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On the motion of a drop of a viscous incompressible fluid in an ideal incompressible fluid

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THE EXISTENCE and uniqueness of local in time solutions of a motion of a drop of a viscous incompressible fluid (described by the Navier-Stokes equation) bounded by a free surface in an ideal incompressible fluid (described by the Euler equations) is shown. The existence is proved in such Sobolev spaces that the equations are satisfied classically. To prove the result there is assumed that density and viscosity of the viscous fluid are much larger than the density of the ideal fluid.

Pokazano istnienie i jednoznaczność lokalnych w czasie rozwiązań ruchu kropli cieczy lepkiej nieściśliwej (opisanej równaniami Naviera-Stokesa) ograniczonej swobodną powierzchnią w cieczy idealnej nieściśliwej (opisanej równaniami Eulera). Istnienie zostało udowodnione w takich przestrzeniach Soboleva, że powyższe równania są spełnione klasycznie. Aby udowodnić powyższy rezultat przyjęto, że gęstość i lepkość cieczy lepkiej są dużo większe niż gęstość cieczy idealnej.

Показаны существование и единственность локальных во времени решений движения капли вязкой несжимаемой жидкости (описанной уравнениями Навье-Стокса) ограниченной свободной поверхностью в идеальной несжимаемой жидкости (описанной уравнениями Эйлера). Существование доказано в таких пространствах Соболева, что вышеупомянутые уравнения удовлетворены классически. Чтобы доказать этот результат принято, что плотность и вязкость вязкой жидкости много больше, чем плотность идеальной жидкости.

1. Introduction

WE CONSIDER the motion of a drop of a viscous incompressible fluid in an ideal incompressible fluid. We assume that the surface of the drop is a free surface. It is the intersurface between these two fluids which is built up of the same moving particles of simultaneously viscous and ideal fluids.

A motion of the viscous incompressible fluid in the drop is described by (see [4])

$$
\begin{aligned}
&\varrho_1(v_{1t}+v_1\cdot\nabla v_1)+\nabla p_1-\nu\nabla v_1=\varrho_1 f_1 &&\text{in}\quad \tilde{\Omega}_1^T,\\
&\operatorname{div} v_1=0 &&\text{in}\quad \tilde{\Omega}_1^T,\\
&v_1|_{t=0}=v_{10} &&\text{in}\quad \Omega_1,
\end{aligned}
\tag{1.1}
$$

where $\tilde{\Omega}_1^T=\bigcup_{t\in(0,T)}\Omega_{1t}\times\{t\}$, Ω_{1t} is a domain of the drop in a moment t and $\Omega_1=\Omega_{10}$ is its initial domain. Moreover, $v_1=v_1(x,t)$ is the velocity of the fluid in the drop, $p_1=p_1(x,t)$ the pressure, $f_1=f_1(x,t)$ the external force field per unit mass, ν the viscosity coefficient, $\varrho_1=\text{const}$ the density.

We consider a motion of an ideal fluid in the exterior domain $\Omega_{2t} = \mathrm{R}^3 \setminus (\Omega_{1t} \cup \partial\Omega_{1t})$. To omit technical difficulties (connected with the behaviour of solutions to the Euler equations at infinity) we assume that the viscous and ideal fluids fill up a bounded domain $\Omega \subset \mathrm{R}^3$ with a boundary Γ ($\Omega_{1t} \bigcup \Omega_{2t} \subset \Omega$). Knowing that only local existence theorems for the Euler equations are known, we can assure that the drop does not meet Γ by assuming that the domain Ω is sufficiently large and the drop is sufficiently far from Γ at the initial moment. Therefore a motion of the ideal fluid in Ω_{2t} which is exterior to Ω_{1t} at time t is described by (see [4])

$$
\begin{aligned}
&\varrho_2(v_{2t} + v_2 \cdot \nabla v_2) + \nabla p_2 = \varrho_2 f_2 && \text{in} \quad \tilde{\Omega}_2^T, \\
&\operatorname{div} v_2 = 0 && \text{in} \quad \tilde{\Omega}_2^T, \\
&v_2|_{t=0} && \text{in} \quad \Omega_2, \\
&v_2 \cdot \bar{n}|_\Gamma = 0 && \text{on} \quad \Gamma^T \equiv \Gamma \times (0, T),
\end{aligned} \tag{1.2}
$$

where $\tilde{\Omega}_2^T = \bigcup_{t \in (0,T)} \Omega_{2t} \times \{t\}$, $\Omega_2 = \Omega_{20}$ is the domain in the initial moment, $v_2 = v_2(x, t)$ is the velocity of the ideal fluid, $p_2 = p_2(x, t)$ the pressure, $f_2 = f_2(x, t)$ the external force field per unit mass, $\bar{n}|_\Gamma$ is the unit outward vector normal to Γ, $\varrho_2 = \text{const}$ the density.

Finally the following boundary conditions are imposed on the free intersurface

$$
[T_1(v_1, p_1) - T_2(v_2, p_2)]\bar{n} = 0 \quad \text{on} \quad \tilde{S}^T, \tag{1.3}
$$

where $\tilde{S}^T = \bigcup_{t \in (0,T)} S_t \times \{t\}$, $S_t = \partial\Omega_{1t}$ is the boundary of the drop in the moment t, $\bar{n}|_{S_t}$ is the unit outward to Ω_{1t} and normal to the S_t vectors, T_1 and T_2 are the stress tensors of viscous and ideal fluids, respectively, such that

$$
T_1(v_1, p_1) = -p_1 I + 2\nu D(v_1), \tag{1.4}
$$

$$
T_2(v_2, p_2) = -p_2 I, \tag{1.5}
$$

where I is the unit matrix,

$$
D_{ij}(v) = \frac{1}{2}\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\right), \quad i, j = 1, 2, 3,
$$

is the deformation tensor.

Let S_t be given by the equation $\varphi(x, t) = 0$. Then $\bar{n}|_{S_t} = \dfrac{\nabla\varphi}{|\nabla\varphi|}$ and the following kinematic condition holds:

$$
v_1 \cdot \bar{n} = v_2 \cdot \bar{n} = -\frac{\varphi_t}{|\nabla\varphi|} \quad \text{on} \quad S_t. \tag{1.6}
$$

Therefore the surface S_t can be constructed either by curves determined by

$$
\frac{dx}{dt} = v_1(x, t), \quad x|_{t=0} = x_0 \in S = S_0 \tag{1.7}
$$

or by

$$(1.8)\qquad \frac{dx}{dt} = v_2(x, t), \quad x|_{t=0} = x_0 \in S,$$

what means that the boundary of the drop is built up of the two kinds of fluid particles.

The initial velocities v_{i0}, $i = 1, 2$, have to satisfy the compatibility conditions

$$(1.9)\qquad \operatorname{div} v_{i0} = 0, \quad i = 1, 2.$$

To prove the existence of solutions of the above problems we introduce the Lagrangian coordinates $\xi = (\xi^1, \xi^2, \xi^3)$ by the relation (see [2])

$$(1.10)\qquad x = \xi + \int_0^t u_1(\xi, z)dz \equiv X_{u_1}(\xi, t), \quad \xi \in \Omega_1,$$

where $x = (x^1, x^2, x^3)$ and $u_1(\xi, t) = v_1(X_{u_1}(\xi, t), t)$. The transformation describes a relation between Eulerian and Lagrangian coordinates. Assume that Ω_1 is given, then $\Omega_{1t} = \{x : x = x(\xi, t),\ \xi \in \Omega_1\}$, where $x(\xi, t) \equiv X_{u_1}(\xi, t)$ is a solution of the following Cauchy problem:

$$(1.11)\qquad \frac{\partial x}{\partial t} = v_1(x, t), \quad x|_{t=0} = \xi \in \Omega_1.$$

Moreover, for $\xi \in S$ we get that $S_t = \{x : x = x(\xi, t),\ \xi \in S\}$.

Using Eq. (1.10) and the notations introduced by Solonnikov in [6, 7, 8], we write the problem (1.1) with the boundary conditions (1.3) in the form

$$(1.12)\qquad \begin{aligned} &\varrho_1 \frac{\partial u_1}{\partial t} - \nu\nabla^2_{u_1} u_1 + \nabla_{u_1} q_1 = \varrho_1 g_1 && \text{in} \quad \Omega_1^T \equiv \Omega_1 \times (0, T), \\ &\nabla_{u_1} \cdot u_1 = 0 && \text{in} \quad \Omega_1^T, \\ &u_1|_{t=0} = v_{10} && \text{in} \quad \Omega_1 \\ &[T_{1u_1}(u_1, q_1)\bar{n}(\xi, t)]|_{\xi \in S} = T_2(q_2)\bar{n}(\xi, t)|_{\xi \in S} && \text{on} \quad S^T, \end{aligned}$$

where

$$\Omega_1^T = \Omega_1 x(0, T), \quad q_1(\xi, t) = p_1(X_{u_1}(\xi, t), t), \quad q_2 = p_2(X_{u_1}(\xi, t), t),$$
$$g_1(\xi, t) = f_1(X_{u_1}(\xi, t), t),$$
$$\nabla_{u_1} = A(\xi, t) \cdot \nabla \equiv \frac{\partial \xi^i}{\partial x} \nabla_{\xi^i},$$

$T_{1u}(u, q)$ is a matrix with elements

$$-q\delta_{ij} + \nu\left(\frac{\partial \xi^m}{\partial x^j}\frac{\partial u^i}{\partial \xi^m} + \frac{\partial \xi^m}{\partial x^i}\frac{\partial u^j}{\partial \xi^m}\right)$$

and the summation over repeated indices is assumed. Since the Jacobi matrix of the transformation $\xi \to X_{u_1}(\xi, t) = x$ is equal to one, elements of matrix A are minors of the matrix

$$\left\{\frac{\partial x^i}{\partial \xi^j}\right\} = \left\{\delta^i_j + \int\limits_0^t \frac{\partial u^i_1(\xi, z)}{\partial \xi^j}\, dz\right\}.$$

Moreover, $\bar{n}(\xi, t) = \bar{n}(X_{u_1}(\xi, t)) \equiv \bar{n}_{u_1}(\xi, t)$.

Therefore, for a given q_2 the problem (1.12) determines v_1, p_1 and Ω_{1t}. Then we can treat the problem (1.2) in a known domain Ω_{2t}. To consider the problem (1.2), we replace it by the following system of problems:

$$\text{(1.13)}\qquad \begin{aligned} v_{2t}+v_2\cdot\nabla v_2 &= -\frac{1}{\varrho_2}\nabla p_2+f_2 \quad \text{in} \quad \tilde{\Omega}^T_2,\\ v_2|_{t=0} &= v_{20} \quad \text{in} \quad \Omega_2, \end{aligned}$$

where p_2 is treated as a given function and

$$\text{(1.14)}\qquad \begin{aligned} \frac{1}{\varrho_2}\Delta p_2 &= -\nabla v_2\cdot\nabla v_2+\operatorname{div} f_2 && \text{in} \quad \Omega_{2t},\\ \frac{1}{\varrho_2}\frac{\partial p_2}{\partial n}\bigg|_{S_t} &= -(v_{2t}+v_2\cdot\nabla v_2)\cdot\bar{n}+f_2\cdot\bar{n} && \text{on} \quad S_t,\\ \frac{1}{\varrho_2}\frac{\partial p_2}{\partial n}\bigg|_{\Gamma} &= +v_2\cdot v_2\cdot\nabla\bar{n}+f_2\cdot\bar{n} && \text{on} \quad \Gamma, \end{aligned}$$

where v_2 is treated as a given function.

