2, describe the two-point boundary value problem. It is more convenient to reformulate the problem in terms of a set of five first-order ordinary differential equations of an initial value problem. The five equations are solved simultaneously by the fourth order Runge–Kutta scheme. Two initial conditions at $\eta = 0$, besides the three conditions given, must be guessed and iterated on to satisfy the remaining boundary conditions at $\eta = \infty$. The gradient method was applied to iterate the corrections to the two guesses. The results were obtained for $F(\eta)$, $F'(\eta)$, $\theta(\eta)$, and $\theta'(\eta)$ for $0.7 \leqslant \text{Pr} \leqslant 10.0$ and $-0.8 \leqslant n \leqslant 1.0$ (in Case 1).

The numerical results, for the two cases of study in Sect. 3, were computed at the University of Alexandria, Faculty of Engineering, Computer model PDP 11/70, and the results are presented and discussed in the following section.

5. Discussions and comments

5.1. Surface-temperature varying with position for $T_w = \frac{4}{A}(Ax+B)^n$

Profiles of dimensionless temperature $\theta(\eta)$ in the boundary layer for different values of the Prandtl number Pr and $n = 1$ (case of linearly increasing surface temperature), are shown in Fig. 2.

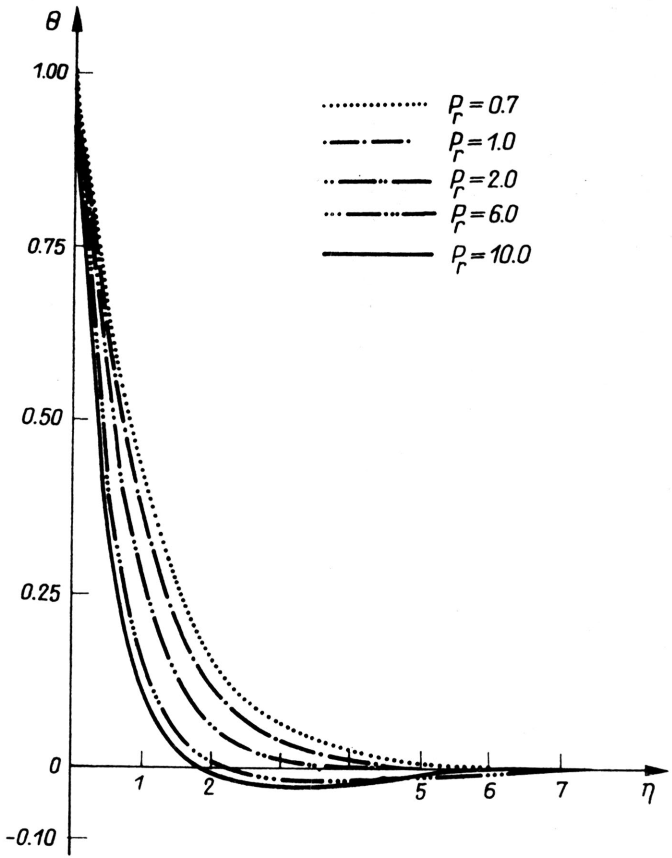

FIG. 2. Calculated dimensionless temperature profiles in the laminar boundary layer on a hot vertical flat plate in free-convection for several values of Pr and $T_w = \frac{4}{A}(Ax+B)$.