To solve Eq. (1.13), we introduce the characteristics

$$\text{(1.15)}\qquad \frac{dy(x, t; s)}{ds} = v_2\big(y(x, t; s), s\big), \quad y(x, t; t) = x,$$

so we write Eq. (1.13) in the form

$$\text{(1.16)}\qquad \begin{aligned} \frac{dv_2\big(y(x, t; s), s\big)}{ds} &= -\frac{1}{\varrho_2}\nabla_y p_2\big(y(x, t; s), s\big)+f_2\big(y(x, t; s), s\big),\\ v_2\big(y(x, t, 0), 0\big) &= v_{20}\big(y(x, t; 0)\big). \end{aligned}$$

Integrating Eq. (1.16) along the characteristics, we obtain

$$\text{(1.17)}\qquad v_2(x, t) = v_{20}\big(y(x, t; 0)\big)+\int\limits_0^t\left[-\frac{1}{\varrho_2}\nabla_y p_2\big(y(x, t; s), s\big)+f_2\big(y(x, t; s), s\big)\right]ds.$$

To solve the problem (1.14) with v_2 determined by Eq. (1.17) and in the correspondence with the problem (1.12), we extend the vector v_1 suitably (we need some reqularity which will be formulated later) on Ω_2. Let $\tilde{v}_1$ be the suitable extension (see (4.3) and the Hestense method [1], Ch. 9) such that

$$\text{(1.18)}\qquad \tilde{v}_1\cdot\bar{n}|_\Gamma = 0.$$

To apply the transformation (1.10) with the vector $\tilde{v}_1$, we introduce

$$\tilde{u}_1(\xi, t) = \tilde{v}_1\big(X_{\tilde{u}_1}(\xi, t), t\big) = \tilde{v}_1\big(x(\xi, t), t\big), \quad \text{where} \quad \xi\in\Omega_2,$$

so $\Omega_{2.} = \{x : x = X_{\tilde{u}_1}(\xi, t),\ \xi \in \Omega_2\}$. Therefore we have the relation

(1.19) $$x = \xi + \int_0^t \tilde{u}_1(\xi, s)\,ds \equiv X_{\tilde{u}_1}(\xi, t), \qquad \xi \in \Omega_2.$$

Using Eq. (1.19) in Eq. (1.17), we obtain

(1.20) $$u_2(\xi, t) \equiv v_2\big(X_{\tilde{u}_1}(\xi, t), t\big) = v_{20}\Big(y\big(X_{\tilde{u}_1}(\xi, t), t; 0\big)\Big) + \int_0^t \Big[-\frac{1}{\varrho_2} \nabla_y p_2\Big(y\big(X_{\tilde{u}_1}(\xi, t), t; s\big), s\Big) + f_2\Big(y\big(X_{\tilde{u}_1}(\xi, t), t; s\big), s\Big)\Big] ds.$$

Applying Eq. (1.19) to Eqs. (1.14) gives

(1.21) $$\begin{aligned} &\frac{1}{\varrho_2} \Delta_{\tilde{u}_1} q_2 = -\nabla_{\tilde{u}_1} u_2 \nabla_{\tilde{u}_1} u_2 + \nabla_{\tilde{u}_1} \cdot g_2, && \xi \in \Omega_2, \\ &\frac{1}{\varrho_2} \bar{n}_{\tilde{u}_1} \cdot \nabla_{\tilde{u}_1} q_2 = -(u_{2,t} + u_2 \cdot \nabla_{\tilde{u}_1} u_2) \cdot \bar{n}_{\tilde{u}_1} + g_2 \cdot \bar{n}_{\tilde{u}_1}, && \xi \in S, \\ &\frac{1}{\varrho_2} \bar{n}_{\tilde{u}_1} \cdot \nabla_{\tilde{u}_1} q_2 = u_2 \cdot u_2 \cdot \nabla_{\tilde{u}_1} \bar{n}_{\tilde{u}_1} + g_2 \cdot \bar{n}_{\tilde{u}_1}, && \xi \in \Gamma, \end{aligned}$$

where

$$q_2(\xi, t) = p_2\big(X_{\tilde{u}_1}(\xi, t), t\big), \qquad \xi \in \Omega_2, \quad \text{and} \quad \Delta_{\tilde{u}_1} = \nabla^2_{\tilde{u}_1}.$$

Finally we introduce some notations. Let Ω be an arbitrary domain in $\mathbb{R}^n$. By $\|\ \|_{l,p,\Omega}$, $l > 0$, $p \geq 1$, we denote the usual norm in Sobolev–Slobodetski space $W_p^l(\Omega)$ and by $|\ |_{p,Q}$ the usual norm in $L_p(Q)$, where Q is equal either to Ω or $\Omega^T = \Omega \times (0, T)$. In the anisotropic Sobolev–Slobodetski space $W^{l,l/2}(\Omega^T)$, $l > 0$, $p \geq 1$, we denote the norm by $\|\cdot\|_{l,p,\Omega^T}$. Moreover we introduce

$$\|\ \|_{L_q(0,T;W_p^l((\Omega)} = \|\ \|_{l,p,q,\Omega^T}$$

and

$$\|\ \|_{L_p(0,T;W_p^4(\Omega)) \cap L_p^1(0,T;W_p^2(\Omega))} = |||\ |||_{a,\Omega^T},$$

where

$$\|u\|_{L_p^1(0,T;W_p^2(\Omega))} = \|D_t u\|_{L_p(0,T;W_p^2(\Omega))},$$

and also

$$\|\ \|_{L_p(0,T;W_p^3(\Omega)) \cap L_p^1(0,T;W_p^1(\Omega))} = |||\ |||_{b,\Omega^T}.$$

Finally we assume

$$\|\ \|_{W_{p,q}^{l,k}(\Omega^T)} = \|\ \|_{l,k,p,q,\Omega^T}.$$

The aim of this paper is to prove the local existence of solutions to the problems (1.1), (1.2), (1.3). The importance of this problem consists in the fact that a free boundary problem to the Euler equations is considered. The author does not know any result about a free boundary problem to the Euler equations. However, the problem is not strictly a free boundary problem for an ideal fluid; the free surface is an intersurface between a viscous and an ideal fluid. The occurrence of the viscous fluid is essential in the proof of the existence. But to prove the existence of the solutions, we have to assume that density of the viscous fluid and a viscosity coefficient must be much larger than the density of the ideal fluid. Therefore the existence is shown for such a fluid in the drop which is very dense and viscous, so the drop behaves similarly to a rigid body.

To prove the existence of the solutions of the problems (1.1), (1.2), (1.3), we replace it by a system of problems (1.12), (1.20), (1.21). In Sect. 2 a result about the existence of solutions to the problem (1.12) for a given q_2 is formulated. This section is based on results of Solonnikov [9]. In Sect. 3 the existence of solutions of the problem (1.21) for given u_1 and u_2 is shown by the method of successive approximations (see Lemma 3.5). Moreover, the regularity of u_2 expressed by the problem (1.20) is shown (see Lemma 3.4). Finally in Sect. 4 the existence of solutions of the problem (1.1)–(1.3) is proved (see Theorem 4.1).

The motions of a viscous incompressible fluid bounded by a free surface were considered by SOLONNIKOV [6, 7, 8].

2. Existence of solutions of the problem (1.12)

To prove the existence of the solutions of the problem (1.12), we write it in the following form:

$$
\begin{aligned}
&\varrho_1 u_{1t} - \nu\nabla_\xi^2 u_1 + \nabla_\xi q_1 = -\nu(\nabla_\xi^2 - \nabla_{u_1}^2)u_1 + (\nabla_\xi - \nabla_{u_1})q_1 + g_1 \equiv F && \text{in} \quad \Omega_1^T,\\
&\operatorname{div}_\xi u_1 = (\operatorname{div}_\xi - \operatorname{div}_{u_1})u_1 \equiv G && \text{in} \quad \Omega_1^T,\\
(2.1)\qquad &u_1|_{t=0} = v_{10} && \text{in} \quad \Omega_1,\\
&\bar\tau_0 D_{u_1}(u_1)\bar n = 0 && \text{on} \quad S^T,\\
&\bar n_0\big(T_{1u_1}(u_1, q_1) - T_2(q_2)\big)\bar n = 0 && \text{on} \quad S^T,
\end{aligned}
$$

where $\bar n_0 = \bar n_0(\xi)$ is the unit outward vector normal to S at ξ, $\bar\tau_0 = \bar\tau_0(\xi)$ is a tangent vector to S, $\bar n(\xi, t) = \dfrac{A\bar n_0(\xi)}{|A\bar n_0(\xi)|}$, the index ξ denotes that the operators are taken in coordinates ξ. To obtain the boundary condition $(2.1)_4$, we have used

$$\Pi\big(T_{1u_1}(u_1, q_1) - T_2(q_2)\big)\bar n = 2\nu D_{u_1}(u_1)\bar n = 0,$$

because $\Pi g = g - \bar n(g\cdot \bar n)$ and Eq. (1.3) is satisfied.

Moreover, knowing that $\dfrac{\partial x^i}{\partial \xi^k} A_{ij} = \delta_{kj}$, we have that $A_{ij,\xi^j} = 0$ so $G = \operatorname{div}_\xi R$. In the above considerations we assumed that $\bar n_0 \cdot \bar n$ is separated from zero, what always holds for sufficiently small time.

To use the result from [9], we rewrite Eqs. (2.1) as follows:

$$
\begin{aligned}
&\varrho_1 u_{1t} - \nu\nabla_\xi^2 u_1 + \nabla_\xi q_1 = F, \quad \operatorname{div}_\xi u_1 = G && \text{in} \quad \Omega_1^T,\\
(2.2)\qquad &u_1|_{t=0} = v_{10} && \text{in} \quad \Omega_1,\\
&\bar\tau_0 D_\xi(u_1)\bar n_0 = \bar\tau_0\big(D_\xi(u_1)\bar n_0 - D_{u_1}(u_1)\bar n\big) \equiv H && \text{on} \quad S^T,\\
&\bar n_0 T_{1\xi}(u_1, q_1)\bar n_0 = \bar n_0\big(T_{1\xi}(u_1, q_1)\bar n_0 - T_{1u_1}(u_1, q_1)\bar n\big) + \bar n_0\cdot\bar n q_2 \equiv K && \text{on} \quad S^T.
\end{aligned}
$$

Using Theorem 3.2, from [9] (see also Theorem 2 in [8]) we have:

THEOREM 2.1. *Assume* $F \in L_p(0, T; W_p^2(\Omega_1^T)), v_{10} \in W^{4-2/p}(\Omega_1), G \in L_p(0, T; W_p^3(\Omega_1))$, $R, R_t \in L_p(0, T; W_p^2(\Omega_1))$, $D_{\xi'}^\alpha, H \in W_p^{1-1/p, 1/2-1/2p}(S^T)$, $\xi' \in S$, $|\alpha| \leqslant 2$, $K \in L_p(0, T; W_p^{3-1/p}(S))$, $S \in W_p^{4-1/p}$. *Moreover, some compatibility conditions are satisfied*

$$\operatorname{div} D_x^\alpha v_0 = 0, \quad D_{x'}^\alpha \bar{\tau}_0 D(v_{10}) \bar{n}_0|_S = 0, \quad |\alpha| \leqslant 2, \quad \text{for} \quad p > 3, \quad x' \in S. \tag{2.3}$$

Then the problem (2.2) has a unique solution such that $D_\xi^\alpha u_1 \in W_p^{2,1}(\Omega_1^T)$, $D_\xi^\alpha q_1 \in L_p(0, T; W_p^1(\Omega_1))$, $D_\xi^\alpha q_1|_S \in W_p^{1-1/p, 1/2-1/2p}(S^T)$, $|\alpha| \leqslant 2$, and

$$\sum_{|\alpha|\leqslant 2} \left(\sigma ||D_\xi^\alpha u_1||_{2, p, \Omega_1^T} + ||D_\xi^\alpha q_1||_{1, p, p, \Omega_1^T} + ||D_\xi^\alpha q_1|_S||_{1-1/p, p, S^T}\right)$$
$$\leqslant c(T)(||F||_{2, p, p, \Omega_1^T} + ||v_{10}||_{4-2/p, p, \Omega_1} + ||G||_{3, p, p, \Omega_1^T} + ||R||_{2, p, p, \Omega_1^T} + ||R_t||_{2, p, p, \Omega_1^T}$$
$$+ \sum_{|\alpha|\leqslant 2} ||D_\xi^\alpha, H||_{1-1/p, p, S^T} + ||K||_{3-1/p, p, p, S^T}), \tag{2.4}$$

where $\xi' \in S$, $\sigma = \min\{\varrho_1, \nu\}$ and $c(T)$ is a nondecreasing function.

Now, by Theorem 3 from [8] one obtains

THEOREM 2.2. *Assume* $D_x^\alpha f_1 \in L_\infty(\mathrm{R}^3 \times (0, T))$ *and are Lipschitz continuous with respect to* x, $|\alpha| \leqslant 2$, $v_0 \in W^{4-2/p}(\Omega_1)$, $q_2|_S \in L_p(0, T; W_p^{3-1/p}(S))$, $S \in W_p^{4-1/p}$ *and the compatibility conditions* (2.3) *are satisfied. Then, for* $t \leqslant T_1$, *where* T_1 *is sufficiently small, there exists a unique solution to the problem* (1.12) *such that* $D_\xi^\alpha u_1 \in W_p^{2,1}(\Omega_1^t)$, $D_\xi^\alpha q_1 \in L_p(0, t; W_p^1(\Omega_2))$, $D_\xi^\alpha q_1|_S \in W_p^{1-1/p, 1/2-1/2p}(S^t)$, $|\alpha| \leqslant 2$ *and*

$$\sum_{|\alpha|\leqslant 2} (\sigma ||D_\xi^\alpha u_1||_{2,p,\Omega_1^t} + ||D_\xi^\alpha q_1||_{1,p,p,\Omega_1^t} + ||D_\xi^\alpha q_1|_S||_{1-1/p,p,S^t})$$
$$\leqslant c(T_1)(||f_1||_{2,\infty,\infty,\mathrm{R}^3\times(0,t)} + ||v_{10}||_{4-2/p,p,\Omega_1} + ||q_2|_S||_{3-1/p,p,p,S^t}). \tag{2.5}$$

REMARK 2.3. To obtain Eq. (2.5), we have used

$$||g_1||_{2,p,p,\Omega_1^T} \leqslant cT^{1/2}||f||_{2,\infty,\infty,\mathrm{R}^3\times(0,T)}(1 + cT^{\frac{p-1}{p}}||u_1||_{2,p,p,\Omega_1^T})^2. \tag{2.6}$$

3. Existence and regularity of solutions of the problems (1.20) and (1.21)

Theorem 2.2 implies that we have to prove the existence of solutions to the problems (1.20) and (1.21) in such a class that $q_2 \in L_p(0, T; W_p^3(\Omega_2))$.

At first we need

LEMMA 3.1. Let $\tilde{u}_1 \in L_1(0, T; W_p^3(\Omega_2))$, $p > 3$ and $\xi = \xi(x, t)$ be the inverse transformation (1.19). Then one has

$$|\xi_x(x, t)| \leqslant cP_2(||\tilde{u}_1||_{2,p,1,\Omega_2^T}), \quad x \in \Omega_{2t}, \quad t \in (0, T), \tag{3.1}$$

$$|\xi_{xx}|_{p,\Omega_2} \leqslant cP_3(||\tilde{\ }_1||_{2,p,1,\Omega_2^T})||\tilde{u}_1||_{2,p,1,\Omega_2^T}, \quad t \in (0, T), \tag{3.2}$$

$$|\xi_{xxx}|_{p,\Omega_2} \leqslant P_5(||\tilde{u}_1||_{2,p,1,\Omega_2^T})||\tilde{u}_1||_{3,p,1,\Omega_2^T}, \quad t \in (0, T), \tag{3.3}$$

where $P_k(x)$ is a polynomial of degree k with respect to x which does not vanish with x.

P r o o f. Knowing that $\{\xi_x\}$ is the inverse Jacobi matrix to $\{x_\xi\}$ with a determinant equal to one, we have

$$|\xi_x| \leqslant c\left(1 + \int_0^t |\tilde{u}_{1\xi}| ds\right)^2 \leqslant c(1 + ||\tilde{u}_1||_{1,\infty,1,\Omega_2^T})^2 \leqslant cP_2(||\tilde{u}_1||_{2,p,1,\Omega_2^T}), \quad p > 3.$$

The form of the Jacobi matrix $\{\xi_x\}$ implies

$$(3.4)\qquad |\xi_{xx}| \leqslant c\Big(1+\int\limits_0^t |\tilde{u}_{1\xi}(\xi, s)|\,ds\Big)^3 \int\limits_0^t |\tilde{u}_{1\xi\xi}(\xi, s)|\,ds.$$

Hence $L_p(\Omega_2)$, $p > 3$, the norm of the relation (3.4) gives the relation (3.2). Differentiating twice, the Jacobi matrix $\{\xi_x\}$ gives

$$(3.5)\quad |\xi_{xxx}| \leqslant c\Big(1+\int\limits_0^t |\tilde{u}_{1\xi}(\xi, s)|ds\Big)^4\Big(\int\limits_0^t |\tilde{u}_{1\xi\xi}(\xi, s)|ds\Big)^2 + c\Big(1+\int\limits_0^t |\tilde{u}_{1\xi}(\xi, s)\,ds\Big)^5 \int\limits_0^t |\tilde{u}_{1\xi\xi\xi}|ds.$$

Taking $L_p(\Omega_2)$, $p > 3$ the norm of the relation (3.5) gives the relation (3.3). This ends the proof.

LEMMA 3.2. Let $\tilde{u}_1 \in L_1(0, T; W_p^2(\Omega_2))$, $p > 3$. Then for the transformation (1.19), one has

$$(3.6)\quad |x_\xi| \leqslant c(1+||\tilde{u}_1||_{2,p,1,\Omega_2^T}), \qquad \xi \in (0, T),$$

$$(3.7)\quad |x_{\xi\xi}|_{p,\Omega_2} \leqslant c||\tilde{u}_1||_{2,p,1,\Omega_2^T}, \qquad t \in (0, T),$$

$$(3.8)\quad ||x_t||_{k,p,\Omega_2} \leqslant c||\tilde{u}_1||_{k,p,\Omega_2}, \qquad t \in (0, T), \qquad k = 0, 1.$$

Proof. The relation (3.6) follows from $|x_\xi| \leqslant 1+\int\limits_0^t |\tilde{u}_{1\xi}(\xi, s)|ds \leqslant c(1+||\tilde{u}_1||_{2,p,1,\Omega_2^T})$, $p > 3$. Differentiating Eq. (1.19) twice with respect to ξ implies $|x_{\xi\xi}| \leqslant \int\limits_0^t |\tilde{u}_{1\xi\xi}(\xi, s)|\,ds$, so taking the L_p norm gives the relation (3.7). From (1.19) we have $x_t = \tilde{u}_1(\xi, t)$, hence the relation (3.8) follows easily. This concludes the proof.

Now we consider the problem

$$(3.9)\qquad \frac{dy(x, t; s)}{ds} = \omega\big(y(x, t; s), s\big), \qquad y(x, t; t) = x, \qquad x \in \Omega_{2t},$$

and $\omega \cdot \bar{n} = v_1 \cdot \bar{n}$ on S_t, $\omega \cdot \bar{n} = 0$ on Γ.

LEMMA 3.3. Let $y = y(x, t; s)$ be a solution of Eq. (3.9), $x \in \Omega_{2t}$ and $x = x(\xi, t)$, $\xi \in \Omega_2$, $t \in (0, T)$ be determined by the transformation (1.19).

Assume that $\tilde{\omega}, \tilde{u}_1 \in L_1(0, T; W_p^2(\Omega_2))$, $p > 3$, where $\tilde{\omega}(\xi, t) = \omega(x(\xi, t), t)$. Then

$$(3.10)\quad |y_x| \leqslant c\exp[P_4(||\tilde{u}_1||_{2,p,1,\Omega_2^T})||\tilde{\omega}||_{2,p,1,\Omega_2^T}],$$

$$(3.11)\quad \Big(\int\limits_{\Omega_2} |y_{xx}\big(x(\xi, t), t; s\big)|^p d\xi\Big)^{1/p} \leqslant P_4(||\tilde{u}_1||_{2,p,1,\Omega_2^T}) P_6^{1/p}(||\tilde{\omega}||_{2,p,1,\Omega_2^T}) \times \exp[P_4(||\tilde{u}_1||_{2,p,1,\Omega_2^T})||\tilde{\omega}||_{2,p,1,\Omega_2^T}],$$

$$(3.12)\quad |y_t| \leqslant ||\tilde{\omega}||_{1,p,\Omega_2} P_2(||\tilde{u}_1||_{2,p,1,\Omega_2^T}) \exp[P_4(||\tilde{u}_1||_{2,p,1,\Omega_2^T})||\tilde{\omega}||_{2,p,1,\Omega_2^T}],$$

$$(3.13)\quad \Big(\int\limits_{\Omega_2} |y_{tx}\big(x(\xi, t), t; s\big)|^p d\xi\Big)^{1/p} \leqslant \varphi_1(||\tilde{u}_1||_{2,p,1,\Omega_2^T}, ||\tilde{\omega}||_{2,p,1,\Omega_2^T}) \times [||\tilde{\omega}||_{2,p,1,\Omega_2^T} + ||\tilde{\omega}||_{1,p,\Omega_2}],$$

where φ_1 is an increasing function, P_k is determined in Lemma 3.1.