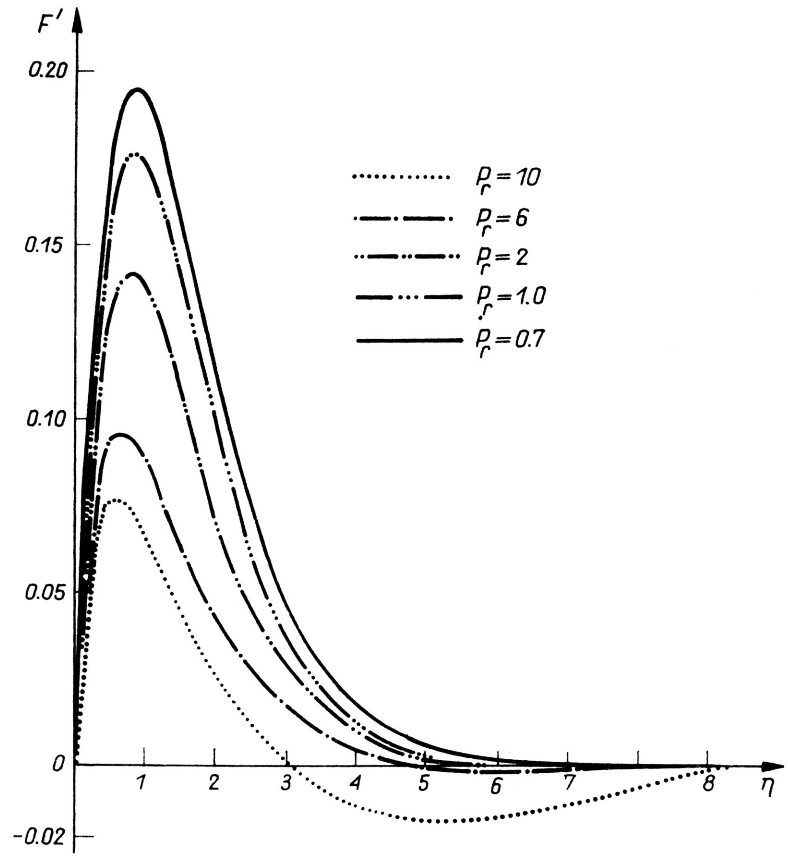

FIG. 3. Calculated dimensionless velocity profiles in the laminar boundary layer on a hot vertical flat plate in free convection for several values of Pr and

$$T_w = \frac{4}{A}(Ax+B)^n.$$

As was expected in the free convection situation, the thickness of the thermal boundary layer, δ_T, decreases as Pr increases. Moreover, it is observed that θ becomes negative in the outer part of the boundary layer. This represents a temperature defect which is clear for Pr = 10.0.

Figure 3 shows the variation of the vertical velocity in the dimensionless form.

Also a flow reversal takes place in the outer part of the velocity boundary layer. At low Prandtl numbers there is a small reversal of flow, while for high Prandtl numbers, the flow reversal is much stronger.

The physical phenomenon of temperature defect and reversal flow in the outer part of the thermal and velocity boundary layers, respectively, occurs when the surface tempera-

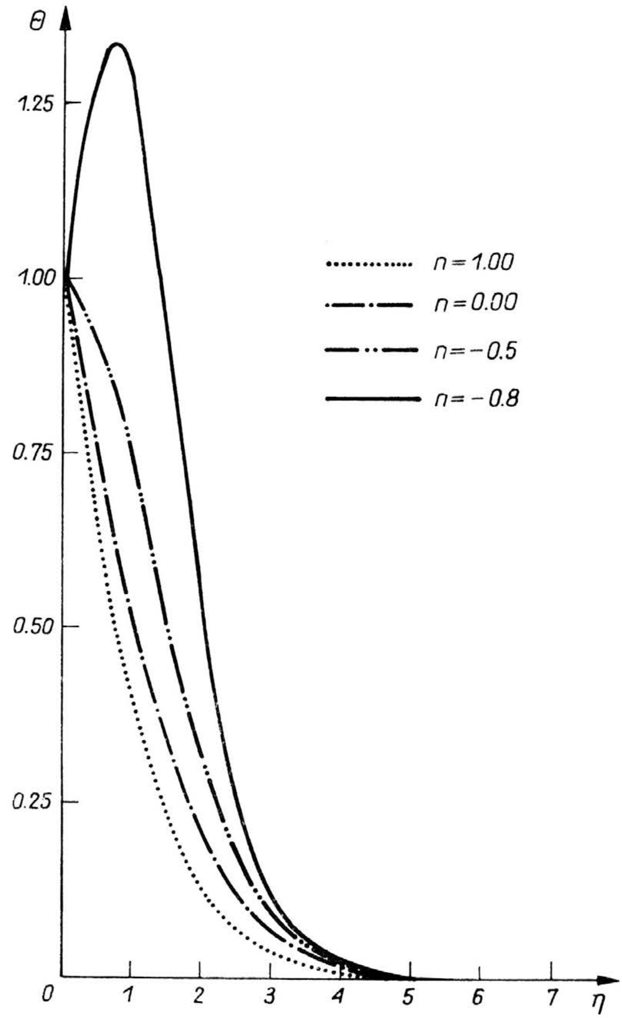

FIG. 4. Calculated dimensionless temperature profiles in the laminar boundary layer on a hot vertical flat plate in free-convection for several values of n, Pr = 0.7, $T_w = \frac{4}{A}(Ax+B)^n$.

ture increases in the streamwise direction (as x increases in the present case). This phenomenon is more pronounced for higher Prandtl numbers.

Profiles of dimensionless temperature $\theta(\eta)$ for different values of n and fixed value of Pr $= 0.7$ are illustrated in Fig. 4 and are identical with the results obtained by SPARROW and GREGG [6].

It is clear that the temperature distribution for $n < 0$ differs notably from that for $n \geqslant 0$. The shape for $n = -0.8$ displays a hill. The shapes of the various velocity profiles in Fig. 5 do not exhibit great differences such as those noted in the temperature profiles of Fig. 4 and are identical with the results obtained by SPARROW and GREGG [6] by a different technique.

From the relation (3.47), the coefficient of heat transfer, q, follows the value of $-\theta'(0)$.

To illustrate the dependence of the coefficient of heat transfer upon the power n

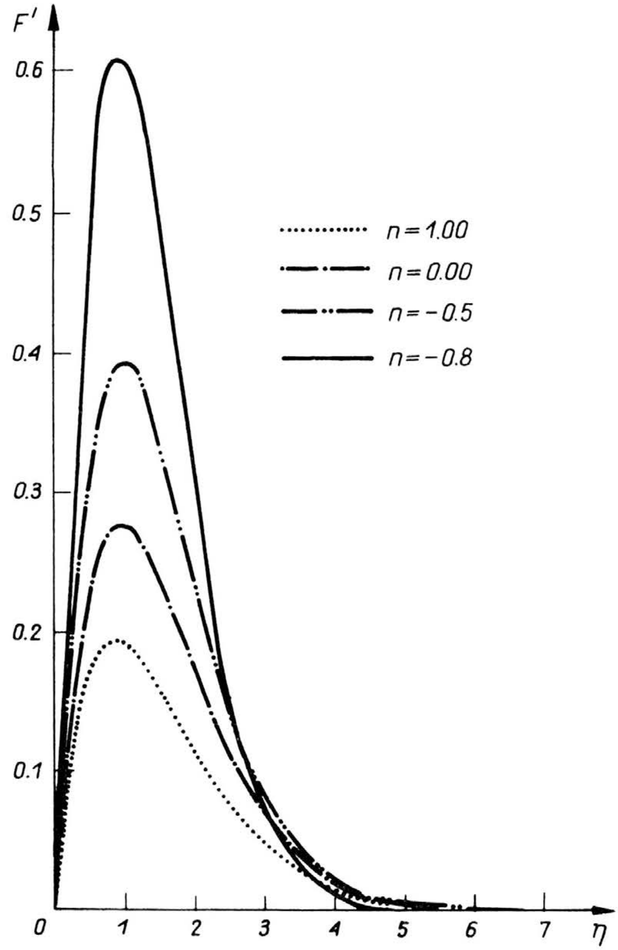

FIG. 5. Calculated dimensionless velocity profiles in the laminar boundary layer on a hot vertical flat plate in free-convection for several values of n, Pr $= 0.7$, $T_w = \frac{4}{A}(Ax+B)^n$.