P r o o f. For solutions of Eq. (3.9) we have

$$(3.14)\qquad |y_x(x,t;s)| \leqslant c\exp\Big[\sup_{x,t}\int_0^t |\omega_y(y(x,t;s)|ds\Big] \leqslant c\exp\Big[\int_0^T \|\omega(\cdot,s)\|_{2,p,\Omega_{2s}}ds\Big],$$

because $y(x,t;s)\in\Omega_{2s}$ and $p>3$. Therefore we consider

$$(3.15)\qquad \|\omega(\cdot,s)\|_{2,p,\Omega_{2s}} = \Big(\int_{\Omega_{2s}}(|\omega_{yy}|^p+|\omega_y|^p+|\omega|^p)dy\Big)^{1/p}$$

$$\leqslant c\Big(\int_{\Omega_2}|\tilde\omega_{\xi\xi}|^p|\xi_y|^{2p}+|\tilde\omega_\xi|^p(|\xi_{yy}|^p+|\xi_y|^p)+|\tilde\omega|^p)d\xi\Big)^{1/p} \leqslant P_4(\|\tilde u_1\|_{2,p,1,\Omega_2^T})\|\tilde\omega\|_{2,p,\Omega}$$

where we used that $\Omega_{2s}\ni y=\xi+\int_0^s\tilde u_1(\xi,\tau)d\tau$, $\xi\in\Omega_2$, $\det\left\{\dfrac{\partial y}{\partial\xi}\right\}=1$ and $\tilde\omega(\xi,t)=$ $=\omega(y(\xi,t),t)$. From the relations (3.14) and (3.15) the relation (3.10) follows.

From Eq. (3.9) we have

$$\frac{d}{ds}y_{xx}(x,t;s)=\omega_{yy}\big(y(x,t;s),s\big)y_x^2(x,t;s)+\omega_y\big(y(x,t;s)y_{xx}(x,t;s)$$

so

$$|y_{xx}|\leqslant c\int_0^t|y_x(x,t;s)|^2|\omega_{yy}(y,s)|ds\exp\int_0^t|\omega_y(y,s)|ds.$$

Using the relation (3.10), we get

$$(3.16)\qquad \Big(\int_{\Omega_2}|y_{xx}\big(x(\xi,t),t;s\big)|^pd\xi\Big)^{1/p}\leqslant c\exp[P_4(\|\tilde u_1\|_{2,p,1,\Omega_2^T})\|\tilde\omega\|_{2,p,1,\Omega_2^T}]\times$$

$$\times\int_0^t ds\Big(\int_{\Omega_2}|\omega_{yy}\big(y\big(x(\xi,t),t;s\big),s\big)|^pd\xi\Big)^{1/p}.$$

The last factor is estimated by

$$(3.17)\qquad \int_0^t ds\Big(\int_{\Omega_{2s}}|\omega_{yy}\big(y\big(x(\xi,t),t;s\big),s\big)|^p|\xi_x||x_y|dy\Big)^{1/p}\leqslant\sup_t(|\xi_x||x_y|)^{1/p}$$

$$\times\int_0^t ds\Big(\int_{\Omega_{2s}}|\omega_{yy}\big(y\big(x(\xi,t),t;s\big),s\big)|^pdy\Big)^{1/p}.$$

Using the relation (3.1) and that $x_y=1+\int_0^t\omega_y(y(x,t;\tau),\tau)d\tau$ which is estimated by

$$(3.18)\qquad |x_y|\leqslant c\Big(1+\int_0^t|\omega_y(y,\tau)|d\tau\Big)\leqslant c(1+P_4(\|\tilde u_1\|_{2,p,1,\Omega_2^T})\|\tilde\omega\|_{2,p,1,\Omega_2^T})$$

(see the relations (3.14) and (3.15)), the relation (3.17) is bounded by

$$cP_6^{1/p}(\|\tilde u\|_{2,p,1,\Omega_2^T})\|\tilde\omega\|_{2,p,1,\Omega_2^T}^{1/p}\int_0^t ds\Big(\int_{\Omega_{2s}}|\omega_{yy}\big(y\big(x(\xi,t),t;s\big),s\big)|^pdy\Big)^{1/p}.$$

Now, to estimate the last integral we use the relation (3.15) so we finally get the relation (3.11).

From Eq. (3.9) we have $y_t = -\omega(x, t) + \int_0^t \omega_y y_t d\tau$, so

$$(3.19)\quad |y_t| \leqslant |\omega| \exp \int_0^t |\omega_y(y, \tau)| d\tau \leqslant |\omega| \exp \int_0^t \|\omega(\cdot, \tau)\|_{2,p,\Omega_{2\tau}} d\tau$$
$$\leqslant P_2(\|\tilde{u}_1\|_{2,p,1,\Omega_2^T}) \|\tilde{\omega}\|_{1,p,\Omega_2} \exp[P_4(\|\tilde{u}_1\|_{2,p,1,\Omega_2^T}) \|\tilde{\omega}\|_{2,p,1,\Omega_2^T}].$$

Hence the relation (3.12) is satisfied. From the expression of y_t we get the problem

$$(3.20)\quad \frac{d}{ds} y_{tx}(x, t; s) = \omega_{yy}(y, s) y_x y_t + \omega_y y_{tx}, \quad y_{tx}|_{s=t} = -\omega_x(x, t).$$

Integrating Eq. (3.20), we obtain

$$(3.21)\quad |y_{tx}(x, t; s)| \leqslant \exp \int_0^t |\omega_y(y(x, t; s), s)| ds \Big[\int_0^t |\omega_{yy}| |y_x| |y_t| ds + |\omega_x(x, t)| \Big].$$

Repeating the considerations from the proof of the relations (3.10) and (3.11), we show the relation (3.13). This concludes the proof.

LEMMA 3.4. Assume that $\tilde{\omega} \in L_1(0, T;\ W_p^2(\Omega_2))$, $q_2, \tilde{u}_1 \in L_1(0, T;\ W_p^3(\Omega_2))$, $v_{20} \in W_p^2(\Omega_2)$, $f_2 \in L_\infty(0, T;\ W_\infty^2(\mathrm{R}^3))$, $p > 3$. Then u_2 described by Eq. (1.20) belongs to $L_\infty(0, T;\ W_p^2(\Omega_2))$ and

$$(3.22)\quad \|u_2\|_{2,p,\infty,\Omega_2^T} \leqslant \Phi_1(\|\tilde{u}_1\|_{2,p,1,\Omega_2^T}, \|\tilde{\omega}\|_{2,p,1,\Omega_2^T}) [\|v_{20}\|_{2,p,\Omega_2}$$
$$+ \frac{1}{\varrho_2} \|q_2\|_{2,p,1,\Omega_2^T} \|\tilde{u}_1\|_{3,p,1,\Omega_2^T} + \frac{1}{\varrho_2} \|q_2\|_{3,p,1,\Omega_2^T} + cT \|f_2\|_{2,\infty,\infty,\mathrm{R}^3\times(0,T)}],$$

where Φ_1(a, b) is an increasing function.

Proof. From Eq. (1.20) we have

$$(3.23)\quad \|u_2\|_{2,p,\Omega_2} \leqslant \|v_{20}(y(x(\xi, t), t; 0))\|_{2,p,\Omega_2} + \Big\| \int_0^t \Big[-\frac{1}{\varrho_2} \nabla_y p_2(y(x(\xi, t), t; s), s)$$
$$+ f_2(y(x(\xi, t), t; s), s)\Big] ds \Big\|_{2,p,\Omega_2},$$

where $x(\xi, t) = X_{\tilde{u}_1}(\xi, t)$, $\xi \in \Omega_2$ and $y = y(x, t; s)$ is determined by Eq. (3.9). At first we consider the first term in the right-hand side of the relation (3.23). It is sufficient to consider the second derivatives only. Knowing that $y(x, t; 0) \in \Omega_2$ for $x \in \Omega_{2t}$, we get

$$(3.24)\quad |v_{20,\xi\xi}|_{p,\Omega_2(\xi)} = |v_{20,yy} y_x^2 x_\xi^2 + v_{20,y}(y_{xx} x_\xi^2 + y_x x_{\xi\xi})|_{p,\Omega_2(\xi)}$$
$$\leqslant \sup_{\xi\in\Omega_2} |y_x|^2 \sup_{\xi\in\Omega_2} |x_\xi|^2 |v_{20,yy}|_{p,\Omega_2(\xi)} + \sup_{\xi\in\Omega_2} |v_{20,y}| \sup_{\xi\in\Omega_2} |x_\xi|^2 |y_{xx}|_{p,\Omega_2(\xi)}$$
$$+ \sup_{\xi\in\Omega_2} |v_{20,y}| \sup_{\xi\in\Omega_2} |y_x| |x_{\xi\xi}|_{p,\Omega_2(\xi)},$$

where $L_p(\Omega_2(\eta))$ denotes the space of integrable functions with respect to the Lebesque measure $d\eta$. Using Lemmas 3.1÷3.3, one obtains

$$|v_{20,\xi\xi}|_{p,\Omega_2(\xi)} \leqslant \Phi_1'(\|\tilde{u}_1\|_{2,p,1,\Omega_2^T}, \|\tilde{\omega}\|_{2,p,1,\Omega_2^T})(|v_{20,yy}|_{p,\Omega_2(\xi)} + \|v_{20,y}\|_{1,p,\Omega_2(\xi)}). \tag{3.25}$$

Changing the variables $\Omega_2 \in \xi \to y(x(\xi, t), t; 0) \in \Omega_2$, we get $d\xi = |\xi_x||x_y|dy$ so by the relation (3.25) we obtain

$$\left\|v_{20}\big(y(x(\xi, t), t; 0)\big)\right\|_{2,p,\Omega_2(\xi)} \leqslant \Phi_1'(\|\tilde{u}_1\|_{2,p,1,\Omega_2^T}, \|\tilde{\omega}\|_{2,p,1,\Omega_2^T})\|v_{20}(y)\|_{2,p,\Omega_2(y)}, \tag{3.26}$$

where $W_p^2(\Omega_2(\eta))$ denotes the space with respect to the variables η.

Now we consider the second term in the right-hand side of Eq. (3.23). Let us consider the first term only

$$\left\|\int_0^t \nabla_y p_2\big(y(x(\xi, t), t; s), s\big)ds\right\|_{2,p,\Omega_2(\xi)}$$
$$\leqslant \int_0^t ds\Big(\int_{\Omega_2} (|\nabla_\xi^2 \nabla_y p_2(y, s)|^p + |\nabla_\xi \nabla_y p_2(y, s)|^p + |\nabla_y p_2(y, s)|^p)d\xi\Big)^{1/p},$$

where $y = y(x(\xi, t), t; s)$, $\xi \in \Omega_2$, hence

$$\leqslant \int_0^t ds\Big(\int_{\Omega_2} (|x_{\xi\xi} y_x \nabla_y^2 p_2|^p + |x_\xi^2 y_{xx} \nabla_y^2 p_2|^p + |x_\xi^2 y_x^2 \nabla_y^3 p_2|^p + |x_\xi y_x \nabla_y^2 p_2|^p + |\nabla_y p_2|^p)d\xi\Big)^{1/p}.$$

Knowing that $y = y(x, t; s) \in \Omega_{2s}$, using Lemmas 3.1÷3.3 and the relation (3.18), after changing the variables $\Omega_2 \in \xi \to y(x(\xi, t), t; s) \in \Omega_{2s}$, we obtain

$$\left\|\int_0^t \nabla_y p_2(y, s)ds\right\|_{2,p,\Omega_2(\xi)} \leqslant \Phi_3'(\|\tilde{u}_1\|_{2,p,1,\Omega_2^T}\|\tilde{\omega}\|_{2,p,1,\Omega_2^T}) \int_0^t ds\|\nabla_y p_2(y, s)\|_{2,p,\Omega_{2s}(y)}. \tag{3.27}$$