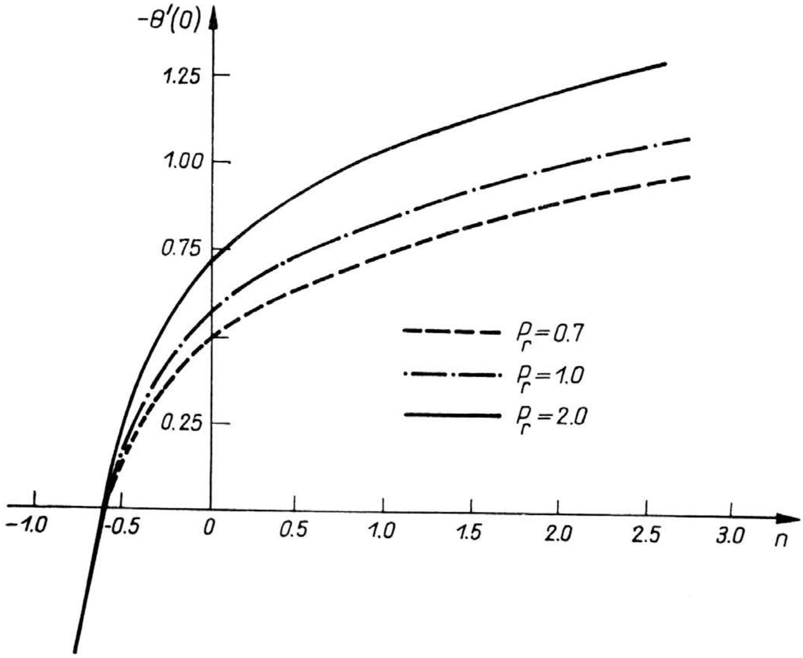

Fig. 6. Plot of the Nusselt–Grashof relation $\frac{Nu_x}{\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}}} = -\theta'(0)$ as a function of n for Pr = 0.7, 1.0 and 2.0, where $T_w = \frac{4}{A}(Ax+B)^n$.

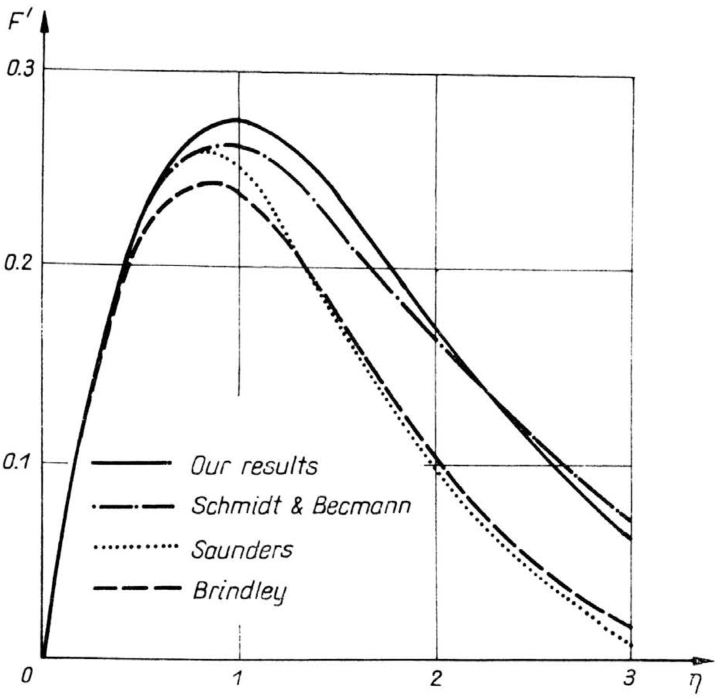

Fig. 7. Comparison of the velocity profile obtained by our calculations with those obtained by earlier workers, for Pr = 0.733 and T_w = constant.

of the surface temperature distribution T_w, the relation between $-\theta'(0)$ and n is plotted in Fig. 6 for Pr = 0.7, 1.0 and 2.0 and is in a good agreement with the results of SPARROW and GREGG [6].