Making use of the fact that there exists a Lagrangian varable $\zeta \in \Omega_2$ such that $y = \zeta + \int_0^s \tilde{u}_1(\zeta, \tau)d\tau$, one has

$$\int_0^t ds\|\nabla_y p_2(y, s)\|_{2,p,\Omega_{2s}} \leqslant \int_0^t ds\Big(\int_{\Omega_2} (|q_{2,\zeta\zeta\zeta}\zeta_y^3 + 3q_{2,\zeta\zeta}\zeta_y\zeta_{yy} + q_{2,\zeta}\zeta_{yyy}|^p$$
$$+ |q_{2,\zeta\zeta}\zeta_y^2 + q_{2,y}\zeta_{yy}|^p + |q_{2,\zeta}\zeta_y|^p)d\zeta\Big)^{1/p},$$

where $q_2(\zeta, s) = p_2(y(\zeta, s), s)$. By Lemma 3.1 the above expression is estimated by

$$\Phi_4'(\|\tilde{u}_1\|_{2,p,1,\Omega_2^T}\|\tilde{\omega}\|_{2,p,1,\Omega_2^T})(1 + \|\tilde{u}_1\|_{3,p,1,\Omega_2^T})\|q_2\|_{3,p,1,\Omega_2^T}.$$

Similar considerations can be applied to the last term in the relation (3.23), where we have made use of the fact that

$$\|g_2\|_{2,p,1,\Omega_2^T} \leqslant cT\|f_2\|_{2,\infty,\infty,R^3\times(0,T)}\Big(1 + cT^{\frac{p-1}{p}}\|\tilde{u}_1\|_{2,p,p,\Omega_2^T}\Big)^2.$$

This concludes the proof.

Now we show the existence of solutions to the problem (1.21). To do this we consider

$$\frac{1}{\varrho_2}\Delta_\xi q_2 = \frac{1}{\varrho_2}(\Delta_\xi - \Delta_{\tilde{u}_1})q_2 - \nabla_{\tilde{u}_1}u_2\nabla_{\tilde{u}_1}u_2 + \operatorname{div} g_2 \equiv H_1 + H_2 + H_3, \quad \xi \in \Omega_2,$$

(3.28)
$$\frac{1}{\varrho_2}\bar{n}_0\cdot\nabla_\xi q_2 = \frac{1}{\varrho_2}(\bar{n}_0\cdot\nabla_\xi - \bar{n}_{\tilde{u}_1}\cdot\nabla_{\tilde{u}_1})q_2 - (u_{2,t} + u_2\cdot\nabla_{\tilde{u}_1}u_2)\bar{n}_{\tilde{u}_1} - g_2\cdot\bar{n}_{\tilde{u}_1} \equiv h_1 + h_2 + h_3, \quad \xi \in S,$$

$$\frac{1}{\varrho_2}\bar{n}_0\cdot\nabla_\xi q_2 = \frac{1}{\varrho_2}(\bar{n}_0\cdot\nabla_\xi - \bar{n}_{\tilde{u}_1}\cdot\nabla_{\tilde{u}_1})q_2 + u_2u_2\nabla_{\tilde{u}_1}\bar{n}_{\tilde{u}_1} + g_2\cdot\bar{n}_{\tilde{u}_1} \equiv k_1 + k_2 + k_3, \quad \xi \in \Gamma,$$

where

$$\bar{n}_{u_1}|_S = \bar{n}_{\tilde{u}_1}|_S = \frac{A\cdot\nabla_\xi\Psi_0}{|A\cdot\nabla_\xi\Psi_0|}, \quad \xi \in S, \quad \Psi_0(\xi) = \Psi(\xi, 0),$$

$$\Psi(\xi, t) = \varphi\big(X_{\tilde{u}_1}(\xi, t), t\big), \quad \xi \in S, \quad \bar{n}_0|_S = \frac{\nabla_\xi\Psi_0}{|\nabla_\xi\Psi_0|}, \quad \bar{n}_0|_\Gamma = \bar{n}_0(\xi)$$

is the unit outward vector normal to Γ at ξ.

Assuming that u_1 and u_2 are known, the existence of solutions to Eq. (3.28) will be proved by the method of successive approximations in $L_p(0, T; W_p^3(\Omega_2))$, $p > 3$.

LEMMA 3.5. Assume

$$\tilde{u}_1 \in L_1\big(0, T; W_p^4(\Omega_2)\big), \quad u_1 \in L_p\big(0, T; W_p^4(\Omega_1)\big) \cap L_p^1(0, T; W_p^2(\Omega_1)),$$

$$u_2 \in L_\infty\big(0, T; W_p^2(\Omega_2)\big), \quad \Psi_0 \in W_p^{3-1/p}(S), \quad \Gamma \in W_p^{1-1/p}, \quad p > 3, \quad f \in L_\infty\big(0, T; W_\infty^2(\mathrm{R}^2)\big)$$

and

(3.29)
$$\int_{\Omega_{2t}} p(x, t)\,dx = \int_{\Omega_2} q_2(\xi, t)\left|\frac{\partial x}{\partial \xi}\right| d\xi = 0.$$

Let T be sufficiently small. Then there exists a unique solution to Eq. (3.28) such that $q_2 \in L_p(0, T; W_p^3(\Omega_2))$ and

(3.30)
$$\frac{1}{\varrho_2}\|q_2\|_{3,p,p,\Omega_2^T} \leqslant \big(1 + \Phi_2(\|u_1\|_{2,p,1,\Omega_2^T}, \|\Psi_0\|_{3-1/p,p,S})\big)|||u_1|||_{a,\Omega_1^T}$$
$$+ \Phi_3(\|u_1\|_{4,p,1,\Omega_1^T}, \|\tilde{u}_1\|_{3,p,1,\Omega_2^T}, \|\Psi_0\|_{3-1/p,p,S}, \|\Gamma\|_{3-1/p,p}, T)\,[T^\alpha(|||u_1|||^2_{a,\Omega_1^T}$$
$$+ \|u_2\|^2_{2,p,\infty,\Omega_1^T}) + T\|f_2\|_{2,\infty,\infty,\mathrm{R}^3\times(0,T)}],$$

where $\alpha > 0$.

Proof. At first we estimate the right-hand sides of Eq. (3.28) (see [3, 1, 5])

$$\|H_1\|_{1,p,p,\Omega_2^T} = \frac{1}{\varrho_2}\|(\delta_{ij} - \xi_{x^i}^l\xi_{x^j}^k)q_{2,\xi^l\xi^k} + \xi_x\partial_\xi(\xi_x)q_{2,\xi}\|_{1,p,p,\Omega_2^T}$$

$$\leqslant \frac{c}{\varrho_2}\left\|\left(\int_0^t \tilde{u}_{1,\xi}\,ds\,q_{2,\xi\xi} + \int_0^t \tilde{u}_{1,\xi\xi}\,ds\,q_{2,\xi}\right)\left(1 + \int_0^t \tilde{u}_{1,\xi}\,ds\right)^3\right\|_{1,p,p,\Omega_2^T}$$

$$\leqslant \frac{c}{\varrho_2}\|\tilde{u}_1\|_{3,p,1,\Omega_2^T}(1 + \|\tilde{u}_1\|_{3,p,1,\Omega_2^T})^3\|q_2\|_{3,p,p,\Omega_2^T},$$

$$||H_2||_{1,p,p,\Omega_2^T} \leqslant P_4(||\tilde{u}_1||_{2,p,1,\Omega_2^T})||u_2||^2_{2,p,2p,\Omega_2^T} \leqslant P_4 T^{1/p}||u_2||^2_{2,p,\infty,\Omega_2^T},$$

$$||h_1||_{2-1/p,p,p,S^T} \leqslant \frac{1}{\varrho_2}\Phi_1'(||\tilde{u}_1||_{3,p,1,\Omega_2^T}||\Psi_0||_{3-1/p,p,S})||\tilde{u}_1||_{3,p,1,\Omega_2^T}||\nabla q_2||_{2,p,p,\Omega_2^T}.$$

Now we consider

$$(3.31)\qquad ||h_2||_{2-1/p,p,p,S^T} = ||(u_1\cdot \bar{n}_{\tilde{u}_1})_{,t} - u_2(\bar{n}_{\tilde{u}_1})_{,t} + u_2\cdot\nabla_{\tilde{u}_1}(u_1\cdot\bar{n}_{\tilde{u}_1}) - u_2\cdot u_2\cdot\nabla_{\tilde{u}_1}\bar{n}_{\tilde{n}_1}||_{2-1/p,p,p,S^T}.$$

We estimate particular terms in the above expression. At the beginning we consider the first term, so we have

$$(3.32)\qquad \left\|u_{1,t}\frac{A\cdot\nabla_\xi\Psi_0}{|A\cdot\nabla_\xi\Psi_0|}\right\|_{2-1/p,p,p,S^T} \leqslant ||u_{1,t}||_{2,p,p,\Omega_1^T} + |\nabla u_{1,t}\cdot\nabla\bar{n}_{u_1} + u_{1,t}\nabla^2\bar{n}_{u_1}|_{p,\Omega_1^T}$$
$$\leqslant [1+\Phi_2'(||u_1||_{2,p,1,\Omega_1^T}, ||\Psi_0||_{3-1/p,p,S})]||u_{1,t}||_{2,p,p,\Omega_1^T},$$

where $\bar{n}_{u_1}$ is defined in Ω_1, and

$$(3.33)\qquad \left\|u_1\left(\frac{A\cdot\nabla_\xi\Psi_0}{|A\cdot\nabla_\xi\Psi_0|}\right)_{,t}\right\|_{2-1/p,p,p,S^T} \leqslant \Phi_3'(||u_1||_{3,p,1,\Omega_1^T}, ||\Psi_0||_{3-1/p,p,S})$$
$$\times[||u_1u_{1,\xi}||_{0,\infty,p,\Omega_1^T} + |u_{1,\xi\xi\xi}u_1|_{p,\Omega_1^T} + |u_{1,\xi\xi}u_{1,\xi}|_{p,\Omega_1^T}].$$

By [3, Ch. 2] and [1, § 18] the terms in the square brackets are estimated in the following way:

$$||u_1u_{1,\xi}||_{0,\infty,p,\Omega_1^T} \leqslant ||u_1||_{0,\infty,pp_1,\Omega_1^T}||u_{1,\xi}||_{0,\infty,pp_1',\Omega_1^T} \leqslant c||u_1||_{2,1,p,q,\Omega_1^T}||u_{1,\xi}||_{2,1,p,q,\Omega_1^T}$$
$$\leqslant cT^{\frac{2(p-q)}{pq}}|||u_1|||^2_{a,\Omega_1^T}$$

where

$$\frac{1}{p}+\frac{1}{q}\leqslant 1,\quad p>3,\quad \frac{1}{p_1}+\frac{1}{p_1'}=1,\quad q<p.$$

$$|u_1u_{1,\xi\xi\xi}|_{p,\Omega_1^T} \leqslant |u_1|_{pp,\Omega_1^T}|u_{1,\xi\xi\xi}|_{pp_1',\Omega_1^T} \leqslant c||u_1||_{2,1,p,q,\Omega_1^T}\,||u_{1,\xi\xi}||_{2,1,p,q,\Omega_1^T}$$
$$\leqslant cT^{\frac{2(p-q)}{pq}}|||u_1|||^2_{a,\Omega_1^T},$$

where $\dfrac{1}{p}+\dfrac{4}{q}\leqslant 3$, $p>3$, $q<p$, $\dfrac{1}{p_1}+\dfrac{1}{p_2'}=1$. Finally

$$|u_{1,\xi}u_{1,\xi\xi}||_{p,\Omega_1^T} \leqslant |u_{1,\xi}|_{pp_1,\Omega_1^T}|u_{1,\xi\xi}|_{pp_1',\Omega_1^T} \leqslant c||u_{1,\xi}||_{2,1,p,q,\Omega_1^T}||u_{1,\xi}||_{2,1,p,q,\Omega_1^T}$$
$$\leqslant cT^{\frac{2(p-q)}{pq}}|||u_1|||^2_{a,\Omega_1^T},$$

where $\dfrac{1}{p}+\dfrac{1}{q}\leqslant 4$, $p>3$, $q<p$, $\dfrac{1}{p_1}+\dfrac{1}{p_1'}=1$.

Similarly we obtain

$$(3.34)\qquad ||-u_2\cdot(\bar{n}_{\tilde{u}_1})_t + u_2\cdot\nabla_{\tilde{u}_1}(u_1\cdot\bar{n}_{\tilde{u}_1})||_{2-1/p,p,p,S^T} \leqslant \Phi_4'(||u_1||_{4,p,1,\Omega_1^T}, ||\Psi_0||_{3-1/p,p,S})$$
$$\cdot T^{\alpha'}||u_2||_{2,p,\infty,\Omega_1^T}, |||u_1|||_{a,\Omega_1^T},$$

where $\alpha' > 0$ and

(3.35) $$||u_2 u_2 \nabla_{\tilde{u}_1} \bar{n}_{\tilde{u}_1}||_{2-1/p,p,p,S^T} \leqslant \Phi_2'(||u_1||_{4,p,1,\Omega_1^T}, ||\Psi_0||_{3-1/p,p,S}) T ||u_2||^2_{2,p,\infty,\Omega_1^T}.$$

Moreover, we have

$$||k_1||_{2-1/p,p,p,\Gamma^T} \leqslant \Phi_6'(||\tilde{u}_1||_{3,p,1,\Omega_2^T}, ||\Gamma||_{3-1/p,p}) ||\tilde{u}_1||_{3,p,1,\Omega_2^T} ||\nabla q_2||_{2,p,p,\Omega_2^T},$$

and

$$||k_2||_{2-1/p,p,p,\Gamma^T} \leqslant \Phi_7'(||\tilde{u}_1||_{3,p,1,\Omega_2^T}, ||\Gamma||_{3-1/p,p}) T ||u_2||^2_{2,p,\infty,\Omega_2^T}.$$

Finally we consider terms with g_2. Therefore we obtain

$$||\mathrm{div}\, g_2||_{1,p,1,\Omega_2^T} + ||g_2 \cdot \bar{n}_{\tilde{u}_1}||_{1,1/p,p,p,S^T} + ||g_2 \cdot {}_{\tilde{u}_1}||_{2-1/p,p,p,\Gamma^T} \leqslant cT ||f_2||_{2,\infty,\infty,\Omega^T} \Phi_8'(||\tilde{u}_1||_{3,p,1,\Omega_2^T}).$$

From the above estimates, Eq. (3.29) and for sufficiently small time T we obtain the estimate (3.30). To prove the existence of solutions of Eq. (3.28) we use the method of successive approximations such that in the left-hand sides of Eq. (3.28) we put $\overset{m}{q}_2$ and in the right-hand sides $\overset{m-1}{q_2}$ in the place of q_2. Assuming $\overset{0}{q}_2 = 0$, we show the existence in L $L_p(0, T; W_p^3(\Omega_2))$ for a sufficiently small T. This concludes the proof.

4. Existence

To prove the existence of solutions to the problems (1.12), (1.20), (1.21), we use the following method of successive approximations:

$$\begin{aligned} &\varrho_1 \overset{m+1}{u}_{1t} - \nu \nabla^2_{\underset{u_1}{m+1}} \overset{m+1}{u_1} + \nabla^2_{\underset{u_1}{m+1}} \overset{m+1}{q_1} = q_1, \quad \nabla_{\underset{u_1}{m+1}} \cdot \overset{m+1}{u_1} = 0 \quad \text{in} \quad \Omega_1^T, \\ (4.1) \qquad &\overset{m+1}{u_1}|_{t=0} = v_{10} \quad \text{in} \quad \Omega_1, \\ &T_{1\underset{u_1}{m+1}}(\overset{m+1}{u_1}, \overset{m+1}{q_1}) \bar{n}_{\underset{u_2}{m+1}}(\xi, t) = T_2(\overset{m}{q_2}) \bar{n}_{\underset{u_1}{m+1}}(\xi, t) \quad \text{on} \quad S^T, \end{aligned}$$

where $\overset{m}{q_2}$ is treated as a given function,

$$\begin{aligned} &\frac{1}{\varrho_2} \Delta_{\underset{u_1}{\tilde{m}}} \overset{m}{q_2} = -\nabla_{\underset{u_1}{\tilde{m}}} \overset{m}{u_2} \nabla_{\underset{u_1}{\tilde{m}}} \overset{m}{u_2} + \mathrm{div}\, g_2 \quad \text{in} \quad \Omega_2, \\ (4.2) \qquad &\frac{1}{\varrho_2} \bar{n}_{\underset{u_1}{\tilde{m}}} \cdot \nabla_{\underset{u_1}{\tilde{m}}} \overset{m}{q_2} = -(\overset{m}{u_1} \cdot \bar{n}_{\underset{u_1}{\tilde{m}}})_{,t} + \overset{m}{u_2} \cdot (\bar{n}_{\underset{u_1}{\tilde{m}}})_{,t} + \overset{m}{u_2} \cdot \nabla_{\underset{u_1}{\tilde{m}}} (\overset{\tilde{m}}{u_1} \cdot \bar{n}_{\underset{u_1}{\tilde{m}}}) + \overset{m}{u_2} \overset{m}{u_2} \cdot \nabla_{\underset{u_1}{\tilde{m}}} \bar{n}_{\underset{u_1}{\tilde{m}}} + g_2 \cdot \bar{n}_{\underset{u_1}{\tilde{m}}} \quad \text{on} \quad S, \\ &\frac{1}{\varrho_2} \bar{n}_{\underset{u_1}{\tilde{m}}} \cdot \nabla_{\underset{u_1}{\tilde{m}}} \overset{m}{q_2} = \overset{m}{u_2} \overset{m}{u_2} \cdot \nabla_{\underset{u_1}{\tilde{m}}} \bar{n}_{\underset{u_1}{\tilde{m}}} + g_2 \cdot \bar{n}_{\underset{u_1}{\tilde{m}}} \quad \text{on} \quad \Gamma, \end{aligned}$$

where $\overset{m}{u_2}, \overset{m}{u_1}, \overset{\tilde{m}}{u_1}$ are treated as given and $\overset{\tilde{m}}{u_1}$ is an extension of $\overset{m}{u_1}$ on Ω_2 such that

$$(4.3)\qquad |||\overset{\tilde m}{u}|||_{a,\Omega_2^T} \leqslant c|||\overset{m}{u_1}|||_{a,\Omega_1^T}.$$

Finally we consider

$$(4.4)\qquad \overset{m+1}{u_2}(\xi, t) = v_{20}\big(\overset{m}{y}(X_{\underset{u_1}{\tilde m}}(\xi, t), t; 0)$$
$$+\int_0^t\Big[-\frac{1}{\varrho_2}\nabla_{\underset{y}{m}}\overset{m}{p_2}\big(\overset{m}{y}(X_{\underset{u_1}{\tilde m}}(\xi, t), t; s), s\big)+f_2\big(\overset{m}{y}(X_{\underset{u_1}{\tilde m}}(\xi, t), t; s), s\big)\Big]ds,$$

where $\overset{m}{u_2}, \overset{\tilde m}{u_1}, \overset{m}{q_2}$ are assumed to be given, $\overset{m}{y}$ is a solution to the problem

$$(4.5)\qquad \frac{d\overset{m}{y}(x, t; s)}{ds} = \overset{m}{v_2}\big(\overset{m}{y}(x, t; s), s\big), \quad \overset{m}{y}(x, t; t) = x.$$

Moreover, we assume that $\overset{0}{u_1} = v_{10}(\xi, t)$, $\overset{0}{u_2} = v_{20}(\xi, t)$, and $m = 0, 1, \ldots$.

By Theorem 2.2 for T sufficiently small there exists a unique solution to the problem (4.1) such that

$$(4.6)\qquad \overset{m+1}{u_1} \in L_p\big(0, T; W_p^4(\Omega_1)\big)\cap L_p^1\big(0, T; W_p^2(\Omega_1)\big), \quad \overset{m+1}{q_1} \in L_p\big(0, T; W_p^3(\Omega_1)\big),$$
$$\sigma|||\overset{m+1}{u_1}|||_{a,\Omega_1^T}+||\overset{m+1}{q_1}||_{3,p,p,\Omega_1^T} \leqslant c(T)[||f_1||_{2,\infty,\infty,\mathrm{R}^3\times(0,T)}+||v_{10}||_{4-2/p,p,\Omega_1}+||\overset{m}{q_2}||_{3,p,p,\Omega_2^T}],$$

where $\sigma = \min\{\varrho_1, \mu\}$. Next, by Lemma 3.4 we have

$$(4.7)\qquad ||\overset{m+1}{u_2}||_{2,p,\infty,\Omega_2^T} \leqslant \Phi_1(T^{\frac{p-1}{p}}||\overset{m}{u_1}||_{2,p,p,\Omega_1^T}, T||\overset{m}{u_2}||_{2,p,\infty,\Omega_2^T})$$
$$\times\Big[||v_{20}||_{2,p,\Omega_2}+||f_2||_{2,\infty,\infty,\mathrm{R}^3\times(0,T)}+\frac{1}{\varrho_2}T^{\frac{p-1}{p}}||q_2||_{3,p,p,\Omega_2^T}\Big(1+T^{\frac{p-1}{p}}||\overset{m}{u_1}||_{3,p,p,\Omega_1^T}\Big)\Big].$$

Finally Lemma 3.5 implies

$$(4.8)\qquad \frac{1}{\varrho_2}||\overset{m}{q_2}||_{3,p,p,\Omega_2^T} \leqslant (1+\Phi_2(||u_1||_{2,p,1,\Omega_2^T}, ||\Psi_0||_{3,1/p,p,S}))|||\overset{m}{u_1}|||_{a,\Omega_1^T}$$
$$+\Phi_4\Big(T^{\frac{p-1}{p}}||\overset{m}{u_1}||_{4,p,p,\Omega_1^T}, ||\Psi_0||_{3-1/p,p,S}||\Gamma||_{3-1/p,p',T}\Big)[T^{\alpha}(|||\overset{m}{u_1}|||^2_{a,\Omega_1^T}+||\overset{m}{u_2}||^2_{2,p,\infty,\Omega_2^T})$$
$$+T||f_2||_{2,\infty,\infty,\mathrm{R}^3\times(0,T)}].$$