It is clear that there is an increase in the coefficient of heat transfer, q, with increasing "n". The negative value of $-\theta'(0)$, which appears for values of $n < -0.6$, corresponds physically to a heat transfer from the fluid to the plate, even though $T_w > T_\infty$, which has been observed by SPARROW and GREGG [6].

Figure 7 illustrates a direct comparison between the results obtained for Pr = 0.733, $n = 0$ and those obtained by SCHMITD and BECKMANN [1], SAUNDERS [30] and BRINDLEY [7]. There is a good agreement with the results of SCHMIDT and BECKMANN [1].

5.2. Surface temperature varying with position for $T_w = \frac{K^4}{r} e^{4rx}$ and $r > 0$

Profiles of dimensionless temperature $\theta(\eta)$ in the boundary layer for different values of the Prandtl number $0.7 \leqslant \mathrm{Pr} \leqslant 10.0$ are shown in Fig. 8.

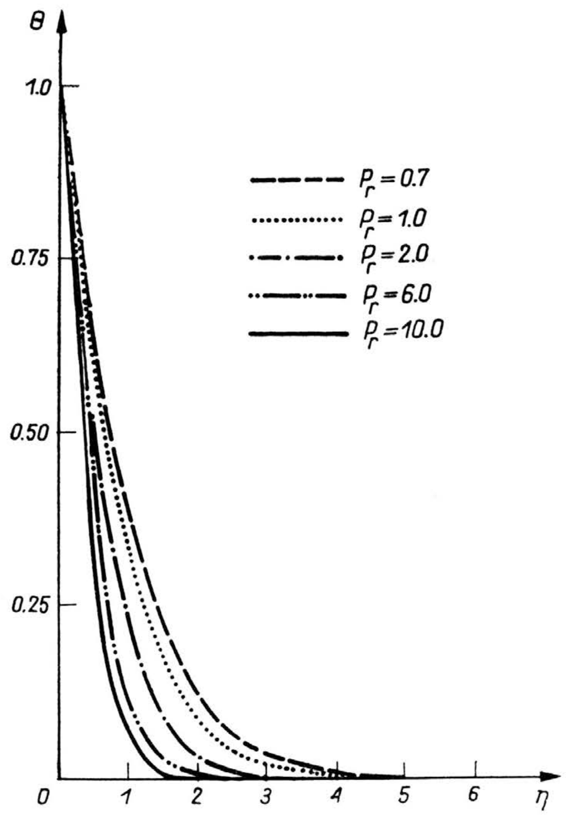

FIG. 8. Calculated dimensionless temperature profiles in the laminar boundary layer on a hot vertical flat plate in free convection for varying values of Pr and $T_w = \frac{K^4}{r} e^{4rx}$.

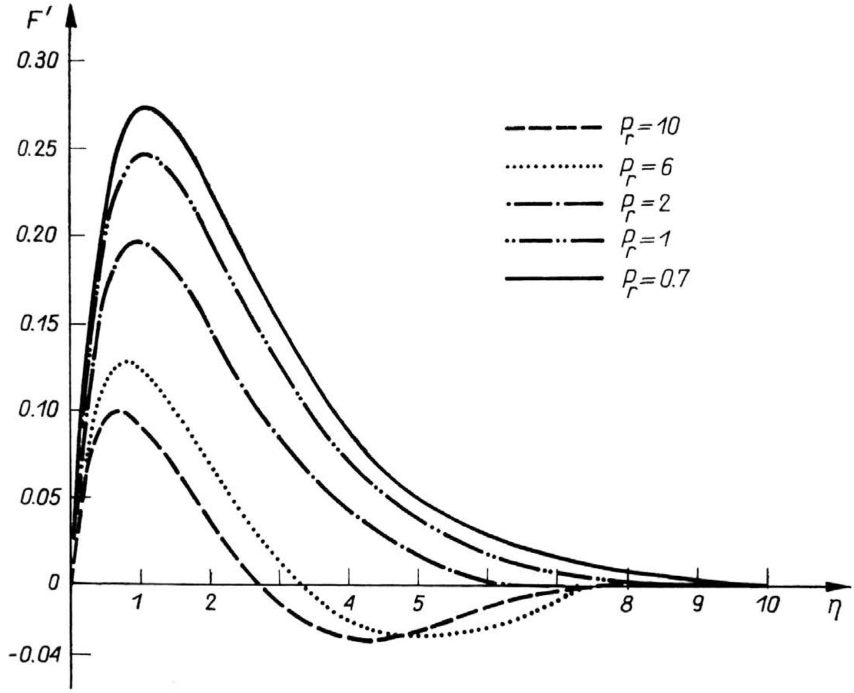

FIG. 9. Calculated dimensionless velocity profiles in the laminar boundary layer on a hot vertical flat plate in free-convection for various values of Pr and $T_w = \dfrac{K^4}{r} e^{4rx}$.

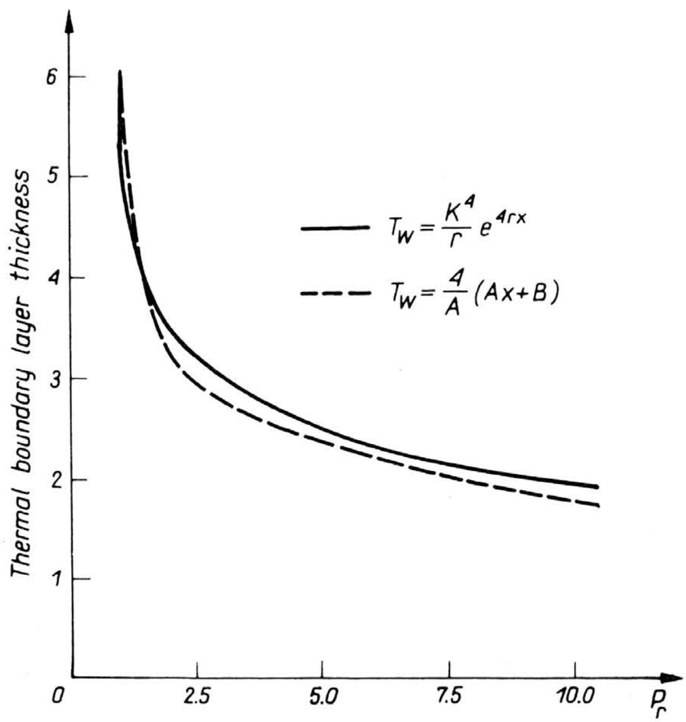

FIG. 10. Effect of the Prandtl number Pr on the thermal boundary layer thickness for two cases of surface temperature distribution.

Figure 9 shows the variation of the vertical velocity in the dimensionless form and is in good agreement with the results obtained by SPARROW and GREGG [6].

The dimensionless temperature and velocity profiles show the phenomenon of temperature defect and flow reversal in the outer part of the thermal and velocity boundary layer, respectively. The phenomenon behaves in a similar manner as in the Case 5.1.

The effect of Prandtl number, Pr, on the thermal boundary layer thickness, δ_T, is shown for the two cases in Fig. 10.

It is clear that the thickness decreases monotonically with the increase of the Prandtl number, Pr, in both cases.

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MATHEMATICS UNIT, DEPARTMENT OF SCIENCES,
THE AMERICAN UNIVERSITY IN CAIRO, CAIRO, EGYPT.

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