Introducing the quantities

$$(4.9)\qquad \begin{aligned}\overset{m}{x} &= |||\overset{m}{u_1}|||_{a,\Omega_1^T}+||\overset{m}{u_2}||_{2,p,\infty\,\Omega_2^T},\\ \alpha &= ||v_{10}||_{4-\frac{2}{p},p,\Omega_1}+||f_1||_{2,\infty,\infty,\mathrm{R}^3\times(0,T)},\\ \beta &= ||v_{20}||_{2,p,\Omega_2}+||f_2||_{2,\infty,\infty,\mathrm{R}^3\times(0,T)},\end{aligned}$$

from the relations (4.6), (4.7), (4.8) we obtain

$$\overset{m+1}{x} \leqslant c(T)\frac{1}{\sigma}(\alpha+||\overset{m}{q_2}||_{3,p,p,\Omega_2^T})+\Phi_5(T^{\varrho}\overset{m}{x})\Big[\beta+\frac{1}{\varrho_2}T^{\frac{p-1}{p}}||\overset{m}{q_2}||_{3,p,p,\Omega_2^T}\Big],$$

$$(4.10)\qquad \frac{1}{\varrho_2}\|\overset{m}{q}_2\|_{3,p,p,\Omega_2^T} \leqslant \Big[1+\Phi_6\Big(T^{\frac{p-1}{p}}\overset{m}{x}\Big)\Big]\overset{m}{x}+\Phi_7\Big(T^{\frac{p-1}{p}}\overset{m}{x}\Big)[T^\alpha\overset{m_2}{x}+T\beta],$$

where $\varrho > 0$, so

$$(4.11)\qquad \overset{m+1}{x} \leqslant c(T)\Big[\frac{1}{\sigma}(\alpha+\varrho_2(1+\Phi_6)\overset{m}{x})+\varrho_2\Phi_7(T^\alpha\overset{m_2}{x}+T\beta)\Big]+\Phi_5\Big[\beta$$
$$+\frac{1}{\varrho_2}T^{\frac{p-1}{p}}[(1+\Phi_6)\overset{m}{x}+\Phi_7(T^\alpha\overset{m_2}{x}+T\beta)]\Big] \equiv G\Big(\alpha,\beta,T,\frac{1}{\sigma},\overset{m}{x}\Big),$$

where the arguments of Φ_i, $i = 5, 6, 7$, are written in the relation (4.10). The function G is a positive continuous nondecreasing function of its arguments. We have that $G(\alpha, \beta, 0, 0, 0) = K_0 > 0$. Then there exist T_2 and σ_0 such that for $T \leqslant T_2$, $\frac{\sigma}{\varrho_2} \geqslant \sigma_0$ there exists $K > K_0$ such that

$$(4.12)\qquad G\Big(\alpha,\beta,T,\frac{\sigma}{\varrho_2},K\Big) \leqslant K.$$

Therefore for $T \leqslant T_2$, $\frac{\sigma}{\varrho_2} \geqslant \sigma_0$ we have

$$(4.13)\qquad \overset{m}{x} \leqslant K,\quad m = 0, 1, \ldots,$$

where $\overset{0}{x} = \|v_{10}\|_{4-2/p,p,\Omega_1}+\|v_{20}\|_{2,p,\Omega_2} \leqslant K$ must be satisfied, too.

Now we show that the sequence $\{\overset{m}{u}_1, \overset{m}{q}_1, \overset{m}{u}_2, \overset{m}{q}_2\}$ converges. For this purpose we consider the following system of problems:

$$\varrho_1\partial_t\overset{m+1}{U}_1-\nu\nabla^2_{\underset{u_1}{m+1}}\overset{m+1}{U}_1+\nabla_{\underset{u_1}{m+1}}\overset{m+1}{Q} = \nu(\nabla^2_{\underset{u_1}{m+1}}-\nabla^2_{\underset{u_1}{m}})\overset{m}{u}_1-(\nabla_{\underset{u_1}{m+1}}-\nabla_{\underset{u_1}{m}})\overset{m}{q}_1$$
$$+f_1(X_{\underset{u_1}{m+1}}(\xi,t),t)-f_1(X_{\underset{u_1}{m}}(\xi,t),t),$$

$$(4.14)\qquad \nabla_{\underset{u_1}{m+1}}\cdot\overset{m+1}{U}_1 = -(\nabla_{\underset{u_1}{m+1}}-\nabla_{\underset{u_1}{m}})\cdot\overset{m}{u}_1,$$

$$\overset{m+1}{U}|_{t=0} = 0,$$

$$\bar{\tau}_0\cdot D_{\underset{u_1}{m+1}}(\overset{m+1}{U}_1)\bar{n}_{\underset{u_1}{m+1}} = -\bar{\tau}_0\cdot D_{\underset{u_1}{m+1}}(\overset{m}{u}_1)\bar{n}_{\underset{u_1}{m+1}}+\bar{\tau}_0\cdot D_{\underset{u_1}{m}}(\overset{m}{u}_1)\cdot\bar{n}_{\underset{u_1}{m}},$$

$$\bar{n}_0\cdot T_{1\underset{u_1}{m+1}}(\overset{m+1}{U}_1,\overset{m+1}{Q}_1)\bar{n}_{\underset{u_1}{m+1}} = -\bar{n}_0 T_{1\underset{u_1}{m+1}}(\overset{m}{u}_1,\overset{m}{q}_1)\bar{n}_{\underset{u_1}{m+1}}+\bar{n}_0 T_{1\underset{u_1}{m}}(\overset{m}{u}_1,\overset{m}{q}_1)\bar{n}_{\underset{u_1}{m}}$$
$$+\bar{n}_0 T_2(\overset{m}{Q}_2)\cdot\bar{n}_{\underset{u_1}{m+1}}+\bar{n}_0 T_2(\overset{m-1}{q_2})(\bar{n}_{\underset{u_1}{m+1}}-\bar{n}_{\underset{u_1}{m}}),$$

where $\overset{m}{U}_i = \overset{m}{u}_i-\overset{m-1}{u}_i$, $\overset{m}{Q}_i = \overset{m}{q}_i-\overset{m-1}{q}_i$, $i = 1, 2$.

$$
\begin{aligned}
\frac{1}{\varrho_2}\underset{u_1}{\Delta_{\tilde m}}\overset{m}{Q}_2 = {} & -(\underset{u_1}{\Delta_{\tilde m}} - \underset{u_1}{\Delta_{\widetilde{m-1}}})\overset{m-1}{q}_2 - (\underset{u_1}{\nabla_{\tilde m}}\overset{m}{U}_2 \underset{u_1}{\nabla_{\tilde m}}\overset{m}{u}_2 + \underset{u_1}{\nabla_{\tilde m}}\overset{m-1}{u}_2 \underset{u_1}{\nabla_{\tilde m}}\overset{m}{U}_2) + \\
& -(\underset{u_1}{\nabla_{\tilde m}}\overset{m-1}{u}_2 \underset{u_1}{\nabla_{\tilde m}}\overset{m-1}{u}_2 - \underset{u_1}{\nabla_{\widetilde{m-1}}}\overset{m-1}{u}_2\underset{u_1}{\nabla_{\widetilde{m-1}}}\overset{m-1}{u}_2) + \underset{u_1}{\nabla_{\tilde m}}\cdot f_2\big(\underset{u_1}{X_{\tilde m}}(\xi,t),t\big) - \underset{u_1}{\nabla_{\widetilde{m-1}}} f_2\big(\underset{u_1}{X_{\widetilde{m-1}}}(\xi,t),t\big),
\end{aligned}
$$

$$
\begin{aligned}
(4.15)\quad \frac{1}{\varrho_2}\underset{u_1}{\bar n_{\tilde m}}\cdot\underset{u_1}{\nabla_{\tilde m}}\overset{m}{Q}_2 = {} & -(\underset{u_1}{\bar n_{\tilde m}}\cdot\underset{u_1}{\nabla_{\tilde m}} - \underset{u_1}{\bar n_{\widetilde{m-1}}}\cdot\underset{u_1}{\nabla_{\widetilde{m-1}}})\overset{m-1}{q}_2 - (\overset{m}{U}_1 \underset{u_1}{\bar n_{\tilde m}})_t - \big(\overset{m-1}{u}_1\cdot(\underset{n_1}{\bar n_{\tilde m}} - \underset{u_1}{\bar n_{\widetilde{m-1}}})\big)_t \\
& + [\overset{m}{u}_2\cdot(\underset{u_1}{\bar n_{\tilde m}})_t - \overset{m-1}{u}_2\cdot(\underset{u_1}{\bar n_{\widetilde{m-}}})_t] - \overset{m}{u}_2\cdot\underset{u_1}{\nabla_{\tilde m}}\big(\overset{\tilde m}{U}_1\underset{u_1}{\bar n_{\tilde m}} + \overset{\widetilde{m-1}}{u}_1(\underset{u_1}{\bar n_{\tilde m}} - \underset{u_1}{\bar n_{\widetilde{m-1}}})\big) \\
& + (\overset{m}{u}_2\cdot\underset{u_1}{\nabla_{\tilde m}} - \overset{m-1}{u}_2\cdot\underset{u_1}{\nabla_{\widetilde{m-1}}})\overset{\widetilde{m-1}}{u}_1 \underset{u_1}{\bar n_{\widetilde{m-1}}} + (\overset{m}{U}_2\overset{m}{u}_2 + \overset{m-1}{u}_2\overset{m}{U}_2)\cdot\underset{u_1}{\nabla_{\tilde m}}\underset{u_1}{\bar n_{\tilde m}} \\
& + \overset{m-1}{u}_2\overset{m-1}{u}_2\cdot(\underset{u_1}{\nabla_{\tilde m}}\underset{u_1}{\bar n_{\tilde m}} - \underset{u_1}{\nabla_{\widetilde{m-1}}}\underset{u_1}{\bar n_{\widetilde{m-1}}}) + f_2\big(\underset{u_1}{X_{\tilde m}}(\xi,t),t\big)\underset{u_1}{\bar n_{\tilde m}} - f_2\big(\underset{u_1}{X_{\widetilde{m-1}}}(\xi,t),t\big)\underset{u_1}{\bar n_{\widetilde{m-1}}},
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\varrho_2}\underset{u_1}{\bar n_{\tilde m}}\cdot\underset{u_1}{\nabla_{\tilde m}}\overset{m}{Q}_2 = {} & -(\underset{u_1}{\bar n_{\tilde m}}\cdot\underset{u_1}{\nabla_{\tilde m}} - \underset{u_1}{\bar n_{\widetilde{m-1}}}\cdot\underset{u_1}{\nabla_{\widetilde{m-1}}})\overset{m-1}{q}_2 + (\overset{m}{U}_2\overset{m}{u}_2 + \overset{m-1}{u}_2\overset{m}{U}_2)\cdot\underset{u_1}{\nabla_{\tilde m}}\underset{u_1}{\bar n_{\tilde m}} \\
& + \overset{m-1}{u}_2\overset{m-1}{u}_2(\underset{u_1}{\nabla_{\tilde m}}\underset{u_1}{\bar n_{\tilde m}} - \underset{u_1}{\nabla_{\widetilde{m-1}}}\underset{u_1}{\bar n_{\widetilde{m-1}}}) + \underset{u_1}{\bar n_{\tilde m}}\cdot f_2\big(\underset{u_1}{X_{\tilde m}}(\xi,t),t\big) - \underset{u_1}{\bar n_{\widetilde{m-1}}} f_2\big(\underset{u_1}{X_{\widetilde{m-1}}}(\xi,t),t\big).
\end{aligned}
$$

Finally from Eq. (4.4) we obtain

$$
\begin{aligned}
(4.16)\quad \overset{m+1}{U}_2(\xi,t) = {} & \nu_{20}\big(\overset{m}{y}(\underset{u_1}{X_{\tilde m}}(\xi,t),t;0)\big) - \nu_{20}\big(\overset{m-1}{y}(\underset{u_1}{X_{\widetilde{m-1}}}(\xi,t),t;0)\big) \\
& + \frac{1}{\varrho_2}\int_0^t \Big[-\nabla_{\underset{y}{m}}\overset{m}{p}_2\big(\overset{m}{y}(\underset{u_1}{X_{\tilde m}}(\xi,t),t;s),s\big) + \nabla_{\underset{y}{m-1}}\overset{m-1}{p}_2\big(\overset{m-1}{y}(\underset{u_1}{X_{\widetilde{m-1}}}(\xi,t),t;s),s\big)\Big]ds \\
& + \int_0^t \Big[f_2\big(\overset{m}{y}(\underset{u_1}{X_{\tilde m}}(\xi,t),t;s),s\big) - f_2\big(\overset{m-1}{y}(\underset{u_1}{X_{\widetilde{m-1}}}(\xi,t),t;s),s\big)\Big]ds.
\end{aligned}
$$

Moreover, $\overset{m}{Y}(x,t;s)$ is a solution to the problem

$$
(4.17)\quad \frac{d}{ds}\overset{m}{Y}(x,t;s) = \overset{m}{V}_2\big(\overset{m}{y}(x,t;s) + \nabla\tilde y_m \overset{m-1}{\nu_2}(\tilde y_m(x,t;s),s)\overset{m}{Y}(x,t;s),\quad \overset{m}{Y}(x,t;t) = 0,
$$

where

$$
\begin{aligned}
\overset{m}{Y}(x,t;s) = \overset{m}{y}(x,t;s) - \overset{m-1}{y}(x,t;s),\quad & \overset{m}{V}_2\big(y(x,t:s),s\big) = \overset{m}{\nu}_2\big(y(x,t;s),s\big) \\
& - \overset{m-1}{\nu_2}\big(y(x,t;s),s\big),\ \tilde y_m(x,t;s) \in [\overset{m}{y}(x,t;s), \overset{m-1}{y}(x,t;s)],
\end{aligned}
$$

$[\alpha,\beta]$ is a segment between α and β.

At first we find an estimate for solutions to the problem (4.14). Using Theorem 2.2 gives

$$
(4.18)\quad \sigma|||\overset{m+1}{U}_1|||_{b,\Omega_1^T} + ||\overset{m+1}{Q}_1||_{2,p,p,\Omega_1^T} \leqslant \varphi_1(||\overset{m+1}{u}_1||_{3,p,p,\Omega_1^T}, ||\overset{m}{u}_1|_{3,p,p,\Omega_1^T})\{||\overset{m+1}{U}_1||_{3,p,1,\Omega_1^T}
$$

(4.18) [cont.]
$$\cdot\,[\|\overset{m}{u}_1\|_{3,p,p,\Omega_1^T}+\|\overset{m}{q}_1\|_{2,p,p,\Omega_1^T}+\|f_1\|_{L\infty(0,T;\,\mathrm{Lip}^1(R^3))}] + \|\overset{m}{Q}_2\|_{2,p,p,\Omega_2^T}+\|\overset{m-1}{q_2}\|_{2,p,p,\Omega_2^T}\|\overset{m+1}{U_1}\|_{2,p,1,\Omega_1^T}\},$$

where we made use of the fact that $f_{1,x}$ is Lipschitz continuous with respect to $x(f_1 \in \in \mathrm{Lip}^1(\mathrm{R}^3))$. From the relation (4.18) for sufficiently small T, one obtains

$$\sigma|||\overset{m+1}{U_1}|||_{b,\Omega_1^T}+\|\overset{m+1}{Q_1}\|_{2,p,p,\Omega_1^T} \leqslant \varphi_2(\|\overset{m+1}{u_1}\|_{3,p,p,\Omega_1^T},\|\overset{m}{u}_1\|_{3,p,p,\Omega_1^T})\cdot\|\overset{m}{Q}_2\|_{2,p,p,\Omega_2^T}. \tag{4.19}$$

Assuming that $f_{2,x}$ is Lipschitz continuous with respect to $x(f_2 \in \mathrm{Lip}^1(\mathrm{R}^3))$, we obtain from Eq. (4.15)

$$\frac{1}{\varrho_2}\|\overset{m}{Q}_2\|_{2,p,p,\Omega_2^T} \leqslant \varphi_3(K)T^a(|||\overset{m}{U}_1|||_{b,\Omega_1^T}+\|\overset{m}{U}_2\|_{1,p,\infty,\Omega_2^T})+\varphi_4(K)|||\overset{m}{U}_1|||_{b,\Omega_1^T}, \tag{4.20}$$

where the function φ_3 depends also on T and $\|f_2\|_{L\infty(0,T;\,\mathrm{Lip}^1(R^3))}$. Moreover, $a > 0$. To estimate Eq. (4.16), we write it in the following form:

$$\begin{aligned}\overset{m+1}{U}_2(\xi,t) &= v_{20,\tilde{y}_m(m)}\underset{u_1}{\overset{m}{Y}}(X_{\tilde{m}})+v_{20,\tilde{y}^1_{m-1}}(m)\overset{m-1}{Y}_{,\tilde{x}_m}\int_0^t \overset{\tilde{m}}{U}_1(\xi;\tau)d\tau\\
&-\int_0^t \frac{1}{\varrho_2}\Big[\nabla^2_{\tilde{y}_m(m)}\overset{m}{p}_2(\tilde{y}_m(m))\overset{m}{Y}(m)+\nabla^2_{\tilde{y}^1_{m-1}(m)}\overset{m}{p}_2(\tilde{y}^1_{m-1}(m))\tilde{y}_{m-1}(m),_{\tilde{x}m}\int_0^s\overset{\tilde{m}}{U}_1(\xi,\tau)d\tau\\
&\qquad+\nabla^{m-1}_{y\,(m-1)}\overset{m}{P}_2(\overset{m-1}{y}(m-1))\Big]ds\\
&+\int_0^t\Big[f_2(\overset{m}{y}(\underset{u_1}{X_{\tilde{m}}}(\xi,t),t;s),s)-f_2(\overset{m-1}{y}(\underset{u_1}{X_{\tilde{m}}}(\xi,t),t;s),s)\Big]ds,\end{aligned} \tag{4.21}$$

where

$$\tilde{y}_m(m)=\tilde{y}_m(\underset{u_1}{X_{\tilde{m}}}(\xi,t),t;s),\quad \tilde{y}_m(x,t;s)\in[\overset{m}{y}(x,t;s),\overset{m-1}{y}(x,t;s)],$$

$$\tilde{X}_m\in[\underset{u_1}{X_{\tilde{m}}}(\xi,t),X_{\tilde{m}-1}(\xi,t)],\quad \overset{m}{y}(m)=\overset{m}{y}(\underset{u_1}{X_{\tilde{m}}}(\xi,t),t;s),\quad \overset{m}{Y}(m)=\overset{m}{y}(m)-\overset{m-1}{y}(m),$$

$$\tilde{y}^1_{m-1}(m)\in[\overset{m-1}{y}(\underset{u_1}{X_{\tilde{m}}}(\xi,t),t;s),\quad \overset{m-1}{y}(\underset{u_1}{X_{\tilde{m}-1}}(\xi,t),t;s)].$$

From Eq. (4.21) one has

$$\begin{aligned}\|\overset{m+1}{U_2}\|_{1,p,\infty,\Omega_2^T} &\leqslant c(K)(\|v_{20}\|_{2,p,\Omega_2}+\frac{1}{\varrho_2}\|\overset{m}{q}_2\|_{3,p,1,\Omega_2^T})(\|\overset{m}{Y}\|_{1,p,\infty,\Omega_2^T}+\|\overset{\tilde{m}}{U}_1\|_{1,p,1,\Omega_2^T})\\
&+\frac{1}{\varrho_2}c(K)\|\overset{m}{Q}_2\|_{2,p,1,\Omega_2^T}+\|f_2\|_{L\infty(0\,T;\mathrm{Lip}^1(R^3))}(\|\overset{m}{Y}\|_{1,p,\infty,\Omega_2^T}+\|\overset{\tilde{m}}{U}_1\|_{1,p,1,\Omega_2^T})\end{aligned} \tag{4.22}$$

and from Eq. (4.17) one obtains

$$\|\overset{m}{Y}\|_{1,p,\infty,\Omega_2^T} \leqslant c(K)\|\overset{m}{U}_2\|_{1,p,1,\Omega_2^T}. \tag{4.23}$$

Finally, from Eqs. (4.18), (4.20), (4.22) and (4.23) it follows that the sequence $\{\overset{m}{u}_1, \overset{m}{u}_2, \overset{m}{q}_1, \overset{m}{q}_2\}$ converges for sufficiently small T and for sufficiently large $\frac{\sigma}{\varrho_2}$ ($\sigma = \min\{\varrho_1, \nu\}$) to a limit $\{u_1, u_2, q_1, q_2\}$ which is a solution of the problem (1.12), (1.20), (1.21). We pass to the limit easily because Eqs. (1.12) and (1.21) are satisfied classically.

By the standard arguments we prove uniqueness for sufficiently small T and large $\frac{\sigma}{\varrho_2}$.

Therefore we have proved

THEOREM 4.1. *Assume that* $v_{10} \in W_p^{4-2/p}(\Omega_1)$, $v_{20} \in W_p^2(\Omega_1)$, f_1, $f_2 \in L_\infty(0, T; W_\infty^2(R^3))$, $S \in W_p^{3-1/p}$, $\Gamma \in W_p^{3-1/p}$, $p > 3$. *Moreover, the following compatibility conditions* $\operatorname{div} v_{i0} = 0$, $i = 1, 2$, $v_{10} \cdot \bar{n}_0|_S = v_{20} \cdot \bar{n}_0|_S$ *and* $v_{20} \cdot \bar{n}_0|_\Gamma = 0$, *where* $\bar{n}_0|_S$ *is the outward vector to* Ω_1 *and normal to* S, $\bar{n}_0|_\Gamma$ *is the unit outward vector normal to* Γ, *hold.*

Then, for sufficiently small T and large $\frac{\sigma}{\varrho_2}$, $\sigma = \min\{\varrho_1, \nu\}$ there exists a unique solution of the problems (1.1), (1.2), (1.3) such that $v_1 \in L_p(0, T; W_p^4(\Omega_1)) \cap L_{pt}^1(0, T; W_p^2(\Omega_1))$, $p_1 \in L_p(0, T; W_p^3(\Omega_1))$, $p_1|_{\tilde{S}^T} \in W_p^{1-1/p, 1/2-1/2p}(\tilde{S}^T)$, $v_2 \in L_\infty(0, T; W_p^2(\Omega_2))$, $p_2 \in L_p(0, T; W_p^3(\Omega_{2t}))$, $t \leqslant T$.

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